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Marianne Akian, Pierre-Alexandre Bliman

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M. Akian , P.-A. Bliman

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On super-high-frequencies in discontinuous 1st-order delay-differential equations

M. Akian* , P.-A. Bliman†

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Abstract: One considers a scalar 1st-order nonlinear differential equation with a delayed relay-output proportional feedback. One shows that, under a boundedness condition on the nonlinearity only, any solution of this equation has, after a finite time, a finite number of zeros on compact sets. An estimate of the time after which the super-high-frequency disappears is provided. This improves some previous works by Shustin. As a consequence, using some works by Fridman *et al.*, any solution of the system under study coincides, after a finite time, with one of the periodic solutions.

Key-words: super-high-frequency oscillations, delay differential equations, periodic solutions, slowly oscillating solutions, relay nonlinearity.

(Résumé : tsvp)

* email: Marianne.Akian@inria.fr
† email: Pierre-Alexandre.Bliman@inria.fr

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : 01 39 63 55 11 - International : +33 1 39 63 55 11
Télécopie : (33) 01 39 63 53 30 - International : +33 1 39 63 53 30

Sur les oscillations à super-haute fréquence d'équations différentielles à retard du 1er ordre discontinues

Résumé : On considère une équation différentielle non-linéaire du 1er ordre avec retard et relais. On montre que, sous une condition de bornitude de la non-linéarité uniquement, toute solution de cette équation admet, au bout d'un temps fini, une nombre fini de zéros sur tout ensemble compact. On fournit de plus une estimation de la durée au bout de laquelle les super-hautes-fréquences disparaissent. On améliore ainsi des résultats obtenus précédemment par Shustin. Ceci permet de prouver, à partir de travaux de Fridman *et al.*, que toute solution du système étudié coïncide au bout d'un temps fini avec une des solutions périodiques.

Mots-clé : oscillations à super-haute fréquence, équations différentielles à retard, solutions périodiques, solutions lentement oscillantes, relais.

1 Introduction

One studies here the scalar delay-differential equation

$$\dot{X} = -\mathrm{sgn}X(t-h) + F(X(t)),\ X_0 = x_0\ , \tag{1}$$

where $h > 0$, $x_0 \in \mathcal{C}((-h,0])$ and $X_t \stackrel{\Delta}{=} X|_{(t-h,t]}$, $t \geq 0$. This equation arises e.g. when controlling the nonlinear system $\dot{x} = F(x)+u$ by use of a proportional (negative) feedback u on the delayed output of a relay sensor. For example, the case where F is linear decreasing comes from an automotive control problem, see Franklin *et al.* [5], Akian *et al.* [1]. Equation (1) is a particular case of a more general class, given by

$$\dot{X} = f(X(t), X(t-h))\ , \tag{2}$$

which arises in a variety of models in the literature, see Diekmann *et al.* [3].

For equation (1), it is shown[1] in Fridman *et al.* [6] that, under the condition:

$$F \text{ is a } \mathcal{C}^1 \text{ function such that } \sup_x |F(x)| < 1\ , \tag{3}$$

if x_0 has a finite number of zeros, then the same holds for X_t, $t \geq 0$. Indeed, denoting in this case $Z \subset [0,+\infty)$ the set of zeros of X with change of sign and $\nu(t) = \mathrm{card}\ Z \cap [t'-h, t')$ where $t' = \inf\ [t,+\infty) \cap Z$, the function[2] ν is well-defined, nonincreasing and even on $[h,+\infty)$ (see however the precise definition of Z in the general case in Section 4 below). For any $n \in \mathbb{N}$, there exists a unique periodic solution with $\nu \equiv 2n$ [6]. The slowest one only (corresponding to $n = 0$) is orbitally stable. It is also the only slowly-oscillating[3] periodic solution. Any solution X departing from an initial condition x_0 with finite number of zeros, is equal, after a finite time, to one of the periodic solutions [6].

Related results have been obtained for various, mostly smooth, systems of type (2), on existence, uniqueness and stability of slowly-oscillating periodic solutions. Mallet-Paret has shown in [11] that, under conditions involving in particular a negative feedback condition in the delay, the dynamical system described by (2) possesses a global integer valued Lyapunov function (just as ν here), which gives rise to a Morse decomposition of the attractor into a finite ordered collection of compact subsets of the phase space, invariant by the flow. Each one of these subsets, called the Morse sets, contains a periodic solution [11, Theorem D]. In the case of equation (1), the Morse sets are the singletons containing the periodic trajectories with $\nu \equiv 2n$, $n \in \mathbb{N}$. Due to the discontinuity of sgn on 0, their number is infinite.

Also, results on existence of chaotic motions have been obtained by several authors. For an overview of these results and a commented bibliography, the reader is refered to

[1] The fields of application of the results by Fridman *et al.* [6] and Shustin [17] are indeed wider, as they also include some nonautonomous systems.

[2] Here we choose a definition of ν in the spirit of Mallet-Paret [11], whereas Shustin's choice [17] $t' = \sup\ [0,t] \cap Z$ necessitates special care for initialization.

[3] that is, such that any two distinct zeros are distant from at least the value h of the delay, see Nussbaum [14], Diekmann *et al.* [3].

the monography by Diekmann *et al.* [3, Chapters XV and XVI]. See also the papers by Peters [15] and Ivanov *et al.* [10] for some results concerning systems of type (2) where f is discontinuous wrt the second variable.

At last, some partial results for 2nd order systems have been obtained by An der Heiden *et al.* [2] and Fridman *et al.* [7].

One is interested here by the case where x_0 is a continuous function with infinite number of zeros (super-high-frequency). In Shustin [17, Theorem 0.5], it is shown that there exists $C > 0$ (dependent on F, but not on x_0 nor h) such that $\nu(t)$ is well-defined and finite for

$$t \geq hC\left(1 + \left(\frac{\delta_0}{h}\right)^{-4}\right) \text{ with } \delta_0 \stackrel{\Delta}{=} \max\{t - t' \,:\, -h \leq t' \leq t \leq 0, x_0 \neq 0 \text{ on } (t', t)\} \,. \quad (4)$$

However, together with (3), a supplementary condition is required in Shustin [17, Formula (0.6)], namely that $(F(hx) - F(0))/x$ be small, together with its first derivative. This can be interpreted as a *small-delay condition.* In Theorem 2, one generalizes Shustin's result and sharpen (4). For the proof, one uses the technique of Shustin [17], that is, for a solution X of (1), one considers a "measure" on the lengths of open intervals where X_t has constant sign and proves that this measure strictly decreases with respect to n, when $t = \alpha - nh$ and α is an accumulation point of the set Z of zeros of X with change of sign. The measure being bounded, this leads to a bound on α, and then on t such that $\nu(t)$ is infinite. In short, one proves that the super-high-frequencies are asymptotically stable for the backward time flow, and then unstable for the forward time flow. Such a technique is also used in Fridman *et al.* [6] and Akian *et al.* [1] to prove the unstability of the non slowly-oscillating periodic solutions.

Some results of Fridman *et al.* [6] and Shustin [17] (namely the fact that ν decreases and the disappearance of super-high-frequency oscillations) have been generalized by Dix [4] in the case of a varying delay such that $\mathrm{Id} - h$ is strictly increasing. One believes that the results presented in the present paper may be generalized in the same way.

An important point in equation (1) is the choice of sgn0. Here, sgn is chosen as a single-valued map, otherwise uniqueness of solution of (1) cannot be expected, due to the delay. Instead of choosing $\mathrm{sgn}0 = 0$ as in Shustin [17], it seems more realistic, in the context of control, to take $\mathrm{sgn}0 \in \{-1, +1\}$: the sign usually models a binary sensor or actuator. Let us define finally $\mathrm{sgn}X$ as follows. Let $z \in L^\infty(-h, +\infty)$ with $|z(t)| \equiv 1$, one denotes:

$$\mathrm{sgn}X(t) = \begin{cases} 1 & \text{if } X(t) > 0 \,, \\ -1 & \text{if } X(t) < 0 \,, \\ z(t) & \text{if } X(t) = 0 \,. \end{cases} \quad (5)$$

This permits to model various policies: e.g. $\mathrm{sgn}0 = 1$, or $\mathrm{sgn}X(t)$ switches as late as possible ... An important consequence is that any solution of (1)-(5) verifies: meas $\{t \geq 0 \,:\, X(t) = 0\} = 0$, see Lemma 5. Moreover, as solving (1) requires regularity of the right-hand side

only in those points where it vanishes, (5) permits to weaken the regularity assumption on F.

The main results are given in Section 2. Theorem 1 states the existence of solutions of (1)-(5), proved in Section 3. Theorem 2 states the disapperance of super-high-frequency oscillations, proved in Section 4. In Corollary 3, the condition $\sup_x |F(x)| < 1$ is weakened. In Corollary 4, boundedness of the solutions of (1)-(5) is proved using Theorem 2 and the results of Fridman *et al.* [6]. The bounds given there are optimal.

2 Main results

Let us first state the existence of solutions to (1).

Theorem 1. *Suppose that F is a measurable function such that $\sup_{x\in\mathbb{R}} |F(x)| < 1$. Then, for any $x_0 \in \mathcal{C}((-h,0])$, there exists at least one function $X \in \mathcal{C}((-h,+\infty))$, absolutely continuous on $[0,+\infty)$ and satisfying (1)-(5) almost everywhere. By definition, such a function is called a* solution of (1)-(5) on $[0,+\infty)$.

