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► **To cite this version:**

Frédéric Blanqui. Higher-order dependency pairs. Eighth International Workshop on Termination - WST 2006, Aug 2006, Seattle, United States. inria-00084821v2

**HAL Id: inria-00084821**

**<https://inria.hal.science/inria-00084821v2>**

Submitted on 11 Sep 2006 (v2), last revised 23 Apr 2018 (v3)

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# 1 Higher-order dependency pairs

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**Abstract.** Arts and Giesl proved that the termination of a first-order rewrite system can be reduced to the study of its “dependency pairs”. We extend these results to rewrite systems on simply typed  $\lambda$ -terms by using Tait’s computability technique.

## 1.1 Introduction

Let  $\mathcal{F}$  be a set of function symbols,  $\mathcal{X}$  be a set of variables and  $\mathcal{R}$  be a set of rewrite rules over the set  $\mathcal{T}(\mathcal{F}, \mathcal{X})$  of first-order terms. Let  $\mathcal{D}$  be the set of symbols occurring at the top of a rule left hand-side and  $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$ . The set  $\mathcal{DP}(\mathcal{R})$  of *dependency pairs* of  $\mathcal{R}$  is the set of pairs  $(l, t)$  such that  $l$  is the left hand-side of a rule  $l \rightarrow r \in \mathcal{R}$  and  $t$  is a subterm of  $r$  headed by some symbol  $f \in \mathcal{D}$ . The term  $t$  represents a potential recursive call. The chain relation is  $\rightarrow_{\mathcal{C}} = \rightarrow_{\mathcal{R}i}^* \rightarrow_{\mathcal{DP}h}$ , where  $\rightarrow_{\mathcal{R}i}^*$  is the reflexive and transitive closure of the restriction of  $\rightarrow_{\mathcal{R}}$  to non-top positions and  $\rightarrow_{\mathcal{DP}h}$  is the restriction of  $\rightarrow_{\mathcal{DP}}$  to top positions. Arts and Giesl prove in [1] that  $\rightarrow_{\mathcal{R}}$  is strongly normalizing (SN) (or terminating, well-founded) iff the chain relation so is. Moreover,  $\rightarrow_{\mathcal{C}}$  is terminating if there is a weak reduction ordering  $>$  such that  $\mathcal{R} \subseteq \geq$  and  $\mathcal{DP}(\mathcal{R}) \subseteq >$  (only dependency pairs need to strictly decrease).

We would like to extend these results to higher-order rewriting. There are several approaches to higher-order rewriting. In Higher-order Rewrite Systems (HRSs) [7], terms and rules are simply typed  $\lambda$ -terms in  $\beta$ -normal  $\eta$ -long form, left hand-sides are patterns à la Miller and matching is modulo  $\beta\eta$ . An extension of dependency pairs for HRSs is studied in [10,9]. In Combinatory Reduction Systems (CRSs) [6], terms are  $\lambda$ -terms, rules are  $\lambda$ -terms with meta-variables, left hand-sides are patterns à la Miller and matching uses  $\alpha$ -conversion and some variable occur-checks. The relation between the two kinds of rewriting is studied in [12]. It appears that the matching algorithms are similar and that, in HRSs, one does more  $\beta$ -reductions after having applied the matching substitution. But, in both cases,  $\beta$ -reduction is used at the meta-level for normalizing right hand-sides after the application of the matching substitution. So, a third more atomic approach is to have no meta-level  $\beta$ -reduction and add  $\beta$ -reduction at the object level. This is the approach that we consider in this paper.

So, we assume given a set  $\mathcal{R}$  of rewrite rules made of simply typed  $\lambda$ -terms and study the termination of  $\rightarrow_{\beta} \cup \rightarrow_{\mathcal{R}}$  when using CRS-like matching. This

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clearly implies the termination of  $\rightarrow_{\mathcal{R}}$  in the corresponding CRS or HRS. Another advantage of this approach is that we can rely on Tait's technique for proving termination [11,3]. This paper explores its use with dependency pairs. This is in contrast with [10,9].

