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# ON LOCAL LINEARIZATION OF CONTROL SYSTEMS

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ABSTRACT. We consider the problem of topological linearization of smooth ( $C^\infty$  or  $C^\omega$ ) control systems, i.e. of their local equivalence to a linear controllable system *via* pointwise transformations on the state and the control that are topological but not necessarily differentiable. In the language of control theory, such transformations reduce to (topological) static feedback transformations. We prove that topological linearization implies smooth linearization, away from singularities. At “regular” singular points, we show that topological linearizability implies local conjugation to a linear system *via* a smooth mapping, although this time the inverse map needs not be smooth. Finally, deciding whether the same is true at “strongly” singular points is tantamount to solve an intriguing open question in differential topology.

Our results entail that control systems sharply differ from ordinary differential equations, in that a naive analog to the Grobman-Hartman theorem (i.e. topological linearizability at generic fixed points) cannot hold.

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## 1. INTRODUCTION

In the theory of dynamical systems, it is well-known that, except in some “degenerate” cases, the flow of a nonlinear differential equation near an equilibrium point “looks like” the flow of its tangent approximation. More precisely, the Grobman-Hartman theorem says that these flows are conjugate around a hyperbolic equilibrium *via* a local homeomorphism. Consequently it is impossible, *locally around an equilibrium*, to distinguish a hyperbolic nonlinear system from a linear one on the basis of *qualitative* phenomena. The conjugating homeomorphism may usually be endowed with additional regularity, but cannot have in general the same degree of smoothness as the ODE (*e.g.* a real analytic change of coordinates if the equation is real analytic). This delicate issue relies upon a refined analysis of resonances and small divisors, and has generated a sizable literature, see section 2 for a small sample.

For control systems, it has been known for some time [12, 9] that smooth<sup>1</sup> linearizability, namely local equivalence to a linear control system by means of a *diffeomorphic* change of variables on the state and the control (also known as *static feedback linearization*), requires highly non-generic conditions. However, in view of the Grobman-Hartman theorem for ODE’s (*i.e.* non-controlled systems), it is natural to ask whether there may exist, in the neighborhood of an equilibrium, a topological although perhaps non-differentiable transformation that conjugates the input-to-state behavior of a “generic” control system to a linear controllable one. If yes, it would mean that a non-linear control system, under appropriate non-degeneracy conditions, cannot be distinguished qualitatively from its linear approximation, at least locally around an equilibrium point. Also, it would imply that systems with linear dynamics, whose nonlinear character lies in their input-to-state and state-to-output transitions only, are generic models for control systems in the neighborhood of an equilibrium.

The present paper provides an essentially negative answer to this question, by showing (*cf.* Theorem 5.2) that a smooth control system which is locally topologically linearizable at some point (that may or may not be an equilibrium) is in fact locally “quasi-smoothly” linearizable. The notion of quasi-smooth linearization is technical, *cf.* Definitions 3.8 and 5.1, but it coincides with smooth linearization on some open dense subset of so-called “regular” points. Singular points themselves fall into two classes. At “regular” singular points, we will show that quasi-smooth linearizability implies the existence of a linearizing homeomorphism which is smooth (although not necessarily a diffeomorphism). Finally, deciding whether this is still true at “strongly” singular points remains an open issue that raises an interesting question in differential topology (*cf.* section 5.1). The proof of Theorem 5.2 relies on classical results concerning orbits of families of smooth vector fields, first given in [23], that we recall and slightly expand in Appendix B.

From this theorem it follows that, unlike ordinary differential equations, control systems will essentially be smoothly linearizable as soon as they are topologically linearizable, that is to say there are no subtle questions involving resonances or small divisors. Thus, one could say in this context that the most remarkable feature of a control system is to be, or not to be linearizable, regardless of smoothness. It is natural to ask whether the results of the present paper remain true if smooth is to mean “of class  $C^k$ ” for some finite  $k > 1$ . We do not know the answer to that question, but our method of proof that relies heavily on [23] runs into difficulties if the vector fields are not infinitely differentiable.

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<sup>1</sup>Hereafter, the word “smooth” means either  $C^\infty$  or  $C^\omega$  (*i.e.* real analytic).

The organization of the paper is as follows. Section 2 briefly recalls classical facts on the local linearization of ordinary differential equations. Section 3 defines conjugation for *control systems* (under a homeomorphism, a diffeomorphism, etc...) and states some basic properties of conjugating maps. Section 4 reviews (topological, smooth, linear) conjugacy between *linear* control systems after [4, 27]. Section 5 contains the main result of the paper (Theorem 5.2, proved in section 6.2), namely that local topological linearizability implies local quasi-smooth linearizability for smooth control systems (smooth meaning either  $C^\infty$  or  $C^\omega$ ), and a discussion on the gap between smooth and quasi-smooth in this result, including geometric characterizations of local smooth and quasi-smooth linearizability (proved in section 6.1) as well as the above mentioned open question. The proof of Theorem 5.2 relies upon section 3 and results from [23] stated in Appendix B plus some technical lemmas from Appendix A.

## 2. LOCAL LINEARIZATION FOR ORDINARY DIFFERENTIAL EQUATIONS

Consider the differential equation

$$\dot{x}(t) = f(x(t)), \quad (1)$$

where  $f \in C^k(U, \mathbb{R}^n)$  with  $U$  an open subset of  $\mathbb{R}^n$  and  $k \in \mathbb{N} \cup \{\infty, \omega\}$ ,  $k \geq 1$ .

It is well known (the “flow box theorem”, see *e.g.* [2]) that, around each  $x_0 \in U$  such that  $f(x_0) \neq 0$ , there is a change of coordinates of class  $C^k$  that conjugates (1) to the equation  $\dot{x}_1 = 1, \dot{x}_2 = 0, \dots, \dot{x}_n = 0$ . Hence all differentiable vector fields are equivalent to each other, at points where they do not vanish, *via* a diffeomorphism having the same degree of smoothness (including real analyticity).

At a point  $x_0 \in U$  such that  $f(x_0) = 0$ , *i.e.* at an equilibrium of the dynamical system (1), its linear approximation is the system

$$\dot{x}(t) = Ax(t) - Ax_0 \quad (2)$$

where  $A = Df(x_0)$  is the derivative of  $f$  at  $x_0$ . The equilibrium  $x_0$  is said to be *hyperbolic* if the matrix  $A$  has no purely imaginary eigenvalue.

The problem of locally linearizing (1) is that of finding a local homeomorphism  $h : V \rightarrow W$  around  $x_0$  mapping the trajectories of (1) in  $V$  onto trajectories of (2) in  $W$  in a time-preserving manner. In other words, if  $\phi_t$  denotes the flow of (1), we should have for each  $x \in V$  that

$$h \circ \phi_t(x) = e^{At}(h(x) - h(x_0)) + h(x_0)$$

provided that  $\phi_\rho(x) \in V$  for  $0 \leq \rho \leq t$ . When this is the case we say that  $h$  conjugates (1) and (2), and we speak of topological, smooth, or analytic linearization depending on the smoothness of  $h$  and  $h^{-1}$ .

Local linearization at an equilibrium is a very old issue. At the beginning of the twentieth century, H. Poincaré already identified the obstructions to the existence of a *formal* change of variables  $h$  that removes all the nonlinear terms when  $f$  is analytic. These are the so-called resonances, see *e.g.* [8, 2]. In fact, resonant monomials of order  $\ell$  are obstructions to linearizing the Taylor expansion of  $f$  at order  $\ell$  and consequently also obstructions to  $C^\ell$  linearization. However, although there exists a formal power series expansion for  $h$  when there are no resonant terms, the existence of a *convergent* power series for  $h$  (analytic linearization) is a delicate issue. When the eigenvalues of the Jacobian belong to the so-called Poincaré domain, the absence of resonances indeed implies analytic linearizability (the Poincaré theorem). If it is not the case, a famous theorem by Siegel gives additional Diophantine conditions on these eigenvalues to the same conclusion. These conditions are generically satisfied in the measure-theoretic sense [2]. If no eigenvalue of the Jacobian is purely imaginary, it turns out [19] that the absence

of resonances is also sufficient for smooth ( $h, h^{-1}$  of class  $C^\infty$ ) but in general not real analytic linearization. This is still valid when  $f$  is merely of class  $C^\infty$ .

In contrast, if one allows conjugation *via* a topological but not necessarily differentiable homeomorphism, the Grobman-Hartman theorem asserts that every ordinary differential equation with no purely imaginary eigenvalue of the Jacobian (hyperbolicity) can be locally linearized around an equilibrium, that is, resonances are no longer an obstruction. A proof of this classical result can be found in [8]:

**Theorem 2.1** (Grobman-Hartman). *Under the assumption that  $x_0$  is a hyperbolic equilibrium point, system (1) is topologically conjugate to system (2) at  $x_0$ .*

In fact, it is proved in [26] that the conjugating homeomorphism  $h$  (together with its inverse  $h^{-1}$ ) can be chosen Hölder-continuous, and even differentiable at  $x_0$  (but not in a neighborhood). This brings additional rigidity to the mapping  $h$ .

The above theorem entails that the only invariant under local topological conjugacy, around a hyperbolic equilibrium, is the number of eigenvalues with positive real part in the Jacobian matrix, counting multiplicity. Indeed, as is well-known (*cf.* [1]), the linear system  $\dot{x} = Ax$  where  $A$  has no pure imaginary eigenvalue is topologically conjugate to  $\dot{x} = DX$ , where  $D$  is diagonal with diagonal entries  $\pm 1$ , the number of  $+1$  being the number of eigenvalues of  $A$  with positive real part.

### 3. PRELIMINARIES ON TOPOLOGICAL EQUIVALENCE FOR CONTROL SYSTEMS

**3.1. Control systems and their solutions.** Consider now two *control systems* ( $n, m, n', m'$  are integers) :

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (3)$$

$$\dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, \quad v \in \mathbb{R}^{m'}, \quad (4)$$

or expanded in coordinates :

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_n, u_1, \dots, u_m) & \dot{z}_1 &= g_1(z_1, \dots, z_{n'}, v_1, \dots, v_{m'}) \\ \vdots & & \vdots & \\ \dot{x}_n &= f_n(x_1, \dots, x_n, u_1, \dots, u_m) & \dot{z}_{n'} &= g_{n'}(z_1, \dots, z_{n'}, v_1, \dots, v_{m'}) \end{aligned}$$

where  $x$  (resp.  $z$ ) is called the *state* and  $u$  (resp.  $v$ ) the *control*. If  $m$  is zero or  $f$  does not depend on  $u$ , equation (3) reduces to the ordinary differential equation (1) which is often called a *non-controlled system* in this context. Of course “genuine” control systems are those whose right hand side does depend on the control. The standing requirements on  $f$  and  $g$  are the following.

**Smoothness:** henceforth we assume that  $f$  and  $g$  are at least continuous; *any additional regularity assumption will be stated explicitly*. In the present section devoted to general background on topological equivalence, we keep such assumptions to a minimum. Our main results, stated in section 5, are however obtained under the hypothesis that  $f$  and  $g$  are either infinitely differentiable or real analytic.

**Domain of definition:** at most places, in particular throughout the present section, we assume that  $f$  and  $g$  are defined on all of  $\mathbb{R}^n \times \mathbb{R}^m$  and  $\mathbb{R}^{n'} \times \mathbb{R}^{m'}$  respectively. This is no real loss of generality to us because all our results are local with respect to  $x, u, z$ , and  $v$ , so that  $f$  and  $g$  can be extended using partitions of unity outside some neighborhood of the arguments under consideration without affecting the conclusions. This extra elbow-room is used at several places to avoid dealing with non-complete vector fields. Of course the extension need not be analytic with  $f$  and  $g$ .

**Definition 3.1.** By a *solution of (3) that remains in an open set*  $\Omega \subset \mathbb{R}^{n+m}$ , we mean a mapping  $\gamma$  defined on a real interval  $I$ , say

$$\begin{aligned} \gamma : I &\rightarrow \Omega \\ t &\mapsto \gamma(t) = (\gamma_{\text{I}}(t), \gamma_{\text{II}}(t)) \end{aligned} \quad (5)$$

with  $\gamma_{\text{I}}(t) \in \mathbb{R}^n$  and  $\gamma_{\text{II}}(t) \in \mathbb{R}^m$ , such that :

- $\gamma$  is measurable, locally bounded, and  $\gamma_{\text{I}}$  is absolutely continuous,
- whenever  $[T_1, T_2] \subset I$ , we have :

$$\gamma_{\text{I}}(T_2) - \gamma_{\text{I}}(T_1) = \int_{T_1}^{T_2} f(\gamma_{\text{I}}(t), \gamma_{\text{II}}(t)) dt. \quad (6)$$

Solutions of (4) that remain in  $\Omega' \subset \mathbb{R}^{n'+m'}$  are likewise defined to be mappings

$$\begin{aligned} \gamma' : I &\rightarrow \Omega' \\ t &\mapsto \gamma'(t) = (\gamma'_{\text{I}}(t), \gamma'_{\text{II}}(t)) \end{aligned} \quad (7)$$

having the corresponding properties with respect to  $g$ .

If  $(\bar{x}, \bar{u})$  is a point in  $\Omega$ ,  $\mathcal{U}$  a neighborhood of  $\bar{u}$  such that  $\{\bar{x}\} \times \mathcal{U} \subset \Omega$ ,  $J$  a real interval, and  $\gamma_{\text{II}} : J \rightarrow \mathcal{U}$  a measurable and locally bounded map, then, by [6, Ch. 2, Theorem 1.1] and the continuity of  $f$ , there exists, on a possibly smaller interval  $I \subset J$ , a solution  $\gamma$  of (3) that remains in  $\Omega$  subject to the initial condition  $\gamma_{\text{I}}(0) = \bar{x}$ . This solution may not be unique without further assumptions on  $f$ , for instance that it is smooth or merely locally Lipschitz in the first argument. This non-uniqueness will, at times, be a difficulty, but we cannot avoid it even if we suppose that  $f$  is smooth because we want to consider control systems obtained on applying to (3) a transformation that is merely continuous.

*Remark 3.2.* Observe that Definition 3.1 assigns a definite value to  $\gamma_{\text{II}}(t)$  for *each*  $t \in I$ . Of course, since  $\gamma_{\text{I}}$  remains a solution to (6) when the control  $\gamma_{\text{II}}$  gets redefined over a set of measure 0, one could identify two control functions whose values agree a.e. on  $I$ , as is customary in integration theory. However, these values are in any case subject to the constraint that  $\gamma(t) \in \Omega$  for *every*  $t \in I$ , and altogether we find it more convenient to adopt Definition 3.1.

**3.2. Feedbacks.** In the terminology of control, a solution in the sense of Definition 3.1 would be termed *open loop* to emphasize that the value of the control at time  $t$  is a function of time only, namely that  $\gamma_{\text{II}}(t)$  bears no relation to the state  $x$  whatsoever. A central concept in control theory, though, is that of *closed loop* or *feedback* control, where the value of the control at time  $t$  is computed from the corresponding value of the state, namely is of the form  $\alpha(x(t))$ . To make a formal definition of a feedback defined on an arbitrary open set, we need one more piece of notation : if  $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$  is open, we let  $\pi_n : \Omega \rightarrow \Omega_{\mathbb{R}^n}$  the natural projection that selects the first  $n$  components, where  $\Omega_{\mathbb{R}^n} = \pi_n(\Omega) \subset \mathbb{R}^n$ .

**Definition 3.3.** Given an open set  $\Omega \subset \mathbb{R}^{n+m}$ , a *feedback* on  $\Omega$  is a continuous mapping  $\alpha : \Omega_{\mathbb{R}^n} \rightarrow \mathbb{R}^m$  such that  $(x, \alpha(x)) \in \Omega$  for all  $x \in \Omega_{\mathbb{R}^n}$ . A *smooth* feedback on  $\Omega$  is one which is a smooth mapping.

A feedback is nothing but a mapping  $\alpha$  such that  $x \mapsto (x, \alpha(x))$  is a continuous section of the natural fibration  $\pi_n : \Omega \rightarrow \Omega_{\mathbb{R}^n}$ . Of course, there are sets  $\Omega$  whose topology prevents the existence of any feedback. However, if there is one there are plenty, among which smooth feedbacks are uniformly dense. This is the content of the next proposition, that will be used in the proof of Theorem 5.2. To fix notations, let us agree throughout that the symbol  $\| \cdot \|$  designates the Euclidean norm on  $\mathbb{R}^\ell$  irrespectively of the positive integer  $\ell$ , while  $B(x, r)$  stands for the open ball centered at  $x$  of radius  $r$  and  $\bar{B}(x, r)$  for the corresponding closed ball.

**Proposition 3.4.** *Let  $\Omega$  be open in  $\mathbb{R}^{n+m}$ , and  $\alpha : \Omega_{\mathbb{R}^n} \rightarrow \mathbb{R}^m$  be a feedback on  $\Omega$ . To each  $\varepsilon > 0$ , there is a smooth feedback  $\beta : \Omega_{\mathbb{R}^n} \rightarrow \mathbb{R}^m$  such that  $\|\alpha(x) - \beta(x)\| < \varepsilon$  for  $x \in \Omega_{\mathbb{R}^n}$ .*

*Proof.* Let  $\emptyset = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \dots \subset \mathcal{K}_k \subset \mathcal{K}_{k+1} \dots$  be an increasing sequence of compact subsets of  $\Omega_{\mathbb{R}^n}$ , each of which contains the previous one in its interior, and whose union is all of  $\Omega_{\mathbb{R}^n}$ . For each  $x \in \Omega_{\mathbb{R}^n}$ , define an integer

$$k(x) \triangleq \min\{k \in \mathbb{N}; x \in \mathcal{K}_k\}. \quad (8)$$

To each  $k$ , by the continuity of  $\alpha$  and the compactness of  $\mathcal{K}_k$ , there is  $\mu_k > 0$  such that

$$x \in \mathcal{K}_k \Rightarrow \begin{cases} \bullet B(x, \mu_k) \times \text{Conv}\{\alpha(B(x, \mu_k))\} \subset \Omega, \\ \bullet \forall u_1, u_2 \in \text{Conv}\{\alpha(B(x, \mu_k))\}, \|u_1 - u_2\| < \varepsilon, \end{cases} \quad (9)$$

where the symbol Conv designates the convex hull. In addition, we may assume that the sequence  $(\mu_k)$  is non increasing.

Denote by  $\overset{\circ}{\mathcal{K}}_k$  the interior of  $\mathcal{K}_k$ , set  $\mathcal{D}_k = \mathcal{K}_k \setminus \overset{\circ}{\mathcal{K}}_{k-1}$  for  $k \geq 1$ , and cover the compact set  $\mathcal{D}_k$  with a finite collection  $\mathcal{B}_k$  of open balls having the following properties :

- each of these balls is centered at a point of  $\mathcal{D}_k$  and is contained in the open set  $\overset{\circ}{\mathcal{K}}_{k+1} \setminus \overset{\circ}{\mathcal{K}}_{k-2}$  (with the convention that  $\mathcal{K}_{-1} = \emptyset$ ),
- each of these balls has radius at most  $\frac{\mu_{k+1}}{2}$ .

The union  $\mathcal{B} = \bigcup_{k \geq 1} \mathcal{B}_k$  is a countable locally finite collection of open balls that covers  $\Omega_{\mathbb{R}^n}$ , and it has the property that *every ball in  $\mathcal{B}$  is included in  $B(x, \mu_{k(x)})$  as soon as it contains  $x$* . Let  $B_j$ , for  $j \in \mathbb{N}$ , enumerate  $\mathcal{B}$ , and  $h_j$  be a smooth partition of unity where  $h_j$  has support  $\text{supp} h_j \subset B_j$ . If we pick  $x_j \in B_j$  for each  $j$ , the map  $\beta : \Omega_{\mathbb{R}^n} \rightarrow \mathbb{R}^m$  defined by

$$\beta(x) = \sum_{j \in \mathbb{N}} h_j(x) \alpha(x_j) \quad (10)$$

is certainly smooth. In addition, since by construction  $x_j$  belongs to  $B(x, \mu_{k(x)})$  whenever  $h_j(x) \neq 0$ , we get that  $\beta(x)$  lies in the convex hull of  $\alpha(B(x, r))$  for some  $r < \mu_{k(x)}$ , and therefore, from (9) and (8), that  $(x, \beta(x)) \in \Omega$  and  $\|\alpha(x) - \beta(x)\| < \varepsilon$ . Hence  $\beta$  is a smooth feedback on  $\Omega$  such that  $\|\alpha(x) - \beta(x)\| < \varepsilon$  for all  $x \in \Omega_{\mathbb{R}^n}$ .  $\square$

**3.3. Conjugacy.** We now turn to the notion of conjugacy for control systems, which is the central topic of the paper. The most general notion is the following.

**Definition 3.5.** Let

$$\begin{aligned} \chi : \quad \Omega &\rightarrow \Omega' \\ (x, u) &\mapsto \chi(x, u) = (\chi_{\text{I}}(x, u), \chi_{\text{II}}(x, u)) \end{aligned} \quad (11)$$

be a bijective mapping between two open subsets of  $\mathbb{R}^{n+m}$  and  $\mathbb{R}^{n'+m'}$  respectively. We say that  $\chi$  *conjugates* systems (3) and (4) if, for any real interval  $I$ , a map  $\gamma : I \rightarrow \Omega$  is a solution of (3) that remains in  $\Omega$  if, and only if,  $\chi \circ \gamma$  is a solution of (4) that remains in  $\Omega'$ .

Although this definition makes sense without any regularity assumption, we only consider the case when  $\chi$  and  $\chi^{-1}$  are at least continuous. Then Brouwer's invariance of the domain (see *e.g.* [16]) implies that  $n' + m' = n + m$  if (3) and (4) are conjugate via such a  $\chi$ . Proposition 3.6 below asserts that more in fact is true.

**Proposition 3.6.** *If the map  $\chi$  in (11) is a homeomorphism that conjugates (3) to (4), then  $n = n'$ ,  $m = m'$ , and  $\chi_I$  depends only on  $x$ :*

$$\chi(x, u) = (\chi_I(x), \chi_{II}(x, u)). \quad (12)$$

Moreover,  $\chi_I : \Omega_{\mathbb{R}^n} \rightarrow \Omega'_{\mathbb{R}^n}$  is a homeomorphism. Here, one should recall the notation  $\Omega_{\mathbb{R}^n}$  that was introduced before Definition 3.3.

*Proof.* Let  $\bar{x}$ ,  $\bar{u}$ ,  $\bar{u}'$  be such that  $(\bar{x}, \bar{u})$  and  $(\bar{x}, \bar{u}')$  belong to  $\Omega$ . Let further  $x(t)$  be a solution<sup>2</sup> to (3) with  $x(0) = \bar{x}$  and

$$\begin{aligned} u(t) &= \bar{u} \text{ if } t \leq 0, \\ u(t) &= \bar{u}' \text{ if } t > 0. \end{aligned}$$

By conjugacy,  $z(t) = \chi_I(x(t), u(t))$  is a solution to (4) with  $v$  given by  $v(t) = \chi_{II}(x(t), u(t))$ , for  $t \in (-\epsilon, \epsilon)$  and some  $\epsilon > 0$ . In particular  $\chi_I(x(t), u(t))$  is continuous in  $t$  so its values at  $0^+$  and  $0^-$  are equal. Hence  $\chi_I(\bar{x}, \bar{u}) = \chi_I(\bar{x}, \bar{u}')$  so that  $\chi_I : \Omega_{\mathbb{R}^n} \rightarrow \Omega'_{\mathbb{R}^n}$  is well defined and continuous. Similarly,  $(\chi^{-1})_I$  induces a continuous inverse  $\Omega'_{\mathbb{R}^n} \rightarrow \Omega_{\mathbb{R}^n}$ . By invariance of the domain  $n = n'$ .  $\square$

In view of this proposition, we will only consider conjugacy between systems having the same number of states and inputs. Hence the distinction between  $(n, m)$  and  $(n', m')$  from now on disappears.

*Remark 3.7.* Taking into account the triangular structure of  $\chi$  in Proposition 3.6, one may describe conjugation as resulting from a change of coordinates in the state-space (upon setting  $z = \chi_I(x)$ ) and then feeding the system with a function both of the state and of a new control variable  $v$  (upon setting  $u = (\chi^{-1})_{II}(z, v)$ ), in such a way that the correspondence  $(x, u) \mapsto (z, v)$  is invertible. In the language of control, this is known as a *static feedback transformation*, and two systems conjugate in the sense of Definition 3.9 would be termed *equivalent under static feedback*.

This notion has received considerable attention (see for instance [11]), albeit only in the *differentiable* case (i.e. when  $\chi$  is a diffeomorphism). Differentiability has the following advantage : when  $\chi_I$  and  $(\chi_I)^{-1}$  are differentiable, systems (3) and (4) are locally smoothly conjugate on some domain if, and only if

$$g(\chi_I(x), \chi_{II}(x, u)) = \frac{\partial \chi_I}{\partial x}(x) f(x, u) \quad (13)$$

holds true on this domain. Hence one may replace Definition 3.9, which is based on *solutions* to (3) and (4), by the equality above expressing the way in which  $\chi$  transforms the *equations*. Note that the differentiability of  $\chi_{II}$  is not required.

Various degrees of regularity for  $\chi$  give rise to corresponding notions of conjugacy in Definition 3.9 below.

**Definition 3.8.** For  $k \in \mathbb{N} \cup \{\infty, \omega\}$ ,  $k \geq 1$ , a map  $\chi$  as in (12) is called a *quasi  $C^k$  diffeomorphism* if and only if it is  $C^0$  homeomorphism and  $\chi_I$  is a  $C^k$  diffeomorphism  $\Omega_{\mathbb{R}^n} \rightarrow \Omega'_{\mathbb{R}^n}$ , i.e.  $\chi_I$  and  $\chi_I^{-1}$  are of class  $C^k$ .

**Definition 3.9.** We say that systems (3) and (4) are *topologically* (resp. *smoothly*, resp. *quasi-smoothly*, resp. *analytically*, resp. *quasi-analytically*) *conjugate over the pair  $\Omega, \Omega'$*  if there exists a homeomorphism (resp.  $C^\infty$  diffeomorphism, resp. quasi  $C^\infty$  diffeomorphism, resp.  $C^\omega$  diffeomorphism, resp. quasi  $C^\omega$  diffeomorphism)  $\chi : \Omega \rightarrow \Omega'$  that conjugates the two systems.

<sup>2</sup>This solution is not necessarily unique since here  $f$  and  $g$  are merely assumed to be continuous.



We say that system (3) is *locally* topologically (smoothly, quasi-smoothly, analytically, quasi-analytically) conjugate to system (4) at  $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$  if<sup>3</sup> the two systems are topologically (smoothly, quasi-smoothly, analytically, quasi-analytically) conjugate over a pair  $\Omega, \Omega'$ , where  $\Omega$  is a neighborhood of  $(\bar{x}, \bar{u})$ .

*Remark 3.10.* All definitions are invariant under linear time re-parameterization, namely : *if  $\chi : \Omega \rightarrow \Omega'$  conjugates systems (3) and (4), then for any  $\lambda \in \mathbb{R}$  (if  $\lambda < 0$ , this reverses time) the map  $\chi$  also conjugates the systems*

$$\dot{x} = \lambda f(x, u) \quad \text{and} \quad \dot{z} = \lambda g(z, v) .$$

Indeed, this is trivial for  $\lambda = 0$ , otherwise, if  $t \mapsto (x(t), u(t))$  is a solution of  $\dot{x} = \lambda f(x, u)$  on a time-interval  $[t_1, t_2]$ , and  $\tilde{x}(t)$  and  $\tilde{u}(t)$  denote respectively  $x(t/\lambda)$  and  $u(t/\lambda)$ , then  $t \mapsto (\tilde{x}(t), \tilde{u}(t))$  is a solution of (3) on  $[\lambda t_1, \lambda t_2]$ , hence  $\chi$  sends  $(\tilde{x}(t), \tilde{u}(t))$  to  $(\tilde{z}(t), \tilde{v}(t))$  satisfying  $\dot{\tilde{z}}(t) = g(\tilde{z}(t), \tilde{v}(t))$ . Consequently,  $\chi$  maps  $(x(t), u(t))$  to  $(z(t), v(t)) = ((\tilde{z}(\lambda t), \tilde{v}(\lambda t)))$ , which is a solution of  $\dot{z} = \lambda g(z, v)$ .