Uniqueness of the solution is guaranteed when x_0 has a finite number of zeros, see Section 3. For general initial conditions, it may be ensured by assuming e.g. F Lispchitz continuous or F nonincreasing. However, there is no evidence that the measurability of F does not guarantee uniqueness too.

Our central result is the following:

Theorem 2. *Suppose that F is a measurable function such that $\sup_{x\in\mathbb{R}} |F(x)| < 1$. Then, for any $x_0 \in \mathcal{C}((-h,0])$, there exists $t_{x_0} \geq 0$ such that, for any solution X of (1)-(5) on $[0,+\infty)$, X_t has a finite number of zeros for $t \geq t_{x_0}$. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ (dependent on $\sup_x |F(x)|$ only), such that*

$$t_{x_0} \leq hC_\varepsilon \left(1 + \left(\frac{\delta}{h}\right)^{-2-\varepsilon}\right)$$

$$\text{with } \delta \stackrel{\Delta}{=} \sup\{\text{meas } I \,:\, I \text{ interval } \subset (-h,0),\ \text{sgn} x_0 \text{ constant a.e. on } I\}\ . \quad (6)$$

The value of δ depends jointly upon x_0 and z, and one shows easily that $\delta \geq \delta_0$.

Corollary 3. *Suppose that F is measurable, and that there exists an open set $\Omega \subset \mathbb{R}$ such that $\sup_{x\in\Omega} |F(x)| < 1$ and such that, for any solution X of (1)-(5) on $[0,+\infty)$, there exists $t_{X,0} > 0$ with $X(t) \in \Omega$ for $t \geq t_{X,0}$. Then, for any solution X of (1)-(5) on $[0,+\infty)$, there exists $t_X > 0$ such that X_t has a finite number of zeros for $t > t_X$.*

The assumptions of Corollary 3 are fulfilled e.g. when F verifies: $xF(x) \leq 0$ on $\mathbb{R}$. In particular, the case treated in Akian *et al.* [1], where $F(x) \equiv -x$, follows from Corollary 3. Proof of Corollary 3 is left to the reader.

When no super-high-frequency may be sustained, the asymptotic behavior is summarized by the periodic solutions. This offers the possibility to express some asymptotic properties of the solutions. As an example, one may prove the estimates contained in the following corollary, which are the best possible. Let us define the increasing functions $\mathcal{F}_\pm$ by

$$\mathcal{F}_\pm(b) \stackrel{\Delta}{=} \int_0^b \frac{dx}{1 \pm F(x)} \ . \tag{7}$$

Corollary 4. *If the assumptions of Corollary* 3 *are fulfilled, then, for any solution* X *of* (1)-(5) *on* $[0,+\infty)$*, there exists* $t_X \geq 0$ *such that, for any* $t \geq t_X$*,* $t \mapsto X(t)$ *is periodic, and*

$$(\mathcal{F}_-)^{-1}(-h) \leq X(t) \leq (\mathcal{F}_+)^{-1}(h) \ , \tag{8}$$

$$\sup\{t'-t \ : \ t' \geq t, \ X \geq 0 \text{ a.e. on } [t,t']\} \leq T_+ \stackrel{\Delta}{=} h + \mathcal{F}_- \circ (\mathcal{F}_+)^{-1}(h) \ , \tag{9}$$

$$\sup\{t'-t \ : \ t' \geq t, \ X \leq 0 \text{ a.e. on } [t,t']\} \leq T_- \stackrel{\Delta}{=} h - \mathcal{F}_+ \circ (\mathcal{F}_-)^{-1}(-h) \ . \tag{10}$$

Moreover, if F *is odd, then, for any solution* X *of* (1)-(5) *on* $[0,+\infty)$*,*

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t X(t) \ dt = 0 \ . \tag{11}$$

The bounds on X in (8) are null when h vanishes. This must be linked with the fact that when $h = 0$, any solution of (1)-(5) vanishes identically after a finite time. The inequalities in (9), (10) are optimal for the slowly-oscillating periodic solution; the period of this latter is equal to $T_+ + T_-$ with $T_\pm$ defined in (9), (10).

Proof of Corollary 4. When F is regular, Fridman *et al.* [6] proved that, when the initial condition x_0 has a finite number of zeros, any solution of (1)-(5) is equal, after a finite time t_{x_0}, to one of the periodic solutions. Using Theorem 2, the restriction on the number of zeros may be removed. Now, it is easy to show that the amplitude and the intervals of constant sign of the slowly-oscillating solution are larger than the same quantities computed for the other periodic solutions. Using this property, one gets (8) to (10). Indeed, for $t \geq t_{x_0}$, one gets

$$\sup\{t'-t \ : \ t' \geq t, \ X \geq 0 \text{ (resp. } X \leq 0) \text{ a.e. on } [t,t']\} \leq T_+ \text{ (resp. } T_-) \ ,$$

where T_+ (resp. T_-), defined by (9) (resp. (10)), is the length of the time intervals on which the slowly-oscillating solution is positive (resp. negative), see the beginning of Section 4.2.1 below. One then deduces that

$$(\mathcal{F}_+ + \mathcal{F}_-)^{-1}(-T_-) \leq X(t) \leq (\mathcal{F}_+ + \mathcal{F}_-)^{-1}(T_+) \ ,$$

see Formula (23) and its interpretation.

When F is odd, then $T_+ = T_-$, and the other periodic solutions have the same symmetry property. More precisely, for any periodic solution X^* of least period $T > 0$, for any $t \in [0, +\infty)$, one has

$$X^*(t + \frac{T}{2}) = -X^*(t) \ .$$

Formula (11) is deduced from this property.

When F is non regular, this proof may be generalized by adapting the proof by Fridman *et al.* [6]. □

3 Proof of Theorem 1 (Existence result)

Let us prove the existence of a solution of (1)-(5) on $[0, h]$. The global existence result is then obtained by induction. Let us denote

$$M \stackrel{\Delta}{=} \{t \in (0, h) \ : \ \mathrm{sgn} x_0(t - h) = -1\} \ .$$

Since M is measurable, there exists a nonincreasing sequence of open sets U_n of $(0, h)$ such that

$$M \subset U_n, \quad \mathrm{meas} \left(\bigcap_{n=1}^{+\infty} U_n \setminus M \right) = 0 \ .$$

As equation (1)-(5) has to be verified a.e., one may suppose that $M = \bigcap_{n=1}^{+\infty} U_n$. Moreover, for any $n \in \mathbb{N}$, U_n is a finite or countable union of disjoint intervals $(t_{n,m}, t'_{n,m})$, and one denotes

$$V_{n,m} \stackrel{\Delta}{=} \bigcup_{m'=1}^{m} (t_{n,m'}, t'_{n,m'}) \ .$$

When $M = V_{n,m}$ for certain $n, m \in \mathbb{N} \setminus \{0\}$, (1)-(5) admits a unique solution $X_{n,m}$ on $[0, h]$. Indeed, assuming that the intervals $(t_{n,m'}, t'_{n,m'})$ have been reordered, that is $t'_{n,m'} \leq t_{n,m'+1}$, and denoting $t'_{n,0} = 0, t_{n,m+1} = h$, one has

$$\begin{cases} X_{n,m}(0) = x_0(0), \\ X_{n,m}(t) = \mathcal{F}_-^{-1} \left(\mathcal{F}_-(X_{n,m}(t'_{n,m'})) - (t - t'_{n,m'}) \right) \text{ if } t \in (t'_{n,m'}, t_{n,m'+1}), \ m' = 0, \ldots, m \\ X_{n,m}(t) = \mathcal{F}_+^{-1} \left(\mathcal{F}_+(X_{n,m}(t_{n,m'})) + (t - t_{n,m'}) \right) \text{ if } t \in (t_{n,m'}, t'_{n,m'}), \ m' = 1, \ldots, m \ , \end{cases}$$

where $\mathcal{F}_\pm$ are defined in (7). Denote

$$\lambda \stackrel{\Delta}{=} \frac{1 - \sup_{x \in \mathbb{R}} |F(x)|}{2} \in \left(0, \frac{1}{2}\right] \ , \tag{12}$$

then

$$\frac{1}{2} < \frac{1}{2(1-\lambda)} = \frac{1}{1+\sup|F|} \leq \mathcal{F}'_{\pm}(b) \leq \frac{1}{1-\sup|F|} = \frac{1}{2\lambda} \ .$$

For any $n, m \in \mathbb{N} \setminus \{0\}$, $X_{n,m} \in \mathcal{C}([0,h])$ and is differentiable a.e., with $|\dot{X}_{n,m}(t)| \leq 2(1-\lambda)$ a.e. Hence, by Ascoli's Theorem, there exist a subsequence also denoted $X_{n,m}$ and a function $X_n \in \mathcal{C}([0,h])$, such that $X_{n,m} \to X_n$ in $\mathcal{C}([0,h])$ when $m \to +\infty$. The limit X_n is Lipschitz continuous with constant $2(1-\lambda)$. As well, there exist a subsequence also denoted X_n and a function $X \in \mathcal{C}([0,h])$, such that $X_n \to X$ when $n \to +\infty$, and X is Lipschitz continuous. Both X_n and X are absolutely continuous. To prove that X is a solution of (1)-(5) on $[0,h]$ is equivalent to show that, for any $t \in [0,h]$,

$$\begin{aligned} t &= \int_0^t \frac{\dot{X}(s)}{-\mathrm{sgn} x_0(s-h) + F(X(s))}\, ds \\ &= \int_{[0,t]\cap M} d\mathcal{F}_+(X(s)) - \int_{[0,t]\cap M^c} d\mathcal{F}_-(X(s)) \\ &= \mathcal{F}_-(X(0)) - \mathcal{F}_-(X(t)) + \int_{[0,t]\cap M} d(\mathcal{F}_+ + \mathcal{F}_-)(X(s)) \ , \end{aligned} \tag{13}$$

where $M^c \stackrel{\Delta}{=} [0,h] \setminus M$.