In Tait's technique, to each type  $T$ , one associates a set  $\llbracket T \rrbracket$  of terms of type  $T$ . Terms of  $\llbracket T \rrbracket$  are said *computable*. Before giving some properties of computable terms, let us introduce a few definitions. The sets  $\text{Pos}^+(T)$  and  $\text{Pos}^-(T)$  of *positive and negative positions* in  $T$  are defined as follows:

- $\text{Pos}^+(B) = \{\varepsilon\}$  and  $\text{Pos}^-(B) = \emptyset$  if  $B$  is a base type,
- $\text{Pos}^\delta(T \Rightarrow U) = 1 \cdot \text{Pos}^{-\delta}(T) \cup 2 \cdot \text{Pos}^\delta(U)$ .

We use  $\mathbf{T}$  to denote a sequence of types  $T_1, \dots, T_n$  of length  $|\mathbf{T}| = n$ . The  $i$ -th argument of a function symbol  $f : \mathbf{T} \Rightarrow B$  is *accessible* if  $B$  occurs only positively in  $T_i$ . Let  $\text{Acc}(f)$  be the set of indexes of the accessible arguments of  $f$ . A base type  $B$  is *basic* if, for all  $f : \mathbf{T} \Rightarrow B$  and  $i \in \text{Acc}(f)$ ,  $T_i$  is a base type. After [3,4], given a relation  $R$ , *computability wrt  $R$*  can be defined so that the following properties are satisfied:

- (1) A computable term is strongly normalizable wrt  $\rightarrow_\beta \cup R$ .
- (2) A term of basic type is computable if it is SN wrt  $\rightarrow_\beta \cup R$ .
- (3) A term  $v^{T \Rightarrow U}$  is computable if, for all  $t^T$  computable,  $vt$  is computable.
- (4) If  $t$  is computable then every reduct of  $t$  is computable.
- (5) A term  $ft$  is computable if all its reducts wrt  $\rightarrow_\beta \cup R$  are computable.
- (6) If  $ft$  is computable then, for all  $i \in \text{Acc}(f)$ ,  $t_i$  is computable.
- (7) If  $t$  contains no  $f \in \mathcal{D}$  and  $\sigma$  is computable, then  $t\sigma$  is computable.
- (8) Every term is computable whenever every  $f \in \mathcal{D}$  is computable.

## 1.2 Admissible rules

An important property of the first-order case is that, given a term  $t$ , a substitution  $\sigma$  and a variable  $x \in \mathcal{V}(t)$ ,  $x\sigma$  is strongly normalizable whenever  $t\sigma$  so is. This is not always true in the higher-order case. So, we need to introduce some restrictions on rules to keep this property.

**Definition 1 (Admissible rules)** A rule  $fl \rightarrow r$  is *admissible* if  $\text{FV}(r) \subseteq \text{PCC}(\mathbf{l})$ , where PCC is defined in Figure 1.1.

The Pattern Computability Closure (PCC) is called *accessibility* in [2]. It includes most usual higher-order patterns [8].

**Lemma 2** If  $fl \rightarrow r$  is admissible,  $\text{dom}(\sigma) \subseteq \text{FV}(\mathbf{l})$  and  $\mathbf{l}\sigma$  is computable, then  $\sigma|_{\text{FV}(r)}$  is computable.

**Proof.** We prove by induction that, for all  $u \in \text{PCC}(\mathbf{t})$  and computable substitution  $\theta$  such that  $\text{dom}(\theta) \subseteq \text{FV}(u) \setminus \text{FV}(\mathbf{t})$ ,  $u\sigma\theta$  is computable.