In case there is no control (*i. e.*  $m = m' = 0$ ) so that neither  $u$  nor  $\chi_{\text{II}}$  appear in (11), Definition 3.9 coincides with the usual notion of local conjugacy for ordinary differential equations, and will serve as a formal definition in this case too.

**3.4. Alternative definitions.** In the literature, there seems to be no general agreement on what should be called a solution of a control system, nor on the concept of equivalence. Let us quickly mention some of the notions in use.

The work [27], devoted to the topological classification of *linear* control systems, is reviewed in section 4.2. It makes for another definition of solutions that privileges the state  $x$  with respect to the control  $u$ , namely a solution there consists merely of  $x(t)$  instead of  $(x(t), u(t))$ . This amounts to replacing Definition 3.1 by one where a solution is a map  $\gamma_{\text{I}}$  such that there exists a map  $\gamma_{\text{II}}$  for which  $\gamma$  is a solution in the sense of that definition. This new set of solutions is the projection on the  $x$  factor of our set of solutions, and if two different  $u(t)$  produce the same  $x(t)$  then Definition 3.1 distinguishes between two solutions whereas this alternative point of view would see only one. It is natural in this case to define a corresponding notion of conjugacy, similar to the one in Definition 3.9, where the conjugating map  $z = h(x)$  conjugates only the state. To avoid confusion in the rare instances where we use this alternative notion, we shall speak then of *x-solution* and *x-conjugacy* respectively. Since for nonlinear control systems local results have to be stated locally with respect both to  $x$  and  $u$ , we favor Definitions 3.5 and 3.9 of conjugacy and *x-conjugacy* will no longer be mentioned, except in Remark 3.12 where we stress the links between *conjugacy* and *x-conjugacy*, after we established some general properties of conjugating maps.

In [7], a control system like (3) is viewed as a flow on the product space  $\mathbb{R}^n \times \mathcal{U}$ , where  $\mathcal{U}$  is a functional space of admissible controls whose dynamics is induced by the time-shift. Let us also mention the reference [5], where control systems are defined abstractly without reference to differential equations as maps  $(x(0), u(\cdot)) \mapsto x(\cdot)$  that satisfy certain axioms, and where the notion of topological equivalence involves transformations on the product  $\mathbb{R}^n \times \mathcal{U}$ . In this setting, one can consider transformations on functional spaces that depend on the *future and the past of the control*, and not just homeomorphisms on finite dimensional spaces *i.e.* pointwise transformations on the state and the control like we do. We refer the reader to [3] for a Grobman-Hartman theorem in this setting, *cf.* Remark 5.11.

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<sup>3</sup>It would be more natural to say that system (3) **at**  $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$  is locally conjugate to system (4) **at**  $(\bar{x}', \bar{u}') \in \mathbb{R}^{n+m}$  if the two systems are conjugate over a pair  $\Omega, \Omega'$ , where  $\Omega$  is a neighborhood of  $(\bar{x}, \bar{u})$  and  $\Omega'$  is a neighborhood of  $(\bar{x}', \bar{u}')$ . However, prescribing  $(\bar{x}', \bar{u}')$  would increase notational burden and add no real information.

**3.5. Some properties of conjugating maps.** Below we derive some technical facts about conjugacy and feedback that are fundamental to the proof of Theorem 5.2, although they are not needed to understand the result itself.

In the proof of Proposition 3.6, we only used conjugacy on a very small class of solutions, namely those corresponding to piecewise constant controls with a single discontinuity. This raises the question whether smaller classes of solutions than prescribed in Definition 3.1 are still sufficiently rich to check for conjugacy. Under mild conditions on  $f$  and  $g$ , as we will see in the forthcoming proposition, conjugacy essentially holds if it is granted for a class of inputs that locally uniformly approximates piecewise continuous functions, and this fact will be of technical use in the proof of Lemma 6.3 which is an ingredient for Theorem 5.2. Let us fix some terminology here, and agree that a function  $I \rightarrow \mathbb{R}^m$ , where  $I$  is a real interval, is called *piecewise continuous* if it is continuous except possibly at *finitely many* interior points of  $I$  where it has limits from both sides and is either right or left continuous. If in addition the function is constant (resp. affine, smooth) on every open interval not containing a discontinuity point, we say that it is *piecewise constant* (resp. *piecewise affine, smooth*).

**Proposition 3.11.** *Assume that  $f$  and  $g$  are continuous  $\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and locally Lipschitz-continuous with respect to their first argument<sup>4</sup>. Let  $\chi : \Omega \rightarrow \Omega'$  be a homeomorphism between two open subsets of  $\mathbb{R}^{n+m}$ , and denote by  $\Omega_{\mathbb{I}}$  and  $\Omega'_{\mathbb{I}}$  respectively the open subsets of  $\mathbb{R}^m$  obtained by projecting  $\Omega$  and  $\Omega'$  onto the second factor. Let further  $\mathcal{C}$  and  $\mathcal{C}'$  be collections of locally bounded measurable functions  $\mathbb{R} \rightarrow \mathbb{R}^m$  whose restrictions  $\mathcal{C}|_J$  and  $\mathcal{C}'|_J$  to any compact interval  $J$  contain in their respective closures, for the topology of uniform convergence, the set of all piecewise continuous functions  $J \rightarrow \Omega_{\mathbb{I}}$  and  $J \rightarrow \Omega'_{\mathbb{I}}$  respectively. If  $\chi$  maps every solution (5) of (3) such that  $\gamma_{\mathbb{I}}(t) \in \mathcal{C}|_I$  to a solution of (4) while, conversely,  $\chi^{-1}$  maps every solution (7) of (4) such that  $\gamma'_{\mathbb{I}}(t) \in \mathcal{C}'|_I$  to a solution of (3), then the restriction of  $\chi$  to any relatively compact open subset  $O \subset \Omega$  conjugates systems (3) and (4) over the pair  $O, \chi(O)$ .*

*Proof.* Let us first show that

$$\left. \begin{array}{l} \text{for any solution } \gamma : I \rightarrow \Omega \text{ of (3) such that } \gamma_{\mathbb{I}} \text{ is} \\ \text{piecewise continuous, } \chi \circ \gamma \text{ is a solution to (4).} \end{array} \right\} \quad (14)$$

Since the property of being a solution is local with respect to time, we may suppose that  $I$  is a compact interval. Then, there is an open set  $\mathcal{O}$  and a compact set  $\mathcal{K}$  such that  $\gamma(I) \subset \mathcal{O} \subset \mathcal{K} \subset \Omega$ . By the hypothesis on  $\mathcal{C}$ , there exists a sequence of functions  $\gamma_{\mathbb{I},k} : I \rightarrow \mathbb{R}^m$  converging uniformly to  $\gamma_{\mathbb{I}}$  such that  $\gamma_{\mathbb{I},k} \in \mathcal{C}|_I$ . Define for each  $k \in \mathbb{N}$  a time-varying vector field  $X^k$  by  $X^k(t, x) = f(x, \gamma_{\mathbb{I},k}(t))$ . By the continuity of  $f$ , this sequence converges uniformly on compact subsets of  $I \times \mathbb{R}^n$  to  $X(t, x) = f(x, \gamma_{\mathbb{I}}(t))$ ; moreover, since  $\gamma_{\mathbb{I}}$  is bounded (being piecewise continuous)  $\gamma_{\mathbb{I},k}$  is also bounded, thus the local Lipschitz character of  $f(x, u)$  with respect to  $x$  implies by compactness that  $X(t, x)$  and  $X^k(t, x)$  are themselves locally Lipschitz with respect to  $x$  on  $I \times \mathcal{O}_{\mathbb{R}^n}$ . Pick  $t_0 \in I$  and apply Lemma A.3 with  $I = [t_1, t_2]$ ,  $x_0 = \gamma_{\mathbb{I}}(t_0)$ , and  $\mathcal{U} = \mathcal{O}_{\mathbb{R}^n}$ . This yields, say for  $k > K$ , that the solution  $\gamma_{\mathbb{I},k}$  to the Cauchy problem

$$\dot{\gamma}_{\mathbb{I},k}(t) = X^k(t, \gamma_{\mathbb{I},k}(t)), \quad \gamma_{\mathbb{I},k}(t_0) = \gamma_{\mathbb{I}}(t_0),$$

maps  $I$  into  $\mathcal{O}_{\mathbb{R}^n}$  and that the sequence  $(\gamma_{\mathbb{I},k})_{k>K}$  converges uniformly on  $I$  to  $\gamma_{\mathbb{I}}$ . Hence, if we let

$$\gamma_k(t) = (\gamma_{\mathbb{I},k}(t), \gamma_{\mathbb{I},k}(t)),$$

<sup>4</sup>This means that each  $(\bar{x}, \bar{u}) \in \Omega$  has a neighborhood  $\mathcal{N}$  such that  $\|f(x', u) - f(x, u)\| \leq c\|x' - x\|$  for some constant  $c$  whenever  $(x, u)$  and  $(x', u)$  lie in  $\mathcal{N}$ .

the sequence  $(\gamma_k)_{k>K}$  converges to  $\gamma$ , uniformly on  $I$ . In particular  $\gamma_k(I) \subset \mathcal{K} \subset \Omega$  for  $k$  large enough.

Now, since  $\gamma_k : I \rightarrow \Omega$  is a solution to (3) with  $\gamma_{\mathbb{I},k} \in \mathcal{C}|_I$ , it follows from the hypothesis that  $\chi \circ \gamma_k$  is a solution to (4) that remains in  $\Omega'$ , *i.e.* with the notations of (11) we have, for  $k$  large enough,

$$\chi_{\mathbb{I}} \circ \gamma_k(t) - \chi_{\mathbb{I}} \circ \gamma_k(t_0) = \int_{t_0}^t g(\chi \circ \gamma_k(s)) ds, \quad t \in I. \quad (15)$$

By the continuity of  $\chi$ , the convergence of  $\gamma_k(t)$  to  $\gamma(t)$ , and the fact that  $g$  remains bounded on the compact set  $\chi(\mathcal{K})$ , we can apply the dominated convergence theorem to the right hand-side of (15) to obtain in the limit, as  $k \rightarrow \infty$ , that

$$\chi_{\mathbb{I}} \circ \gamma(t) - \chi_{\mathbb{I}} \circ \gamma(t_0) = \int_{t_0}^t g(\chi \circ \gamma(s)) ds, \quad t \in I.$$

Thus  $\chi \circ \gamma : I \rightarrow \mathbb{R}^{n+m}$  is a solution to (4) that remains in  $\Omega'$ , thereby proving (14).

The next step is to observe from (14) that, since piecewise constant controls are in particular piecewise continuous, the proof of Proposition 3.6 applies to show that  $\chi : \Omega \rightarrow \Omega'$  has a triangular structure of the form (12).

With (14) and (12) at our disposal, let us now prove the proposition in its generality. Choose an arbitrary open subset  $\mathcal{O}$  with compact closure  $\overline{\mathcal{O}}$  in  $\Omega$ , and fix two compact subsets  $\mathcal{K}$  and  $\mathcal{K}_1$  of  $\Omega$  such that

$$\mathcal{O} \subset \overline{\mathcal{O}} \subset \overset{\circ}{\mathcal{K}} \subset \mathcal{K} \subset \overset{\circ}{\mathcal{K}}_1 \subset \mathcal{K}_1 \subset \Omega.$$

where  $\overset{\circ}{\mathcal{K}}$  stands for the *interior* of  $\mathcal{K}$ .

Let  $\gamma : I \rightarrow \mathcal{O}$  be a solution of (3). We need to prove that  $\chi \circ \gamma$  is a solution to (4) and again, since the property of being a solution is local with respect to time, we may suppose that  $I$  is compact. Notations being as in (5), it follows by definition of a solution that  $\gamma_{\mathbb{I}}$  is a bounded measurable function  $I \rightarrow \mathbb{R}^m$ . We shall proceed as before in that we again approximate  $\gamma$  by a sequence  $\gamma_k$  of trajectories of (3) that are mapped by  $\chi$  to trajectories of (4). This time, however, the approximation process is slightly more delicate, because it is no longer granted by the hypothesis on  $\mathcal{C}$  but it will rather depend on general point-wise approximation properties to measurable functions by continuous ones.

By the compactness of  $\mathcal{K}$ , there is  $\varepsilon_{\mathcal{K}} > 0$  such that

$$(x, u) \in \mathcal{K} \Rightarrow B((x, u), \varepsilon_{\mathcal{K}}) \subset \overset{\circ}{\mathcal{K}}_1. \quad (16)$$

Let  $u_{\gamma_{\mathbb{I}}} : I \rightarrow \mathbb{R}^m$  be an auxiliary function with the following properties :

- (i)  $u_{\gamma_{\mathbb{I}}}$  is piecewise constant on  $I$ ,
- (ii)  $(\xi(t), u_{\gamma_{\mathbb{I}}}(t)) \in \overset{\circ}{\mathcal{K}}_1$  for all  $t \in I$  and every map  $\xi : I \rightarrow \mathbb{R}^n$  that satisfies

$$\sup_{t \in I} \|\xi(t) - \gamma_{\mathbb{I}}(t)\| < \varepsilon_{\mathcal{K}}/2. \quad (17)$$

Such a function  $u_{\gamma_{\mathbb{I}}}$  certainly exists. Indeed, by definition of a solution,  $\gamma_{\mathbb{I}}$  is absolutely continuous thus *a fortiori* continuous  $I \rightarrow \mathbb{R}^n$ , and therefore we know for each  $t \in I$  that the set

$$\gamma_{\mathbb{I}}^{-1}(B(\gamma_{\mathbb{I}}(t), \varepsilon_{\mathcal{K}}/2))$$

is an open neighborhood of  $t$  in  $I$ , hence a disjoint union of open intervals in  $I$  one of which contains  $t$ ; call this particular interval  $U_t$ . By the compactness of  $I$ , we may cover the latter with finitely many intervals  $U_{t_j}$  for  $1 \leq j \leq \nu$ . Let now  $j(t)$  denote, for each  $t \in I$ , the smallest index  $j \in \{1, \dots, \nu\}$  such that  $t \in U_{t_j}$ . Then, the map

$$u_{\gamma_{\mathbb{I}}}(t) = \gamma_{\mathbb{I}}(t_{j(t)})$$

clearly satisfies (i), and since  $(\gamma_I(t_{j(t)}), \gamma_{II}(t_{j(t)})) \in \mathcal{O} \subset \mathcal{K}$ , it follows from (16) and the fact that  $\|\gamma_I(t) - \gamma_I(t_{j(t)})\| < \varepsilon_{\mathcal{K}}/2$  by definition of  $j(t)$  that  $u_{\gamma_I}$  also satisfies (ii).

Next, recall that  $\gamma_{II}$  is a bounded measurable function  $I \rightarrow \mathbb{R}^m$  so, by Lusin's theorem [20, Theorem 2.23] applied component-wise, there is, for every integer  $k \geq 1$ , a continuous function  $h_k : I \rightarrow \mathbb{R}^m$  that coincides with  $\gamma_{II}$  outside some set  $\mathcal{T}_k \subset I$  of Lebesgue measure strictly less than  $1/k^2$ , and in addition such that

$$\sup_{t \in I} \|h_k(t)\| \leq \sqrt{m} \sup_{t \in I} \|\gamma_{II}(t)\|. \quad (18)$$

Put

$$E_k = \{t \in I; (\gamma_I(t), h_k(t)) \notin \overset{\circ}{\mathcal{K}}\}.$$

Since  $h_k$  is continuous  $E_k$  is compact, and since  $\gamma(I) \subset \mathcal{O} \subset \overset{\circ}{\mathcal{K}}$  it is clear that  $E_k \subset \mathcal{T}_k$  hence  $E_k$  has Lebesgue measure strictly less than  $1/k^2$ . Consequently, by the outer regularity of Lebesgue measure,  $E_k$  can be covered by finitely many open real intervals  $I_{k,1}, \dots, I_{k,N_k}$  whose lengths add up to no more than  $1/k^2$ .

We now define the sequence of functions  $\gamma_{II,k}$  on  $I$  by setting, for  $k \geq 1$ ,

$$\begin{aligned} \gamma_{II,k}(t) &= h_k(t) \text{ if } t \in I \setminus \bigcup_{j=1}^{N_k} I_{k,j}, \\ \gamma_{II,k}(t) &= u_{\gamma_I}(t) \text{ if } t \in \bigcup_{j=1}^{N_k} I_{k,j}. \end{aligned} \quad (19)$$

By construction  $\gamma_{II,k}$  is piecewise continuous, and uniformly bounded independently of  $k$  in view of (18) and the fact that  $u_{\gamma_I}$ , being piecewise constant, is bounded. Moreover, as  $\sum_{k \geq 1} 1/k^2 < \infty$ , the measure of the set  $\bigcup_{j=1}^{N_k} I_{k,j}$  is the general term, indexed by  $k$ , of a convergent series, hence almost every  $t \in I$  belongs at most to finitely many of these sets so that  $\gamma_{II,k}$  converges point-wise a.e. to  $\gamma_{II}$  on  $I$  as  $k \rightarrow \infty$ .

Redefine now  $X^k(t, x) = f(x, \gamma_{II,k}(t))$ ,  $X(t, x) = f(x, \gamma_{II}(t))$ , and observe from what we just said and the continuity of  $f$  that  $X^k(t, x)$  converges to  $X(t, x)$  when  $k \rightarrow \infty$ , locally uniformly with respect to  $x \in \mathcal{O}_{\mathbb{R}^n}$ , as soon as  $t \notin E$  where  $E \subset I$  is a set of zero measure which is independent of  $k$ . Moreover, again from the boundedness of  $\gamma_{II,k}$ ,  $\gamma_{II}$  and the local Lipschitz character of  $f$ , we have that  $X^k(t, x)$ ,  $X(t, x)$  are locally Lipschitz with respect to  $x$ . Pick  $t_0 \in I$  and apply Lemma A.3 with  $\mathcal{U} = \mathcal{O}_{\mathbb{R}^n}$ ,  $I = [t_1, t_2]$ , and  $x_0 = \gamma_I(t_0)$ . We get, say for  $k > K$ , that the solution  $\gamma_{I,k}$  to the Cauchy problem

$$\dot{\gamma}_{I,k}(t) = X^k(t, \gamma_{I,k}(t)), \quad \gamma_{I,k}(t_0) = \gamma_I(t_0),$$

is defined over  $I$ , maps the latter into  $\mathcal{O}_{\mathbb{R}^n}$ , and that the sequence  $(\gamma_{I,k})_{k > K}$  converges uniformly on  $[t_1, t_2]$  to  $\gamma_I$ .

We claim that  $\gamma_k(t) = (\gamma_{I,k}(t), \gamma_{II,k}(t))$  lies in  $\overset{\circ}{\mathcal{K}}_1$  for all  $t \in I$  when  $k$  is so large that

$$\sup_{t \in I} \|\gamma_{I,k}(t) - \gamma_I(t)\| < \varepsilon_{\mathcal{K}}/2. \quad (20)$$

Indeed, if  $t \in \bigcup_j I_{k,j}$ , this follows automatically from definition (19) by property (ii) of  $u_{\gamma_I}$ ; if  $t \notin \bigcup_j I_{k,j}$ , then  $(\gamma_I(t), h_k(t)) \in \overset{\circ}{\mathcal{K}}$  by the very definition of  $\bigcup_j I_{k,j}$ , and since  $\gamma_k(t) = (\gamma_{I,k}(t), h_k(t))$  in this case, we deduce from (16) and (20) that  $\gamma_k(t) \in \overset{\circ}{\mathcal{K}}_1$ . This proves the claim.

Altogether, we have shown that  $\gamma_k : I \rightarrow \overset{\circ}{\mathcal{K}}_1$  is a solution of (3) as soon as  $k$  is large enough, with  $\gamma_{II,k}$  a piecewise continuous function on  $I$  by construction. By (14), we now deduce that, for  $k$  large enough,  $\gamma'_k = \chi \circ \gamma_k$  is a solution of (4) that stays in  $\Omega'$ . Let us block-decompose  $\gamma'_k$  into

$$\gamma'_{I,k}(t) = \chi_I(\gamma_{I,k}(t)), \quad \gamma'_{II,k}(t) = \chi_{II}(\gamma_{I,k}(t), \gamma_{II,k}(t)),$$

where we have taken into account the triangular structure of  $\chi$ . That  $\gamma'_k : I \rightarrow \Omega'$  is a solution of (4) means exactly that

$$\gamma'_{I,k}(t) - \gamma'_{I,k}(t_0) = \int_{t_0}^t g(\gamma'_{I,k}(s), \gamma'_{II,k}(s)) ds, \quad t \in I. \quad (21)$$

Due to the continuity of  $\chi$ , the functions  $\gamma'_{I,k}$  and  $\gamma'_{II,k}$  respectively converge uniformly and point-wise almost everywhere to  $\gamma'_I = \chi_I \circ \gamma_I$  and  $\gamma'_{II} = \chi_{II} \circ \gamma_{II}$  on  $I$ . Since  $g$  is bounded on the compact set  $\chi(\mathcal{K}_1)$  that contains  $\gamma_k(I)$  for  $k$  large enough, we get on the one hand, by dominated convergence, that the right-hand side of (21) converges, as  $k \rightarrow \infty$ , to  $\int_{t_0}^t g(\gamma'_I(s), \gamma'_{II}(s)) ds$ , and on the other hand that the left-hand side converges to  $\gamma'_I(t) - \gamma'_I(t_0)$ . Therefore  $(\gamma'_I, \gamma'_{II}) = \chi \circ \gamma : I \rightarrow \Omega'$  is a solution of (4).

This way we have shown that  $\chi$  maps any solution of (3) that stays in a relatively compact open subset  $\mathcal{O}$  of  $\Omega$  to a solution of (4) that stays in  $\Omega'$ . This achieves the proof, for the converse is obtained symmetrically upon swapping  $f$  and  $g$ ,  $\mathcal{C}$  and  $\mathcal{C}'$ , and replacing  $\chi$  by  $\chi^{-1}$ .  $\square$

*Remark 3.12.* In Section 3.4, we introduced the notion of  $x$ -conjugacy, which is somewhat different from the conjugacy used in the present paper. The two propositions established so far now allow for us to further compare these two notions.

To begin with, conjugacy implies  $x$ -conjugacy by Proposition 3.6. The converse is not true in general, in particular because conjugacy preserves the number of states and inputs by the proposition just quoted, whereas  $x$ -conjugacy preserves the dimension of the state (by invariance of domain), but may not preserve the dimension of the input: any control system  $\dot{x} = f(x, u)$  with state  $x \in \mathbb{R}^n$  and input  $u \in \mathbb{R}^m$  is  $x$ -conjugate to the control system  $\dot{z} = f(z, \pi_m(v))$  with state  $z \in \mathbb{R}^n$  and input  $v \in \mathbb{R}^{m+k}$  where, we recall,  $\pi_m$  denotes the projection onto the first  $m$  components. It is therefore necessary to *assume* that  $n = n'$  and  $m = m'$  in order to compare conjugacy and  $x$ -conjugacy in a nontrivial way. Under this assumption,  $x$ -conjugacy *via a continuously differentiable homeomorphism  $\phi$*  implies conjugacy *locally around points where  $f$  and  $g$  are Lipschitz-continuous with respect to their first argument and where  $\frac{\partial f}{\partial u}$  and  $\frac{\partial g}{\partial v}$  exist and have constant rank*. This follows easily from the inverse function theorem, thanks to Proposition 3.11 that shows it is enough to check local conjugacy on piecewise smooth inputs. It is unclear how much these conditions can be relaxed, but the following example shows that conjugacy does not imply  $x$ -conjugacy if the constant rank assumption fails, even if  $f, g, \phi$  are smooth.

Indeed, consider the two systems  $\dot{x} = u^3 - xu$  and  $\dot{z} = v$  with  $x, u, z, v$  scalars, i.e.  $m = n = m' = n' = 1$ . Any absolutely continuous  $x(t)$  or  $z(t)$  is a  $x$ -solution of the corresponding system : for an associated control, take  $v(t)$  to be  $\dot{z}(t)$  and  $u(t)$  to be for instance the smallest root of  $u(t)^3 - x(t)u(t) - \dot{x}(t) = 0$ . Hence the set of  $x$ -solutions is the same for these two systems and they are  $x$ -conjugate, *via  $z = x$* . However they are not locally topologically conjugate at  $(0, 0)$ . Indeed, using Proposition 3.6, suppose that a transformation  $(z, v) = (\chi_I(x), \chi_{II}(x, u))$  conjugates these two systems on a neighborhood of  $(0, 0)$ . To any strictly positive  $x$ , there are three distinct values  $u_1(x), u_2(x), u_3(x)$  for which  $u_i(x)^3 - xu_i(x) = 0$ , and these go to zero with  $x$ . Consequently, the three constant-maps  $t \mapsto (x, u_i(x))$  for  $1 \leq i \leq 3$  are three trajectories of the first system whose image will be contained in an arbitrary small neighborhood of  $(0, 0)$  if  $x$  is small enough. Since  $\chi$  is one-to-one, the constant maps  $t \mapsto (\chi_I(x), \chi_{II}(x, u_i(x)))$  should be three distinct trajectories of the second system for strictly positive but sufficiently small  $x$ , which is clearly impossible.

The triangular structure of conjugating homeomorphisms asserted by Proposition 3.6 is to the effect that any such homeomorphism  $\chi : \Omega \rightarrow \Omega'$  is a fiber preserving map from the bundle  $\Omega \rightarrow \Omega_{\mathbb{R}^n}$  to the bundle  $\Omega' \rightarrow \Omega'_{\mathbb{R}^n}$ . Since feedbacks are naturally associated to sections of these bundles by Definition 3.3,  $\chi$  gives rise to a natural transformation from feedbacks on  $\Omega$  to feedbacks on  $\Omega'$ . This transformation will prove important enough to deserve a notation : to any feedback  $\alpha$  on  $\Omega$ , we associate a feedback  $\chi \blacksquare \alpha$  on  $\Omega'$  by the formula

$$\chi \blacksquare \alpha(z) \triangleq \chi_{\text{II}}(\chi_{\text{I}}^{-1}(z), \alpha(\chi_{\text{I}}^{-1}(z))). \quad (22)$$

We leave it to the reader to check that the properties of an action are satisfied, and in particular that

$$\chi^{-1} \blacksquare (\chi \blacksquare \alpha) = \alpha. \quad (23)$$

Naturally associated to a control system (3) and a feedback  $\alpha$  is the following continuous vector field  $f_\alpha$  on  $\Omega_{\mathbb{R}^n}$  :

$$f_\alpha(x) = f(x, \alpha(x)). \quad (24)$$

If the homeomorphism  $\chi$  in (12) conjugates system (3) to system (4), then it is clear that  $\chi_{\text{I}}$  maps the solutions of the ordinary differential equation  $\dot{x} = f_\alpha(x)$  to the solutions of the ordinary differential equation  $\dot{z} = g_{\chi \blacksquare \alpha}(z)$ . Indeed if  $x(t)$  is a solution of the former, then  $(x(t), \alpha(x(t)))$  is a solution of the control system (3) in the sense of Definition 3.1 so the conjugacy assumption implies that  $(\chi_{\text{I}}(x(t)), \chi_{\text{II}}(x(t), \alpha(x(t))))$  is a solution of (4), and setting  $z(t) = \chi_{\text{I}}(x(t))$  one clearly has  $\chi_{\text{II}}(x(t), \alpha(x(t))) = \chi \blacksquare \alpha(z(t))$ ; hence  $z(t)$  is a solution to  $\dot{z} = g_{\chi \blacksquare \alpha}(z)$  because  $(z(t), \chi \blacksquare \alpha(z(t)))$  is a solution of (4).