Suppose first that $M = U_n$ (that is $X_n = X$) for a certain $n \in \mathbb{N} \setminus \{0\}$. Let $\varepsilon > 0$, there exists $\underline{m}$ such that meas $(U_n \setminus V_{n,\underline{m}}) \leq \varepsilon$. One has, for $t \in [0,h]$:

$$t + \mathcal{F}_-(X_{n,m}(t)) - \mathcal{F}_-(X_{n,m}(0)) = \int_{[0,t]\cap V_{n,m}} d(\mathcal{F}_+ + \mathcal{F}_-)(X_{n,m}(s)) \ .$$

The left-hand side of the previous formula tends to $t + \mathcal{F}_-(X_n(t)) - \mathcal{F}_-(X_n(0))$ when $m \to +\infty$. Now, if $m \geq \underline{m}$, then

$$\begin{aligned} &\left| \int_{[0,t]\cap V_{n,m}} d(\mathcal{F}_+ + \mathcal{F}_-)(X_{n,m}(s)) - \int_{[0,t]\cap U_n} d(\mathcal{F}_+ + \mathcal{F}_-)(X_n(s)) \right| \\ &\leq \frac{1}{\lambda}\Big(\text{meas}\,(V_{n,m} \setminus V_{n,\underline{m}}) + \text{meas}\,(U_n \setminus V_{n,\underline{m}})\Big) \\ &\quad + \left| \int_{[0,t]\cap V_{n,\underline{m}}} d(\mathcal{F}_+ + \mathcal{F}_-)(X_{n,m}(s)) - d(\mathcal{F}_+ + \mathcal{F}_-)(X_n(s)) \right| \\ &\leq \frac{2\varepsilon}{\lambda} + \left| \int_{[0,t]\cap V_{n,\underline{m}}} d(\mathcal{F}_+ + \mathcal{F}_-)(X_{n,m}(s)) - d(\mathcal{F}_+ + \mathcal{F}_-)(X_n(s)) \right| \ , \end{aligned}$$

where the last integral goes to 0 when $m \to +\infty$, $\underline{m}$ fixed, due to the continuity of $\mathcal{F}_+, \mathcal{F}_-$ and the convergence of $X_{n,m}$ towards X_n. This being true for any $\varepsilon > 0$, one gets

$$t + \mathcal{F}_-(X_n(t)) - \mathcal{F}_-(X_n(0)) = \int_{[0,t]\cap U_n} d(\mathcal{F}_+ + \mathcal{F}_-)(X_n(s)) \ .$$

The left-hand side of the last formula tends to $t+\mathcal{F}_-(X(t))-\mathcal{F}_-(X(0))$ when $n\to+\infty$. Let $\varepsilon>0$, there exist $\underline{n},\underline{m}$ such that meas $(U_{\underline{n}}\setminus M)\le\varepsilon$ and meas $(U_{\underline{n}}\setminus V_{\underline{n},\underline{m}})\le\varepsilon$. If $n\ge\underline{n}$, then one has

$$\begin{aligned}
&\left|\int_{[0,t]\cap U_n} d(\mathcal{F}_+ + \mathcal{F}_-)(X_n(s)) - \int_{[0,t]\cap M} d(\mathcal{F}_+ + \mathcal{F}_-)(X(s))\right| \\
&\quad\le \frac{1}{\lambda}\Big(\text{meas }(U_n \mathbin{\triangle} V_{\underline{n},\underline{m}}) + \text{meas }(M \mathbin{\triangle} V_{\underline{n},\underline{m}})\Big) \\
&\qquad + \left|\int_{[0,t]\cap V_{\underline{n},\underline{m}}} d(\mathcal{F}_+ + \mathcal{F}_-)(X_n(s)) - d(\mathcal{F}_+ + \mathcal{F}_-)(X(s))\right| \\
&\quad\le \frac{4\varepsilon}{\lambda} + \left|\int_{[0,t]\cap V_{\underline{n},\underline{m}}} d(\mathcal{F}_+ + \mathcal{F}_-)(X_n(s)) - d(\mathcal{F}_+ + \mathcal{F}_-)(X(s))\right| ,
\end{aligned}$$

where $A\mathbin{\triangle}B\stackrel{\Delta}{=}A\setminus B\cup B\setminus A$, and using the fact that $M\subset U_n\subset U_{\underline{n}}$ and $V_{\underline{n},\underline{m}}\subset U_{\underline{n}}$. So, by the same arguments as before, X satisfies (13) for any $t\in[0,h]$ and hence (1)-(5) a.e.

4 Proof of Theorem 2 (Disappearance of the super-high-frequency oscillations)

In order to prove Theorem 2, one follows the technique of Shustin [17], except that one considers the set Z of zeros with change of sign instead of the set of all zeros. This is possible because of our choice of sgn0, which has as a consequence that the set of all zeros of X on $[0,+\infty)$ is zero-measured (see Lemma 5). Also, the measure on the set of open intervals which is used is different, and the inequalities are sharper.

Let X be a solution of (1)-(5) on $[0,+\infty)$. The precise definition of Z is as follows:

$$Z\stackrel{\Delta}{=}\{t\ge0\,:\,X(t)=0 \text{ and } \forall\varepsilon>0,\exists t'\in[t-\varepsilon,t),t''\in(t,t+\varepsilon],\ X(t')X(t'')<0\}\ . \quad (14)$$

Let us denote $\mathrm{Acc}Z$ the set of accumulation points of Z.

If $t\ge h$ and $Z\cap[t-h,t]$ is infinite, that is if X_t has an infinite number of zeros with change of sign, then, by compacity, there exists at least one accumulation point $\alpha\in\mathrm{Acc}Z$ in $[t-h,t]$ (conversely, if $\alpha\in\mathrm{Acc}Z$ and $\alpha\ge h$, then X_t has an infinite number of zeros with change of sign for $\alpha<t<\alpha+h$). Hence, if $\mathrm{Acc}Z$ is empty or bounded, then X_t has a finite number of zeros with change of sign for $t>\max\mathrm{Acc}Z+h$ (with $\max\emptyset=0$), and X_t has a finite number of zeros for $t>\max\mathrm{Acc}Z+2h$. Therefore, in order to prove Theorem 2, one shall bound $\mathrm{Acc}Z$ (when it is not empty) by $hC_\varepsilon(1+(\delta/h)^{-2-\varepsilon})$.

4.1 Properties of the zeros of the solution

Lemma 5. *The following properties are true:*

$$\text{meas } \{t \geq 0 : X(t) = 0\} = 0 \ ,$$
$$t \notin Z \Rightarrow X > 0 \text{ or } X < 0 \text{ a.e. in a neighborhood of } t \ .$$

Proof. Denote $N \subset \mathbb{R}^+$ a set of measure zero such that X is differentiable and (1)-(5) is fulfilled outside N. For any $\varepsilon > 0$, there exists an open set U such that $N \subset U$ and meas $U < \varepsilon$. Denote $S \stackrel{\Delta}{=} \{t \geq 0 : X(t) = 0\}$. It is clear that $S \setminus N$ has no accumulation point, otherwise on such a point t, one would have $X(t) = 0$, $\dot{X}(t) = 0$, so $|F(X(t))| = 1$. As $S \setminus U \subset S \setminus N$, the set $S \setminus U$ has no accumulation point. Being closed (by the continuity of X), it is then finite or countable. Hence, meas $(S \setminus U) = 0$, so meas $S =$ meas $(S \cap U) < \varepsilon$ for any $\varepsilon > 0$. This proves the first property. The second property is deduced from the first one and (14). □

Lemma 6. *The set Z is closed.*

Proof. Let t_i be a sequence of elements of Z converging to t. There exists a subsequence, also denoted t_i, which converges e.g. from below. Let $t_i' < t_i < t_i'' < t$ be such that $X(t_i')X(t_i'') < 0$, together with $t_i'' - t_i' \to 0$ when $i \to +\infty$ (see (14)). Then t_i', t_i'' tend to t.

Let $\varepsilon > 0$ be fixed. There exists i such that $t_i', t_i'' \in (t - \varepsilon, t)$. Now, by Lemma 5, the set of all zeros of X on $[0, +\infty)$ is zero-measured, due to the choice of sgn0. Hence, there exists $t'' \in (t, t + \varepsilon)$ such that $X(t'') \neq 0$. One has $X(t_i')X(t'') < 0$ or $X(t_i'')X(t'') < 0$, which proves that $t \in Z$ and completes the proof. □

The following result is proved in Shustin [17] too.