**Fig. 1.1.** Pattern Computability Closure [2]

(arg)	$t_i \in \text{PCC}(\mathbf{t})$
(acc)	$\frac{gu \in \text{PCC}(\mathbf{t}) \quad i \in \text{Acc}(g)}{u_i \in \text{PCC}(\mathbf{t})}$
(lam)	$\frac{\lambda y u \in \text{PCC}(\mathbf{t}) \quad y \notin \text{FV}(\mathbf{t})}{u \in \text{PCC}(\mathbf{t})}$
(app-left)	$\frac{uy \in \text{PCC}(\mathbf{t}) \quad y \notin \text{FV}(\mathbf{t}) \cup \text{FV}(u)}{u \in \text{PCC}(\mathbf{t})}$
(app-right)	$\frac{y^{U \Rightarrow T \Rightarrow U} u \in \text{PCC}(\mathbf{t}) \quad y \notin \text{FV}(\mathbf{t}) \cup \text{FV}(u)}{u \in \text{PCC}(\mathbf{t})}$

- (arg) Since  $\text{dom}(\theta) = \emptyset$ ,  $l_i \sigma \theta = l_i \sigma$  is computable by assumption.
- (acc) By induction hypothesis,  $gu \sigma$  is computable. Thus, by property (6),  $u_i \sigma$  is computable.
- (lam) Let  $\theta' = \theta|_{\text{dom}(\theta) \setminus \{y\}}$ . Wlog, we can assume that  $y \notin \text{codom}(\sigma \theta)$ . Hence,  $(\lambda y u) \sigma \theta' = \lambda y u \sigma \theta'$ . Now, since  $\text{dom}(\theta) \subseteq \text{FV}(u) \setminus \text{FV}(\mathbf{t})$ ,  $\text{dom}(\theta') \subseteq \text{FV}(\lambda y u) \setminus \text{FV}(\mathbf{t})$ . Thus, by induction hypothesis,  $\lambda y u \sigma \theta'$  is computable. Since  $y \theta$  is computable, by (3),  $(\lambda y u \sigma \theta') y \theta$  is computable and, by (4),  $u \sigma \theta' \{y \mapsto y \theta\}$  is computable. Finally, since  $y \notin \text{dom}(\sigma \theta') \cup \text{codom}(\sigma \theta')$ ,  $u \sigma \theta' \{y \mapsto y \theta\} = u \sigma \theta$ .
- (app-left) Let  $v : T_y$  computable and  $\theta' = \theta \cup \{y \mapsto v\}$ . Since  $\text{dom}(\theta) \subseteq \text{FV}(u) \setminus \text{FV}(\mathbf{t})$  and  $y \notin \text{FV}(\mathbf{t})$ ,  $\text{dom}(\theta') = \text{dom}(\theta) \cup \{y\} \subseteq \text{FV}(uy) \setminus \text{FV}(\mathbf{t})$ . Thus, by induction hypothesis,  $(uy) \sigma \theta' = u \sigma \theta' v$  is computable. Since  $y \notin \text{FV}(u)$ ,  $u \sigma \theta' = u \sigma \theta$ . Thus,  $u \sigma \theta$  is computable.
- (app-right) Let  $v = \lambda x^U \lambda y^T x$  and  $\theta' = \theta \cup \{y \mapsto v\}$ . By (3),  $v$  is computable. Since  $\text{dom}(\theta) \subseteq \text{FV}(u) \setminus \text{FV}(\mathbf{t})$  and  $y \notin \text{FV}(\mathbf{t})$ ,  $\text{dom}(\theta') \subseteq \text{FV}(yu) \setminus \text{FV}(\mathbf{t})$ . Thus, by induction hypothesis,  $(yu) \sigma \theta' = v u \sigma \theta'$  is computable. Since  $y \notin \text{FV}(u)$ ,  $u \sigma \theta' = u \sigma \theta$ . Thus, by (4),  $u \sigma \theta$  is computable.  $\square$

### 1.3 Higher-order dependency pairs

In the following, we assume given a set  $\mathcal{R}$  of admissible rules. The sets  $\text{FAP}(t)$  of *full application positions* of a term  $t$  and the *level* of a term  $t$  are defined as follows:

- $\text{FAP}(x) = \emptyset$  and  $\text{level}(x) = 0$
- $\text{FAP}(\lambda x t) = 1 \cdot \text{FAP}(t)$  and  $\text{level}(\lambda x t) = \text{level}(t)$