Now, if  $\alpha_1$  and  $\alpha_2$  are two feedbacks on  $\Omega$ , and the two vector fields  $f_{\alpha_1}$  and  $f_{\alpha_2}$  are defined on  $\Omega_{\mathbb{R}^n}$  by (24), we denote their difference by  $\delta f_{\alpha_1, \alpha_2}$  :

$$\delta f_{\alpha_1, \alpha_2} = f_{\alpha_1} - f_{\alpha_2}. \quad (25)$$

Such vector fields are similar to the difference vector fields used in [14], except that we consider arbitrary feedbacks instead of constant ones. To us, these vector fields will play an essential role. The next proposition states that a homeomorphism that conjugates two control systems also conjugates the integral curves of such difference vector fields.

**Proposition 3.13.** *Suppose that  $f$  and  $g$  in (3) and (4) are continuous and locally Lipschitz continuous with respect to their first argument. Assume they are locally topologically conjugate at  $(0, 0)$  over the pair  $\Omega, \Omega'$ . Then, notations for  $\chi_{\text{I}}$  and  $\chi_{\text{II}}$  being as in Proposition 3.6, we have for every pair of feedbacks  $\alpha_1, \alpha_2$  on  $\Omega$  that  $\chi_{\text{I}}$  conjugates any solution of*

$$\dot{x} = \delta f_{\alpha_1, \alpha_2}(x) \quad (26)$$

that remains in  $\Omega_{\mathbb{R}^n}$  to a solution of

$$\dot{z} = \delta g_{\chi \blacksquare \alpha_1, \chi \blacksquare \alpha_2}(z) \quad (27)$$

that remains in  $\Omega'_{\mathbb{R}^n}$ .

It is perhaps worth emphasizing that the solutions of (26) and (27) need not be unique since  $\alpha$  is merely assumed to be continuous. The following proof would carry over to arbitrary linear combinations with constant coefficients of the  $f_\alpha$ 's, but we only need (in Section 5) the result for differences.

*Proof.* Let  $\eta : [t_1, t_2] \rightarrow \Omega_{\mathbb{R}^n}$  be an integral curve of  $\delta f_{\alpha_1, \alpha_2}$ , and set

$$u_1(t) = \alpha_1(\eta(t)) \quad , \quad u_2(t) = \alpha_2(\eta(t)). \quad (28)$$

Let further  $\widehat{f}: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  be bounded, continuous and Lipschitz continuous with respect to its first argument, and coincide with  $f$  on some compact neighborhood of

$$\eta([t_1, t_2]) \times \left( \alpha_1(\eta([t_1, t_2])) \cup \alpha_2(\eta([t_1, t_2])) \right).$$

Such a  $\widehat{f}$  is easily obtained upon multiplying  $f$  by a smooth function with compact support. For  $\ell \in \mathbb{N}$ , let  $\eta^\ell$  be the solution to the Cauchy problem

$$\eta^\ell(t) = \eta(t_1) + \int_{t_1}^t G_\ell(\tau, \eta^\ell(\tau)) d\tau, \quad (29)$$

with

$$\begin{aligned} G_\ell(t, x) &= 2\widehat{f}(x, u_1(t)) \\ &\quad \text{if } t \in [t_1 + \frac{j}{\ell}(t_2 - t_1), t_1 + (\frac{j}{\ell} + \frac{1}{2\ell})(t_2 - t_1)), \\ G_\ell(t, x) &= -2\widehat{f}(x, u_2(t)) \\ &\quad \text{if } t \in [t_1 + (\frac{j}{\ell} + \frac{1}{2\ell})(t_2 - t_1), t_1 + \frac{j+1}{\ell}(t_2 - t_1)), \\ G_\ell(t_2, x) &= -2\widehat{f}(x, u_2(t_2)), \quad 0 \leq j \leq \ell - 1. \end{aligned} \quad (30)$$

The definition of  $\eta^\ell$  is valid because, since  $G_\ell(t, x)$  is bounded and locally Lipschitz with respect to the variable  $x$ , the solution to (29) uniquely exists.

From Lemma A.4 applied to the case where  $X^{1,\ell}(t, x) = \widehat{f}(x, u_1(t))$  and  $X^{2,\ell}(t, x) = \widehat{f}(x, u_2(t))$  are in fact independent of  $\ell$ , any accumulation point of the sequence  $(\eta^\ell)$ , say  $\eta^\infty$ , is a solution to

$$\dot{\eta}^\infty(t) = \widehat{f}(\eta^\infty(t), u_1(t)) - \widehat{f}(\eta^\infty(t), u_2(t)), \quad \eta^\infty(t_1) = \eta(t_1).$$

Since  $\widehat{f}$  is locally Lipschitz continuous with respect to its first argument, the solution to this Cauchy problem is unique and, since  $f$  and  $\widehat{f}$  coincide at all points  $(\eta(t), u_1(t))$  and  $(\eta(t), u_2(t))$ , this entails  $\eta^\infty = \eta$ . Thus  $(\eta^\ell)$  converges uniformly to  $\eta$  on  $[t_1, t_2]$  and, for  $\ell$  large enough,  $\eta^\ell$  remains a solution of (29) if  $\widehat{f}$  is replaced by  $f$  in (30). Moreover,  $\eta^\ell([t_1, t_2]) \subset \Omega_{\mathbb{R}^n}$  for  $\ell$  large since the same is true of  $\eta$ . Since  $\chi$  conjugates the two systems, hence also by Remark 3.10 the systems where  $f$  and  $g$  are multiplied by 2 or  $-2$ , the map  $\chi_I \circ \eta^\ell: [t_1, t_2] \rightarrow \Omega'_{\mathbb{R}^n}$  is, for  $\ell$  large enough, a solution to

$$\chi_I \circ \eta^\ell(t) = \chi_I \circ \eta(t_1) + \int_{t_1}^t \widetilde{G}_\ell(\tau, \chi_I \circ \eta^\ell(\tau)) d\tau \quad (31)$$

with

$$\begin{aligned} \widetilde{G}_\ell(t, z) &= 2g(z, \chi_{\text{II}}(\chi_{\text{I}}^{-1}(z), u_1(t))) \\ &\quad \text{if } t \in [t_1 + \frac{j}{\ell}(t_2 - t_1), t_1 + (\frac{j}{\ell} + \frac{1}{2\ell})(t_2 - t_1)), \\ \widetilde{G}_\ell(t, z) &= -2g(z, \chi_{\text{II}}(\chi_{\text{I}}^{-1}(z), u_2(t))) \\ &\quad \text{if } t \in [t_1 + (\frac{j}{\ell} + \frac{1}{2\ell})(t_2 - t_1), t_1 + \frac{j+1}{\ell}(t_2 - t_1)), \\ \widetilde{G}_\ell(t_2, z) &= -2g(z, \chi_{\text{II}}(\chi_{\text{I}}^{-1}(z), u_2(t_2))). \end{aligned} \quad (32)$$

Since  $(\chi_I \circ \eta^\ell)$  converges uniformly to  $\chi_I \circ \eta$  by the continuity of  $\chi$ , replacing  $g$  by a bounded and continuous  $\widehat{g}: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  that coincides with  $g$  on a compact neighborhood of

$$\chi_I \circ \eta([t_1, t_2]) \times \left( \chi_{\text{II}}(\eta([t_1, t_2]), \alpha_1(\eta([t_1, t_2]))) \cup \chi_{\text{II}}(\eta([t_1, t_2]), \alpha_2(\eta([t_1, t_2]))) \right)$$

does not affect the validity of (31)-(32) for  $\ell$  large enough. Lemma A.4 now implies that all accumulation points of the sequence  $(\chi_I \circ \eta^\ell)$  in the uniform topology on  $[t_1, t_2]$  are solutions of

$$\dot{z} = g(z, \chi_{\text{II}}(\chi_{\text{I}}^{-1}(z), u_1(t))) - g(z, \chi_{\text{II}}(\chi_{\text{I}}^{-1}(z), u_2(t))).$$

Because  $\chi_I \circ \eta$  is such an accumulation point, it is by (28) a solution to

$$\dot{z} = g(z, \chi_{II}(\chi_I^{-1}(z)), \alpha_1(\chi_I^{-1}(z))) - g(z, \chi_{II}(\chi_I^{-1}(z)), \alpha_2(\chi_I^{-1}(z))),$$

which is nothing but (27).  $\square$

#### 4. THE CASE OF LINEAR CONTROL SYSTEMS

**4.1. Kronecker indices.** A linear control systems is a special instance of (3), of the form

$$\dot{x} = Ax + Bu \quad (33)$$

where  $A$  and  $B$  are constant matrices of dimension  $n \times n$  and  $n \times m$  respectively. When dealing with linear systems, it is natural to consider an equivalence relation similar to that of Definition 3.9, but where  $\chi$  is restricted to be a linear isomorphism :

**Definition 4.1.** Two linear systems

$$\dot{x} = Ax + Bu \quad \text{and} \quad \dot{z} = \tilde{A}z + \tilde{B}v$$

are *linearly conjugate* if and only if any of the following two equivalent properties is satisfied :

- (1) There is a nonempty open set  $\Omega \subset \mathbb{R}^{n+m}$ , and a linear isomorphism  $\chi$  of  $\mathbb{R}^{n+m}$  whose restriction  $\Omega \rightarrow \chi(\Omega)$  conjugates the two systems in the sense of Definition 3.9.
- (2) There exist matrices  $P \in \mathbb{R}^{n \times n}$ ,  $Q \in \mathbb{R}^{m \times m}$  and  $K \in \mathbb{R}^{n \times m}$ , with  $P$  and  $Q$  invertible, such that

$$\begin{aligned} \tilde{A} &= P(A - BK)P^{-1}, \\ \tilde{B} &= PBQ^{-1}. \end{aligned} \quad (34)$$

Since, by Proposition 3.6, a linear conjugating homeomorphism is necessarily of the form  $(x, u) \mapsto (Px, Kx + Qu)$ , the equivalence between properties (1) and (2) follows at once from differentiating the solutions. Provided it exists,  $\Omega$  plays absolutely no role in this context since (34) implies that the two systems are in fact linearly conjugate on all of  $\mathbb{R}^{n+m}$ .

Linear conjugacy actually defines an equivalence relation on linear control systems or equivalently on pairs  $(A, B)$ , for which (34) can be read as “ $(A, B)$  is equivalent to  $(\tilde{A}, \tilde{B})$ ”. The classification of linear systems under this equivalence relation is well-known [4], and goes as follows. Each equivalence class contains a pair  $(A_c, B_c)$  of the form (block matrices) :

$$A_c = \begin{pmatrix} A_0^c & 0 & \cdots & 0 \\ 0 & A_1^c & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_m^c \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 & \cdots & 0 \\ b_1^c & \ddots & \vdots \\ 0 & \ddots & 0 \\ \vdots & 0 & b_m^c \end{pmatrix} \quad (35)$$

where

$$A_i^c = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & 0 \\ & & & \ddots & 1 \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}_{(\kappa_i \times \kappa_i)}, \quad b_i^c = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{(\kappa_i \times 1)}, \quad 1 \leq i \leq m. \quad (36)$$



The integers  $(\kappa_1, \dots, \kappa_m)$  are called the controllability indices of the control system, also known as the Kronecker indices of the matrix pencil  $(A, B)$ , while  $A_0^c$  is a square matrix of dimension  $n - (\kappa_1 + \dots + \kappa_m)$  that may be assumed in Jordan canonical form. Note that  $\kappa_1 + \dots + \kappa_m \leq n$ , and if  $\kappa_1 + \dots + \kappa_m = n$  there is no  $A_0^c$ ; also, it may well happen that  $\kappa_i = 0$ , in which case  $A_i^c$  and  $b_i^c$  are empty and do not occur in (35) to the effect that there are less than  $m$  blocks beyond  $A_0^c$ . Normalizing so that

$$\kappa_1 \geq \dots \geq \kappa_m \geq 0,$$

and ordering the Jordan blocks arbitrarily, there is one and only one such normal form per equivalence class. A complete set of invariants is then the list of Kronecker indices and the spectral invariants of the matrix  $A_0^c$ .

With the natural partition  $z = (Z_0, Z_1, \dots, Z_m)$  corresponding to the block decomposition (35), the control system associated to the pair  $(A_c, B_c)$  reads

$$\dot{Z}_0 = A_0 Z_0, \quad \dot{Z}_1 = A_1 Z_1 + u_1 b_1^c, \quad \dots, \quad \dot{Z}_m = A_m Z_m + u_m b_m^c,$$

where  $Z_0$  is missing if  $\kappa_1 + \dots + \kappa_m = n$  and  $Z_i$  is missing if  $\kappa_i = 0$ . Because it is not influenced at all by the controls,  $Z_0$  is sometimes called the non-controllable part of the state. In this paper, we are only interested in controllable linear systems, namely :

**Definition 4.2.** A linear control system (33) is said to be *controllable* if, and only if, the following two equivalent properties are satisfied :

- (1) There is no bloc  $A_0^c$  in the associated normal form (35).
- (2) Kalman's criterion for controllability :

$$\text{Rank}(B, AB, \dots, A^{n-1}B) = n.$$

To see the equivalence of the two properties, observe that the  $n - \kappa_1 - \dots - \kappa_m$  first rows of the matrix  $P$  that puts  $(A, B)$  into canonical form (i.e.  $z = Px$ ) form a basis of the smallest dual subspace that annihilates the columns of  $B$  and at the same time is invariant under right multiplication by  $A$ , i.e. they are a basis of the left kernel of  $(B, AB, \dots, A^{n-1}B)$ . For controllable linear systems, the only invariant under linear conjugacy is thus the ordered list of Kronecker indices. These can be computed from  $(B, AB, \dots, A^{n-1}B)$  as follows : if we put

$$\begin{aligned} r_j &= \text{Rank}(B, AB, \dots, A^{j-1}B), \quad j \geq 1, & r_0 &= 0, \quad r_{-1} = -m, \\ s_j &= r_j - r_{j-1}, \quad j \geq 1, & s_0 &= m, \end{aligned} \quad (37)$$

then  $s_j$  does not increase with  $j$  and a moment's thinking will convince the reader that the number of Kronecker indices that are equal to  $i$  is  $s_i - s_{i+1}$ , or equivalently that  $s_k$  is the number of  $\kappa_j$ 's that are no smaller than  $k$ .

To us, it will be more convenient to use as normal form the following permutation of the previous one. Let  $\rho$  be the smallest integer such that  $s_\rho = 0$ , so that

$$0 = s_\rho < s_{\rho-1} \leq s_{\rho-2} \leq \dots \leq s_1 \leq s_0 = m,$$

with  $\sum_{j \geq 1} s_j = n$ . From these we define, for  $0 \leq i \leq \rho$  :

$$\sigma_i = \sum_{j \geq i} s_j = n - r_{i-1}, \quad (38)$$

so that in particular  $\sigma_\rho = 0$ ,  $\sigma_{\rho-1} = s_{\rho-1} > 0$ ,  $\sigma_1 = n$  and  $\sigma_0 = n + m$ . Note that, from (37),  $\sigma_i = n - r_{i-1}$  for  $i \geq 1$ . We shall write our controllable canonical form

as  $\dot{z} = A_c z + B_c v$  with

$$A_c = \left( \begin{array}{c|c|c|c|c|c} \hline 0 & J_{s_{\rho-2}}^{s_{\rho-1}} & & & & \\ \hline & 0 & J_{s_{\rho-3}}^{s_{\rho-2}} & & & \\ \hline & & 0 & J_{s_{\rho-4}}^{s_{\rho-3}} & & \\ \hline & & & \ddots & \ddots & \\ \hline & & & & \ddots & \\ \hline & & & & & \ddots & \\ \hline & & & & & & 0 & J_{s_1}^{s_2} & & \\ \hline & & & & & & & & & \\ \hline & & & & & & & & & 0 \\ \hline & & & & & & & & & J_{s_0}^{s_1} \\ \hline \end{array} \right), \quad B_c = \left( \begin{array}{c} 0 \\ \\ \\ \\ \\ \\ \\ 0 \\ \\ J_{s_0}^{s_1} \end{array} \right) \quad (39)$$

where for any integers  $r$  and  $s$  with  $s \leq r$ ,  $J_r^s$  is the  $s \times r$  matrix

$$J_r^s = \left( \begin{array}{c|c} I_s & 0 \\ \hline & \end{array} \right) \quad (40)$$

where  $I_s$  is the  $s \times s$  identity matrix.

**4.2. Topological classification of linear control systems.** In [27], which is devoted to the topological classification of *linear* control systems and uses the notion of *x*-conjugacy rather than conjugacy (*cf* Section 3.4 and Remark 3.12), the following result is proved:

**Theorem 4.3** (Willems [27]). *If two linear control systems  $\dot{x} = Ax + Bu$  and  $\dot{z} = \tilde{A}z + \tilde{B}v$  are topologically *x*-conjugate, then they have the same list of Kronecker indices, and the non-controllable blocks  $A_0^c$  and  $\tilde{A}_0^c$  in their respective canonical forms (35) are such that the two linear differential equations  $\dot{X}_0 = A_0^c X_0$  and  $\dot{Z}_0 = \tilde{A}_0^c Z_0$  are topologically equivalent.*

As pointed out in Remark 3.12, topological conjugacy implies topological *x*-conjugacy but not conversely. However, for linear control systems having the same number  $m$  of inputs, Theorem 4.3 implies that these notions are equivalent. Indeed, if two systems are respectively brought into their canonical form (35) by a linear change of variable on  $\mathbb{R}^{n+m}$ , and if in addition they are *x*-conjugate, then their non-controllable parts are topologically equivalent while the remaining blocks are identical by equality of the Kronecker indices. Hence, both in the above theorem and in the corollary below, one may use indifferently “*x*-conjugate” or “conjugate”

**Corollary 4.4.** *If two linear systems  $\dot{x} = Ax + Bu$  and  $\dot{z} = \tilde{A}z + \tilde{B}v$  are topologically conjugate and one of them is controllable, then the other one is controllable too and they are linearly conjugate.*

*Proof.* Controllability is preserved, since Kronecker indices are by the theorem. Linear conjugacy follows, as we saw that the list of Kronecker indices is a complete invariant for controllable systems under linear conjugacy.  $\square$

In some sense, the results of section 5 can be viewed as a generalization of Corollary 4.4 to a local setting where only one of the two systems is linear.

## 5. LOCAL LINEARIZATION FOR CONTROL SYSTEMS

In this section, we consistently assume that the map  $f$  defining system (3) is either of class  $C^\infty$ , *i.e.* infinitely differentiable, or of class  $C^\omega$ , *i.e.* real analytic.

For a control system, *local linearizability* means local conjugacy to a linear *controllable* system. Various degrees of smoothness for the conjugating map give rise to corresponding notions of linearizability in Definition 5.1 below.

**Definition 5.1.** The system (3) is said to be *locally topologically* (resp. *smoothly*, resp. *quasi-smoothly*, resp. *analytically*, resp. *quasi-analytically*) *linearizable at*  $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$  if it is locally topologically (resp. smoothly, resp. quasi-smoothly, resp. analytically, resp. quasi-analytically) conjugate, in the sense of Definition 3.9, to a linear controllable system  $\dot{z} = Az + Bv$  (*cf.* Definition 4.2).

This definition of smooth or analytic linearizability coincides with linearizability by smooth or analytic static feedback as described in the textbooks [10, 18]. In subsection 5.2, we recall classical necessary and sufficient geometric conditions for a system to be smoothly or analytically linearizable, and complement them with a characterization of quasi-smooth and quasi-analytic linearizability.

**5.1. Main result.** If a smooth control system is locally topologically linearizable, then the conjugating homeomorphism has a lot more regularity than required *a priori*. This is in contrast with the Grobman-Hartman theorem for ODE's and constitutes the central result of the paper:

**Theorem 5.2.** *Assume that  $f$  is of class  $C^\infty$  (resp.  $C^\omega$ ) on an open set  $\Omega \subset \mathbb{R}^{n+m}$ . Then system (3) is locally topologically linearizable at  $(\bar{x}, \bar{u}) \in \Omega$  if, and only if it is locally quasi-smoothly (resp. quasi-analytically) linearizable at  $(\bar{x}, \bar{u})$ .*

The proof of Theorem 5.2 will occupy section 6.2. The theorem does not say that local topological linearizability is equivalent to local smooth linearizability, and this is actually false; an example of a system which is quasi-analytically yet not smoothly linearizable at the origin is given in section 5.3, equation (56).

However, smooth and quasi-smooth linearizability appear closely related through Theorems 5.7 and 5.8 of section 5.2, so it is natural to ask how big the gap between topological and smooth linearizability. We now discuss this matter in some details. First we observe from (13), which is valid as soon as  $\chi$  is a homeomorphism of the form (12) such that  $\chi_I$  is a  $C^1$  diffeomorphism (this includes smooth, quasi-smooth, analytic or quasi-analytic conjugacy), that such a  $\chi$  is a linearizing homeomorphism if and only if it satisfies

$$\frac{\partial \chi_I}{\partial x}(x) f(x, u) = A \chi_I(x) + B \chi_{II}(x, u). \quad (41)$$

This makes quasi-smooth linearizability easier to characterize than topological linearizability, that relies on conjugating *solutions* instead of equations, and stresses the usefulness of Theorem 5.2. An interesting consequence is the following.

**Proposition 5.3.** *If  $\chi : \Omega \rightarrow \Omega'$  is a quasi-smooth (resp. quasi-analytic) diffeomorphism that conjugates the smooth (resp. analytic) system (3) to the linear controllable system  $\dot{z} = Az + Bv$ , it must satisfy :*

- (1) *the map  $B \chi_{II} : \Omega \rightarrow \mathbb{R}^m$  is smooth (resp. analytic),*
- (2) *for any  $(x, u) \in \Omega$  in the neighborhood of which the rank of  $\partial f / \partial u$  is constant, one has  $\text{Rank } \frac{\partial f}{\partial u}(x, u) = \text{Rank } B$ .*

(3) for any open subset  $O$  of  $\Omega$ , one has  $\sup_{(x', u') \in O} \text{Rank} \frac{\partial f}{\partial u}(x', u') = \text{Rank} B$ .

*Proof.* Point (1) is direct consequence of (41) and the smoothness (resp. analyticity) of  $\chi_I$  and  $f$ . To establish (2) and (3), differentiate (41) with respect to  $u$  to obtain

$$\frac{\partial \chi_I}{\partial x}(x) \frac{\partial f}{\partial u}(x, u) = \frac{\partial (B \chi_{II})}{\partial u}(x, u). \quad (42)$$

Let  $\mathcal{V} \subset \Omega$  be open and such that  $\text{Rank} \partial f / \partial u(x, u) = \rho$  some integer  $\rho$  and all  $(x, u) \in \mathcal{V}$ . Define  $\phi : \mathcal{V} \rightarrow \mathbb{R}^{n+m}$  by  $\phi(x, u) = (\chi_I(x), B \chi_{II}(x, u))$ . On the one hand, since  $\chi_I$  is a diffeomorphism, (42) implies that the rank of the Jacobian of  $\phi$  is  $n + \rho$ , hence, by the constant rank theorem,  $\phi(\mathcal{V})$  is a  $(n + \rho)$  dimensional immersed submanifold of  $\mathbb{R}^{n+m}$ ; on the other hand, since  $\chi$  is open,  $\phi(\mathcal{V})$  is an open subset of the  $(n + \text{Rank} B)$ -dimensional linear range of  $I_n \times B$ ; hence  $\rho = \text{Rank} B$ . This proves point 2, and at the same time point 3 because any  $O \subset \Omega$  contains an open part on which the rank of  $\partial f / \partial u$  is constant while (42) clearly implies that, for all  $(x, u) \in \Omega$ , the rank of  $\partial f / \partial u(x, u)$  is no larger than  $\text{Rank} B$ .  $\square$

Based on Proposition 5.3, let us divide the points of  $\Omega$  into three classes.

- A point  $(\bar{x}, \bar{u}) \in \Omega$  is called *regular* if it has a neighborhood on which  $\partial f / \partial u$  has constant rank. It is easy to see that regular points form an open dense subset of  $\Omega$ .
- If  $(\bar{x}, \bar{u})$  is not regular, it is termed *weakly singular* if each neighborhood  $O \subset \Omega$  of this point satisfies

$$\sup_{(x, u) \in O} \text{Rank} \frac{\partial f}{\partial u}(x, u) = m. \quad (43)$$

- A point  $(\bar{x}, \bar{u}) \in \Omega$  which is neither regular nor weakly singular is said to be *strongly singular*. This means it has a neighborhood  $O \subset \Omega$ , such that

$$\sup_{(x, u) \in O} \text{Rank} \frac{\partial f}{\partial u}(x, u) = m' < m, \quad \text{Rank} \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) < m'. \quad (44)$$

The distinction between topological and smooth linearizability may now be approached *via* the following theorem.

**Theorem 5.4.** *Assume  $f$  is of class  $C^\infty$  (resp.  $C^\omega$ ) on an open set  $\Omega \subset \mathbb{R}^{n+m}$ .*

- *System (3) is locally smoothly (resp. analytically) linearizable at  $(\bar{x}, \bar{u}) \in \Omega$  if, and only if it is locally topologically linearizable at  $(\bar{x}, \bar{u})$  and the latter is a regular point.*
- *If (3) is locally topologically linearizable at  $(\bar{x}, \bar{u})$  and the latter is a weakly singular point, then a linearizing homeomorphism around  $(\bar{x}, \bar{u})$  may be chosen to be smooth (resp. analytic), although not necessarily a diffeomorphism (because its inverse may fail to be smooth).*

*Proof.* The first assertion is a consequence of Theorem 5.2 together with Theorems 5.7 and 5.8 to come, observing that condition (2') in the latter will automatically hold at a regular point by the constant rank theorem. Next, assume that  $\chi : \Omega \rightarrow \Omega'$  is a quasi smooth diffeomorphism that conjugates the smooth (resp. analytic) system (3) to the linear controllable system  $\dot{z} = Az + Bv$  at some weakly singular point  $(\bar{x}, \bar{u})$ . By (3) of Proposition 5.3, the rank of  $B$  is  $m$  hence it is left invertible; by (1) of the same proposition,  $\chi_{II}$  is indeed smooth (resp. analytic).  $\square$

Whether Theorem 5.4 remains true if “weakly singular” gets replaced by “strongly singular” is unknown to the authors. This turns out to be equivalent to the following question in differential topology which is of interest in its own right and seems to have no answer so far.

**Open Question 5.5.** *Let  $O$  be a neighborhood of the origin in  $\mathbb{R}^{p+q}$  and  $F : O \rightarrow \mathbb{R}^p$  a smooth map. Suppose  $G : O \rightarrow \mathbb{R}^q$  is a continuous map such that  $F \times G : O \rightarrow \mathbb{R}^p \times \mathbb{R}^q$  is a local homeomorphism at 0.*

*Does there exist another neighborhood  $O' \subset O$  of the origin and a smooth map  $H : O' \rightarrow \mathbb{R}^q$  such that  $F \times H : O' \rightarrow \mathbb{R}^p \times \mathbb{R}^q$  is also a local homeomorphism at 0 ?*

If the answer to the open question was yes, then Definitions 5.1 and 3.8 of quasi-smooth linearizability might equivalently require  $\chi$  to be smooth because, assuming the linear system is in normal form (39)-(40), one could set  $F = \pi_{n+s_1} \circ \chi$  and smoothly redefine the last  $m - s_1$  components of  $\chi$ .

If the answer to the open question was no, then Definition 5.1 would really be more general than the one obtained by restricting  $\chi$  to be smooth. Indeed, if  $F$  provides a counterexample to the open question, we may consider on  $\mathbb{R}^p \times O$  the control system

$$\dot{x} = F(u) \quad , \quad x \in \mathbb{R}^p, \quad u \in \mathbb{R}^{p+q} \quad (45)$$

which is locally quasi-smoothly linearizable at the origin since the local homeomorphism

$$(x, u) \mapsto (z, v) = (x, F(u), G(u))$$

conjugates (45) to

$$\dot{z} = Bv, \quad \text{with } B = (I_p \mid 0). \quad (46)$$

However, no *smooth* homeomorphism

$$\chi : (x, u) \mapsto (z, v) = (\chi_I(x), \chi_{II}(x, u))$$

exists that quasi-smoothly linearizes (45) at 0: if this was the case, by Corollary 4.4 we may assume up to a linear change of variables that  $\chi$  conjugates (45) to (46). Then conjugacy would imply

$$\frac{\partial \chi_I}{\partial x}(x) F(u) = B \chi_{II}(x, u)$$

whence in particular

$$F(u) = \left( \frac{\partial \chi_I}{\partial x}(0) \right)^{-1} B \chi_{II}(0, u),$$

and the last  $q$  components of  $\chi_{II}(0, u)$  would yield a smooth  $H$  such that  $F \times H$  is a local homeomorphism at 0 in  $\mathbb{R}^{p+q}$ , contrary to the assumption.