Lemma 7. *If $t \in \text{Acc}Z$, $t \geq h$, then $t - h \in \text{Acc}Z$.*

Proof. Indeed, due to Lemma 5, if $t \in \text{Acc}Z$, then there exists for instance an increasing sequence $t_i \in Z$ with X increasing around the points t_{2i} and decreasing around the points t_{2i+1} and such that $t_i \to t$ when $i \to +\infty$. Then, $\text{sgn}X(t_i - h) = (-1)^{i+1}$, and there exists $t_i' \in (t_i, t_{i+1})$ such that $t_i' - h \in Z$. Hence, $t - h = \lim_{i \to +\infty} t_i' - h \in \text{Acc}Z$. □

4.2 Principle of the proof

4.2.1 Preliminary notations

Let $t, b > 0$ be such that $X(t) = X(t + b) = 0$. When $X > 0$ on $(t, t + b)$ and X is increasing and then decreasing on $[t, t + b]$, one denotes $\mathcal{G}_-^P(b)$ (resp. $\mathcal{G}_+^P(b)$) the time length of the increasing (resp. decreasing) phase. When $X < 0$ on $(t, t + b)$ and X is decreasing and then increasing, one denotes $\mathcal{G}_-^N(b)$ (resp. $\mathcal{G}_+^N(b)$) the time length of the decreasing (resp. increasing) phase. Simple computations lead to:

$$\mathcal{G}_\pm^P \stackrel{\Delta}{=} \mathcal{F}_\mp \circ (\mathcal{F}_+ + \mathcal{F}_-)^{-1} \ , \quad \mathcal{G}_\pm^N(b) \stackrel{\Delta}{=} -\mathcal{G}_\mp^P(-b) \ ,$$

where $\mathcal{F}_\pm$ are defined in (7) (P, N stand for *positive, negative*). Remark that $\mathcal{G}^P_\pm$ (resp. $\mathcal{G}^N_\pm$) depend only upon $F|_{\mathbb{R}^+}$ (resp. $F|_{\mathbb{R}^-}$). When F is odd, then $\mathcal{G}^P_\pm = \mathcal{G}^N_\pm$.

Below are listed some useful properties, whose proof is straightforward (λ is defined in (12)).

$$(\mathcal{G}^P_\pm)' = \frac{1 \pm F}{2} \circ (\mathcal{F}_+ + \mathcal{F}_-)^{-1} ,$$
$$0 < \lambda \leq (\mathcal{G}^a_\pm)' \leq 1 - \lambda < 1 , \quad a \in \{P, N\} ,$$
$$\mathcal{G}^a_- + \mathcal{G}^a_+ = \mathrm{Id} , \quad a \in \{P, N\} .$$

4.2.2 Measures on the set of open intervals

For any $\mathcal{C}^1$ convex function φ on $\mathbb{R}^+$ such that $\varphi(0) = \varphi'(0) = 0$, one denotes

$$\varphi^a(b) \stackrel{\Delta}{=} \int_0^b \varphi' \circ \mathcal{G}^a_-(b')\, db' , \quad a \in \{P, N\},\ b \in \mathbb{R}^+ .$$

One proves easily that for $a \in \{P, N\}$, φ^a is convex, $\varphi^a(0) = (\varphi^a)'(0) = 0$, and

$$\frac{1}{\lambda}\varphi(\lambda b) \leq \varphi^a(b) \leq \varphi(b) \quad \forall b \geq 0 .$$

Definition 8. *Let μ_φ denote the (unique) function on the set of bounded open intervals $I \subset [0, +\infty)$, such that*

$$\mu_\varphi(I) = \varphi^P(\mathrm{meas}\ I) \quad \textit{if } X > 0 \textit{ a.e. on } I ,$$
$$\mu_\varphi(I) = \varphi^N(\mathrm{meas}\ I) \quad \textit{if } X < 0 \textit{ a.e. on } I ,$$
$$\mu_\varphi(I) = \sum_{\substack{J \text{ connected} \\ \text{component of } I \setminus Z}} \mu_\varphi(J) \quad \textit{for all } I .$$

For any bounded open interval I, one denotes

$$\mu_p(I) \stackrel{\Delta}{=} \mu_{\{s \mapsto s^p\}}(I) , \ p > 1 ,$$

$$\|I\|_p \stackrel{\Delta}{=} \left(\sum_{\substack{J \text{ connected} \\ \text{component of } I \setminus Z}} (\mathrm{meas}\ J)^p \right)^{\frac{1}{p}} , \ p \in [1, +\infty) ,$$

$$\|I\|_\infty \stackrel{\Delta}{=} \sup_{\substack{J \text{ connected} \\ \text{component of } I \setminus Z}} \mathrm{meas}\ J .$$

For any open interval I, the set $I \setminus Z$ is open, since Z is closed. Hence, it has at most a countable number of connected components J, which are open intervals. Since $\mu_\varphi(J) \geq 0$ and meas $J \geq 0$, the sums on J in Definition 8 do not depend upon the ordering of the connected components. Hence, $\mu_\varphi(I)$ and $\|I\|_p$ are well-defined. Moreover, if I is bounded, then meas $I < +\infty$, $\|I\|_p \leq \|I\|_1 =$ meas $I < +\infty$ (as meas $Z = 0$ by Lemma 5), and $\mu_\varphi(I) \leq \sum_J \varphi(\text{meas } J) \leq \varphi(\text{meas } I) < +\infty$ (by the convexity of φ and $\varphi(0) = 0$).

Note that $(\mu_p(I))^{\frac{1}{p}}$ and $\|I\|_p$ are "equivalent":

$$\lambda^{1-\frac{1}{p}} \|I\|_p \leq (\mu_p(I))^{\frac{1}{p}} \leq \|I\|_p \ . \tag{15}$$

In the sequel, we shall denote also $\|\cdot\|_p$ the usual l^p-norm of *sequences*, when no ambiguity is possible.

4.2.3 The central intermediate result

In order to bound Acc$Z$, let us fix $\alpha \in \text{Acc}Z$, $\alpha \geq 2h$. By Lemma 7, $\alpha - kh \in \text{Acc}Z$ for all $k \in \mathbb{N}$, $k \leq \frac{\alpha}{h}$. In the following, one exhibits a function μ of $\alpha \in \text{Acc}Z \cap [h, +\infty)$ (depending on Z), such that on the one hand $\mu(\alpha - kh)$ decreases and "tends to 0 when $k \to +\infty$", and on the other hand, $\mu(\alpha - k_0 h)$ is lower bounded by some constant (depending on δ), for $k_0 = \lfloor \frac{\alpha}{h} \rfloor - 1$. Here, $\lfloor \frac{\alpha}{h} \rfloor$ denotes the integer part of $\frac{\alpha}{h}$. This furnishes a bound for k_0 and then for α.

Proposition 9. *Let φ be as before. Let $\gamma \in (0, 1]$, $p \in (1, +\infty]$. Then, there exists $\theta > 0$ depending on γ, p and $\sup_{x \in \mathbb{R}} |F(x)|$ only, such that, for any open interval $I \subset [h, +\infty)$ with bounds in* Acc$Z$ *and length* meas $I \leq h$,

$$\mu_\varphi(I - h) \leq \mu_\varphi(I) \ , \tag{16}$$

$$\mu_{\gamma+1}(I - h) \leq \mu_{\gamma+1}(I) - \theta \frac{\|I\|_p^{\frac{3}{p-1}+2+\gamma}}{(\text{meas } I)^{\frac{3}{p-1}+1}} \ . \tag{17}$$

In Proposition 9 and in the sequel, one denotes $A \pm t = \{t' \pm t \ : \ t' \in A\}$ for any $t \in \mathbb{R}$ and A subset of $\mathbb{R}$.

Applying (17) to

$$I_k \stackrel{\Delta}{=} (\alpha - (k+1)h, \alpha - kh) \ , \quad k = 0, \ldots, k_0 - 1 \ ,$$

one gets

$$\mu_{\gamma+1}(I_{k_0}) \leq \mu_{\gamma+1}(I_0) - \frac{\theta}{h^{\frac{3}{p-1}+1}} \sum_{k=0}^{k_0-1} \|I_k\|_p^{\frac{3}{p-1}+2+\gamma} \ ,$$

Using formulas (15) and (16) for $p \in (1, +\infty)$, one obtains

$$\begin{aligned}\sum_{k=0}^{k_0-1} \|I_k\|_p^{\frac{3}{p-1}+2+\gamma} &\geq \sum_{k=0}^{k_0-1} (\mu_p(I_k))^{\frac{1}{p}\left(\frac{3}{p-1}+2+\gamma\right)} \\ &\geq k_0 \left(\mu_p(I_{k_0})\right)^{\frac{1}{p}\left(\frac{3}{p-1}+2+\gamma\right)} \\ &\geq k_0 \lambda^{\frac{1-\gamma}{p}+2+\gamma} \|I_{k_0}\|_p^{\frac{3}{p-1}+2+\gamma} .\end{aligned}$$

Passing to the limit, the resulting inequality is also valid for $p = +\infty$. Since $\mu_{\gamma+1}(I_0) \leq \|I_0\|_{\gamma+1}^{\gamma+1} \leq \|I_0\|_1^{\gamma+1} = h^{\gamma+1}$, one gets

$$k_0 \leq \frac{1}{\theta \lambda^{\frac{1-\gamma}{p}+2+\gamma}} \left(\frac{h}{\|I_{k_0}\|_p}\right)^{\frac{3}{p-1}+2+\gamma} .$$