If  $f \in \mathcal{D}$  then:

- $\text{level}(f t_1 \dots t_n) = 1 + \max\{\text{level}(t_i) \mid 1 \leq i \leq n\}$

$$- \text{FAP}(ft_1 \dots t_n) = \{\varepsilon\} \cup \bigcup_{i=1}^n 1^{n-i} 2 \cdot \text{FAP}(t_i)$$

If  $t \neq ft_1 \dots t_n$  with  $f \in \mathcal{D}$ , then  $\text{FAP}(tu) = 1 \cdot \text{FAP}(t) \cup 2 \cdot \text{FAP}(u)$  and  $\text{level}(tu) = \max\{\text{level}(t), \text{level}(u)\}$ .

**Definition 3 (Dependency pairs)** The set of *dependency pairs* is  $\mathcal{DP} = \{l \rightarrow r|_p \mid l \rightarrow r \in \mathcal{R}, p \in \text{FAP}(r)\}$ . The *chain relation* is  $\rightarrow_C = \rightarrow_{\mathcal{R}_i}^* \rightarrow_{\mathcal{DP}_h}$ , where  $\rightarrow_{\mathcal{R}_i}$  is the restriction of  $\rightarrow_{\mathcal{R}}$  to non-top positions, and  $\rightarrow_{\mathcal{DP}_h}$  is the restriction of  $\rightarrow_{\mathcal{DP}}$  to top positions.

Since  $\rightarrow_C \subseteq \rightarrow_{\mathcal{R}}^+ \supseteq$ ,  $\rightarrow_{\beta_C}$  is terminating whenever  $\rightarrow_{\beta_{\mathcal{R}}}$  so is.

**Theorem 4** Assume that, for all  $l \rightarrow r \in \mathcal{R}$  and  $p \in \text{FAP}(r)$ ,  $\text{FV}(r|_p) \subseteq \text{FV}(r)$  and  $r|_p$  has the type of  $l$  (\*). Then,  $\rightarrow_{\beta_{\mathcal{R}}}$  is terminating if  $\rightarrow_{\beta_C}$  so is.

**Proof.** By (1), this is so if every term is computable wrt  $\rightarrow_{\mathcal{R}}$ . By (8), this is so if every  $f^{\mathbf{T} \Rightarrow B} \in \mathcal{D}$  is computable. By (3), this is so if, for all  $\mathbf{t} : \mathbf{T}$  computable,  $f\mathbf{t}$  is computable. We prove it by induction on  $(f\mathbf{t}, \mathbf{t})$  with  $(\rightarrow_C, (\rightarrow_{\beta_{\mathcal{R}}})_{\text{lex}})_{\text{lex}}$  as well-founded ordering (H1). Indeed, by (1),  $\mathbf{t}$  are strongly normalizable wrt  $\rightarrow_{\beta_{\mathcal{R}}}$ . By (5), it suffices to prove that every reduct of  $f\mathbf{t}$  is computable. If  $\mathbf{t} \rightarrow_{\beta_{\mathcal{R}}} \mathbf{t}'$  then, by (H1),  $f\mathbf{t}'$  is computable since, by (4),  $\mathbf{t}'$  are computable and  $\rightarrow_{\beta_C}(f\mathbf{t}') = \rightarrow_{\beta_C}(f\mathbf{t})$ . Now, assume that there is  $f\mathbf{l} \rightarrow r \in \mathcal{R}$  and  $\sigma$  such that  $\mathbf{t} = \mathbf{l}\sigma$ . Since rules are admissible, by Lemma 2,  $\sigma' = \sigma|_{\text{FV}(r)}$  is computable. We now prove that  $r\sigma'$  is computable by induction on the level  $n$  of  $r$  (H2). Let  $p_1, \dots, p_k$  be the positions in  $r$  of the subterms of level  $n-1$ ;  $\mathbf{y}^i$  be the variables of  $\text{FV}(r|_{p_i}) \setminus \text{FV}(r)$ ;  $x_1, \dots, x_k$  be distinct variables not occurring in  $r$ ;  $r'$  be the term obtained by replacing  $r|_{p_i}$  by  $x_i \mathbf{y}^i$  in  $r$ ; and  $\theta = \{x_i \mapsto \lambda \mathbf{y}^i r|_{p_i} \sigma'\}$ . We have  $\text{level}(r') = 0$  and  $r'\sigma'\theta \rightarrow_{\beta}^* r\sigma'$ . If  $\theta$  is computable then, by (7),  $r'\sigma'\theta$  is computable and we are done. By (\*),  $\{\mathbf{y}^i\} = \emptyset$  and it suffices to prove that  $r_{p_i} \sigma'$  is computable. For all  $i \leq k$ ,  $r|_{p_i}$  is of the form  $g\mathbf{u}$  with  $\text{level}(u_j) < n$ . By (H2),  $\mathbf{u}\sigma'$  are computable and, since  $f\mathbf{t} \rightarrow_C r|_{p_i} \sigma'$ , by (H1),  $x_i \theta$  is computable.  $\square$