**5.2. Geometric characterization of (quasi) smooth and (quasi) analytic linearization.** Local smooth or analytic linearization were characterized in [12, 9] for systems that are affine with respect to the control, and more generally for smooth systems in [25]. Theorem 5.7 below reformulates these classical results in a suitable form for us. Quasi smooth and quasi analytic linearization were apparently never studied before. We characterize them in Theorem 5.8 in a way that parallels Theorem 5.7 to make for an easy comparison between smooth (resp. analytic) and quasi smooth (resp. quasi analytic) linearizability.

We first introduce a few notations. Let  $\mathcal{X}$  and  $\mathcal{U}$  be two open subsets of  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively, and assume that  $f$  is defined on  $\mathcal{X} \times \mathcal{U}$ . For each  $u \in \mathbb{R}^m$ , let  $f_u$  be the vector field on  $\mathcal{X}$  defined by :

$$f_u(x) = f(x, u). \quad (47)$$

Also, for each  $(x, u) \in \mathcal{X} \times \mathcal{U}$ , we will define a subspace  $D(x, u)$ , that coincides with the range of the linear mapping  $\partial f / \partial u(x, u)$  when its dimension is locally constant.

First, we consider the subset  $\mathcal{L}_{x,u} \subset \mathbb{R}^n$  (not a vector subspace) given by :

$$y \in \mathcal{L}_{x,u} \Leftrightarrow \exists (w_n) \in \mathcal{U}^N, \quad \lim_{n \rightarrow \infty} w_n = u \quad (48)$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{f(x, w_n) - f(x, u)}{\|f(x, w_n) - f(x, u)\|} = y;$$

subsequently we put

$$D(x, u) = \overline{\text{Span}_{\mathbb{R}} \mathcal{L}_{x,u}}. \quad (49)$$

In words,  $D(x, u)$  is the vector space spanned by all limit directions of straight lines through  $f(x, u)$  and  $f(x, u')$  as  $u'$  approaches  $u$  in  $\mathbb{R}^m$ ; it is of common use in stratified geometry to generalize the notion of tangent space. Note that the set  $\mathcal{L}_{x,u}$  depends on the norm used in (48), but the subspace  $D(x, u)$  does not.

**Proposition 5.6.** *If  $f$  is of class  $C^\infty$  and if we denote by  $\text{Ran } L$  the range of a linear map  $L$ , we have that*

$$D(x, u) \supset \text{Ran } \frac{\partial f}{\partial u}(x, u) \quad (50)$$

and equality holds at every  $(x, u)$  where the rank of  $\partial f / \partial u(x, u)$  is locally constant with respect to  $u$ .

*Proof.* The inclusion (50) is clear since any element of  $\text{Ran } \partial f / \partial u(x, u)$  can be written  $\partial f / \partial u(x, u).h$  with  $h \in \mathbb{R}^m$ , and, if it is nonzero, one has

$$\frac{\partial f / \partial u(x, u).h}{\|\partial f / \partial u(x, u).h\|} = \lim_{t \rightarrow 0^+} \frac{f(x, u + th) - f(x, u)}{\|f(x, u + th) - f(x, u)\|}.$$

To get the reverse inclusion when the rank of  $\partial f / \partial u(x, u)$  is locally constant, we use the constant rank-theorem to the effect that the partial map  $u' \mapsto f(x, u')$  may, in suitable coordinates around  $u$ , be factored as  $\varphi \circ i_{r,n} \circ \pi_r$  where  $\varphi$  is a local diffeomorphism around  $f(x, u)$  in  $\mathbb{R}^n$ ,  $r$  is the rank of  $\partial f / \partial u(x, u)$ , the map  $\pi_r : \mathbb{R}^m \rightarrow \mathbb{R}^r$  is the canonical projection on the first  $r$  components and the inclusion  $i_{r,n} : \mathbb{R}^r \rightarrow \mathbb{R}^n$  maps  $(w_1, \dots, w_r)$  to  $(w_1, \dots, w_r, 0, \dots, 0)$ . Therefore, up to a local change of coordinates around  $u$  in  $\mathbb{R}^m$  which clearly leaves  $D(x, u)$  and  $\text{Ran } \partial f / \partial u(x, u)$  unchanged, we can assume that  $f(x, u')$  depends only on  $\pi_r(u')$  and that

$$\|f(x, u') - f(x, u)\| \geq c \|\pi_r(u' - u)\| \quad (51)$$

for some constant  $c > 0$ . Take  $y \in D(x, u)$ . By definition there is a sequence  $(h_n)$  satisfying (48). Since  $f(x, u + h_n)$  depends only (for  $x$  fixed) on  $\pi_r(u + h_n)$ , we may replace  $h_n$  by any element of  $\mathbb{R}^m$  having the same  $r$  first components : assume that the  $m - r$  last components of  $h_n$  are zero, so that  $\|h_n\| = \|\pi_r(h_n)\|$ . Then, we can write

$$y = \lim_{n \rightarrow \infty} \frac{f(x, u + h_n) - f(x, u)}{\|h_n\|} \frac{\|\pi_r(h_n)\|}{\|f(x, u + h_n) - f(x, u)\|}. \quad (52)$$

Let  $h$  be a limit point of  $h_n / \|h_n\|$  in  $\mathbb{R}^m$ . By definition of the derivative, we see, up to a subsequence, that the first ratio on the right-hand side of (52) converges to  $\partial f / \partial u(x, u).h$ . Moreover the second ratio is bounded from above by virtue of (51), and therefore, extracting another subsequence if necessary, we may assume it converges to some real number. It is now apparent that  $y \in \text{Ran } \partial f / \partial u(x, u)$  as desired.  $\square$

We can now state the following characterization of smooth or analytic linearizability. It is essentially a reformulation of [12, 9, 25].

**Theorem 5.7** (smooth or analytic linearizability). *Assume that  $f$  is smooth (resp. analytic) on an open subset  $\Omega \subset \mathbb{R}^{n+m}$ . The control system (3) is locally smoothly (resp. analytically) linearizable at  $(\bar{x}, \bar{u}) \in \Omega$  if, and only if there are open neighborhoods  $\mathcal{X}$  and  $\mathcal{U}$  of  $\bar{x}$  and  $\bar{u}$  in  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , with  $\mathcal{X} \times \mathcal{U} \subset \Omega$ , such that the following conditions are satisfied.*

- (1)  $D(x, u)$  does not depend on  $u$  for  $(x, u) \in \mathcal{X} \times \mathcal{U}$ .
- (2) The rank of  $\frac{\partial f}{\partial u}(x, u)$  is constant in  $\mathcal{X} \times \mathcal{U}$ .
- (3) Defining on  $\mathcal{X}$  the distribution  $\Delta_0$  by  $\Delta_0(x) = D(x, u)$  — this is possible if point (1) holds true — and inductively the flag of distributions  $(\Delta_k)$  by :

$$\Delta_{k+1} = \Delta_k + [f_{\bar{u}}, \Delta_k] \quad (53)$$

where  $[ , ]$  denotes the Lie bracket, then each  $\Delta_k$  for  $0 \leq k \leq n-1$  is integrable (i.e. has constant dimension over  $\mathbb{R}$  and is closed under Lie bracket) and the rank of  $\Delta_{n-1}$  is  $n$ .

Considering Proposition 5.6 and the necessity of point (2), the use of  $D(x, u)$  instead of  $\text{Ran } \partial f / \partial u(x, u)$  appears as a useless complication in this theorem. However,  $D(x, u)$  is the relevant object to use for quasi smooth or quasi analytic linearizability and it allows the following characterization to run parallel to the above.

**Theorem 5.8** (quasi smooth or quasi analytic linearizability). *Assume that  $f$  is smooth (resp. analytic) on an open subset  $\Omega \subset \mathbb{R}^{n+m}$ . The control system (3) is locally quasi smoothly (resp. quasi analytically) linearizable at  $(\bar{x}, \bar{u}) \in \Omega$  if, and only if there are open neighborhoods  $\mathcal{X}$  and  $\mathcal{U}$  of  $\bar{x}$  and  $\bar{u}$  in  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , with  $\mathcal{X} \times \mathcal{U} \subset \Omega$ , such that conditions (1) and (3) of Theorem 5.7 are met and, instead of condition (2), it holds that*

- (2') The integer  $r_1 \leq m$  being the constant rank of the distribution  $\Delta_0$ , the smooth mapping

$$\begin{aligned} F : \mathcal{X} \times \mathcal{U} &\rightarrow \mathcal{X} \times \mathbb{R}^n \\ (x, u) &\mapsto (x, f(x, u)) \end{aligned} \quad (54)$$

restricts to a  $C^0$  fibration<sup>5</sup>  $\mathcal{W} \rightarrow F(\mathcal{W})$  with fiber  $\mathbb{R}^{m-r_1}$  on some neighborhood  $\mathcal{W}$  of  $(\bar{x}, \bar{u})$  in  $\mathcal{X} \times \mathcal{U}$ .

Theorems 5.7 and 5.8 will be proved in section 6.1. Since analyticity does not appear in the conditions, an immediate corollary is the following, which should be held in contrast with the discussion in section 2 on the local linearization of ODEs.

**Corollary 5.9.** *Assume that  $f$  is real analytic some open set  $\Omega \subset \mathbb{R}^{n+m}$ . If the control system (3) is locally smoothly (resp. quasi-smoothly) linearizable at  $(\bar{x}, \bar{u}) \in \Omega$ , then it is also analytically (resp. quasi-analytically) linearizable there.*

**5.3. Linearization and equivalence to the linear approximation.** For a control system, smooth linearizability at an equilibrium implies conjugacy to its linear approximation. This is the content of the next proposition. In contrast, this is not the case if the control system is only topologically linearizable, that is, a smooth control system may well be quasi smoothly conjugate to a controllable linear system which is *not* equivalent to its linear approximation. We give examples below.

<sup>5</sup> A  $C^0$  fibration with fiber  $\mathcal{F}$  over  $\mathcal{B}$  is a continuous map  $g : \mathcal{E} \rightarrow \mathcal{B}$  for which every  $\xi \in \mathcal{B}$  has a neighborhood  $\mathcal{O}$  in  $\mathcal{B}$  such that  $g^{-1}(\mathcal{O}) \subset \mathcal{E}$  is homeomorphic to  $\mathcal{O} \times \mathcal{F}$ , the so-called *trivializing* homeomorphism  $\psi : g^{-1}(\mathcal{O}) \rightarrow \mathcal{O} \times \mathcal{F}$  being such that  $\pi \circ \psi = g$  where  $\pi : \mathcal{O} \times \mathcal{F} \rightarrow \mathcal{O}$  is the natural projection onto the first factor.

**Proposition 5.10.** *Let  $(\bar{x}, \bar{u})$  be an equilibrium point of (3), i.e.  $f(\bar{x}, \bar{u}) = 0$ , and let  $\bar{A} = \partial f / \partial x(\bar{x}, \bar{u})$ ,  $\bar{B} = \partial f / \partial u(\bar{x}, \bar{u})$  so that :*

$$f(x, u) = \bar{A}(x - \bar{x}) + \bar{B}(u - \bar{u}) + \varepsilon(x - \bar{x}, u - \bar{u}), \quad (55)$$

where  $\varepsilon$  is little  $o(\|x - \bar{x}\| + \|u - \bar{u}\|)$ .

If system (3) is locally smoothly linearizable at  $(\bar{x}, \bar{u})$ , then

1. its linear approximation  $(\bar{A}, \bar{B})$  is controllable (cf. Definition 4.2), and
2. the system is smoothly conjugate to  $(\bar{A}, \bar{B})$  at  $(\bar{x}, \bar{u})$ .

*Proof.* Let  $\chi$  be a local diffeomorphism conjugating system (3) to  $\dot{z} = Az + Bv$  at  $(\bar{x}, \bar{u})$ , and observe from (13) in Remark 3.7 that smooth linearizability translates into (41). If we write  $f$  as in (55), and if we set  $\bar{P} = \frac{\partial \chi_{\mathbb{R}^n}}{\partial x}(\bar{x})$ ,  $\bar{K} = \frac{\partial \chi_{\mathbb{R}^m}}{\partial x}(\bar{x}, \bar{u})$ ,  $\bar{Q} = \frac{\partial \chi_{\mathbb{R}^m}}{\partial u}(\bar{x}, \bar{u})$ , we get by differentiating (41) with respect to  $x$  and  $u$  at  $(\bar{x}, \bar{u})$ , using the relation  $f(\bar{x}, \bar{u}) = 0$ , that

$$\bar{P}\bar{A} = A\bar{P} + B\bar{K}, \quad \bar{P}\bar{B} = B\bar{Q}.$$

Since  $\bar{P}$  and  $\bar{Q}$  are square invertible matrices by the triangular structure of  $\chi$  displayed in (12), this implies that the linear systems  $(A, B)$  and  $(\bar{A}, \bar{B})$  are linearly conjugate, see (34). Since  $(A, B)$  is controllable by definition so is  $(\bar{A}, \bar{B})$ , thereby achieving the proof.  $\square$

Let us give now examples of quasi analytically linearizable systems which are not conjugate to their linear approximation. This is the case of

$$\dot{x} = u^3 \quad u \in \mathbb{R}, x \in \mathbb{R}, \quad (56)$$

whose linear approximation at the origin is not controllable whereas the system is nevertheless quasi analytically conjugate to the linear controllable system  $\dot{z} = v$  via the quasi analytic diffeomorphism  $(x, u) \mapsto (z, v) = (x, u^3)$ . Apart from such degenerate cases, there also exist systems that are quasi analytically linearizable at some point with controllable linear approximation there, and still they are not conjugate to this linear approximation. An example when  $m = n = 2$  is given by :

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = x_1 + u_2^3,$$

This system is quasi analytically conjugate at  $(0, 0)$  to

$$\dot{z}_1 = v_1, \quad \dot{z}_2 = v_2, \quad (57)$$

via  $z = x$ ,  $v_1 = u_1$ ,  $v_2 = u_2^3 + x_1$ . However, its linear approximation at the origin is  $\dot{x}_1 = u_1$ ,  $\dot{x}_2 = x_1$ , which is controllable yet not conjugate to (57) (cf Theorem 4.3).

**5.4. Non-genericity of (quasi-) smooth linearizability.** Except when  $m \geq n$  or  $(n, m) = (2, 1)$ , the conditions of Theorem 5.7 require a certain number of equalities (involving  $f$  and its partial derivatives) to hold *everywhere*. For example, the integrability of a distribution entails that all Lie brackets be linearly dependent on the original vector fields, i.e. certain determinants must be identically zero. This makes smooth (resp. analytic) linearizability of a smooth (resp. analytic) control system non-generic in any reasonable sense, since written in proper jet spaces it is contained in a set of infinite co-dimension. Moreover, small perturbations of a system that does not satisfy these condition will not satisfy them either, while most perturbations of a system which satisfies them will fail to do so. Compare for instance [24] where it is shown that the equivalence class of any system affine in the control has infinite codimension in some Whitney topology.

Quasi-smooth linearizability is more general than smooth linearizability, but by Theorem 5.4 it imposes that the same equalities as before hold on an open dense set, although this time some singularities are allowed. This is no more “generic”



than smooth linearizability. In view of Theorem 5.2, this means that topological linearizability is highly non-generic for control systems, as opposed to ODE's for which the Grobman-Hartman theorem allows one to linearize around an equilibrium as soon as it is hyperbolic.

*Remark 5.11.* In [3], it is proved that any control system (3) whose linear tangent approximation is controllable (this is true at generic points for generic systems) is locally conjugate in some sense to a linear controllable one. This does not contradict the present results because the notion of conjugacy there is much weaker : it is obtained via a homeomorphism that acts on the product of the state-space with a functional space of admissible controls (for instance, the “new” state depends on the original state and on the past and future values of the control).

## 6. PROOFS

**6.1. Proof of Theorems 5.7 and 5.8.** We begin with a lemma whose cumbersome index arrangement will be rewarded later when constructing the Kronecker indices of the linearized system.

**Lemma 6.1.** *Let  $\Delta_0$  be a smooth (resp. real analytic) distribution and  $f_{\bar{u}}$  be a smooth (resp. real analytic) vector field on a connected open set  $\mathcal{X} \subset \mathbb{R}^n$ . Let further  $\Delta_i$ ,  $i \geq 0$ , be the distributions defined according to (53). Assume they satisfy point (3) of Theorem 5.7 or 5.8. Put*

$$r_i = \text{Rank}\Delta_{i-1}, \quad i > 0, \quad r_0 = 0, \quad r_{-1} = -m, \quad (58)$$

so that  $r_i = n$  for some  $i \leq n-1$ ; let  $\rho \in \{3, \dots, n+1\}$  be the smallest integer such that  $r_{\rho-1} = n$ . Define also

$$s_i = r_i - r_{i-1}, \quad \sigma_i = \sum_{j=i}^{\rho} s_j = n - r_{i-1}, \quad 0 \leq i \leq \rho \quad (59)$$

(note that  $s_{\rho} = \sigma_{\rho} = 0$ ).

Then, there exists around each  $x \in \mathcal{X}$  some smooth (resp. real analytic) coordinates  $\chi_1, \dots, \chi_n$  such that

- $\chi_1, \dots, \chi_{\sigma_i}$  are independent first integrals of  $\Delta_{i-2}$  for  $i \in \{2, \dots, \rho\}$ ,
- $\chi_{\sigma_i+j} = f_{\bar{u}}\chi_{\sigma_{i+1}+j}$  for all integers  $i, j$ ,  $2 \leq i \leq \rho-1$ ,  $1 \leq j \leq s_i$

( $f_{\bar{u}}\chi_{\sigma_{i+1}+j}$  is the Lie derivative of the function  $\chi_{\sigma_{i+1}+j}$  along the vector field  $f_{\bar{u}}$ ).

*Proof.* The Frobenius theorem provides us with  $n-r$  independent smooth (resp. analytic) first integrals for a smooth (resp. analytic) integrable distribution of rank  $r$ . This accounts for the regularity of the coordinates if we construct them as follows.

First pick  $n - r_{\rho-2} = \sigma_{\rho-1}$  independent first integrals of  $\Delta_{\rho-3}$  and call them  $\chi_1, \dots, \chi_{\sigma_{\rho-1}}$ ; define further  $\chi_{1+\sigma_{\rho-1}}, \dots, \chi_{2\sigma_{\rho-1}}$  by  $\chi_{\sigma_{\rho-1}+j} = f_{\bar{u}}\chi_j$  for  $1 \leq j \leq \sigma_{\rho-1} = s_{\rho-1}$ . Clearly,  $\chi_1, \dots, \chi_{\sigma_{\rho-1}+s_{\rho-1}}$  satisfy the conditions for  $i \geq \rho-1$ . Then proceed inductively : assume that, for some  $i_0 \in \{3, \dots, \rho-1\}$ , the functions  $\chi_1, \dots, \chi_{\sigma_{i_0}+s_{i_0}}$  have been constructed and satisfy the conditions for  $i \geq i_0$ . We claim that the differentials  $d\chi_{\ell}$  are linearly independent at each point of  $\mathcal{X}$ . Indeed, assume that there is  $\bar{x} \in \mathcal{X}$  and real coefficients  $\mu_j$  and  $\lambda_k$  such that

$$\sum_{j=1}^{\sigma_{i_0}} \mu_j d\chi_j(\bar{x}) + \sum_{k=1+\sigma_{i_0+1}}^{\sigma_{i_0}} \lambda_k d(f_{\bar{u}}\chi_k)(\bar{x}) = 0. \quad (60)$$

Put  $\omega_1 = \sum \mu_j d\chi_j$  and  $\omega_2 = \sum \lambda_k d\chi_k$ . Since  $d$  commutes with the Lie derivative, we may rewrite (60) as  $\omega_1(\bar{x}) + f_{\bar{u}}\omega_2(\bar{x}) = 0$ . In particular, for any smooth vector

field  $X$  in  $\Delta_{i_0-2}$ , we get as  $\omega_1(X) \equiv 0$  that  $f_{\bar{u}}\omega_2(X)(\bar{x}) = 0$ . Now, by virtue of the formula

$$f_{\bar{u}}(\omega_2(X)) = f_{\bar{u}}\omega_2(X) + \omega_2([f_{\bar{u}}, X]), \quad (61)$$

we obtain since  $\omega_2(X) \equiv 0$  that  $\omega_2([f_{\bar{u}}, X])(\bar{x}) = 0$ , that is,  $\omega_2$  annihilates  $\Delta_{i_0-1}$  at  $\bar{x}$ . But  $d\chi_1(\bar{x}), \dots, d\chi_{\sigma_{i_0+1}}(\bar{x})$  are a basis of the orthogonal space to  $\Delta_{i_0-1}(\bar{x})$  by the induction hypothesis, whereas  $\omega_2(\bar{x}) = 0$  is a linear combination of the  $d\chi_k$  for  $\sigma_{i_0+1} < k \leq \sigma_{i_0}$ . Therefore, since we know by the induction hypothesis that the  $d\chi_\ell$  are pointwise independent for  $1 \leq \ell \leq \sigma_{i_0}$ , we get that the  $\lambda_k$  are zero and then the  $\mu_j$  are also zero by (60). *This proves the claim.* Next, recall that  $\chi_1, \dots, \chi_{\sigma_{i_0}}$  are first integrals of  $\Delta_{i_0-2}$ , thus *a fortiori* of  $\Delta_{i_0-3}$ . For  $X$  a smooth vector field in the latter we deduce from (61), where  $\omega_2$  is replaced by  $d\chi_\ell$  with  $1 + \sigma_{i_0+1} \leq \ell \leq \sigma_{i_0}$ , that  $\chi_{1+\sigma_{i_0}}, \dots, \chi_{\sigma_{i_0}+\sigma_{i_0}}$  are also first integrals of  $\Delta_{i_0-3}$ . In case  $\sigma_{i_0} + \sigma_{i_0} < \sigma_{i_0-1}$ , pick  $\chi_{\sigma_{i_0}+\sigma_{i_0}+1}, \dots, \chi_{\sigma_{i_0-1}}$  so that  $\chi_\ell$  for  $1 \leq \ell \leq \sigma_{i_0-1}$  is a complete set of independent integrals of  $\Delta_{i_0-3}$ , and define  $\chi_{\sigma_{i_0-1}+j} = f_{\bar{u}}\chi_{\sigma_{i_0}+j}$  for  $1 \leq j \leq \sigma_{i_0-1}$  in order to complete the induction step.  $\square$

*Proof of Theorems 5.7 and 5.8.* The two proofs run parallel to each other.

We first prove necessity. Since analyticity does not appear in the conclusions, the analytic case is a consequence of the smooth case. Assume local (quasi) smooth linearizability, *cf.* Definitions 5.1 and 3.8. Without loss of generality, we assume that  $\Omega = \mathcal{X} \times \mathcal{U}$  where  $\mathcal{X}$  and  $\mathcal{U}$  are open neighborhoods of  $\bar{x}$  and  $\bar{u}$  in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively. Let  $\chi : \mathcal{X} \times \mathcal{U} \rightarrow \Omega' \subset \mathbb{R}^{n+m}$  be as in (12); recall that  $\chi_I$  is a smooth diffeomorphism  $\mathcal{X} \rightarrow \chi_I(\mathcal{X})$ . We may also assume, after composing  $\chi$  with a linear invertible map, that the pair  $(A, B)$  is in canonical form (39)-(40), but we still write  $A, B$  rather than  $A_c, B_c$ . Denote by  $B_0, \dots, B_m$  the columns of  $B$  and define the vector fields  $b_0, \dots, b_m$  on  $\mathbb{R}^n$  by

$$b_i(z) = B_i, \quad 1 \leq i \leq m, \quad b_0(z) = Az + B\bar{u} \quad (62)$$

and the distributions  $\Lambda_i$  by

$$\Lambda_0(z) = \text{Span}_{\mathbb{R}}\{b_1(z), \dots, b_m(z)\} = \text{Ran}B \quad \Lambda_{i+1} = \Lambda_i + [b_0, \Lambda_i], \quad 1 \leq i \leq m. \quad (63)$$

From (41), we have

$$\frac{\partial \chi_I}{\partial x}(x) f(x, u) = b_0(\chi_I(x)) + B(\chi_{II}(x, u) - \chi_{II}(x, \bar{u})) \quad (64)$$

Then, since the variations of  $\chi_{II}(x, u)$  when  $u$  varies around  $\bar{u}$  cover on open set,  $\mathcal{L}_{x,u}$  defined by (48) contains an open set in  $\left(\frac{\partial \chi_I}{\partial x}(x)\right)^{-1} \text{Ran}B$ ; hence

$$D(x, u) = \left(\frac{\partial \chi_I}{\partial x}(x)\right)^{-1} D(x, u) = \text{Ran}B$$

This proves point 1, and also proves that the distribution  $\Delta_0$  in point 3 is the pull-back of  $\Lambda_0$  by the diffeomorphism  $\chi_I$ , *i.e.*  $(\chi_I)_* \Delta_0 = \Lambda_0$ . Since (64) also implies  $(\chi_I)_* f_{\bar{u}} = b_0$ , we have  $(\chi_I)_* \Delta_i = \Lambda_i$  for all  $i$ . This gives point 3 because it is obviously true with  $\Lambda_i$  instead of  $\Delta_i$ , and integrability and ranks are preserved by conjugation by the smooth diffeomorphism  $\chi_I$ . In the case of smooth linearizability, point 2 is easily obtained by differentiating (64) with respect to  $u$  and using invertibility of  $\partial \chi_{II} / \partial u(x, u)$ .

To conclude the proof of necessity, let us prove point 2' in the case of quasi-smooth linearizability. Let

$$\mathcal{M} = \left\{ (x, y) \in \mathcal{X} \times \mathbb{R}^n; \frac{\partial \chi_I}{\partial x}(x) y - A\chi_I(x) \in \text{Ran}B \right\}.$$

This is a smooth embedded sub-manifold of  $\mathcal{X} \times \mathbb{R}^m$  of dimension  $n + r_1$ , where  $r_1 = \text{Rank } B \leq m$ . If we define  $F$  as in (54), it is clear from (41) that

$$F(\mathcal{X} \times \mathcal{U}) \subset \mathcal{M}.$$

Now, take some  $(m - r_1) \times m$  matrix  $C$  whose rows complement  $r_1$  independent rows of  $B$  into a basis of  $\mathbb{R}^m$ . Pick matrices  $E_1$  and  $E_2$  of appropriate sizes such that

$$E_1 B + E_2 C = I_m.$$

By (41) we get

$$E_1 \left[ \frac{\partial \chi_{\text{I}}}{\partial x}(x) f(x, u) - A \chi_{\text{I}}(x) \right] + E_2 C \chi_{\text{II}}(x, u) = \chi_{\text{II}}(x, u). \quad (65)$$

Define

$$\psi : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{M} \times \mathbb{R}^{m-r_1}$$

by the formula:

$$\psi(x, u) = (x, f(x, u), C \chi_{\text{II}}(x, u)).$$

From (65), this mapping has an inverse given by

$$\begin{aligned} \psi^{-1} : \psi(\mathcal{X} \times \mathcal{U}) &\rightarrow \mathcal{X} \times \mathcal{U} \\ (x, y, z) &\mapsto \chi^{-1} \left( \chi_{\text{I}}(x), E_1 \left[ \frac{\partial \chi_{\text{I}}}{\partial x}(x) y - A \chi_{\text{I}}(x) \right] + E_2 z \right) \end{aligned}$$

so that  $\psi$  defines a homeomorphism from  $\mathcal{X} \times \mathcal{U}$  onto its image which is open in  $\mathcal{M} \times \mathbb{R}^{m-r_1}$  by invariance of the domain. Let  $\mathcal{O}$  be a neighborhood of  $(\bar{x}, f(\bar{x}, \bar{u}))$  in  $\mathcal{M}$  and  $\mathcal{S}$  an open ball centered at  $C \chi_{\text{II}}(\bar{x}, \bar{u})$  in  $\mathbb{R}^{m-r_1}$  such that  $\mathcal{O} \times \mathcal{S} \subset \psi(\mathcal{X} \times \mathcal{U})$ , and take  $\mathcal{W} = \psi^{-1}(\mathcal{O} \times \mathcal{S})$ . Then  $F : \mathcal{W} \rightarrow F(\mathcal{W}) = \mathcal{O}$  is a  $C^0$  fibration with fiber  $\mathcal{S}$  and trivializing homeomorphism  $\psi : \mathcal{W} \rightarrow \mathcal{O} \times \mathcal{S}$ . Since  $\mathcal{S}$  is homeomorphic to  $\mathbb{R}^{m-r_1}$ , condition 2' follows.