Let us bound $\|I_{k_0}\|_p$ from below wrt δ. The connected components of $Z^c = \mathbb{R}^+ \setminus Z$ are the maximal intervals where X has a constant sign a.e. Let I be a bounded interval of $(-h, +\infty)$ with e.g. $\mathrm{sgn}X(t) = 1$ a.e. on I. Then, X decreases in $I + h$, which implies that either X has constant sign on $I + h$, or X is positive on the first part of the interval $I + h$ and negative on the other part. Hence, $I + h$ is included in the union of the closure of two connected components of Z^c. This implies that δ verifies:

$$\begin{aligned}\delta &= \sup\{\mathrm{meas}\, I : I \text{ interval } \subset (-h, 0),\ \mathrm{sgn}X \text{ constant a.e. on } I\} \\ &\leq 2\sup\{\mathrm{meas}\, I : I \text{ connected component of } (0, h) \setminus Z\} \\ &\leq 4\sup\{\mathrm{meas}\, I : I \text{ connected component of } (\alpha - (k_0+1)h, \alpha - k_0 h) \setminus Z\} \\ &= 4\|I_{k_0}\|_\infty \leq 4\|I_{k_0}\|_p .\end{aligned}$$

The second inequality is deduced from the fact that $(\alpha - (k_0 + 1)h, \alpha - k_0 h) \subset (0, 2h)$. Therefore,

$$k_0 \leq \frac{1}{\theta \lambda^{\frac{1-\gamma}{p}+2+\gamma}} \left(\frac{4h}{\delta}\right)^{\frac{3}{p-1}+2+\gamma} ,$$

so $\max \mathrm{Acc} Z \leq (k_0 + 2)h \leq hC_{p,\gamma}(1 + (h/\delta)^{\frac{3}{p-1}+2+\gamma})$ for a certain constant $C_{p,\gamma} > 0$ depending on p, γ and λ only. The proof of Theorem 2 is achieved taking $p = +\infty$ and $\gamma = \varepsilon$.

4.2.4 Overview of the proof of Proposition 9

In order to end the proof of Theorem 2, it only remains to prove Proposition 9. This is done in four steps, in Sections 4.3 to 4.6. Let us give an overview of this proof.

When I and $(I - h)$ do not contain accumulation points of Z, their connected components may be ordered, together with the sums defining the measures involved in (16) and (17). For

the particular case where X changes only one time its sense of variation on any subinterval of I where its sign is constant (that is when any connected component of $I \setminus Z$ intersects no more than two connected components of $I \setminus (Z + h)$), the proof is treated in Section 4.3.

When some connected components of $I \setminus Z$ intersect more than two connected components of $I \setminus (Z+h)$, but always a *finite* number of them, which is the case when $I-h$ does not contain accumulation points of Z, the computations are more cumbersome. They are presented in Section 4.4. Note that this part does not use Section 4.3: the particular case presented therein is simpler, and one provides it for pedagogical purpose rather.

The next step (in Section 4.5) consists in allowing an infinite number of changes of variation on subintervals of I where X has a constant sign, that is to eliminate the restrictions on $I - h$. The proof in this case follows from the results of Section 4.4, by an approximation process.

Last, the case where I contains accumulation points of Z is treated in Section 4.6: a summation of contributions coming from each connected component of $I \setminus \mathrm{Acc}Z$, that are treated by the results of Section 4.5, is achieved. This ends the proof of Proposition 9.

Independently, one presents in Section 4.7 a simpler variant of Sections 4.4 and 4.5, in the case where F is odd. It consists in proving that, given the set $I \setminus Z$, the value of $\mu_\varphi(I-h)$ obtained in the particular case of Section 4.3 is indeed the maximal that may be obtained. The technique uses Schur convexity as a central tool.

4.3 Proof of Proposition 9 in a particular case where I and $I - h$ do not contain accumulation points of Z

If I is an open interval with bounds in $\mathrm{Acc}Z$ such that $I \cap \mathrm{Acc}Z = \emptyset$, then the points of $Z \cap I$ may be ordered in increasing order. Therefore, the connected components of the open set $U \stackrel{\Delta}{=} I \setminus Z$ may be ordered in increasing order too. Let us denote them by $U_j, j \in \mathcal{J} \subset \mathbb{Z}$ (U_j increasing wrt j), and put $U_j = \emptyset$ for $j \in \mathbb{Z} \setminus \mathcal{J}$. Since, by the definition of Z, the sign of X is constant a.e. in U_j and alternating wrt j, one imposes e.g. $\mathrm{sgn}X = (-1)^j$ a.e. on U_j.

Let us define the sequences a and b by: $a_j \stackrel{\Delta}{=} P$ if j is even, $a_j \stackrel{\Delta}{=} N$ if j is odd, $j \in \mathbb{Z}$, and $b_j \stackrel{\Delta}{=} \mathrm{meas}\, U_j$ for $j \in \mathbb{Z}$. One has $\mathrm{meas}\, I = \|b\|_1$, $\|I\|_p = \|b\|_p$, and $\mu_\varphi(I) = \mu_\varphi(b) \stackrel{\Delta}{=} \sum_{j\in\mathbb{Z}} \varphi^{a_j}(b_j)$.

Suppose now that $I - h$ does not contain accumulation points of Z too. Denote $\tilde{U}_j$, $j \in \tilde{\mathcal{J}} \subset \mathbb{Z}$, the connected components of $\tilde{U} \stackrel{\Delta}{=} I \setminus (Z + h)$, increasingly ordered as the U_j, and put $\tilde{U}_j = \emptyset$ for $j \in \mathbb{Z} \setminus \tilde{\mathcal{J}}$. Let $\tilde{b}$ be the sequence defined by: $\tilde{b}_j \stackrel{\Delta}{=} \mathrm{meas}\, \tilde{U}_j$ for $j \in \mathbb{Z}$. One has $\mu_\varphi(I - h) = \mu_\varphi(\tilde{b}) = \sum_{j\in\mathbb{Z}} \varphi^{a_j}(\tilde{b}_j)$. Since X increases (resp. decreases) around $t \in I$ if and only if $X(. - h) < 0$ (resp. $X(. - h) > 0$) a.e. around t, the sets $\tilde{U}_j$, $j \in \tilde{\mathcal{J}}$, are exactly the connected components of the open subset of I containing all the points around which X is monotonous. The sequence $\tilde{b}$ verifies the following property: there exists $\tilde{X} \in W^{1,\infty}(I)$

such that, for all $j \in \mathbb{Z}$,

$$\dot{\tilde{X}} = (-1)^{j-1} + F(\tilde{X}) \text{ a.e. on } \tilde{U}_j \ , (-1)^j \tilde{X} > 0 \text{ a.e. on } U_j \ . \tag{18}$$

Any sequence verifying this property is called a *predecessor* of b. A particularly simple predecessor, denoted b', is obtained by using a function $\tilde{X}$ satisfying (18) and being either increasing/decreasing, or decreasing/increasing, on any interval U_j. Let us denote X' the corresponding value of $\tilde{X}$. One shows easily (see Figure 1) that this corresponds to choose

$$b'_{2j-1} = \mathcal{G}_-^P(b_{2j}) + \mathcal{G}_+^N(b_{2j-1}) \ , \quad b'_{2j} = \mathcal{G}_-^N(b_{2j+1}) + \mathcal{G}_+^P(b_{2j}) \ , \tag{19}$$

that is

$$b'_j = \mathcal{G}_-^{a_{j+1}}(b_{j+1}) + \mathcal{G}_+^{a_j}(b_j) = b_j + \mathcal{G}_-^{a_{j+1}}(b_{j+1}) - \mathcal{G}_-^{a_j}(b_j) \ .$$

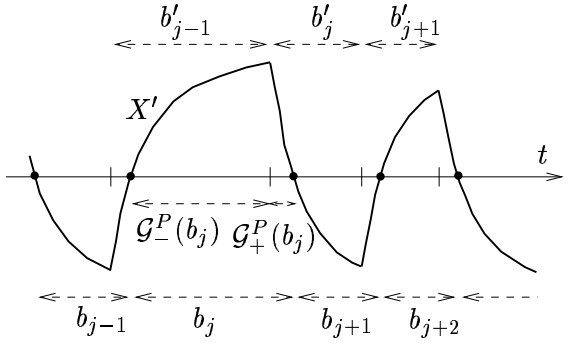


Figure 1: The predecessor b' (the indexation shown corresponds to an even integer j)

In this section, one proves Proposition 9 in the particular case where $\tilde{b} = b'$.
Using the identity

$$\varphi^a(s_1 + s_2 - s_3) = \varphi^a(s_1) + [\varphi(s)]_{s_3}^{s_2} + \int_{s_3}^{s_2} \left(\varphi' \circ \mathcal{G}_-^a(s + s_1 - s_3) - \varphi'(s) \right) \, ds \ , \tag{20}$$

one gets

$$\varphi^{a_j}(b'_j) = \varphi^{a_j}(b_j) + [\varphi(s)]_{\mathcal{G}_-^{a_j}(b_j)}^{\mathcal{G}_-^{a_{j+1}}(b_{j+1})} + \int_{\mathcal{G}_-^{a_j}(b_j)}^{\mathcal{G}_-^{a_{j+1}}(b_{j+1})} \left(\varphi' \circ \mathcal{G}_-^{a_j}(s + \mathcal{G}_+^{a_j}(b_j)) - \varphi'(s) \right) \, ds \ . \tag{21}$$