The condition on free variables is an important restriction since it is not satisfied by function calls with bound variables like in  $(\lim F) + x \rightarrow \lim \lambda n (Fn + x)$ .

**Theorem 5** An higher-order reduction pair is two relations  $(>, \geq)$  such that:

- $>$  is well-founded and stable by substitution,
- $\geq$  is a reflexive and transitive rewrite relation containing  $\rightarrow_{\beta}$ ,
- $\geq \circ > \subseteq >$ .

In the conditions of Theorem 4,  $\rightarrow_{\beta_C}$  is well-founded whenever  $\mathcal{R} \subseteq \geq$  and  $\mathcal{DP} \subseteq >$ .

**Proof.** By (1), this is so if every term is computable wrt  $\rightarrow_C$ . By (8), this is so if every  $f^{\mathbf{T} \Rightarrow B} \in \mathcal{D}$  is computable. By (3), this is so if, for all

$t : T$  computable,  $ft$  is computable. We prove it by induction on  $(ft, t)$  with  $(>, (\rightarrow_{\beta\mathcal{R}})_{\text{lex}})_{\text{lex}}$  as well-founded ordering (H1). Indeed, by (1) and Theorem 4,  $t$  are strongly normalizable wrt  $\rightarrow_{\beta\mathcal{R}}$ . By (5), it suffices to prove that every reduct of  $ft$  is computable. If  $t \rightarrow_{\beta\mathcal{R}} t'$  then, by (H1),  $ft'$  is computable since, by (4),  $t'$  are computable and  $>(ft') \subseteq >(ft)$  since  $\rightarrow_{\beta\mathcal{R}} \subseteq \geq$  and  $\geq \circ > \subseteq >$ . Now, assume that there is  $fl \rightarrow r \in \mathcal{DP}$  and  $\sigma$  such that  $t = l\sigma$ . Since rules are admissible, by Lemma 2,  $\sigma' = \sigma|_{\text{FV}(r)}$  is computable. Since  $\mathcal{DP} \subseteq >$  and  $>$  is stable by substitution,  $ft > r\sigma'$ . Thus, by (H1),  $r\sigma'$  is computable.  $\square$

An example of reduction pair can be given by using the higher-order recursive path ordering  $>_{\text{horpo}}$  [5]. Take  $\geq = (\rightarrow_{\beta} \cup >_{\text{horpo}})^+$  and  $\geq = (\rightarrow_{\beta} \cup >_{\text{horpo}})^*$ . The study of these two relations has to be done. However,  $>_{\text{horpo}}$  does not take advantage of the fact that  $>$  does not need to be monotonic. Such a relation is given by the weak higher-order recursive computability ordering  $>_{\text{whorco}}$ , whose monotonic closure strictly contains  $>_{\text{horpo}}$  [4]. Moreover,  $>_{\text{whorco}}$  is transitive, which is not the case of  $>_{\text{horpo}}$ . It would therefore be interesting to look for reduction pairs built from  $>_{\text{whorco}}$ .

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