We turn to sufficiency. Assume that  $f$  is smooth (resp. analytic) and that points 1, 3, and either 2 or 2' are satisfied. Then, one has, for all  $x \in \mathcal{X}$ ,

$$\Delta_0(x) = \text{Span}_{\mathbb{R}}\{f(x, w) - f(x, u), (u, w) \in \mathcal{U} \times \mathcal{U}\}. \quad (66)$$

Indeed the right-hand side always contains  $D(x, u)$  because it contains all the differences  $f(x, w_n) - f(x, u)$  in (48), and point 1 implies the reverse inclusion because  $f(x, w) - f(x, u)$  can be computed as the integral on the segment  $[u, w] \subset \mathcal{U}$  of a function that, thanks to Proposition 5.6, belongs constantly to  $\Delta_0(x)$ .

From (66),  $\Delta_0$  is a smooth (resp. analytic) distribution. Since, obviously,  $f_{\bar{u}}$  is a smooth (resp. analytic) vector field, we may, considering point 3, apply Lemma 6.1. We thus obtain some, with  $r_i, s_i$  and  $\sigma_i$  the integers defined by (58) and (59), some smooth (resp. analytic) coordinates  $\chi_1, \dots, \chi_n$  on a neighborhood of  $\bar{x}$  possibly smaller than  $\mathcal{X}$  (but that we continue to denote by  $\mathcal{X}$ ), i.e. a diffeomorphism  $\chi_{\text{I}} : \mathcal{X} \rightarrow \chi_{\text{I}}(\mathcal{X})$ , with  $\chi_{\text{I}} = (\chi_1, \dots, \chi_n)$ , meeting the conclusions of Lemma 6.1. In particular,  $\chi_1, \dots, \chi_{n-r_1}$  are first integrals of the distribution  $\Delta_0$ , and from (66), this implies that  $\partial \chi_i / \partial x(x) f(x, u)$  does not depend on  $u$ , and is therefore equal to its value for  $u = \bar{u}$ :

$$\frac{\partial \chi_i}{\partial x}(x) f(x, u) = f_{\bar{u}} \chi_i(x), \quad 1 \leq i \leq n - r_1. \quad (67)$$

For larger  $i$ , the left-hand side depends on  $x$  and  $u$ : define  $\lambda : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^{m_1}$  by

$$\lambda(x, u) = \left( \frac{\partial \chi_{n-r_1+1}}{\partial x}(x) f(x, u), \dots, \frac{\partial \chi_n}{\partial x}(x) f(x, u) \right). \quad (68)$$

Then, defining coordinates  $z_1, \dots, z_n$  by  $z = \chi_I(x)$ . The equations of system (3) are as follows (the first line gives the derivatives of the  $n - r_1$  first coordinates and the second line the last  $r_1$  ones) :

$$\begin{aligned} \dot{z}_{\sigma_{i+1}+j} &= z_{\sigma_i+j}, & 2 \leq i \leq \rho - 1, 1 \leq j \leq s_i, \\ \dot{z}_\ell &= \lambda_{n-\ell}(\chi_I^{-1}(z), u), & n - m_1 + 1 \leq \ell \leq n. \end{aligned} \quad (69)$$

If point 2 is satisfied, the rank of the map  $(x, u) \mapsto (\chi_I(x), \frac{\partial \chi_I}{\partial(x)} f(x, u))$  is constant and thus, according to (66), it is equal to  $n + r_1$ ,  $r_1$  being the rank of  $\Delta_0$ . From (67), the map  $(x, u) \mapsto (\chi_I(x), \lambda(x, u))$  has the same constant rank  $n + r_1$ . Hence there exists  $\phi : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}^{m-r_1}$  such that

$$(x, u) \mapsto (\chi_I(x), \lambda(x, u), \phi(x, u)) \quad (70)$$

is a smooth (resp. analytic) diffeomorphism. Obviously, defining  $\chi_{II}$  by  $\chi_{II}(x, u) = (\lambda(x, u), \phi(x, u))$  yields a smooth (resp. analytic) diffeomorphism  $\chi$  that conjugates (3) to a linear controllable system  $\dot{z} = Az + Bu$ . This proves sufficiency in Theorem 5.7.

If point 2' is satisfied instead, let  $\psi : \mathcal{W} \rightarrow F(\mathcal{W}) \times \mathbb{R}^{n-r_1}$  be the ‘‘trivializing’’ homeomorphism. Recall that, with  $\pi : F(\mathcal{W}) \times \mathbb{R}^{n-r_1} \rightarrow F(\mathcal{W})$  the natural projection, one has  $\pi \circ \psi = F$ ; call  $\phi : \mathcal{W} \rightarrow \mathbb{R}^{n-r_1}$  the map such that  $\psi = F \times \phi$ . Composing  $F$  with  $(x, \xi) \mapsto (\chi_I(x), \frac{\partial \chi_I}{\partial(x)} \xi)$ , one gets that  $(x, u) \mapsto \chi(x, u) = (\chi_I(x), \lambda(x, u), \phi(x, u))$  is a homeomorphism. It clearly conjugates (3) to a linear controllable system  $\dot{z} = Az + Bu$ . This proves sufficiency in Theorem 5.8.  $\square$

**6.2. Proof of Theorem 5.2.** Without loss of generality, we suppose that  $(\bar{x}, \bar{u}) = (0, 0)$ . Assume there exists a homeomorphism  $\chi$  from a neighborhood of the origin in  $\mathbb{R}^{n+m}$  to an open subset of  $\mathbb{R}^{n+m}$  that conjugates system (3) to the linear controllable system

$$\dot{z} = Az + Bv \quad (71)$$

with  $z \in \mathbb{R}^n$  and  $v \in \mathbb{R}^m$ . Composing  $\chi$  with a linear invertible map allows us to suppose that the pair  $(A, B)$  is in canonical form (39)-(40), i.e. that (71) can be read

$$\dot{z}_{\sigma_i+k} = z_{\sigma_{i-1}+k}, \quad 2 \leq i \leq \rho, 1 \leq k \leq s_{i-1}, \quad (72)$$

where the integers  $s_i$  and  $\sigma_i$  were defined in (37) and (38) and where, for notational compactness, we have set :

$$z_{n+k} \triangleq v_k; \quad (73)$$

recall here that  $s_0 = m$ , and notice that  $s_1 < m$  may well occur as it simply means that  $\text{Rank } B < m$ , in which case some of the controls do not appear in the canonical form. With the aggregate notation :

$$Z_j \triangleq \begin{pmatrix} z_{\sigma_{j+1}+1} \\ \vdots \\ z_{\sigma_j} \end{pmatrix}, \quad 1 \leq j \leq \rho - 1, \quad Z_0 \triangleq \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}, \quad (74)$$

and the matrices  $J_r^s$  defined in (40), system (72) can be rewritten as

$$\begin{aligned} \dot{Z}_{\rho-1} &= J_{s_{\rho-2}}^{s_{\rho-1}} Z_{\rho-2} \\ \dot{Z}_{\rho-2} &= J_{s_{\rho-3}}^{s_{\rho-2}} Z_{\rho-3} \\ &\vdots \\ \dot{Z}_2 &= J_{s_1}^{s_2} Z_1 \\ \dot{Z}_1 &= J_{s_0}^{s_1} Z_0 \end{aligned} \quad (75)$$

and is viewed as a control system with state  $(Z_{\rho-1}, \dots, Z_1)$  and control  $Z_0$ . We also make the convention, similar to (73), that

$$x_{n+k} \triangleq u_k, \quad (76)$$

and we use for the controls the aggregate notation :

$$X_0 \triangleq \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}. \quad (77)$$

Let us now prove that property  $\mathcal{P}_\ell$  below is true for  $0 \leq \ell \leq \rho - 1$ .

**Property  $\mathcal{P}_\ell$  :** *there exists a smooth local change of coordinates around 0 in  $\mathbb{R}^n$ , say*

$$(x_1, \dots, x_n) \mapsto (\widehat{X}, X_\ell, \dots, X_2, X_1),$$

with  $\widehat{X} \in \mathbb{R}^{\sigma_{\ell+1}}$  and  $X_i \in \mathbb{R}^{s_i}$  for  $0 \leq i \leq \ell$  (if  $\ell = 0$  there are no  $X_i$ 's beyond  $X_0$  whereas if  $\ell = \rho - 1$  there is no  $\widehat{X}$ ), after which system (3) reads:

$$\begin{aligned} \dot{\widehat{X}} &= \widehat{F}(\widehat{X}, X_\ell) \\ \dot{X}_\ell &= F_\ell(\widehat{X}, X_\ell, X_{\ell-1}) \\ &\vdots \\ \dot{X}_2 &= F_2(\widehat{X}, X_\ell, \dots, X_1) \\ \dot{X}_1 &= F_1(\widehat{X}, X_\ell, \dots, X_1, X_0) \end{aligned}, \quad (78)$$

and such that (78), viewed as a control system with state  $(\widehat{X}, X_\ell, \dots, X_1)$  and control  $X_0$ , is locally topologically conjugate at  $(0, 0)$  to system (75) via a local homeomorphism

$$(\widehat{X}, X_\ell, \dots, X_1, X_0) \mapsto (Z_{\rho-1}, \dots, Z_0)$$

which is, together with its inverse, of the block triangular form :

$$\begin{aligned} (Z_{\rho-1}, \dots, Z_{\ell+1}) &= \widehat{\Phi}(\widehat{X}) & \widehat{X} &= \widehat{\Psi}(Z_{\rho-1}, \dots, Z_{\ell+1}) \\ Z_\ell &= \Phi_\ell(\widehat{X}, X_\ell) & X_\ell &= \Psi_\ell(Z_{\rho-1}, \dots, Z_\ell) \\ &\vdots & &\vdots \\ Z_1 &= \Phi_1(\widehat{X}, X_\ell, \dots, X_1) & X_1 &= \Psi_1(Z_{\rho-1}, \dots, Z_1) \\ Z_0 &= \Phi_0(\widehat{X}, X_\ell, \dots, X_1, X_0) & X_0 &= \Psi_0(Z_{\rho-1}, \dots, Z_1, Z_0) \end{aligned}$$

where  $\Phi_i$  and  $\Psi_i$  are, for  $1 \leq i \leq \ell$ , continuously differentiable with respect to  $X_i$  and  $Z_i$  respectively, have an invertible derivative, and satisfy for  $1 \leq i \leq \ell$  the relation :

$$\begin{aligned} F_i(\widehat{X}, X_\ell, \dots, X_i, X_{i-1}) &= F_i(\widehat{X}, X_\ell, \dots, X_i, 0) \\ &+ \left( \frac{\partial \Phi_i}{\partial X_i}(\widehat{X}, X_\ell, \dots, X_i) \right)^{-1} J_{s_{i-1}}^{s_i} \left( \right. \\ &\quad \left. \Phi_{i-1}(\widehat{X}, X_\ell, \dots, X_i, X_{i-1}) - \Phi_{i-1}(\widehat{X}, X_\ell, \dots, X_i, 0) \right); \end{aligned} \quad (79)$$

furthermore, the partial homeomorphism

$$(\widehat{X}, X_\ell) \mapsto (Z_{\rho-1}, \dots, Z_\ell) \quad (80)$$

locally topologically conjugates, at  $(0, 0) \in \mathbb{R}^{\sigma_{\ell+1} + s_\ell}$ , the reduced control system

$$\dot{\widehat{X}} = \widehat{F}(\widehat{X}, X_\ell), \quad (81)$$

with state  $\widehat{X}$  and input  $X_\ell$ , to the reduced linear control system

$$\begin{aligned}\dot{Z}_{\rho-1} &= J_{s_{\rho-2}}^{s_{\rho-1}} Z_{\rho-2}, \\ &\vdots \\ \dot{Z}_{\ell+1} &= J_{s_\ell}^{s_{\ell+1}} Z_\ell\end{aligned}\tag{82}$$

with state  $(Z_{\rho-1}, \dots, Z_{\ell+1})$  and input  $Z_\ell$ .

Indeed,  $\mathcal{P}_0$  is merely the original assumption on local topological conjugacy of systems (3) and (75), where the triangular structure (12) of the conjugating homeomorphism was taken into account; note that, in  $\mathcal{P}_0$ , (79) is empty and that the reduced system (81) is the original system. Next, supposing that  $\mathcal{P}_\ell$  holds for some  $\ell \geq 0$ , we apply Lemmas 6.2 and 6.3 (see below) to the reduced systems (81), (82), and to the partial homeomorphism (80), with

$$\begin{aligned}d &= \sigma_{\ell+1}, \quad r = s_\ell, \quad s = s_{\ell+1}, \quad U = X_\ell, \quad (x_1, \dots, x_d) = \widehat{X}, \\ Z^1 &= (Z_{\rho-1}, \dots, Z_{\ell+2}), \quad Z^2 = Z_{\ell+1}, \quad \text{and } V = Z_\ell,\end{aligned}$$

and then, upon renaming  $\widetilde{X}^2$  as  $X_{\ell+1}$ ,  $\widetilde{f}^2$  as  $F_{\ell+1}$ , and choosing  $\widetilde{X}^1$  to be the new  $\widehat{X}$ , we get  $\mathcal{P}_{\ell+1}$ .

Now,  $\mathcal{P}_{\rho-1}$ , where we specialize (79) to  $i = 1$ , provides a *smooth* change of variables around 0 in  $\mathbb{R}^n$ :

$$(x_1, \dots, x_n) \mapsto (X_{\rho-1}, \dots, X_2, X_1)$$

with  $X_i \in \mathbb{R}^{s_i}$  such that, in the new coordinates, system (3) reads

$$\begin{aligned}\dot{X}_{\rho-1} &= F_{\rho-1}(X_{\rho-1}, X_{\rho-2}) \\ \dot{X}_{\rho-2} &= F_{\rho-2}(X_{\rho-1}, X_{\rho-2}, X_{\rho-3}) \\ &\vdots \\ \dot{X}_2 &= F_2(X_{\rho-1}, \dots, X_1) \\ \dot{X}_1 &= F_1(X_{\rho-1}, \dots, X_1, X_0),\end{aligned}\tag{83}$$

and also such that the local homeomorphism  $\Phi$  that topologically conjugates system (83) to system (75) at  $(0, 0)$  is, together with its inverse  $\Psi$ , of the triangular form :

$$\begin{aligned}Z_{\rho-1} &= \Phi_{\rho-1}(X_{\rho-1}) & X_{\rho-1} &= \Psi_{\rho-1}(Z_{\rho-1}) \\ Z_{\rho-2} &= \Phi_{\rho-2}(X_{\rho-1}, X_{\rho-2}) & X_{\rho-2} &= \Psi_{\rho-2}(Z_{\rho-1}, Z_{\rho-2}) \\ &\vdots & &\vdots \\ Z_1 &= \Phi_1(X_{\rho-1}, \dots, X_1) & X_1 &= \Psi_1(Z_{\rho-1}, \dots, Z_1) \\ Z_0 &= \Phi_0(X_{\rho-1}, \dots, X_1, X_0) & X_0 &= \Psi_0(Z_{\rho-1}, \dots, Z_1, Z_0),\end{aligned}\tag{84}$$

where the following three properties hold :

- (1) Each  $\Phi_k$  and  $\Psi_k$  for  $k \geq 1$  is continuously differentiable with respect to  $X_k$  and  $Z_k$  respectively; in particular,  $\partial\Phi_k/\partial X_k$  is invertible throughout the considered neighborhood.
- (2) For  $k \geq 2$ ,  $\text{Rank} \frac{\partial F_k}{\partial X_{k-1}}(0, \dots, 0) = s_k$ , i.e. this rank is maximum, equal to the number of rows.
- (3)  $F_1$  satisfies

$$\begin{aligned}F_1(X_{\rho-1}, \dots, X_1, X_0) &= F_1(X_{\rho-1}, \dots, X_1, 0) \\ &+ \left( \frac{\partial \Phi_1}{\partial X_1}(X_{\rho-1}, \dots, X_1) \right)^{-1} J_m^{s_1} \left( \right. \\ &\quad \left. \Phi_0(X_{\rho-1}, \dots, X_1, X_0) - \Phi_0(X_{\rho-1}, \dots, X_1, 0) \right).\end{aligned}\tag{85}$$

From the maximum rank assumption on  $\partial F_{\rho-1}/\partial X_{\rho-2}$ , it is possible to define  $Y_{\rho-2}$  whose first  $s_{\rho-1}$  entries are those of  $F_{\rho-1}(X_{\rho-1}, X_{\rho-2})$  and whose remaining  $s_{\rho-2} - s_{\rho-1}$  entries are suitable components of  $X_{\rho-2}$ , in such a way that

$$(X_{\rho-1}, \dots, X_1) \mapsto (X_{\rho-1}, Y_{\rho-2}, X_{\rho-3}, \dots, X_1)$$

is a local *smooth* change of coordinates around 0 in  $\mathbb{R}^n$ . After performing this change of coordinates and setting  $Y_{\rho-1} = X_{\rho-1}$  for notational homogeneity, system (83) reads

$$\begin{aligned} \dot{Y}_{\rho-1} &= J_{s_{\rho-2}}^{s_{\rho-1}} Y_{\rho-2} \\ \dot{Y}_{\rho-2} &= \tilde{F}_{\rho-2}(Y_{\rho-1}, Y_{\rho-2}, X_{\rho-3}) \\ &\vdots \\ \dot{X}_2 &= \tilde{F}_2(Y_{\rho-1}, Y_{\rho-2}, X_{\rho-3}, \dots, X_1) \\ \dot{X}_1 &= \tilde{F}_1(Y_{\rho-1}, Y_{\rho-2}, X_{\rho-3}, \dots, X_1, X_0) \end{aligned}$$

where the  $\tilde{F}$ 's enjoy the same properties than the  $F$ 's, in particular the maximality of  $\text{Rank } \partial \tilde{F}_k / \partial X_{k-1}(0, \dots, 0)$  for  $\rho - 2 \geq k \geq 2$ . One may iterate this procedure, limited only by the fact that the maximum rank property mentioned above only holds for  $k \geq 2$  but not necessarily for  $k = 1$ . Altogether, this yields a *smooth* local change of coordinates around 0 in  $\mathbb{R}^n$  :

$$(X_{\rho-1}, \dots, X_1) \mapsto (Y_{\rho-1}, \dots, Y_1),$$

after which system (83) is of the form

$$\begin{aligned} \dot{Y}_{\rho-1} &= J_{s_{\rho-2}}^{s_{\rho-1}} Y_{\rho-2} \\ &\vdots \\ \dot{Y}_2 &= J_{s_1}^{s_2} Y_1 \\ \dot{Y}_1 &= F_1(Y_{\rho-1}, \dots, Y_1, X_0), \end{aligned} \tag{86}$$

where we abuse the notation  $F_1$  for simplicity because, although it needs not be the same as in (83), this new  $F_1$  enjoys the same property (85) for some suitably redefined  $\Phi_1$  and  $\Phi_0$ . Now, we may rewrite (85) as

$$F_1(Y_{\rho-1}, \dots, Y_1, X_0) = J_m^{s_1} H(Y_{\rho-1}, \dots, Y_1, X_0) \tag{87}$$

where  $H$ , in the aggregate notation  $Y = (Y_{\rho-1}, \dots, Y_1)$ , is defined by

$$H(Y, X_0) = \begin{pmatrix} F_1(Y, 0) \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial \Phi_1}{\partial Y_1}(Y)^{-1} & 0 \\ 0 & I_{m-s_1} \end{pmatrix} (\Phi_0(Y, X_0) - \Phi_0(Y, 0)).$$

Since  $\Phi$  has the triangular structure displayed in (84), the map  $X_0 \mapsto \Phi_0(Y, X_0)$  is injective for fixed  $Y = (Y_{\rho-1}, \dots, Y_1)$  in the neighborhood of 0 where it is defined in  $\mathbb{R}^m$ . Consequently,  $(Y, X_0) \mapsto (Y, H(Y, X_0))$  is also injective in the neighborhood of 0 where it is defined in  $\mathbb{R}^{n+m}$ ; since it is continuous, it is a local homeomorphism of  $\mathbb{R}^{n+m}$  at  $(0, 0)$  by invariance of the domain, and then (86), (87) make it clear that system (83) is locally quasi-smoothly linearizable at this point.

Since (83) is smoothly conjugate to the original system (3), this proves local quasi-smooth linearizability of the latter hence the theorem.

**Two lemmas.** The following two lemmas are applied recursively in the above proof of Theorem 5.2 to obtain the forms (83), (75), and (84). Although these lemmas team up into a single result in the above-mentioned proof, they have been stated here separately for the sake of clarity.

We will consider two control systems with state in  $\mathbb{R}^d$  and control in  $\mathbb{R}^r$ . Expanded in coordinates, the first system reads

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_d, x_{d+1}, \dots, x_{d+r}) \\ &\vdots \\ \dot{x}_d &= f_d(x_1, \dots, x_d, x_{d+1}, \dots, x_{d+r}), \end{aligned} \quad (88)$$

with state variable  $(x_1, \dots, x_d)$  and control variable  $(x_{d+1}, \dots, x_{d+r}) \in \mathbb{R}^r$ , the functions  $f_1, \dots, f_d$  being smooth  $\mathbb{R}^{d+r} \rightarrow \mathbb{R}$ . The second system has state variable  $(z_1, \dots, z_d)$  and control variable  $(z_{d+1}, \dots, z_{d+r}) \in \mathbb{R}^r$ , and it assumes the special form :

$$\begin{aligned} \dot{z}_1 &= g_1(z_1, \dots, z_d) \\ &\vdots \\ \dot{z}_{d-s} &= g_{d-s}(z_1, \dots, z_d) \\ \dot{z}_{d-s+1} &= z_{d+1} \\ &\vdots \\ \dot{z}_d &= z_{d+s}, \end{aligned} \quad (89)$$

where  $0 < s \leq d$  and  $s \leq r$  while  $g_1, \dots, g_{d-s}$  are again smooth  $\mathbb{R}^d \rightarrow \mathbb{R}$ . Nothing prevents us here from having  $s < r$ , in which case some of the controls do not enter the equation. It will be convenient to use the aggregate notations

$$\begin{aligned} X &\triangleq (x_1, \dots, x_d), & U &\triangleq (x_{d+1}, \dots, x_{d+r}), \\ Z &\triangleq (z_1, \dots, z_d), & V &\triangleq (z_{d+1}, \dots, z_{d+r}), \end{aligned}$$

and to further split  $Z$  into  $(Z^1, Z^2)$  with

$$Z^1 \triangleq (z_1, \dots, z_{d-s}), \quad Z^2 \triangleq (z_{d-s+1}, \dots, z_d), \quad (90)$$

so as to write (88) in the form

$$\dot{X} = f(X, U) \quad (91)$$

and (89) as

$$\begin{aligned} \dot{Z}^1 &= g^1(Z^1, Z^2) \\ \dot{Z}^2 &= J_r^s V, \end{aligned} \quad (92)$$

with  $J_r^s$  the  $s \times r$  matrix, defined in (40), that selects the first  $s$  entries of a vector.

**Lemma 6.2.** *Let  $d, r$  and  $s$  be strictly positive integers with  $s \leq d$  and  $s \leq r$ . Suppose, for some  $\varepsilon > 0$ , that*

$$\varphi : (-\varepsilon, \varepsilon)^{d+r} \rightarrow \mathbb{R}^{d+r}$$

*is a homeomorphism onto its image, with inverse  $\psi$ , that conjugates system (91) to system (92). Then, there exists  $0 < \varepsilon' < \varepsilon$  and a smooth local change of coordinates around  $0 \in \mathbb{R}^d$  :*

$$\theta : (-\varepsilon', \varepsilon')^d \rightarrow \theta((-\varepsilon', \varepsilon')^d) \subset (-\varepsilon, \varepsilon)^d$$

*that fixes the origin and is such that, in the new coordinates  $\tilde{X} = \theta^{-1}(X)$ , both the system (91) and the conjugating homeomorphism  $\tilde{\varphi} = \varphi \circ (\theta \times \text{id})$  assume a block triangular structure with respect to the partition  $\tilde{X} = (\tilde{X}^1, \tilde{X}^2)$ , where  $\tilde{X}^1 \triangleq (\tilde{x}_1, \dots, \tilde{x}_{d-s})$  and  $\tilde{X}^2 \triangleq (\tilde{x}_{d-s+1}, \dots, \tilde{x}_d)$ ; that is to say, on  $(-\varepsilon', \varepsilon')^{d+r}$ , we have that*



- system (88) reads :

$$\begin{aligned}\dot{\tilde{X}}^1 &= \tilde{f}^1(\tilde{X}^1, \tilde{X}^2) \\ \dot{\tilde{X}}^2 &= \tilde{f}^2(\tilde{X}^1, \tilde{X}^2, U),\end{aligned}\tag{93}$$

- On their respective domains of definition, the homeomorphism  $\tilde{\varphi}$  and its inverse  $\tilde{\psi} = (\theta^{-1} \times \text{id}) \circ \psi$  read :

$$\begin{aligned}Z^1 &= \tilde{\varphi}^1(\tilde{X}^1) & \tilde{X}^1 &= \tilde{\psi}_1(Z^1) \\ Z^2 &= \tilde{\varphi}^2(\tilde{X}^1, \tilde{X}^2) & \tilde{X}^2 &= \tilde{\psi}_2(Z^1, Z^2) \\ V &= \tilde{\varphi}^3(\tilde{X}^1, \tilde{X}^2, U) & U &= \tilde{\psi}_3(Z^1, Z^2, V).\end{aligned}\tag{94}$$

**Lemma 6.3.** *Let*

$$\tilde{\varphi} : (-\varepsilon', \varepsilon')^{d+r} \rightarrow \mathbb{R}^{d+r}$$

be a homeomorphism onto its image, having the block triangular structure displayed in (94), and assume that it conjugates the smooth system (93) to the smooth system (92). Necessarily then,  $\tilde{\varphi}$  has the following properties :

- (1) The map  $\tilde{\varphi}^2$  is continuously differentiable with respect to its second argument  $\tilde{X}^2$ , and  $\frac{\partial \tilde{\varphi}^2}{\partial \tilde{X}^2}(0, 0)$  is invertible.

- (2) On some neighborhood of  $0 \in \mathbb{R}^{d+r}$  included in  $(-\varepsilon', \varepsilon')^{d+r}$ , one has :

$$\begin{aligned}\tilde{f}^2(\tilde{X}^1, \tilde{X}^2, U) &= \\ \tilde{f}^2(\tilde{X}^1, \tilde{X}^2, 0) &+ \left( \frac{\partial \tilde{\varphi}^2}{\partial \tilde{X}^2}(\tilde{X}^1, \tilde{X}^2) \right)^{-1} J_r^s \left( \tilde{\varphi}^3(\tilde{X}^1, \tilde{X}^2, U) - \tilde{\varphi}^3(\tilde{X}^1, \tilde{X}^2, 0) \right)\end{aligned}\tag{95}$$

- (3) On some neighborhood of  $0 \in \mathbb{R}^d$  included in  $(-\varepsilon', \varepsilon')^d$ , the partial homeomorphism

$$(\tilde{X}^1, \tilde{X}^2) \mapsto (\tilde{\varphi}^1(\tilde{X}^1), \tilde{\varphi}^2(\tilde{X}^1, \tilde{X}^2))\tag{96}$$

conjugates the control system

$$\dot{\tilde{X}}^1 = \tilde{f}^1(\tilde{X}^1, \tilde{X}^2),\tag{97}$$

with state  $\tilde{X}^1$  and control  $\tilde{X}^2$ , to the control system

$$\dot{Z}^1 = g^1(Z^1, Z^2)\tag{98}$$

with state  $Z^1$  and input  $Z^2$ .