Let us study the last term of (21). If e.g. $\mathcal{G}_-^{a_j}(b_j) \leq \mathcal{G}_-^{a_{j+1}}(b_{j+1})$, then one has, for any $s \in [\mathcal{G}_-^{a_j}(b_j), \mathcal{G}_-^{a_{j+1}}(b_{j+1})]$,

$$\mathcal{G}_-^{a_j}(b_j) \leq s \Rightarrow \mathcal{G}_+^{a_j}(b_j) \leq \mathcal{G}_+^{a_j} \circ (\mathcal{G}_-^{a_j})^{-1}(s) \Rightarrow s + \mathcal{G}_+^{a_j}(b_j) \leq (\mathcal{G}_-^{a_j})^{-1}(s) \Rightarrow \mathcal{G}_-^{a_j}(s + \mathcal{G}_+^{a_j}(b_j)) \leq s \ ,$$

using $\mathcal{G}_-^a + \mathcal{G}_+^a = \mathrm{Id}$ and the fact that the functions $\mathcal{G}_\pm^a$ are increasing. More precisely, using the bounds on the derivatives, one gets:

$$\begin{aligned} \mathcal{G}_-^{a_j}(s + \mathcal{G}_+^{a_j}(b_j)) - s &= \mathcal{G}_-^{a_j}(s + \mathcal{G}_+^{a_j}(b_j)) - \mathcal{G}_-^{a_j}(s + \mathcal{G}_+^{a_j} \circ (\mathcal{G}_-^{a_j})^{-1}(s)) \\ &\leq \frac{\lambda^2}{(1-\lambda)} \left(\mathcal{G}_-^{a_j}(b_j) - s\right) \leq 0 \ . \end{aligned}$$

Then, by the convexity of φ, one obtains

$$\varphi' \circ \mathcal{G}_-^{a_j}(s + \mathcal{G}_+^{a_j}(b_j)) - \varphi'(s) \leq 0 \ ,$$

and the last term of (21) is nonpositive. A similar proof holds when $\mathcal{G}_-^{a_j}(b_j) \geq \mathcal{G}_-^{a_{j+1}}(b_{j+1})$. Hence, one obtains by addition of (21): $\mu_\varphi(b') \leq \mu_\varphi(b)$, that is (16), using the fact that $\varphi(\mathcal{G}_-^{a_j}(b_j)) \to 0$ when $|j| \to +\infty$.

To prove (17), consider the sequence $c_j \stackrel{\Delta}{=} \mathcal{G}_-^{a_j}(b_j)$. The previous computations lead to

$$\mu_\varphi(b') \leq \mu_\varphi(b) + \sum_{j \in \mathbb{Z}} \int_{c_j}^{c_{j+1}} \left(\varphi'(s + \frac{\lambda^2}{(1-\lambda)}(c_j - s)) - \varphi'(s) \right) \, ds \ .$$

For $\varphi(s) = s^{\gamma+1}$, bounding φ'' from below, one deduces:

$$\mu_\varphi(b') \leq \mu_\varphi(b) - \gamma(\gamma+1)(\sup_{j \in \mathbb{Z}} c_j)^{\gamma-1} \frac{\lambda^2}{(1-\lambda)} \sum_{j \in \mathbb{Z}} \frac{(c_j - c_{j+1})^2}{2} \ .$$

Let us now use (with $q = 2$) the following interpolation inequality, which is a discrete analog of a result by Gagliardo [8] and Nirenberg [13]. A proof is given in Appendix.

Lemma 10. *Let $1 < p \leq +\infty$ and $1 < q < +\infty$. Then, there exists $C_{p,q} > 0$ such that, for any nonzero real sequence $(b_j)_{j \in \mathbb{Z}}$, the following formula is true, as soon as the involved norms are finite:*

$$\|b_j - b_{j-1}\|_q^q \geq C_{p,q} \frac{\|b\|_p^{\frac{p(2q-1)}{p-1}}}{\|b\|_1^{\frac{pq-p+q}{p-1}}} \ .$$

One gets

$$\sum_{j \in \mathbb{Z}} (c_j - c_{j+1})^2 \geq C_{p,2} \frac{\|c\|_p^{\frac{3p}{p-1}}}{\|c\|_1^{\frac{p+2}{p-1}}} \ .$$

As $(\sup c_j)^{\gamma-1} \geq \|c\|_p^{\gamma-1}$, one deduces

$$\mu_{\gamma+1}(b') \leq \mu_{\gamma+1}(b) - \theta' \frac{\|c\|_p^{\frac{3}{p-1}+2+\gamma}}{\|c\|_1^{\frac{3}{p-1}+1}} , \tag{22}$$

where $\theta' > 0$ depends on γ, p and λ. Using the inequalities on $(\mathcal{G}_-^a)'$ and the fact that $\mathcal{G}_-^a(0) = 0$, one gets

$$\|c\|_p \geq \lambda \|b\|_p , \quad \|c\|_1 \leq (1-\lambda)\|b\|_1 ,$$

which gives (17) in the case where $\tilde{b} = b'$.

4.4 Proof of Proposition 9 for general I such that I and $I-h$ do not contain accumulation points of Z

One studies here the case where $I \cap \mathrm{Acc}Z = I \cap (\mathrm{Acc}Z + h) = \emptyset$, so the connected components of $U = I \setminus Z$ and $\tilde{U} = I \setminus (Z+h)$ may be ordered as in Section 4.3, but one supposes that the predecessor $\tilde{b}$ of b is different (up to a translation) from b' defined in (19).

Let $U_j, b_j, \tilde{U}_j, \tilde{b}_j, a_j$ be defined as in Section 4.3. The function $\tilde{X} = X|_I$ satisfies (18). Let us denote t_j the right boundary of the interval $\tilde{U}_j$ (the t_j are exactly the local extrema of $\tilde{X}$). Let us define also

$$d_j \stackrel{\Delta}{=} |(\mathcal{F}_+ + \mathcal{F}_-)(\tilde{X}(t_j))| . \tag{23}$$

The quantity d_j is the time necessary for a solution of (1)-(5) to go from 0 to $\tilde{X}(t_j)$ and back to 0 with only one change of the sense of variation. For any $j \in \mathbb{Z}$, let us denote $\tilde{\jmath} = \tilde{\jmath}(j)$ the unique index such that $\tilde{U}_{\tilde{\jmath}} \cap U_j \neq \emptyset$ and $\tilde{U}_{\tilde{\jmath}} \cap U_{j+1} \neq \emptyset$. The parity of j and $\tilde{\jmath}$ is the same. The maximal integer l such that $\tilde{U}_{\tilde{\jmath}-l} \subset U_j$ is even and denoted $2\bar{l} = 2\bar{l}(j)$. One has $\tilde{U}_{\tilde{\jmath}-l} \subset U_j$ for $l = 1, \ldots, 2\bar{l}$, and $\tilde{\jmath}(j) - 2\bar{l}(j) - 1 = \tilde{\jmath}(j-1)$. For all the notations, see Figure 2.

To compare, as in Section 4.3, $\mu_\varphi(\tilde{b})$ and $\mu_\varphi(b)$, one remarks that

$$\sum_{j\in\mathbb{Z}} \varphi^{a_j}(\tilde{b}_j) = \sum_{j\in\mathbb{Z}} \sum_{l=0}^{2\bar{l}(j)} \varphi^{a_{j-l}}(\tilde{b}_{\tilde{\jmath}(j)-l}) .$$

Then, one is led to compare $\sum_{l=0}^{2\bar{l}(j)} \varphi^{a_{j-l}}(\tilde{b}_{\tilde{\jmath}(j)-l})$ and $\varphi^{a_j}(b_j)$. One has (see Figure 2)

$$\begin{aligned}
\tilde{b}_{\tilde{\jmath}} &= \mathcal{G}_-^{a_{j+1}}(d_{\tilde{\jmath}}) + \mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-1}) = d_{\tilde{\jmath}-1} + \mathcal{G}_-^{a_{j+1}}(d_{\tilde{\jmath}}) - \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-1}) \\
\tilde{b}_{\tilde{\jmath}-l} &= \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l}) - \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1}) \quad \text{for } l \text{ odd in } \{1, \ldots, 2\bar{l}-1\} \\
\tilde{b}_{\tilde{\jmath}-l} &= -\mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-l}) + \mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-l-1}) \\
&= d_{\tilde{\jmath}-l-1} - d_{\tilde{\jmath}-l} + \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l}) - \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1}) \quad \text{for } l \text{ even in } \{2, \ldots, 2\bar{l}\} .
\end{aligned}$$

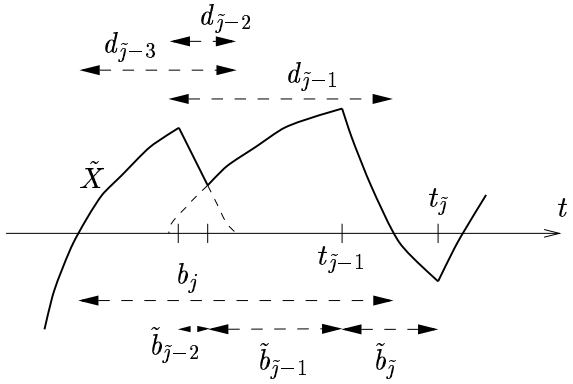


Figure 2: A predecessor $\tilde{b}$ (here j is even, $\bar{l}(j) = 1$)