Note that (97) and (98) are reduced systems from (93) and (92).

*Proof of Lemma 6.2.* Since the homeomorphism  $\varphi$  conjugates (91) to (92), we know, by Proposition 3.6, that  $\varphi$  and  $\psi$  split component-wise into :

$$\begin{aligned}Z &= \varphi_{\text{I}}(X) & X &= \psi_{\text{I}}(Z) \\ V &= \varphi_{\text{II}}(X, U) & U &= \psi_{\text{II}}(Z, V).\end{aligned}\tag{99}$$

Consider the map  $f : (-\varepsilon, \varepsilon)^{d+r} \rightarrow \mathbb{R}^d$  given in (91), and let us define  $g : \varphi((-\varepsilon, \varepsilon)^{d+r}) \rightarrow \mathbb{R}^d$  analogously from (92), namely  $g$  is the concatenated map whose first  $d-s$  components are given by  $g^1(Z)$  and whose last  $s$  components are given by  $J_r^s V$ . Define two families of continuous vector fields  $\mathcal{F}'$  and  $\mathcal{G}'$ , on  $(-\varepsilon, \varepsilon)^d$  and  $\varphi_{\text{I}}((-\varepsilon, \varepsilon)^d)$  respectively, by the following formulas (compare (142)) :

$$\mathcal{F}' = \{ \delta f_{\alpha_1, \alpha_2} ; \alpha_1, \alpha_2 \text{ feedbacks on } (-\varepsilon, \varepsilon)^{d+r} \},\tag{100}$$

$$\mathcal{G}' = \{ \delta g_{\beta_1, \beta_2} ; \beta_1, \beta_2 \text{ feedbacks on } \varphi((-\varepsilon, \varepsilon)^{d+r}) \}.\tag{101}$$

Applying Proposition 3.13 twice, first to  $\chi = \varphi$  and then to  $\chi = \psi$ , we see that each integral curve of a vector field in  $\mathcal{F}'$  is mapped by  $\varphi_{\text{I}}$  to some integral curve of a

vector field in  $\mathcal{G}'$  and *vice-versa* upon replacing  $\varphi_I$  by  $\psi_I$ . This shows in particular that uniqueness of solutions to the Cauchy problem associated to vector fields is preserved, i.e. if we define the families of vector fields (compare (143)) :

$$\mathcal{F}'' = \{ Y \in \mathcal{F}', Y \text{ has a flow} \}, \quad (102)$$

$$\mathcal{G}'' = \{ Y \in \mathcal{G}', Y \text{ has a flow} \}, \quad (103)$$

we also have that each integral curve of a vector field in  $\mathcal{F}''$  is mapped by  $\varphi_I$  to an integral curve of a vector field in  $\mathcal{G}''$  and *vice-versa* upon replacing  $\varphi_I$  by  $\psi_I$ . By concatenation, using Proposition B.5, it follows that

$$\left. \begin{array}{l} \text{for any } X \in (-\varepsilon, \varepsilon)^d, \varphi_I \text{ defines a homeomorphism,} \\ \text{for the orbit topologies, from the orbit of } \mathcal{F}'' \text{ through } X \\ \text{onto the orbit of } \mathcal{G}'' \text{ through } \varphi_I(X), \end{array} \right\} \quad (104)$$

where the orbit topology as described in Proposition B.5 (by definition the restriction of  $\varphi_I$  is bi-continuous for the topologies induced by the ambient space; bi-continuity for the orbit topologies requires the description of these topologies as given in Proposition B.5).

Now, the vector fields  $\delta g_{\beta_1, \beta_2}$  appearing in (101) inherit from the structure of  $g$ , displayed in (92), the following particular form :

$$\delta g_{\beta_1, \beta_2}(Z) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_{1,1}(Z) - \beta_{2,1}(Z) \\ \vdots \\ \beta_{1,s}(Z) - \beta_{2,s}(Z) \end{pmatrix}, \quad (105)$$

where  $\beta_{i,1}, \dots, \beta_{i,s}$  designate, for  $i = 1, 2$ , the first  $s$  component of the feedback  $\beta_i$ . This will allow for us to describe explicitly the orbits of  $\mathcal{G}''$ , namely :

$$\left. \begin{array}{l} \text{the orbit of } \mathcal{G}'' \text{ through } Z_0 = (c_1, \dots, c_d) \\ \text{is the connected component containing } Z_0 \text{ of the set} \\ \{ Z \in \varphi_I((-\varepsilon, \varepsilon)^d), z_1 = c_1, \dots, z_{d-s} = c_{d-s} \}. \end{array} \right\} \quad (106)$$

Indeed, the orbit in question is contained in this set, because it is connected, and because all the vector fields in  $\mathcal{G}''$  have their first  $d - s$  components equal to zero by (105).

To prove the reverse inclusion, it is enough to show that the orbit of  $\mathcal{G}''$  through  $Z_0$ , denoted hereafter by  $\mathcal{O}_{\mathcal{G}'', Z_0}$ , contains all the points sufficiently close to  $Z_0$  having the same first  $d - s$  coordinates as  $Z_0$ . Indeed, since  $Z_0$  was arbitrary, this will imply that the connected component defined by (106) splits into a disjoint union of open orbits hence consists of a single one by connectedness. That is to say, putting  $Z_0 = (Z_0^1, Z_0^2)$  according to (90), 106 will follow from the existence of a  $\rho > 0$  such that

$$\{Z_0^1\} \times B(Z_0^2, \rho) = B(Z_0, \rho) \cap \mathcal{O}_{\mathcal{G}'', Z_0}. \quad (107)$$

Now, it follows from Remark B.3 that, for sufficiently small  $\rho$ , each connected component of  $B(Z_0, \rho) \cap \mathcal{O}_{\mathcal{G}'', Z_0}$  is an embedded sub-manifold of  $B(Z_0, \rho)$ . Then, the connected component of  $B(Z_0, \rho) \cap \mathcal{O}_{\mathcal{G}'', Z_0}$  containing  $Z_0$  is, by inclusion, an embedded sub-manifold of the linear manifold  $\{Z_0^1\} \times B(Z_0^2, \rho)$ . In particular, since no strict sub-manifold can be densely embedded in a given manifold, we see that (107) will hold is only we can prove that

$$\left. \begin{array}{l} \text{The connected component containing } Z_0 \text{ of } B(Z_0, \rho) \cap \mathcal{O}_{\mathcal{G}'', Z_0} \\ \text{is dense in } \{Z_0^1\} \times B(Z_0^2, \rho) \text{ for the Euclidean topology.} \end{array} \right\} \quad (108)$$

To prove (108), pick  $V_0$  such that  $(Z_0, V_0) \in \varphi((-\varepsilon, \varepsilon)^{d+r})$  and observe, since the latter is an open set, that shrinking  $\rho$  further, if necessary, allows us to assume  $\overline{B}(Z_0, \rho) \times \overline{B}(V_0, \rho) \subset \varphi((-\varepsilon, \varepsilon)^{d+r})$ . We claim that any continuous map  $\overline{B}(Z_0, \rho) \rightarrow \overline{B}(V_0, \rho)$  extends to a feedback on  $\varphi((-\varepsilon, \varepsilon)^{d+r})$ . Indeed, in view of the one-to-one correspondence  $\beta \rightarrow \psi \blacksquare \beta$  between feedbacks on  $\varphi((-\varepsilon, \varepsilon)^{d+r})$  and feedbacks on  $(-\varepsilon, \varepsilon)^{d+r}$  (cf the discussion leading to (22)-(23)), it is enough to prove that every continuous map  $\psi_I(\overline{B}(Z_0, \rho)) \rightarrow (-\varepsilon, \varepsilon)^r$  extends to a continuous map  $(-\varepsilon, \varepsilon)^d \rightarrow (-\varepsilon, \varepsilon)^r$ , and this in turn follows from the Tietze extension theorem since  $\psi_I(\overline{B}(Z_0, \rho))$  is closed in  $(-\varepsilon, \varepsilon)^d$  and since  $(-\varepsilon, \varepsilon)^r$  is a poly-interval. This proves the claim.

From the claim, it follows that the restriction to  $\overline{B}(Z_0, \rho)$  of the  $\mathbb{R}^s$ -valued vector field  $J_r^s(\beta_1(Z) - \beta_2(Z))$ , accounting for the lower half of the right-hand side in (105), can be assigned *arbitrarily*, by choosing adequately the feedbacks  $\beta_1$  and  $\beta_2$ , among *continuous* vector fields  $\overline{B}(Z_0, \rho) \rightarrow \overline{B}(0, \rho)$  (take  $\beta_2$  to extend the constant map  $V_0$  on  $\overline{B}(Z_0, \rho)$ ). Of course, the corresponding vector field  $\delta g_{\beta_1, \beta_2}$  in (105) belongs to  $\mathcal{G}'$  but not necessarily to  $\mathcal{G}''$  since continuous vector fields need not have a flow. However, since  $\delta g_{\beta_1, \beta_2}$  has a flow at least when  $\beta_1$  and  $\beta_2$  are smooth, we deduce from Proposition 3.4 that the restriction to  $\overline{B}(Z_0, \rho)$  of the vector fields in  $\mathcal{G}''$  are of the form  $\{0\} \times Y$ , where  $Y$  ranges over a uniformly dense subset  $\Upsilon$  of all  $\mathbb{R}^s$ -valued continuous maps  $\overline{B}(Z_0, \rho) \rightarrow \overline{B}(0, \rho)$ . Now, every point in  $B(Z_0^2, \rho)$  can be attained from  $Z_0^2$  upon integrating, *within*  $B(Z_0^2, \rho)$ , a constant vector field of arbitrary small norm. By Lemma A.2 applied with  $\mathcal{U} = B(Z_0^2, \rho)$  and  $K = \{Z_0^2\}$ , the corresponding trajectory can be approximated uniformly by integral curves that remain in  $B(Z_0^2, \rho)$  of vector fields in  $\Upsilon$ . Therefore, every point in  $\{z_0^1\} \times B(Z_0^2, \rho)$  is the limit of endpoints of integral curves of  $\mathcal{G}''$  that remain in  $\{z_0^1\} \times B(Z_0^2, \rho)$ , which proves (108) and thus (106). In particular, the orbits of  $\mathcal{G}''$  are *embedded* sub-manifolds in  $\varphi_I((-\varepsilon, \varepsilon)^d)$ .

Next, we turn to the orbits of  $\mathcal{F}''$ , and we designate by  $\mathcal{O}_{\mathcal{F}'', p}$  the orbit of  $\mathcal{F}''$  in  $] -\varepsilon, \varepsilon[^d$  through the point  $p$ . On the one hand, Proposition B.5 and Theorem B.2 show that  $\mathcal{O}_{\mathcal{F}'', p}$  is a smooth immersed sub-manifold of  $] -\varepsilon, \varepsilon[^d$ . On the other hand, by (104), this immersed sub-manifold is sent homeomorphically by  $\varphi_I$ , both for the orbit topology and the ambient topology, onto  $\mathcal{O}_{\mathcal{G}'', \varphi_I(p)}$  which is a smooth *embedded*  $s$ -dimensional sub-manifold of  $\varphi_I((-\varepsilon, \varepsilon)^d)$ , as we saw from (106). This entails that all orbits of  $\mathcal{F}''$  in  $] -\varepsilon, \varepsilon[^d$  are *embedded* sub-manifolds of dimension  $s$ . Consequently, still from Proposition B.5 and Theorem B.2, there are coordinates  $(\xi_1, \dots, \xi_d)$  defined on an open neighborhood  $W_0$  of the origin in  $] -\varepsilon, \varepsilon[^d$  — this neighborhood may be assumed to be of the form  $\{(\xi_1, \dots, \xi_d), |\xi_i| < \varepsilon'\}$  — such that, in these coordinates,

$$W_0 \cap \mathcal{O}_{\mathcal{F}'', 0} = \{(\xi_1, \dots, \xi_d), \text{ with } (\xi_{s+1}, \dots, \xi_d) \in T\},$$

with  $T$  a subset of  $] -\varepsilon', \varepsilon'^{[d-s}$  containing  $(0, \dots, 0)$ , the tangent space to  $W_0 \cap \mathcal{O}_{\mathcal{F}'', 0}$  at each of its points being *spanned* by  $\partial/\partial\xi_1, \dots, \partial/\partial\xi_s$ , while at any point  $p \in W_0$  the vector fields  $\partial/\partial\xi_1, \dots, \partial/\partial\xi_s$  *belong* to the tangent space of  $\mathcal{O}_{\mathcal{F}'', p}$ . But since we saw that *all* orbits are smooth sub-manifolds of dimension  $s$ , these vector fields actually *span* the tangent space to the orbit at every point. Hence all the vector fields  $\delta f_{\alpha_1, \alpha_2}$  in  $\mathcal{F}''$  have their last  $d - s$  components equal to zero on  $W_0$  in the  $\xi$  coordinates, and this holds in particular when  $\alpha_1, \alpha_2$  range over all constant feedbacks  $(-\varepsilon, \varepsilon)^d \rightarrow (-\varepsilon, \varepsilon)^r$ . This implies, by the very definition of  $\delta f_{\alpha_1, \alpha_2}$ , that  $(\dot{\xi}_{s+1}, \dots, \dot{\xi}_d)$  — as computed from (91) upon performing the change of variable  $X \mapsto (\xi_1, \dots, \xi_d)$  — does not depend on the control variable  $U$ . Choose for  $\tilde{X}$  the  $\xi$  coordinates arranged in reverse order, and let  $\tilde{f}$  be the analog of  $f$  in the new

coordinates  $(\tilde{X}, U)$ . Then the first  $d - s$  components of  $\tilde{f}$  do not depend on  $U$  so that (93) holds. Moreover, if  $\tilde{\varphi}$  denotes the new homeomorphism that conjugates (93) to (92) over  $(-\varepsilon, \varepsilon)^{d+r}$ ,  $\tilde{\varphi}((-\varepsilon, \varepsilon)^{d+r})$ , and if  $\tilde{\psi}$  denotes its inverse, it follows from (104) and the above characterization of the orbits that  $\tilde{\varphi}_I$  maps the sets where  $\tilde{x}_1, \dots, \tilde{x}_{d-s}$  are constant to those where  $z_1, \dots, z_{d-s}$  are constant, thus the functions  $\tilde{\varphi}_1, \dots, \tilde{\varphi}_{d-s}$  and  $\tilde{\psi}_1, \dots, \tilde{\psi}_{d-s}$  depend only on their  $d - s$  first arguments whence (94) follows.  $\square$

*Proof of Lemma 6.3.* We use again the concatenated notation  $\tilde{\varphi}_I = (\tilde{\varphi}^1, \tilde{\varphi}^2)$ ,  $\tilde{\psi}_I = (\tilde{\psi}^1, \tilde{\psi}^2)$ , these partial homeomorphisms being inverse of each other. Let  $(Z_0, V_0) \in \tilde{\varphi}((-\varepsilon', \varepsilon')^{d+r})$  and  $\varepsilon''$  be so small that the product neighborhood  $(Z_0, V_0) + (-\varepsilon'', \varepsilon'')^{d+r}$  lies entirely within  $\tilde{\varphi}((-\varepsilon', \varepsilon')^{d+r})$ . The restriction to  $(Z_0, V_0) + (-\varepsilon'', \varepsilon'')^{d+r}$  of  $\tilde{\psi}$  conjugates (92) to (93). Consequently, for any  $\bar{V} \in (-\varepsilon'', \varepsilon'')^r$ , we may apply Proposition 3.13 to this restriction and to the constant feedbacks  $\alpha_1(Z) = V_0 + \bar{V}$  and  $\alpha_2(Z) = V_0$ ; this yields that  $\tilde{\psi}_I$ , given by

$$(Z^1, Z^2) \mapsto (\tilde{X}^1, \tilde{X}^2) = (\tilde{\psi}^1(Z^1), \tilde{\psi}^2(Z^1, Z^2)),$$

maps every solution of

$$\dot{Z}^1 = 0, \quad \dot{Z}^2 = J_r^s \bar{V} \quad (109)$$

that remains in  $Z_0 + (-\varepsilon'', \varepsilon'')^d$  to a solution of

$$\begin{aligned} \dot{\tilde{X}}^1 = 0, \quad \dot{\tilde{X}}^2 = & \tilde{f}^2(\tilde{X}^1, \tilde{X}^2, \tilde{\psi}^3(\tilde{\varphi}^1(\tilde{X}^1), \tilde{\varphi}^2(\tilde{X}^1, \tilde{X}^2), V_0 + \bar{V})) \\ & - \tilde{f}^2(\tilde{X}^1, \tilde{X}^2, \tilde{\psi}^3(\tilde{\varphi}^1(\tilde{X}^1), \tilde{\varphi}^2(\tilde{X}^1, \tilde{X}^2), V_0)) \end{aligned} \quad (110)$$

that remains in  $\tilde{\psi}_I(Z_0 + (-\varepsilon'', \varepsilon'')^d)$ , and *vice versa* upon applying Proposition 3.13 in the other direction.

Integrating (109) explicitly with initial condition  $Z(0) = Z_0$ , we get that

$$t \mapsto \begin{pmatrix} \tilde{\psi}^1(Z_0^1) \\ \tilde{\psi}^2(Z_0^1, Z_0^2 + tJ_r^s \bar{V}) \end{pmatrix}$$

solves (110) for sufficiently small  $t$ , hence  $\tilde{\psi}^2(Z^1, Z^2)$  is differentiable at  $Z_0$  with respect to its second argument in the direction  $J_r^s \bar{V}$ , with directional derivative

$$\begin{aligned} \frac{\partial \tilde{\psi}^2}{\partial Z^2}(Z_0^1, Z_0^2) J_r^s \bar{V} = & \tilde{f}^2(\tilde{\psi}^1(Z_0^1), \tilde{\psi}^2(Z_0^1, Z_0^2), \tilde{\psi}^3(Z_0^1, Z_0^2, V_0 + \bar{V})) \\ & - \tilde{f}^2(\tilde{\psi}^1(Z_0^1), \tilde{\psi}^2(Z_0^1, Z_0^2), \tilde{\psi}^3(Z_0^1, Z_0^2, V_0)). \end{aligned} \quad (111)$$

In particular, since  $Z_0$  can be any member of  $\tilde{\varphi}_I((-\varepsilon', \varepsilon')^d)$  while  $J_r^s \bar{V}$  can be assigned arbitrarily in  $(-\varepsilon'', \varepsilon'')^s$ , we conclude that  $\partial \tilde{\psi}^2 / \partial Z^2(Z^1, Z^2)$  exists and is continuous since this holds for the partial derivatives. Next we prove that  $\partial \tilde{\psi}^2 / \partial Z^2$  is invertible at every point by showing that its kernel reduces to zero. In fact, if the left-hand side of (111) vanishes, so does the right-hand side which is also the value of the right-hand side of (110) for  $\tilde{X} = \tilde{\psi}_I(Z_0)$ . Therefore the constant map  $t \mapsto \tilde{\psi}_I(Z_0)$  is a solution to (110) over a suitable time interval, and by conjugation the constant map  $t \mapsto Z_0$  is a solution to (109) over that time interval which clearly entails  $J_r^s \bar{V} = 0$ , as desired. Now, since  $\partial \tilde{\psi}^2 / \partial Z^2$  is invertible at every  $(Z^1, Z^2) \in \tilde{\varphi}_I((-\varepsilon', \varepsilon')^d)$ , the triangular structure of (94) and the inverse function theorem together imply that

$$\frac{\partial \tilde{\varphi}^2}{\partial \tilde{X}^2}(\tilde{X}^1, \tilde{X}^2) = \left( \frac{\partial \tilde{\psi}^2}{\partial Z^2}(\tilde{\varphi}^1(\tilde{X}^1), \tilde{\varphi}^2(\tilde{X}^1, \tilde{X}^2)) \right)^{-1} \quad (112)$$

continuously exists and is invertible for  $(\tilde{X}^1, \tilde{X}^2) \in (-\varepsilon', \varepsilon')^d$ . This proves point 1.

Let us turn to point 2. Select an open neighborhood  $\mathcal{W}$  of 0 having compact closure in  $(-\varepsilon', \varepsilon')^d$ , so there is  $\eta > 0$  such that  $\tilde{\varphi}(\tilde{X}, 0) + (-\eta, \eta)^{d+r}$  is included in  $\tilde{\varphi}((-\varepsilon', \varepsilon')^{d+r})$  whenever  $\tilde{X} \in \mathcal{W}$ . If  $\bar{V} \in (-\eta, \eta)^r$ , we can apply (111) to  $(Z_0, V_0) = \tilde{\varphi}(\tilde{X}, 0)$  with  $\tilde{X} \in \mathcal{W}$ , and we obtain in view of (112) :

$$\begin{aligned} \left( \frac{\partial \tilde{\varphi}^2}{\partial \tilde{X}^2}(\tilde{X}^1, \tilde{X}^2) \right)^{-1} J_r^s \bar{V} &= -\tilde{f}^2(\tilde{X}^1, \tilde{X}^2, 0) \\ &+ \tilde{f}^2 \left( \tilde{X}^1, \tilde{X}^2, \tilde{\psi}^3(\tilde{\varphi}^1(\tilde{X}^1), \tilde{\varphi}^2(\tilde{X}^1, \tilde{X}^2), \tilde{\varphi}^3(\tilde{X}^1, \tilde{X}^2, 0) + \bar{V}) \right). \end{aligned} \quad (113)$$

Set

$$U = \tilde{\psi}^3(\tilde{\varphi}^1(\tilde{X}^1), \tilde{\varphi}^2(\tilde{X}^1, \tilde{X}^2), \tilde{\varphi}^3(\tilde{X}^1, \tilde{X}^2, 0) + \bar{V}) \quad (114)$$

and observe that  $(\tilde{X}, \bar{V}) \mapsto (\tilde{X}, U) = \tilde{\psi}(\tilde{\varphi}(\tilde{X}, 0) + (0, \bar{V}))$  defines a continuous map  $h : \mathcal{W} \times (-\eta, \eta)^r \rightarrow (-\varepsilon', \varepsilon')^{d+r}$ , such that  $h(0) = 0$ , which is injective. By invariance of the domain,  $h$  is a homeomorphism onto some open neighborhood of 0, say  $\mathcal{N} \subset (-\varepsilon', \varepsilon')^{d+r}$ . For  $(\tilde{X}, U) \in \mathcal{N}$ , (114) can be inverted as

$$\bar{V} = \tilde{\varphi}^3(\tilde{X}, U) - \tilde{\varphi}^3(\tilde{X}, 0), \quad (115)$$

and substituting (114) and (115) in (113) yields (95).

Finally we prove point 3, keeping in mind the previous definitions and properties of  $h$ ,  $\mathcal{W}$ ,  $\eta$  and  $\mathcal{N}$ . For  $\tilde{X} = (\tilde{X}^1, \tilde{X}^2) \in (-\varepsilon', \varepsilon')^d$ , define  $\bar{V}(\tilde{X}) \in \mathbb{R}^s \times \{0\} \subset \mathbb{R}^r$  by the formula :

$$J_r^s \bar{V}(\tilde{X}) = \frac{\partial \tilde{\varphi}^2}{\partial \tilde{X}^2}(\tilde{X}^1, \tilde{X}^2)(\tilde{f}^2(0, 0, 0) - \tilde{f}^2(\tilde{X}^1, \tilde{X}^2, 0)). \quad (116)$$

Clearly  $\bar{V} : (-\varepsilon', \varepsilon')^d \rightarrow \mathbb{R}^r$  is continuous and  $\bar{V}(0) = 0$ , so there exists an open neighborhood  $\mathcal{V} \subset \mathcal{W}$  of 0 in  $\mathbb{R}^d$  such that  $\bar{V}(\tilde{X}) \in (-\eta, \eta)^r$  as soon as  $\tilde{X} \in \mathcal{V}$ ; then, if we set  $h(\tilde{X}, \bar{V}(\tilde{X})) = (\tilde{X}, U(\tilde{X})) \in \mathcal{N}$ , it follows from (116), (115), and (95) that

$$\tilde{f}^2(\tilde{X}^1, \tilde{X}^2, U(\tilde{X})) = \tilde{f}^2(0, 0, 0), \quad \tilde{X} \in \mathcal{V}. \quad (117)$$

We will show, using Proposition 3.11, that the restriction of  $\tilde{\varphi}_I$  to any relatively compact open subset  $\mathcal{X}$  of  $\mathcal{V}$  conjugates (97) and (98) over  $\mathcal{X}$ ,  $\tilde{\varphi}(\mathcal{X})$ , and this will achieve the proof. To this effect, let  $\mathcal{C}$  to be the collection of all piecewise affine maps  $\mathbb{R} \rightarrow \mathbb{R}^s$  with constant slope  $\tilde{f}^2(0, 0, 0)$  (cf the discussion before Proposition 3.11) and note that, for any open set  $\mathcal{O} \subset \mathbb{R}^s$  and any compact interval  $J \subset \mathbb{R}$ , the restriction of  $\mathcal{C}$  to  $J$  contains, in its uniform closure, the set all piecewise continuous maps  $J \rightarrow \mathcal{O}$ . Now, consider a solution  $\gamma : I \rightarrow \mathcal{V}$  of the control system :

$$\dot{\tilde{X}}^1 = \tilde{f}^1(\tilde{X}^1, \Upsilon) \quad (118)$$

with state  $\tilde{X}^1$  and control  $\Upsilon$ ; hereafter,  $\mathcal{V}_I \subset \mathbb{R}^{d-s}$  and  $\mathcal{V}_{II} \subset \mathbb{R}^s$  will indicate the projections of  $\mathcal{V}$  onto the first  $d-s$  and the last  $s$  components respectively, and similarly for any other open set in  $\mathbb{R}^d$ . Assume that the control function  $\gamma_{II} : I \rightarrow \mathcal{V}_{II}$  is the restriction to  $I$  of some member of  $\mathcal{C}$ . By definition, if  $a, b$  are the endpoints of  $I$  (that may belong to  $I$  or not), there are time instants  $a = t_0 < t_1 < \dots < t_N = b$ , and vectors  $\tilde{\xi}_1, \dots, \tilde{\xi}_N \in \mathbb{R}^s$  such that, for  $1 \leq j < N$ , one has

$$t_{j-1} < t < t_j \Rightarrow \gamma_{II}(t) = \tilde{\xi}_j + t\tilde{f}^2(0, 0, 0), \quad (119)$$

while at the points  $t_j$  themselves  $\gamma_{II}$  is either right or left continuous when  $1 < j < N$ . We claim that  $\tilde{\varphi}_I(\gamma(t))$  is a solution that remains in  $\tilde{\varphi}_I(\mathcal{V})$  of the control system :

$$\dot{Z}^1 = g^1(Z^1, \Gamma) \quad (120)$$

with state  $Z^1$  and control  $\Gamma$ . In fact, since  $\gamma_I$  is continuous by definition of a solution, so is  $\tilde{\varphi}^1(\gamma_I)$  and therefore, as  $\tilde{\varphi}_I(\gamma(t))$  lies in  $\tilde{\varphi}_I(\mathcal{V})$  for all  $t \in I$  by construction, it is enough to check that

$$\tilde{\varphi}^1(\gamma_I(T_2)) - \tilde{\varphi}^1(\gamma_I(T_1)) = \int_{T_1}^{T_2} g^1(\tilde{\varphi}^1(\gamma_I(t)), \tilde{\varphi}^2(\gamma_I(t), \gamma_{II}(t))) dt \quad (121)$$

whenever  $t_{j-1} < T_1 < T_2 < t_j$  for some  $j > 1$ . However, the restriction of  $\gamma(t)$  to  $(t_{j-1}, t_j)$  is a solution that remains in  $\mathcal{V}$  of the differential equation :

$$\begin{aligned} \dot{\gamma}_I &= \tilde{f}^1(\gamma_I, \gamma_{II}) \\ \dot{\gamma}_{II} &= \tilde{f}^2(0, 0, 0), \end{aligned}$$

hence  $(\gamma(t), U(\gamma(t)))$  is, by (117), a solution of (93) that remains in  $\mathcal{N}$ , and therefore (121) follows from the triangular structure (94) of  $\tilde{\varphi}$  and the fact that it conjugates system (93) to system (92). *This proves the claim.*

In the other direction, we observe since it is included in  $\mathcal{W}$  that  $\mathcal{V}$  has compact closure in  $(-\varepsilon', \varepsilon')^d$ , and therefore that  $\tilde{\varphi}_I(\mathcal{V})$  in turn has compact closure in  $\tilde{\varphi}_I((-\varepsilon', \varepsilon')^d)$ . Pick  $\eta' > 0$  such that  $\tilde{\varphi}_I(\mathcal{V}) \times (-\eta', \eta')^r \subset \tilde{\varphi}((-\varepsilon', \varepsilon')^{d+r})$ , and let  $\mathcal{C}'$  denote the collection of all piecewise smooth maps  $\mathbb{R} \rightarrow \mathbb{R}^s$  whose derivative is strictly bounded by  $\eta'$  component-wise. The restriction of  $\mathcal{C}'$  to any compact real interval  $J$  is uniformly dense in the set all piecewise continuous maps  $J \rightarrow \mathcal{O}$ , for any open set  $\mathcal{O} \subset \mathbb{R}^s$ . Clearly, any solution  $\gamma' : I \rightarrow \tilde{\varphi}_I(\mathcal{V})$  of system (120), whose control function  $\gamma'_{II} : I \rightarrow (\tilde{\varphi}_I(\mathcal{V}))_{II}$  is the restriction to  $I$  of some member of  $\mathcal{C}'$ , satisfies the differential equation

$$\begin{aligned} \dot{\gamma}'_I &= g^1(\gamma'_I, \gamma'_{II}) \\ \dot{\gamma}'_{II} &= J_r^s(d\gamma'_{II}/dt, 0) \end{aligned}$$

on every interval where it is smooth. By the very definition of  $\eta'$  and  $\mathcal{C}'$ , it follows that  $(\gamma'(t), (d\gamma'_{II}(t)/dt, 0))$  is, on such intervals, a solution to (92) that remains in  $\tilde{\varphi}((-\varepsilon', \varepsilon')^{d+r})$  and, since  $\tilde{\psi}$  conjugates system (92) to system (93), we argue as before to the effect that  $\psi_I(\gamma')$  is a solution to system (118) that remains in  $\mathcal{V}$ . Appealing to Proposition 3.11, we conclude that  $\tilde{\varphi}_I$  conjugates system (118) to system (120) on relatively compact open subsets of  $\mathcal{V}$ , as desired.  $\square$

#### APPENDIX A. FOUR LEMMAS ON ODES

Throughout this section, we let  $\mathcal{U}$  be an open subset of  $\mathbb{R}^d$ . We say that a continuous vector field  $X : \mathcal{U} \rightarrow \mathbb{R}^d$  has a flow if the Cauchy problem  $\dot{x}(t) = X(x(t))$  with initial condition  $x(0) = x_0$  has a unique solution, defined for  $t \in (-\varepsilon, \varepsilon)$  with  $\varepsilon = \varepsilon(x_0) > 0$ . The flow of  $X$  at time  $t$  is denoted by  $X_t$ , in other words we have with the preceding notations that  $X_t(x_0) = x(t)$ . It is easy to see that the domain of definition of  $(t, x) \mapsto X(t, x)$  is open in  $\mathbb{R} \times \mathcal{U}$ .

**Lemma A.1.** *If  $X : \mathcal{U} \rightarrow \mathbb{R}^d$  is a continuous vector field that has a flow, the map  $(t, x) \mapsto X_t(x)$  is continuous on the open subset of  $\mathbb{R} \times \mathcal{U}$  where it is defined.*

*Proof.* This is an easy consequence of the Ascoli-Arzelà theorem, and actually a special case of [8, chap. V, Theorem 2.1].  $\square$

**Lemma A.2.** *Assume that the sequence of continuous vector fields  $X^k : \mathcal{U} \rightarrow \mathbb{R}^d$  converges to  $X$ , uniformly on compact subsets of  $\mathcal{U}$ , and that all the  $X^k$  as well as  $X$  itself have a flow. Suppose that  $X_t(x)$  is defined for all  $(t, x) \in [0, T] \times K$  with  $T > 0$  and  $K \subset \mathcal{U}$  compact. Then  $X_t^k(x)$  is also defined on  $[0, T] \times K$  for  $k$  large enough, and the sequence of mappings  $(t, x) \mapsto X_t^k(x)$  converges to  $(t, x) \mapsto X_t(x)$ , uniformly on  $[0, T] \times K$ .*

*Proof.* By assumption,

$$K_1 = \{X_t(x); (t, x) \in [0, T] \times K\}$$

is a well-defined subset of  $\mathcal{U}$  that contains  $K$ , and it is compact by Lemma A.1. Let  $K_0$  be another compact subset of  $\mathcal{U}$  whose interior contains  $K_1$ , and put  $d(K_1, \mathcal{U} \setminus K_0) = \eta > 0$  where  $d(E_1, E_2)$  indicates the distance between two sets  $E_1, E_2$ . From the hypothesis there is  $M > 0$  such that  $\|X^k\| \leq M$  on  $K_0$  for all  $k$ , hence the maximal solution to  $\dot{x}(t) = X^k(x(t))$  with initial condition  $x(0) = x_0 \in K$  remains in  $K_0$  as long as  $t \leq \eta/2M$ . Consequently the flow  $(t, x) \mapsto X_t^k(x)$  is defined on  $[0, \eta/2M] \times K$  for all  $k$ , with values in  $K_0$ . We claim that it is a bounded equicontinuous sequence of functions there. Boundedness is clear since these functions are  $K_0$ -valued, so we must show that, to every  $(t, x) \in [0, \eta/2M] \times K$  and every  $\varepsilon > 0$ , there is  $\alpha > 0$  such that  $\|X^k(t', x') - X^k(t, x)\| < \varepsilon$  for all  $k$  as soon as  $|t - t'| + \|x - x'\| < \alpha$ . By the mean-value theorem and the uniform majorization  $\|X^k(X_t^k(x))\| \leq M$ , it is sufficient to prove this when  $t = t'$ . Arguing by contradiction, assume for some subsequence  $k_l$  and some sequence  $x_l$  converging to  $x$  in  $K$  that

$$\|X_t^{k_l}(x) - X_t^{k_l}(x_l)\| \geq \varepsilon \text{ for all } l \in \mathbb{N}. \quad (122)$$

Then, by Lemma A.1, the index  $k_l$  tends to infinity with  $l$ . Next consider the sequence of maps  $F_l : [0, \eta/2M] \rightarrow K_0$  defined by  $F_l(t) = X_t^{k_l}(x_l)$ . Again, by the mean value theorem, it is a bounded equicontinuous family of functions and, by the Ascoli-Arzelà theorem, it is relatively compact in the topology of uniform convergence (compare [8, chap. II, Theorem 3.2]). But if  $\Phi : [0, \eta/2M] \rightarrow K_0$  is the uniform limit of some subsequence  $F_{l_j}$ , and since  $X^{k_{l_j}}$  converges uniformly to  $X$  on  $K_0$  as  $j \rightarrow \infty$ , taking limits in the relation

$$X_t^{k_{l_j}}(x_{l_j}) = x_{l_j} + \int_0^t X^{k_{l_j}}(X_s^{k_{l_j}}(x_{l_j})) ds$$

gives us

$$\Phi(t) = x + \int_0^t X(\Phi(s)) ds$$

so that  $\Phi(t) = X_t(x)$  since  $X$  has a flow. Altogether  $F_l(t)$  converges uniformly to  $X_t(x)$  on  $[0, \eta/2M]$  because this is the only accumulation point, and then (122) becomes absurd. *This proves the claim.* From the claim it follows, using the Ascoli-Arzelà theorem again, that the family of functions  $(t, x) \mapsto X_t^k(x)$  is relatively compact for the topology of uniform convergence  $[0, \eta/2M] \times K \rightarrow K_0$ , and in fact it converges to  $(t, x) \mapsto X_t(x)$  because, by the same limiting argument as was used to prove the claim, every accumulation point  $\Phi(t, x)$  must be a solution to

$$\Phi(t, x) = x + \int_0^t X(s, \Phi(s, x)) ds$$

hence for fixed  $x$  is an integral curve of  $X$  with initial condition  $x$ . In particular, by definition of  $K_1$ , we shall have that  $d(X_t^k(x), K_1) < \eta/2$  for all  $(t, x) \in [0, \eta/2M] \times K$  as soon as  $k$  is large enough. For such  $k$  the flow  $(t, x) \mapsto X_t^k(x)$  will be defined on  $[0, \eta/M] \times K$  with values in  $K_0$ , and we can repeat the whole argument again to the effect that  $X_t^k(x)$  converges uniformly to  $X_t(x)$  there. Proceeding inductively, we obtain after  $\lceil 2TM/\eta \rceil + 1$  steps at most that  $(t, x) \mapsto X_t^k(x)$  is defined on  $[0, T] \times K$  with values in  $K_0$  for  $k$  large enough, and converges uniformly to  $(t, x) \mapsto X_t(x)$  there, as was to be shown.  $\square$

The next lemma stands analogous to Lemma A.2 for time-dependent vector fields, assuming that the convergence holds boundedly almost everywhere in time.

The assumption that the vector fields have a flow is replaced here by a local Lipschitz condition that we now comment upon.

By definition, a time-dependent vector field  $X : [t_1, t_2] \times \mathcal{U} \rightarrow \mathbb{R}^d$  is locally Lipschitz with respect to the second variable if every  $(t_0, x_0) \in [t_1, t_2] \times \mathcal{U}$  has a neighborhood there such that  $\|X(t, x') - X(t, x)\| < c\|x' - x\|$ , for some constant  $c$ , whenever  $(t, x)$  and  $(t, x')$  belong to that neighborhood. This of course entails that  $X$  is bounded on compact subsets of  $[t_1, t_2] \times \mathcal{U}$ . Next, by the compactness of  $[t_1, t_2]$ , the local Lipschitz character of  $X$  strengthens to the effect that each  $x_0 \in \mathcal{U}$  has a neighborhood  $\mathcal{N}_{x_0}$  such that  $\|X(t, x') - X(t, x)\| < c_{x_0}\|x' - x\|$ , for some constant  $c_{x_0}$ , whenever  $x, x' \in \mathcal{N}_{x_0}$  and  $t \in [t_1, t_2]$ . If now  $\mathcal{K} \subset \mathcal{U}$  is compact, we can cover it by finitely many  $\mathcal{N}_{x_{0,k}}$  as above and find  $\varepsilon > 0$  such that  $x, x' \in \mathcal{K}$  and  $\|x - x'\| < \varepsilon$  is impossible unless  $x, x'$  lie in some common  $\mathcal{N}_{x_0}$ . Consequently there is  $c_{\mathcal{K}} > 0$  such that  $\|X(t, x') - X(t, x)\| < c_{\mathcal{K}}\|x' - x\|$  whenever  $x, x' \in \mathcal{K}$  and  $t \in [t_1, t_2]$ , because if  $\|x - x'\| < \varepsilon$  we can take  $c_{\mathcal{K}} \geq \max_k c_{x_{0,k}}$ , whereas if  $\|x - x'\| \geq \varepsilon$  it is enough to take  $c_{\mathcal{K}} > 2M/\varepsilon$  where  $M$  is a bound for  $\|X\|$  on  $[t_1, t_2] \times \mathcal{K}$ . Finally, if  $X(t, x)$  happens to vanish identically for  $x$  outside some compact  $\mathcal{K}' \subset \mathcal{U}$ , we can choose  $\mathcal{K}$  such that

$$\mathcal{K}' \subset \overset{\circ}{\mathcal{K}} \subset \mathcal{K} \subset \mathcal{U}$$

and construct  $c_{\mathcal{K}}$  as before except that we also pick  $\varepsilon > 0$  so small that  $\|x - x'\| < \varepsilon$  is impossible for  $x \in \mathcal{K}'$  and  $x' \notin \mathcal{K}$ . Then it holds that  $\|X(t, x') - X(t, x)\| < c_{\mathcal{K}}\|x' - x\|$  for all  $x, x' \in \mathcal{U}$  and all  $t \in [t_1, t_2]$ , that is to say  $X(t, x)$  becomes globally Lipschitz with respect to  $x$ . These remarks will be used in the proof to come.

**Lemma A.3.** *Let  $t_1 < t_2$  be two real numbers and  $X^k : [t_1, t_2] \times \mathcal{U} \rightarrow \mathbb{R}^d$  a sequence of time-dependent vector fields, measurable with respect to  $t$ , locally Lipschitz continuous with respect to  $x \in \mathcal{U}$ , and bounded on compact subsets of  $[t_1, t_2] \times \mathcal{U}$  independently of  $k$ . Let  $X : [t_1, t_2] \times \mathcal{U} \rightarrow \mathbb{R}^d$  be another time-dependent vector field, measurable with respect to  $t$ , locally Lipschitz continuous with respect to  $x \in \mathcal{U}$ , and assume that, to each compact  $\mathcal{K} \subset \mathcal{U}$ , there is  $E_{\mathcal{K}} \subset [t_1, t_2]$  of zero measure such that, whenever  $t \notin E_{\mathcal{K}}$ , the sequence  $X^k(t, x)$  converges to  $X(t, x)$  as  $k \rightarrow \infty$ , uniformly with respect to  $x \in \mathcal{K}$ . Suppose finally that  $\gamma : [t_1, t_2] \rightarrow \mathcal{U}$  is, for some  $(t_0, x_0) \in [t_1, t_2] \times \mathcal{U}$ , a solution to the Cauchy problem*

$$\dot{\gamma}(t) = X(t, \gamma(t)), \quad \gamma(t_0) = x_0. \quad (123)$$

*Then, for  $k$  large enough, there is a unique solution  $\gamma_k : [t_1, t_2] \rightarrow \mathcal{U}$  to the Cauchy problem*

$$\dot{\gamma}_k(t) = X^k(t, \gamma_k(t)), \quad \gamma_k(t_0) = x_0, \quad (124)$$

*and the sequence  $(\gamma_k)$  converges to  $\gamma$ , uniformly on  $[t_1, t_2]$ .*

*Proof.* Upon multiplying  $X^k(t, x)$  and  $X(t, x)$  by a smooth function  $\varphi(x)$  which is compactly supported  $\mathcal{U} \rightarrow \mathbb{R}$  and identically 1 on a neighborhood of  $\gamma([t_1, t_2])$ , we may assume in view of the discussion preceding the lemma that  $X(t, x)$  and  $X^k(t, x)$  are defined and bounded  $[t_1, t_2] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  independently of  $k$ , measurable with respect to  $t$ , and (globally) Lipschitz continuous with respect to  $x$ .

Then, by classical results [21, Proposition C 3.8., Theorem 54], the solution to (124), say  $\gamma_k$  uniquely exists  $[t_1, t_2] \rightarrow \mathbb{R}^d$  for each  $k$  :

$$\gamma_k(t) = x_0 + \int_{t_0}^t X^k(s, \gamma_k(s)) ds, \quad t \in [t_1, t_2]. \quad (125)$$

From the boundedness of  $X^k$ , it is clear that  $\gamma_k$  is an equicontinuous and bounded family of functions, hence it is relatively compact in the topology of uniform convergence on  $[t_1, t_2]$ . All we have to prove then is that every accumulation point



of  $\gamma_k$  coincides with  $\gamma$ . Extracting a subsequence if necessary, let us assume that  $\gamma_k$  converges to some  $\bar{\gamma}$ , uniformly on  $[t_1, t_2]$ . Let  $\mathcal{K} \subset \mathbb{R}^d$  be a compact set containing  $\gamma_k([t_1, t_2])$  for all  $k$ ; such a set exists by the boundedness of  $\gamma_k$ . If we let  $E_{\mathcal{K}} \subset [t_1, t_2]$  be the set of zero measure granted by the hypothesis, there exists to each  $s \in [t_1, t_2] \setminus E_{\mathcal{K}}$  and each  $\varepsilon > 0$  an integer  $k_{s,\varepsilon}$  such that  $\|X^k(s, x) - X(s, x)\| < \varepsilon$  as soon as  $x \in \mathcal{K}$  and  $k > k_{s,\varepsilon}$ . In another connection, the Lipschitz character of  $X$  with respect to the second argument and the uniform convergence of  $\gamma_k$  to  $\bar{\gamma}$  shows that  $\|X(s, \gamma_k(s)) - X(s, \bar{\gamma}(s))\| < \varepsilon$  for  $k$  large enough. Altogether, by a 2- $\varepsilon$  majorization, we find that

$$\lim_{k \rightarrow \infty} \|X^k(s, \gamma_k(s)) - X(s, \bar{\gamma}(s))\| = 0,$$

that is to say the integrand in the right-hand side of (125) converges point-wise almost everywhere to  $X(s, \bar{\gamma}(s))$ . Since  $X^k$  is bounded we can apply the dominated convergence theorem and, taking limits on both sides of (125) as  $k \rightarrow \infty$ , we find that  $\bar{\gamma}$  is a solution to (123) whereas the latter is unique. Hence  $\bar{\gamma} = \gamma$  as desired.  $\square$

The following averaging lemma for continuous vector fields is less classical than in the locally Lipschitz case, where uniqueness of solutions to the Cauchy problem prevails.

**Lemma A.4.** *Let  $t_1 < t_2$  be real numbers and  $(X^{1,\ell})_{\ell \in \mathbb{N}}, (X^{2,\ell})_{\ell \in \mathbb{N}}$ , be two sequences of continuous time-dependent vector fields  $[t_1, t_2] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ , uniformly bounded with respect to  $\ell$ , that converge uniformly on compact subsets of  $[t_1, t_2] \times \mathbb{R}^d$  to some vector fields  $X^1$  and  $X^2$  respectively. Denoting by  $L = t_2 - t_1$  the length of the time interval, define, for each  $\ell \in \mathbb{N}$ , the ‘‘average’’ vector field  $G_\ell : [t_1, t_2] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  by :*

$$\begin{aligned} t \in [t_1 + \frac{j}{\ell}L, t_1 + \frac{2j+1}{2\ell}L) &\Rightarrow G_\ell(t, x) = X^{1,\ell}(t, x), \\ t \in [t_1 + \frac{2j+1}{2\ell}L, t_1 + \frac{j+1}{\ell}L) &\Rightarrow G_\ell(t, x) = X^{2,\ell}(t, x), \end{aligned} \quad (126)$$

for  $j \in \{0, \dots, \ell - 1\}$  and, say,  $G_\ell(t_2, x) = X^{2,\ell}(t_2, x)$  for definiteness.

Let  $\gamma_\ell : [t_1, t_2] \rightarrow \mathbb{R}^d$  be a solution to

$$\gamma_\ell(t) - \bar{x} = \int_{t_1}^t G_\ell(\tau, \gamma_\ell(\tau)) d\tau. \quad (127)$$

Then the sequence  $(\gamma_\ell)$  is compact in  $C^0([t_1, t_2], \mathbb{R}^d)$ , and every accumulation point  $\gamma_\infty$  is a solution to

$$\gamma_\infty(t) - \bar{x} = \frac{1}{2} \int_{t_1}^t (X^1(\tau, \gamma_\infty(\tau)) + X^2(\tau, \gamma_\infty(\tau))) d\tau. \quad (128)$$

*Proof.* Let

$$M = \sup_{t,x,i,\ell} \|X^{i,\ell}(t, x)\|. \quad (129)$$

From (126)-(127), it is clear that  $M$  is a Lipschitz constant for  $\gamma_\ell$ , regardless of  $\ell$ . In particular  $\gamma_\ell(t)$  stays in a fixed compact ball  $B$  of radius  $ML$ , and the family  $(\gamma_\ell)$  is equicontinuous. From Ascoli-Arzelà's theorem this implies compactness of the sequence  $(\gamma_\ell)$  in the uniform topology on  $[t_1, t_2]$ .

Rewrite (127) as

$$\begin{aligned} \gamma_\ell(t) - \bar{x} &= \int_{t_1}^t \left( G_\ell(\tau, \gamma_\ell(\tau)) - \frac{X^{1,\ell}(\tau, \gamma_\ell(\tau)) + X^{2,\ell}(\tau, \gamma_\ell(\tau))}{2} \right) d\tau \\ &+ \int_{t_1}^t \left( \frac{X^{1,\ell}(\tau, \gamma_\ell(\tau)) + X^{2,\ell}(\tau, \gamma_\ell(\tau))}{2} - \frac{X^1(\tau, \gamma_\ell(\tau)) + X^2(\tau, \gamma_\ell(\tau))}{2} \right) d\tau \\ &+ \int_{t_1}^t \frac{X^1(\tau, \gamma_\ell(\tau)) + X^2(\tau, \gamma_\ell(\tau))}{2} d\tau. \end{aligned} \quad (130)$$

By the uniform convergence of  $X^{i,\ell}$  to  $X^i$ , it will clearly follow that any accumulation point  $\gamma_\infty$  of  $(\gamma_\ell)$  satisfies (128) if only we can show that the first integral in the right-hand side of (130) converges to zero as  $\ell \rightarrow \infty$ .

To prove this, we compute, from the definition of  $G_\ell$  :

$$\begin{aligned} &\int_{t_1 + \frac{j}{\ell}L}^{t_1 + \frac{j+1}{\ell}L} \left( G_\ell(\tau, \gamma_\ell(\tau)) - \frac{X^{1,\ell}(\tau, \gamma_\ell(\tau)) + X^{2,\ell}(\tau, \gamma_\ell(\tau))}{2} \right) d\tau \\ &= \int_{t_1 + \frac{j}{\ell}L}^{t_1 + \frac{2j+1}{2\ell}L} \frac{X^{1,\ell}(\tau, \gamma_\ell(\tau)) - X^{2,\ell}(\tau, \gamma_\ell(\tau))}{2} d\tau \\ &\quad - \int_{t_1 + \frac{2j+1}{2\ell}L}^{t_1 + \frac{j+1}{\ell}L} \frac{X^{1,\ell}(\tau, \gamma_\ell(\tau)) - X^{2,\ell}(\tau, \gamma_\ell(\tau))}{2} d\tau \\ &= \int_{t_1 + \frac{j}{\ell}L}^{t_1 + \frac{2j+1}{2\ell}L} (\Delta_\ell(\tau, \gamma_\ell(\tau)) - \Delta_\ell(\tau + \frac{L}{2\ell}, \gamma_\ell(\tau + \frac{L}{2\ell}))) d\tau \end{aligned} \quad (131)$$

with  $\Delta_\ell = \frac{1}{2}(X^{1,\ell} - X^{2,\ell})$ . On the compact set  $[t_1, t_2] \times B$ , the vector field  $\Delta_\ell$  is uniformly continuous with a modulus of continuity that does not depend on  $\ell$ ; consequently, by the uniform Lipschitz property of  $\gamma_\ell$ , we see for arbitrary  $\varepsilon > 0$  that the norm of the last integral is less than  $\varepsilon/2\ell$  as soon as  $\ell$  is large enough, independently of  $j$ .

Now, the first integral in (130) can be decomposed into a sum of at most  $\ell$  integrals like these we just studied plus an integral over an interval of length smaller than  $1/\ell$ . Since the norm of the integrand is bounded by  $2M$ , the norm of the last term is less than  $2M/\ell$ . Summing over  $j$ , the above estimates tell us that, for  $t \in [t_1, t_2]$  and for  $\ell$  is large enough,

$$\int_{t_1}^t \left( G_\ell(\tau, \gamma_\ell(\tau)) - \frac{X^1(\tau, \gamma_\ell(\tau)) + X^2(\tau, \gamma_\ell(\tau))}{2} \right) d\tau \leq \frac{\varepsilon}{2} + \frac{2M}{\ell}.$$

This achieves the proof since  $\varepsilon > 0$  was arbitrary.  $\square$

## APPENDIX B. ORBITS OF FAMILIES OF VECTOR FIELDS

In the proof of lemma 6.2 we need results from [23] on orbits<sup>6</sup> of families of smooth vector fields, that were recently exposed in the textbook [13, chapter II]. We recall them below, in a slightly expanded form.

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<sup>6</sup>One of the motivations in [23] was to generalize the notion of integral manifolds to vector fields that are smooth but not real analytic. Note that the orbits of a family of *real analytic* vector fields actually coincide with the maximal integral manifolds of the closure of this family under Lie brackets [23, 15, 17]. However, even if we assume the control system (3) to be real analytic, integral manifolds are of no help to us because topological conjugacy does not preserve tangency nor Lie brackets. Using orbits of families of vector fields instead is much more efficient, because topological conjugacy does preserve integral curves.

Let  $\mathcal{F}$  be a family of smooth vector fields defined on an open subset  $U$  of  $\mathbb{R}^d$ . For any positive integer  $N$  and vector fields  $X^1, \dots, X^N$  belonging to  $\mathcal{F}$ , given  $m \in U$ , consider the map  $F$  given by

$$(t_1, \dots, t_N) \mapsto X_{t_1}^1(X_{t_2}^2(\dots(X_{t_N}^N(m))\dots)) \quad (132)$$

where the standard notation  $X_t(x)$  indicates the flow of  $X$  from  $x$  at time  $t$ ; of course,  $F$  depends on the choice of the vector fields  $X^j$  and of the point  $m$ . This map is defined on some open connected neighborhood of the origin, hereafter denoted by  $\text{dom}(F)$ , and takes values in  $U$ . In fact,  $(t_1, \dots, t_N) \in \text{dom}(F)$  if, and only if, for every  $j \in \{1, \dots, N\}$ , the solution  $x(\tau)$  to  $\dot{x} = X^j(x)$ , with initial condition  $x(0) = X_{t_j-1}^{j-1}(\dots(X_{t_1}^1(m))\dots)$ , exists in  $U$  for all  $\tau \in [0, t_j]$  (or  $[t_j, 0]$  if  $t_j < 0$ ).

The *orbit* of the family  $\mathcal{F}$  through a point  $m \in U$  is the set of all points that lie in the image of  $F$  for at least one choice of the vector fields  $X^1, \dots, X^N$ . In words, the orbit of the family  $\mathcal{F}$  through  $m$  is the set of points that may be linked to  $m$  in  $U$  upon concatenating finitely many integral curves of vector fields in the family. We shall denote by  $\mathcal{O}_{\mathcal{F}, p}$  the orbit of  $\mathcal{F}$  through  $m$ .

Note that the definition depends on  $U$  in a slightly subtle manner : if  $\mathcal{F}$  defines by restriction a family of vector fields  $\mathcal{F}|_V$  on a smaller open set  $V \subset U$  and if  $m \in V$ , then

$$V \cap \mathcal{O}_{\mathcal{F}, m} \supset \mathcal{O}_{\mathcal{F}|_V, m}, \quad (133)$$

but the inclusion is generally strict because of the requirement that the integral curves used to construct  $\mathcal{O}_{\mathcal{F}|_V, m}$  should lie entirely in  $V$ .

We turn to topological considerations. The topology of  $U$  is the usual Euclidean topology. The topology of  $\mathcal{O}_{\mathcal{F}, m}$  as an orbit is the finest that makes all the maps  $F$ , arising from (132), continuous on their respective domains of definition, the latter being endowed with the Euclidean topology. The classical smoothness of the flow implies that each  $F$  is continuous  $\text{dom}(F) \rightarrow \mathbb{R}^d$ , hence the topology of  $\mathcal{O}_{\mathcal{F}, m}$  as an orbit is finer than the Euclidean topology induced by the ambient space  $U$ . It can be strictly finer, and this is why we speak of the *orbit topology*, as opposed to the *induced topology*.

Starting from  $\mathcal{F}$ , one defines a larger family of vector fields  $P_{\mathcal{F}}$ , consisting of all the push-forwards<sup>7</sup> of vector fields in  $\mathcal{F}$  through all local diffeomorphisms of the form  $X_{t_1}^1 \circ X_{t_2}^2 \circ \dots \circ X_{t_N}^N$  where  $X^1, \dots, X^N$  belong to  $\mathcal{F}$ . That is to say, vector fields in  $P_{\mathcal{F}}$  are of the form

$$(X_{t_1}^1 \circ \dots \circ X_{t_N}^N)_* X^0 \quad (134)$$

where  $X^0, X^1, \dots, X^N$  belong to  $\mathcal{F}$ .