4.4.1 Rewriting of $\mu_\varphi(\tilde{b})$ using identity (20)

First, one has, for $l = 0$:

$$\varphi^{a_j}(\tilde{b}_{\tilde{\jmath}}) = \varphi^{a_j}(d_{\tilde{\jmath}-1}) + [\varphi(s)]_{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-1})}^{\mathcal{G}_-^{a_j+1}(d_{\tilde{\jmath}})} + r_{\tilde{\jmath}} \ ,$$

where the rest $r_{\tilde{\jmath}}$ is given by:

$$r_{\tilde{\jmath}} \stackrel{\Delta}{=} \int_{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-1})}^{\mathcal{G}_-^{a_j+1}(d_{\tilde{\jmath}})} \left(\varphi' \circ \mathcal{G}_-^{a_j}(s + \mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-1})) - \varphi'(s) \right) \ ds \ .$$

Second, for odd index l, one gets:

$$\varphi^{a_{j-l}}(\tilde{b}_{\tilde{\jmath}-l}) = [\varphi(s)]_{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})}^{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l})} + r_{\tilde{\jmath}-l} \ ,$$

where, for l odd

$$r_{\tilde{\jmath}-l} \stackrel{\Delta}{=} \int_{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})}^{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l})} \left(\varphi' \circ \mathcal{G}_-^{a_j+1}(s - \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})) - \varphi'(s) \right) \ ds \ .$$

Third, for l nonzero and even, one has

$$\begin{aligned} \varphi^{a_{j-l}}(\tilde{b}_{\tilde{\jmath}-l}) = \varphi^{a_j}(d_{\tilde{\jmath}-l-1} - d_{\tilde{\jmath}-l}) + [\varphi(s)]_{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})}^{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l})} \\ + \int_{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})}^{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l})} \left(\varphi' \circ \mathcal{G}_-^{a_j}(s + \mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-l-1}) - d_{\tilde{\jmath}-l}) - \varphi'(s) \right) \ ds \ . \end{aligned}$$

Adding the null term

$$\varphi^{a_j}\left(\sum_{l'=0}^{l}(-1)^{l'}d_{\tilde{\jmath}-l'-1}\right)-\varphi^{a_j}\left(\sum_{l'=0}^{l-2}(-1)^{l'}d_{\tilde{\jmath}-l'-1}\right)$$
$$-\int_{d_{\tilde{\jmath}-l}}^{d_{\tilde{\jmath}-l-1}}\varphi'\circ\mathcal{G}_-^{a_j}\left(\sum_{l'=0}^{l-1}(-1)^{l'}d_{\tilde{\jmath}-l'-1}+s\right)\,ds\ ,$$

one gets:

$$\varphi^{a_j-l}(\tilde{b}_{\tilde{\jmath}-l})=\varphi^{a_j}\left(\sum_{l'=0}^{l}(-1)^{l'}d_{\tilde{\jmath}-l'-1}\right)-\varphi^{a_j}\left(\sum_{l'=0}^{l-2}(-1)^{l'}d_{\tilde{\jmath}-l'-1}\right)+[\varphi(s)]_{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})}^{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l})}+r_{\tilde{\jmath}-l}\ ,$$

where, for l nonzero and even

$$r_{\tilde{\jmath}-l}\stackrel{\Delta}{=}\int_{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})}^{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l})}\left(\varphi'\circ\mathcal{G}_-^{a_j}(s+\mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-l-1})-d_{\tilde{\jmath}-l})-\varphi'(s)\right)\,ds$$
$$+\int_{d_{\tilde{\jmath}-l}}^{d_{\tilde{\jmath}-l-1}}\left(\varphi'\circ\mathcal{G}_-^{a_j}(s-d_{\tilde{\jmath}-l})-\varphi'\circ\mathcal{G}_-^{a_j}\left(\sum_{l'=0}^{l-1}(-1)^{l'}d_{\tilde{\jmath}-l'-1}+s\right)\right)\,ds=\int_{d_{\tilde{\jmath}-l}}^{d_{\tilde{\jmath}-l-1}}\psi_{j,l}(s)\,ds$$

and

$$\psi_{j,l}(s)\stackrel{\Delta}{=}(\mathcal{G}_-^{a_j})'(s)\left[\varphi'\circ\mathcal{G}_-^{a_j}(s)-\varphi'\circ\mathcal{G}_-^{a_j}\left(\mathcal{G}_-^{a_j}(s)-d_{\tilde{\jmath}-l}+\mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-l-1})\right)\right]$$
$$+\varphi'\circ\mathcal{G}_-^{a_j}(s-d_{\tilde{\jmath}-l})-\varphi'\circ\mathcal{G}_-^{a_j}\left(\sum_{l'=0}^{l-1}(-1)^{l'}d_{\tilde{\jmath}-l'-1}+s\right)\ .$$

By summation over all the indexes l, one gets:

$$\sum_{l=0}^{2\bar{l}}\varphi^{a_j-l}(\tilde{b}_{\tilde{\jmath}-l})=\varphi^{a_j}\left(\sum_{l=0}^{2\bar{l}}(-1)^{l}d_{\tilde{\jmath}-l-1}\right)+[\varphi(s)]_{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-2\bar{l}-1})}^{\mathcal{G}_-^{a_j+1}(d_{\tilde{\jmath}})}+\sum_{l=0}^{2\bar{l}}r_{\tilde{\jmath}-l}\ .$$

As (cf. Figure 2)

$$\sum_{l=0}^{2\bar{l}}(-1)^{l}d_{\tilde{\jmath}-l-1}=b_j\ ,$$

and

$$\tilde{\jmath}(j)-2\bar{l}(j)-1=\tilde{\jmath}(j-1)\ ,$$

one finally may write:

$$\mu_\varphi(\tilde{b})=\sum_{j\in\mathbb{Z}}\sum_{l=0}^{2\bar{l}(j)}\varphi^{a_j-l}(b_{\tilde{\jmath}(j)-l})=\mu_\varphi(b)+\sum_{j\in\mathbb{Z}}\sum_{l=0}^{2\bar{l}(j)}r_{\tilde{\jmath}(j)-l}\ .$$

4.4.2 Bounding of the rest

Let us consider the sequence $(c_{\frac{j}{3}})_{j\in\mathbb{Z}}$, defined for $j \in \mathbb{Z}$ by:

$$\begin{aligned} c_j &\stackrel{\Delta}{=} \mathcal{G}_-^{a_j}(b_j) \quad \text{(as in Section 4.3)} , \\ c_{j+\frac{1}{3}} &\stackrel{\Delta}{=} \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}(j)-1}) \leq c_j \quad \text{(since } d_{\tilde{\jmath}(j)-1} \leq b_j) , \\ c_{j-\frac{1}{3}} &\stackrel{\Delta}{=} \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}(j)-2\bar{l}(j)-1}) \leq c_j \quad \text{(since } d_{\tilde{\jmath}(j)-2\bar{l}(j)-1} \leq b_j) . \end{aligned}$$

Lemma 11. *For any $l = 0, \ldots, 2\bar{l}$, $r_{\tilde{\jmath}-l} \leq 0$. Moreover, when $\varphi(s) = s^{\gamma+1}$, there exists $C_{\gamma,\lambda} > 0$, such that for all $j \in \mathbb{Z}$,*

$$r_{\tilde{\jmath}(j)} \leq -C_{\gamma,\lambda} \max\{c_j, c_{j+1}\}^{\gamma-1}(c_{j+\frac{2}{3}} - c_{j+\frac{1}{3}})^2 , \tag{24}$$

$$\sum_{l=1}^{\bar{l}(j)} r_{\tilde{\jmath}(j)-2l} \leq -C_{\gamma,\lambda}\, c_j^{\gamma-1}(c_{j+\frac{1}{3}} - c_j)^2 , \tag{25}$$

$$\sum_{l=1}^{2\bar{l}(j)} r_{\tilde{\jmath}(j)-l} \leq -C_{\gamma,\lambda}\, c_j^{\gamma-1}(c_j - c_{j-\frac{1}{3}})^2 . \tag{26}$$

From Lemma 11, all the $r_{\tilde{\jmath}-l}$ are nonpositive, then $\mu_\varphi(\tilde{b}) \leq \mu_\varphi(b)$ and (16) is deduced.

Now, for $\varphi(s) = s^{\gamma+1}$, adding the three inequalities of Lemma 11, and then summing on $j \in \mathbb{Z}$, one gets

$$2\sum_{j\in\mathbb{Z}}\sum_{l=0}^{2\bar{l}(j)} r_{\tilde{\jmath}(j)-l} \leq -C_{\gamma,\lambda}\|c_j\|_\infty^{\gamma-1} \sum_{j\in\frac{1}{3}\mathbb{Z}} (c_j - c_{j-\frac{1}{3}})^2 \leq -2\theta''\|c_j\|_\infty^{\gamma-1}\frac{\|c_{\frac{j}{3}}\|_p^{\frac{3}{p^{-1}}+3}}{\|c_{\frac{j}{3}}\|_1^{\frac{3}{p^{-1}}+1}} ,$$

by Lemma 10, where $\theta'' > 0$ depends upon λ, γ and p. Since $\|c_{\frac{j}{3}}\|_p \geq \|c_j\|_p$, $\|c_{\frac{j}{3}}\|_1 \leq 3\|c_j\|_1$ (as $c_j \geq c_{j+\frac{1}{3}}, c_{j-\frac{1}{3}}$), and $\sup_{j\in\mathbb{Z}} c_j \leq \|c_j\|_p$, one obtains:

$$\mu_{\gamma+1}(\tilde{b}) \leq \mu_{\gamma+1}(b) - \frac{\theta''}{3^{\frac{3}{p^{-1}}+1}}\frac{\|c_j\|_p^{\frac{3}{p^{-1}}+2+\gamma}}{\|c_j\|_1^{\frac{3}{p^{-1}}+1}} ,$$

which is analogous to (22), and one ends the proof as in Section 4.3.