*Remark B.1.* Note that a member of  $P_{\mathcal{F}}$  is defined on an open set which is generally a strict subset of  $U$ , whereas members of  $\mathcal{F}$  are defined over the whole of  $U$ , and it is understood that a curve  $\gamma : I \rightarrow U$ , where  $I$  is a real interval, will be called an integral curve of  $Y \in P_{\mathcal{F}}$  only when  $\gamma(I)$  is included in the domain of definition of  $Y$ .

For  $x \in U$ , we denote by  $P_{\mathcal{F}}(x)$  the subspace of  $\mathbb{R}^d$  spanned by all the vectors  $Y(x)$ , where  $Y \in P_{\mathcal{F}}(x)$  is defined in a neighborhood of  $x$ .

Theorem B.2 below, which is the central result in this appendix, describes the topological nature of the orbits. To interpret the statement correctly, it is necessary to recall (see for instance [22]) that an *immersed* sub-manifold of a manifold is a subset of the latter which is a manifold in its own right, and is such that the

<sup>7</sup>Recall that the push-forward of a vector field  $X : V \rightarrow \mathbb{R}^d$  through a diffeomorphism  $\varphi : V \rightarrow \varphi(V)$  is the vector field  $\varphi_*X$  on  $\varphi(V)$  whose flow at each time is the conjugate of the flow of  $X$  under the diffeomorphism  $\varphi$ ; it can be defined as  $\varphi_*X(\varphi(x)) = D\varphi(x)X(x)$ , where  $D\varphi(x)$  is the derivative of  $\varphi$  at  $x \in V$ .

inclusion map is an immersion. This allows one to naturally identify the tangent space to an immersed sub-manifold at a given point with a linear subspace of the tangent space to the ambient manifold at the same point. The topology of an immersed sub-manifold is in general finer than the one induced by the ambient manifold; when these two topologies coincide, the sub-manifold is called *embedded*.

**Theorem B.2** (Orbit Theorem, Sussmann [23]). *Let  $\mathcal{F}$  be a family of smooth vector fields defined on an open set  $U \subset \mathbb{R}^d$ , and  $m$  be a point in  $U$ . If  $\mathcal{O}_{\mathcal{F},m}$  denotes the orbit of  $\mathcal{F}$  through  $m$ , then:*

- (i) *Endowed with the orbit topology,  $\mathcal{O}_{\mathcal{F},m}$  has a unique differential structure that makes it a smooth connected immersed sub-manifold of  $U$ , for which the maps (132) are smooth.*
- (ii) *The tangent space to  $\mathcal{O}_{\mathcal{F},m}$  at  $x \in \mathcal{O}_{\mathcal{F},m}$  is  $P_{\mathcal{F}}(x)$ .*
- (iii) *There exists an open neighborhood  $W$  of  $m$  in  $U$ , and smooth local coordinates  $\xi : W \rightarrow (-\eta, \eta)^d \subset \mathbb{R}^d$ , with  $\xi(m) = 0$ , such that*
  - (a) *in these coordinates,  $W \cap \mathcal{O}_{\mathcal{F},m}$  is a product :*

$$W \cap \mathcal{O}_{\mathcal{F},m} = (-\eta, \eta)^q \times T \quad (135)$$

where  $\eta > 0$ ,  $q$  is the dimension of  $\mathcal{O}_{\mathcal{F},m}$ , and  $T$  is some subset of  $(-\eta, \eta)^{d-q}$  containing the origin. The orbit topology of  $\mathcal{O}_{\mathcal{F},m}$  induces on  $W \cap \mathcal{O}_{\mathcal{F},m}$  the product topology where  $(-\eta, \eta)^q$  is endowed with the usual Euclidean topology and  $T$  with the discrete topology.

- (b) *if  $\gamma : [t_1, t_2] \rightarrow W \cap \mathcal{O}_{\mathcal{F},m}$  is an integral curve of a vector field  $Y \in P_{\mathcal{F}}$  (see remark B.1), then  $t \mapsto \xi_i(\gamma(t))$ ,  $q + 1 \leq i \leq d$ , are constant mappings,*
- (c) *the tangent space to  $\mathcal{O}_{\mathcal{F},m}$  at each point  $p \in W \cap \mathcal{O}_{\mathcal{F},m}$  is spanned by the vector fields  $\partial/\partial\xi_1, \dots, \partial/\partial\xi_q$ ,*
- (d) *at any point  $p \in W$ , the vector fields  $\partial/\partial\xi_1, \dots, \partial/\partial\xi_q$  belong to the tangent space to the orbit of  $\mathcal{F}$  through  $p$ .*

*Remark B.3.* Another description of the product topology in point (iii) – (a) is as follows. The connected components of  $W \cap \mathcal{O}_{\mathcal{F},m}$  are the sets

$$S_{W,a} = (-\eta, \eta)^q \times \{a\} \quad (136)$$

for  $a \in T$ , and the topology on each of these connected components is the topology induced by the ambient Euclidean topology. In particular each  $S_{W,a}$  is an *embedded* sub-manifold of  $U$ .

*Proof of Theorem B.2.* Assertion (i) is the standard form of the orbit theorem (cf e.g. [13, Chapter 2, Theorem 1]), while assertion (ii) is a rephrasing of [23, Theorem 4.1, point (b)]. Assertion (iii) apparently cannot be referenced exactly in this form, but we shall deduce it from the previous ones as follows.

By point (ii), the tangent space to  $\mathcal{O}_{\mathcal{F},m}$  at  $m \in S$  is the linear span over  $\mathbb{R}$  of  $Y^1(m), \dots, Y^q(m)$ , where  $Y^1, \dots, Y^q$  are  $q$  vector fields belonging to  $P_{\mathcal{F}}$ , defined on some neighborhood of  $m$ , and such that  $Y^1(m), \dots, Y^q(m)$  are linearly independent (recall that  $q$  is the dimension of  $\mathcal{O}_{\mathcal{F},m}$ ). Let us write

$$Y^j = \left( X_{t_{j,1}}^{j,1} \circ \dots \circ X_{t_{j,N_j}}^{j,N_j} \right)_* X^{j,0}, \quad 1 \leq j \leq q,$$

where  $X^{j,k} \in \mathcal{F}$  for  $0 \leq k \leq N_j$ , and where the  $t_{j,k}$ 's are real numbers for which the concatenated flow exists, locally around  $m$  (compare (134)).

Since  $Y^1(m), \dots, Y^q(m)$  are linearly independent, one may complement them into a basis of  $\mathbb{R}^d$  by adjunction of  $d - q$  independent vectors that may, without

loss of generality, be regarded as values at  $m$  of  $d - q$  smooth vector fields in  $U$ , say  $Y^{q+1}, \dots, Y^d$ . Then, the smooth map

$$L(\xi_1, \dots, \xi_d) = \left( Y_{\xi_1}^1 \circ \dots \circ Y_{\xi_q}^q \circ Y_{\xi_{q+1}}^{q+1} \circ \dots \circ Y_{\xi_d}^d \right) (m) \quad (137)$$

defines a diffeomorphism from some poly-interval  $\mathcal{I}_\eta = \{(\xi_1, \dots, \xi_d), |\xi_i| < \eta\}$  onto an open neighborhood  $W$  of  $m$  in  $U$ , simply because the derivative of  $L$  is invertible at the origin as  $Y^1(m), \dots, Y^d(m)$  are linearly independent by construction. Let  $\xi : W \rightarrow \mathcal{I}_\eta$  denote its inverse.

By the characteristic property of push-forwards, we locally have, for  $1 \leq j \leq q$ , that

$$Y_{\xi_j}^j = X_{t_{j,1}}^{j,1} \circ \dots \circ X_{t_{j,N_j}}^{j,N_j} \circ X_{\xi_j}^{j,0} \circ X_{-t_{j,N_j}}^{j,N_j} \circ \dots \circ X_{-t_{j,1}}^{j,1}. \quad (138)$$

This implies that, in (137), the images under  $L$  of those  $d$ -tuples sharing a common value of  $\xi_{q+1}, \dots, \xi_d$  all lie in the same orbit  $\mathcal{O}_{\mathcal{F}, L(0, \dots, 0, \xi_{q+1}, \dots, \xi_d)}$ . In particular, the map

$$\tau_1, \dots, \tau_q \mapsto \left( Y_{\tau_1 + \xi_1}^1 \circ \dots \circ Y_{\tau_q + \xi_q}^q \circ Y_{\xi_{q+1}}^{q+1} \circ \dots \circ Y_{\xi_d}^d \right) (m)$$

is defined  $\prod_{j=1}^q (-\eta - \xi_j, \eta - \xi_j) \rightarrow W \cap \mathcal{O}_{\mathcal{F}, L(\xi_1, \dots, \xi_d)}$ , and this map is smooth from the Euclidean to the orbit topology by (138) and point (i). If we compose it with the immersive injection  $J_W : W \cap \mathcal{O}_{\mathcal{F}, L(\xi_1, \dots, \xi_d)} \rightarrow W$  (keeping in mind that  $W \cap \mathcal{O}_{\mathcal{F}, L(\xi_1, \dots, \xi_d)}$  is open in  $\mathcal{O}_{\mathcal{F}, L(\xi_1, \dots, \xi_d)}$  since the orbit topology is finer than the Euclidean one), and if we subsequently apply  $\xi$ , we get the affine map

$$\tau_1, \dots, \tau_q \mapsto (\tau_1 + \xi_1, \dots, \tau_q + \xi_q, \xi_{q+1}, \dots, \xi_d). \quad (139)$$

Thus the derivative of (139) factors through the derivative of  $\xi \circ J_W$  at  $L(\xi_1, \dots, \xi_d)$ , which implies (d); from this (c) follows, because  $q$  is the dimension of the orbit through  $m$ . If  $Y \in P_{\mathcal{F}}$  is defined over an open subset of  $W$ , and if we write in the  $\xi$  coordinates  $Y(\xi) = \sum_i a_i(\xi) \partial / \partial \xi_i$ , then, since  $Y(\xi)$  is tangent to  $\mathcal{O}_{\mathcal{F}, \xi}$  by (ii), we deduce from (c), that the functions  $a_{q+1}, \dots, a_d$  vanish on  $\mathcal{O}_{\mathcal{F}, m}$ , whence (b) holds.

We finally prove (a). Considering (137) and (138), a moment's thinking will convince the reader that  $W \cap \mathcal{O}_{\mathcal{F}, m}$  consists exactly, in the  $\xi$  coordinates, of those  $(\xi_1, \dots, \xi_d)$  such that

$$\left( Y_{\xi_{q+1}}^{q+1} \circ \dots \circ Y_{\xi_d}^d \right) (m) \in \mathcal{O}_{\mathcal{F}, m}, \quad (140)$$

which accounts for (135) where  $T$  is the set of  $(d - q)$ -tuples  $(\xi_{q+1}, \dots, \xi_d)$  such that (140) holds. To prove that the orbit topology is the product topology on  $(-\eta, \eta)^q \times T$  where  $T$  is discrete, consider a map  $F$  as in (132), and pick  $\bar{t} = (\bar{t}_1, \dots, \bar{t}_N) \in \text{dom}(F)$  such that  $F(\bar{t}) \in W$  (hence  $F(\bar{t}) \in W \cap \mathcal{O}_{\mathcal{F}, m}$ ); then  $F$  is continuous at  $\bar{t}$  for the product topology because, for  $t$  close enough to  $\bar{t}$ , the values  $\xi_{q+1}(F(t)), \dots, \xi_d(F(t))$  do not depend on  $t$  by (b) (moving  $t_i$  means following the flow of a vector field in  $P_{\mathcal{F}}$ , namely the push-forward of  $X^i$  through  $X_{t_1}^1 \circ \dots \circ X_{t_{i-1}}^{i-1}$ ) while  $\xi_1(F(t)), \dots, \xi_q(F(t))$  vary continuously with  $t$  according to the continuous dependence on time and initial conditions of solutions to differential equations. Since this is true for all maps  $F$ , the orbit topology on  $W \cap \mathcal{O}_{\mathcal{F}, m}$  is finer than the product topology. To show that it cannot be strictly finer, it is enough to prove that the orbit topology coincides with the Euclidean topology on each set  $S_{W,a}$  defined in (136), a basis of which consists of the sets  $O \times \{a\}$  where  $O$  is open in  $(-\eta, \eta)^q$ . Being open for the product topology, these sets are open the orbit topology as well by what precedes and, since  $\mathcal{O}_{\mathcal{F}, m}$  is a manifold by (i), each point  $(y, a) \in O \times \{a\}$  has, in the orbit topology, a neighborhood  $\mathcal{N}_y \subset O \times \{a\}$  which is homeomorphic to an open ball of  $\mathbb{R}^q$  via some coordinate map. When viewed in these coordinates, the injection  $\mathcal{N}_y \rightarrow O \times \{a\}$  from the orbit topology to the Euclidean topology is a continuous injective map from an open ball in  $\mathbb{R}^q$  into  $\mathbb{R}^q$ ,

and therefore it is a homeomorphism onto its image by invariance of the domain. As  $(y, a)$  was arbitrary in  $O \times \{a\}$ , this shows the latter is a union of open sets for the orbit topology, as desired.  $\square$

Consider now the control system :

$$\dot{x} = f(x, u), \quad (141)$$

with state  $x \in \mathbb{R}^d$  and control  $u \in \mathbb{R}^r$ , the function  $f$  being smooth on  $\mathbb{R}^d \times \mathbb{R}^r$ . Let  $\Omega$  be an open subset of  $\mathbb{R}^d \times \mathbb{R}^r$  and, following the notation introduced in section 3, put  $\Omega_{\mathbb{R}^d}$  to denote its projection onto the first factor. In the proof of Theorem 5.2, we shall be concerned with the following family of vector fields on  $\Omega_{\mathbb{R}^d}$  :

$$\mathcal{F}' = \{ \delta f_{\alpha_1, \alpha_2}, \alpha_1, \alpha_2 \text{ feedbacks on } \Omega \}, \quad (142)$$

where feedbacks on  $\Omega$  were introduced in Definition 3.3 and the notation  $\delta f_{\alpha_1, \alpha_2}$  was fixed in (24), (25).

Since feedbacks are only required to be *continuous*,  $\mathcal{F}'$  is a family of continuous *but not necessarily differentiable* vector fields on  $\Omega_{\mathbb{R}^d}$  and, though the existence of solutions to differential equations with continuous right-hand side makes it still possible to define the orbit as the collection of endpoints of all concatenated integrations like (132), Theorem B.2 does not apply in this case. In fact, if one thinks of the possible non-uniqueness of solutions to the Cauchy problem, a moment's thinking will convince him that the orbits of a family of continuous vector fields have no chance of being topological manifolds in general.

To overcome this difficulty, we will consider instead of  $\mathcal{F}'$  the smaller family :

$$\mathcal{F}'' = \{ X \in \mathcal{F}', X \text{ has a flow} \}, \quad (143)$$

where the sentence “ $X$  has a flow” means, as in appendix A, that the Cauchy problem  $\dot{x}(t) = X(x(t))$ ,  $x(0) = x_0$ , has a unique solution, defined for  $|t| < \varepsilon_0$  where  $\varepsilon_0$  may depend on  $x_0$ , whenever  $x_0$  lies in the domain of definition of  $X$ . Let us consider the orbit  $\mathcal{O}_{\mathcal{F}'', m}$  of  $\mathcal{F}''$  through  $m \in \Omega_{\mathbb{R}^d}$ , which is still defined as the union of images of all maps (132) where  $X^j \in \mathcal{F}''$ , the domain of each such map  $F$  being again a connected open neighborhood  $\text{dom}(F)$  of the origin in  $\mathbb{R}^N$  by repeated application of Lemma A.1. As before, we define the orbit topology on  $\mathcal{O}_{\mathcal{F}'', m}$  to be the finest that makes all the maps (132) continuous, and since uniqueness of solutions implies continuous dependence on initial conditions (see [8, chap. V, Theorem 2.1] or Lemma A.1), the orbit topology is again finer than the Euclidean topology. *A priori*, we know very little about  $\mathcal{O}_{\mathcal{F}'', m}$  and its orbit topology as Theorem B.2 does not apply. However, Proposition B.5 below will establish that these notions coincide with those arising from the family  $\mathcal{F}$  of *smooth* vector fields obtained by setting :

$$\mathcal{F} = \{ \delta f_{\alpha_1, \alpha_2}, \alpha_1, \alpha_2 \text{ smooth feedbacks on } \Omega \}. \quad (144)$$

Note that, from the definitions (142), (143) and (144), we obviously have

$$\mathcal{F} \subset \mathcal{F}'' \subset \mathcal{F}', \quad (145)$$

hence the orbits of these families through a given point obey the same inclusions.

*Remark B.4.* It may of course happen that the family  $\mathcal{F}'$  is empty because  $\Omega$  admits no feedback at all. However, if  $\mathcal{F}'$  is not empty, then  $\mathcal{F}$  is not empty either by Proposition 3.4.

**Proposition B.5.** *Suppose that  $f : \mathbb{R}^d \times \mathbb{R}^r \rightarrow \mathbb{R}^d$  is smooth, and let  $\Omega$  be an open subset of  $\mathbb{R}^d \times \mathbb{R}^r$ . Let  $\mathcal{F}''$  be defined by (142)-(143).*

*For any  $m \in \Omega_{\mathbb{R}^d}$ , the orbit  $\mathcal{O}_{\mathcal{F}'', m}$  of  $\mathcal{F}''$  through  $m$  coincides with the orbit through  $m$  of the family  $\mathcal{F}$  of smooth vector fields defined by (144), and the topology*

of  $\mathcal{O}_{\mathcal{F}'',m}$ , as an orbit of  $\mathcal{F}$ , coincides with its topology as an orbit of  $\mathcal{F}''$ . In particular, the conclusions of Theorem B.2 hold if we replace  $\mathcal{F}$  by  $\mathcal{F}''$  and  $U$  by  $\Omega_{\mathbb{R}^d}$ .

*Remark B.6.* With a limited amount of extra-work, it is possible to show that the orbits of  $\mathcal{F}'$  also coincide with those of  $\mathcal{F}$ . Hence they turn out to be manifolds despite the possible non-uniqueness of solutions to the Cauchy problem. However, (132) is no longer convenient to define the orbit topology in this case because the maps  $F$  may be multiply-valued when  $X^j \in \mathcal{F}'$ , and it is simpler to work with the family  $\mathcal{F}''$  anyway.

The proof of the proposition is based on the following lemma.

**Lemma B.7.** *For  $m \in \Omega_{\mathbb{R}^d}$  and  $X^1, \dots, X^N \in \mathcal{F}''$ , let  $F : \text{dom}(F) \rightarrow \Omega_{\mathbb{R}^d}$  be defined by (132). Fix  $\bar{t} = (\bar{t}_1, \dots, \bar{t}_N) \in \text{dom}(F)$  and set  $\bar{m} = F(\bar{t})$ .*

*Then, there is a neighborhood  $\mathcal{T}$  of  $\bar{t}$  in  $\text{dom}(F)$ , with  $F(\mathcal{T}) \subset \mathcal{O}_{\mathcal{F},\bar{m}}$ , such that  $F : \mathcal{T} \rightarrow \mathcal{O}_{\mathcal{F},\bar{m}}$  is continuous from the Euclidean topology to the orbit topology.*

Assuming the lemma for a while, we first prove the proposition.

*Proof of Proposition B.5.* We noticed already from (145) that the orbit of  $\mathcal{F}''$  through  $m$  contains the orbit of  $\mathcal{F}$  through  $m$ . To get the reverse inclusion, consider the map  $F$  defined by (132) for some vector fields  $X^1, \dots, X^N$  belonging to  $\mathcal{F}''$ . Then, observe from Lemma B.7 that  $F$  takes values in a disjoint union of orbits of  $\mathcal{F}$ , and that it is continuous if each orbit in this union is endowed with the orbit topology. Since  $\text{dom}(F)$  is connected,  $F$  takes values in a single orbit, which can be none but  $\mathcal{O}_{\mathcal{F},m}$ . As  $F$  was arbitrary, we conclude that  $\mathcal{O}_{\mathcal{F}'',m} \subset \mathcal{O}_{\mathcal{F},m}$  and therefore the two orbits agree as sets. Moreover, since each map  $F$  was continuous  $\text{dom}(F) \rightarrow \mathcal{O}_{\mathcal{F},m}$ , the orbit topology of  $\mathcal{O}_{\mathcal{F}'',m}$  is by definition finer than the orbit topology of  $\mathcal{O}_{\mathcal{F},m}$ ; but since it is also coarser, by definition of the orbit topology on  $\mathcal{O}_{\mathcal{F},m}$ , because  $\mathcal{F} \subset \mathcal{F}''$ , the two topologies in turn agree as desired.  $\square$

*Proof of Lemma B.7.* Theorem B.2 applied to the family  $\mathcal{F}$ , at the point  $\bar{m} = F(\bar{t})$ , yields an open neighborhood  $W$  of  $\bar{m}$  in  $\Omega_{\mathbb{R}^d}$  and smooth local coordinates  $(\xi_1, \dots, \xi_d) : W \rightarrow (-\eta, \eta)^d$  satisfying properties (iii) – (a) to (iii) – (d) of that theorem. For  $\varepsilon > 0$  denote by  $\mathcal{T}_\varepsilon$  the compact poly-interval :

$$\mathcal{T}_\varepsilon = \{t = (t_1, \dots, t_N) \in \mathbb{R}^N, |t_i - \bar{t}_i| \leq \varepsilon\}.$$

By Lemma A.1,  $F$  is continuous  $\text{dom}(F) \rightarrow \Omega_{\mathbb{R}^d}$  and, since  $\text{dom}(F)$  is an open neighborhood of  $\bar{t}$  in  $\mathbb{R}^N$ , we can pick  $\varepsilon > 0$  such that

$$\mathcal{T}_\varepsilon \subset \text{dom}(F) \quad \text{and} \quad F(\mathcal{T}_\varepsilon) \subset W.$$

As  $X^1, \dots, X^N$  belong to  $\mathcal{F}'' \subset \mathcal{F}'$ , we can write

$$X^\ell = \delta f_{\alpha_1^\ell, \alpha_2^\ell}, \quad 1 \leq \ell \leq N$$

for some collection of feedbacks  $\alpha_1^\ell, \alpha_2^\ell$  on  $\Omega$ . From Proposition 3.4, there exists for each  $(\ell, l) \in \{1, \dots, N\} \times \{1, 2\}$  a sequence of smooth feedbacks on  $\Omega$ , say  $(\beta_i^{\ell,k})_{k \in \mathbb{N}}$ , converging to  $\alpha_i^\ell$  uniformly on  $\Omega_{\mathbb{R}^d}$ . Subsequently, we let  $Y^{\ell,k}$  denote, for  $1 \leq \ell \leq N$  and  $k \in \mathbb{N}$ , the smooth vector field on  $\Omega_{\mathbb{R}^d}$

$$Y^{\ell,k} = \delta f_{\beta_1^{\ell,k}, \beta_2^{\ell,k}}.$$

Clearly  $Y^{\ell,k} \in \mathcal{F}$  and, for each  $\ell$ , we have that  $Y^{\ell,k}$  converges to  $X^\ell$  as  $k \rightarrow \infty$ , uniformly on compact subsets of  $\Omega_{\mathbb{R}^d}$ .

Now, pick  $j \in \{1, \dots, N\}$  and consider a  $N$ -tuple  $t^{(j)} \in \mathcal{T}_\varepsilon$  of the form :

$$t^{(j)} = (\bar{t}_1, \dots, \bar{t}_{j-1}, t_j, \dots, t_N), \quad |t_\ell - \bar{t}_\ell| \leq \varepsilon \quad \text{for } j \leq \ell \leq N.$$

Let also  $\mathbf{1}_j$  designate, for simplicity, the  $N$ -tuple  $(0, \dots, 1, \dots, 0)$  with zero entries except for the  $j$ -th one which is 1. Then, for  $|\lambda| \leq \varepsilon$ , we have that

$$t^{(j)} + \lambda \mathbf{1}_j = (\bar{t}_1, \dots, \bar{t}_{j-1}, \bar{t}_j + \lambda, t_{j+1}, \dots, t_N) \in \mathcal{T}_\varepsilon,$$

and a simple computation allows us to rewrite  $F(t + \lambda \mathbf{1}_j)$  as :

$$F(t^{(j)} + \lambda \mathbf{1}_j) = X_{\bar{t}_1}^1 \circ \dots \circ X_{\bar{t}_{j-1}}^{j-1} \circ X_\lambda^j \circ X_{-\bar{t}_{j-1}}^{j-1} \circ \dots \circ X_{-\bar{t}_1}^1(F(t)).$$

Let us set

$$A_k(\lambda) = Y_{\bar{t}_1}^{1,k} \circ \dots \circ Y_{\bar{t}_{j-1}}^{j-1,k} \circ Y_\lambda^{j,k} \circ Y_{-\bar{t}_{j-1}}^{j-1,k} \circ \dots \circ Y_{-\bar{t}_1}^{1,k}(F(t)).$$

Repeated applications of Lemmas A.1 and A.2 show that, for fixed  $j$  and  $t^{(j)}$ , the map  $\lambda \mapsto A_k(\lambda)$  is well-defined  $[-\varepsilon, \varepsilon] \rightarrow W$  as soon as the integer  $k$  is sufficiently large, and moreover that  $A_k(\lambda)$  converges to  $F(t^{(j)} + \lambda \mathbf{1}_j)$  as  $k \rightarrow +\infty$ , uniformly with respect to  $\lambda \in [-\varepsilon, \varepsilon]$ . Now, by the characteristic property push forwards,  $\lambda \mapsto A_k(\lambda)$  is an integral curve of the smooth vector field

$$Z^k = \left( Y_{\bar{t}_1}^{1,k} \circ \dots \circ Y_{\bar{t}_{j-1}}^{j-1,k} \right)_* Y^{j,k},$$

which is defined on a neighborhood of  $\{F(t^{(j)} + \lambda \mathbf{1}_j); |\lambda| \leq \varepsilon\}$  in  $W$ . Since  $Z^k \in P_{\mathcal{F}}$  (cf equation (134)), it follows from point (iii) – (b) of Theorem B.2 that, for  $k$  large enough,

$$\xi_i \circ A_k(\lambda) = \xi_i \circ A_k(0), \quad \forall \lambda \in [-\varepsilon, \varepsilon], \quad i \in \{q+1, \dots, d\}.$$

It is clear from the definition that  $A_k(0) = F(t^{(j)})$ ; hence, using the continuity of  $\xi_i$  and taking, in the above equation, the limit as  $k \rightarrow +\infty$ , we get

$$\xi_i \circ F(t^{(j)} + \lambda \mathbf{1}_j) = \xi_i \circ F(t^{(j)}), \quad \forall \lambda \in [-\varepsilon, \varepsilon], \quad i \in \{q+1, \dots, d\}. \quad (146)$$

Since  $\xi_{q+1} \circ F(\bar{t}) = \dots = \xi_d \circ F(\bar{t}) = 0$  by definition of  $W$ , successive applications of (146) for  $j = N, \dots, 1$  lead us to the conclusion that

$$\xi_{q+1} \circ F(t) = \dots = \xi_d \circ F(t) = 0, \quad \forall t \in \mathcal{T}_\varepsilon. \quad (147)$$

Equation (147) means that, in the  $\xi$ -coordinates,  $F(\mathcal{T}_\varepsilon) \subset (-\eta, \eta)^q \times \{0\}$ . Hence, from the local description of the orbits in (135) (where  $m$  is to be replaced by  $\bar{m}$ ), we deduce that  $F(\mathcal{T}_\varepsilon) \subset \mathcal{O}_{\mathcal{F}, \bar{m}}$ . Actually, with the notations of (136), we even get the stronger conclusion that

$$F(\mathcal{T}_\varepsilon) \subset S_{W,0}$$

which achieves the proof of the lemma, with  $\mathcal{T} = \mathcal{T}_\varepsilon$ , because the orbit topology on  $S_{W,0}$  is the Euclidean topology by Remark B.3.  $\square$

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