Proof of Lemma 11.

• When $l = 0$, by the same proof as in Section 4.3, one may show that $r_{\tilde{\jmath}} \leq 0$ for any ($\mathcal{C}^1$ convex) φ and that, for $\varphi(s) = s^{\gamma+1}$:

$$\begin{aligned} r_{\tilde{\jmath}} &\leq -\gamma(\gamma+1)\frac{\lambda^2}{2(1-\lambda)} \max\{\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-1}), \mathcal{G}_-^{a_{j+1}}(d_{\tilde{\jmath}})\}^{\gamma-1} \, (\mathcal{G}_-^{a_{j+1}}(d_{\tilde{\jmath}}) - \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-1}))^2 \\ &\leq -\gamma(\gamma+1)\frac{\lambda^2}{2(1-\lambda)} \max\{c_{j+\frac{1}{3}}, c_{j+\frac{2}{3}}\}^{\gamma-1} \, (c_{j+\frac{2}{3}} - c_{j+\frac{1}{3}})^2 . \end{aligned}$$

Formula (24) is then deduced from the inequalities: $c_{j+\frac{2}{3}} \leq c_{j+1}$, $c_{j+\frac{1}{3}} \leq c_j$.
• If l is odd, then $d_{\tilde{\jmath}-l} \geq d_{\tilde{\jmath}-l-1}$ and, for $s \in [\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1}), \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l})]$, one has

$$0 \leq \mathcal{G}_-^{a_{j+1}}(s - \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})) \leq (1-\lambda)(s - \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})) \leq s \ .$$

Hence, $r_{\tilde{\jmath}-l} \leq 0$ for any φ, and, when $\varphi(s) = s^{\gamma+1}$, using the previous inequalities, $\mathcal{G}_-^{a_j}(0) = 0$ and $\lambda < 1$, one gets:

$$\begin{aligned}
r_{\tilde{\jmath}-l} &\leq -\gamma(\gamma+1)(\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l}))^{\gamma-1}\left[\frac{\lambda}{2}\left(\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l}) - \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})\right)^2\right. \\
&\quad \left. + \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})\left(\mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l}) - \mathcal{G}_-^{a_j}(d_{\tilde{\jmath}-l-1})\right)\right] \\
&\leq -\gamma(\gamma+1)(\mathcal{G}_-^{a_j}(b_j))^{\gamma-1}\lambda^2\left(\frac{\lambda}{2}(d_{\tilde{\jmath}-l} - d_{\tilde{\jmath}-l-1})^2 + d_{\tilde{\jmath}-l-1}(d_{\tilde{\jmath}-l} - d_{\tilde{\jmath}-l-1})\right) \\
&\leq -\gamma(\gamma+1)\frac{\lambda^3}{2}c_j^{\gamma-1}(d_{\tilde{\jmath}-l}^2 - d_{\tilde{\jmath}-l-1}^2) \ .
\end{aligned}$$

• If l is nonzero and even, then $d_{\tilde{\jmath}-l} \leq d_{\tilde{\jmath}-l-1}$. Let us rewrite

$$\begin{aligned}
\psi_{j,l}(s) = (\mathcal{G}_-^{a_j})'(s)&\left[\varphi' \circ \mathcal{G}_-^{a_j}(s) - \varphi' \circ \mathcal{G}_-^{a_j}\left(\sum_{l'=0}^{l-1}(-1)^{l'} d_{\tilde{\jmath}-l'-1} + s\right)\right. \\
&\left. + \varphi' \circ \mathcal{G}_-^{a_j}(s - d_{\tilde{\jmath}-l}) - \varphi' \circ \mathcal{G}_-^{a_j}\left(\mathcal{G}_-^{a_j}(s) - d_{\tilde{\jmath}-l} + \mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-l-1})\right)\right] \\
&+ (1 - (\mathcal{G}_-^{a_j})'(s))\left[\varphi' \circ \mathcal{G}_-^{a_j}(s - d_{\tilde{\jmath}-l}) - \varphi' \circ \mathcal{G}_-^{a_j}\left(\sum_{l'=0}^{l-1}(-1)^{l'} d_{\tilde{\jmath}-l'-1} + s\right)\right] \ .
\end{aligned}$$

One has $0 \leq \sum_{l'=0}^{l''}(-1)^{l'} d_{\tilde{\jmath}-l'-1} \leq b_j$ for $l'' = l-1, l-2$, and $s - \mathcal{G}_-^{a_j}(s) - \mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-l-1}) = \mathcal{G}_+^{a_j}(s) - \mathcal{G}_+^{a_j}(d_{\tilde{\jmath}-l-1}) \leq 0$ for $s \in [d_{\tilde{\jmath}-l}, d_{\tilde{\jmath}-l-1}]$. Therefore, $\psi_{j,l}(s)$ is a sum of nonpositive terms and $r_{\tilde{\jmath}-l} \leq 0$ for any φ.

Using the fact that in the expression of $r_{\tilde{\jmath}-l}$, φ' is applied to numbers $s \in [0, c_j]$, one obtains for $\varphi(s) = s^{\gamma+1}$:

$$\begin{aligned} r_{\tilde{\jmath}-l} &\leq -\gamma(\gamma+1)\lambda^2 c_j^{\gamma-1}\left[\left(\sum_{l'=0}^{l-1}(-1)^{l'}d_{\tilde{\jmath}-l'-1}\right)\ (d_{\tilde{\jmath}-l-1}-d_{\tilde{\jmath}-l})\right. \\ &\quad \left. +\frac{\lambda}{2}(d_{\tilde{\jmath}-l-1}-d_{\tilde{\jmath}-l})^2+\left(\sum_{l'=0}^{l-2}(-1)^{l'}d_{\tilde{\jmath}-l'-1}\right)\ (d_{\tilde{\jmath}-l-1}-d_{\tilde{\jmath}-l})\right] \\ &\leq -\gamma(\gamma+1)\lambda^2 c_j^{\gamma-1}\left[\frac{\lambda}{2}(d_{\tilde{\jmath}-l-1}-d_{\tilde{\jmath}-l})^2+\left(\sum_{l'=0}^{l-2}(-1)^{l'}d_{\tilde{\jmath}-l'-1}\right)(d_{\tilde{\jmath}-l-1}-d_{\tilde{\jmath}-l})\right] \\ &\leq -\gamma(\gamma+1)\frac{\lambda^3}{2}c_j^{\gamma-1}\left[\left(\sum_{l'=0}^{l}(-1)^{l'}d_{\tilde{\jmath}-l'-1}\right)^2-\left(\sum_{l'=0}^{l-2}(-1)^{l'}d_{\tilde{\jmath}-l'-1}\right)^2\right]\ , \end{aligned}$$

as $\lambda < 1$.

• Summing the last inequality over all nonzero even indexes, one obtains

$$\begin{aligned} \sum_{l=1}^{\bar{l}} r_{\tilde{\jmath}-2l} &\leq -\gamma(\gamma+1)\frac{\lambda^3}{2}c_j^{\gamma-1}(b_j^2-d_{\tilde{\jmath}-1}^2) \\ &\leq -\gamma(\gamma+1)\frac{\lambda^3}{2}c_j^{\gamma-1}(b_j-d_{\tilde{\jmath}-1})^2 \\ &\leq -\gamma(\gamma+1)\frac{\lambda^3}{2(1-\lambda)^2}c_j^{\gamma-1}(c_j-c_{j+\frac{1}{3}})^2\ , \end{aligned}$$

that is (25).

• At last, summing the contributions over all nonzero indexes, one gets:

$$\begin{aligned} \sum_{l=1}^{2\bar{l}} r_{\tilde{\jmath}-l} &\leq -\gamma(\gamma+1)\frac{\lambda^3}{2}c_j^{\gamma-1}\left(b_j^2-d_{\tilde{\jmath}-1}^2+(d_{\tilde{\jmath}-1}^2-d_{\tilde{\jmath}-2}^2)\right. \\ &\quad \left. +(d_{\tilde{\jmath}-3}^2-d_{\tilde{\jmath}-4}^2)+\cdots+(d_{\tilde{\jmath}-2\bar{l}+1}^2-d_{\tilde{\jmath}-2\bar{l}}^2)\right) \\ &\leq -\gamma(\gamma+1)\frac{\lambda^3}{2}c_j^{\gamma-1}(b_j^2-d_{\tilde{\jmath}-2\bar{l}-1}^2)\ , \quad \text{as } d_{\tilde{\jmath}-2l-1}\geq d_{\tilde{\jmath}-2l} \text{ for } l=1,\ldots\bar{l}. \\ &\leq -\gamma(\gamma+1)\frac{\lambda^3}{2(1-\lambda)^2}c_j^{\gamma-1}(c_j-c_{j-\frac{1}{3}})^2\ , \end{aligned}$$

that is (26). □

4.5 Proof of Proposition 9 for general I such that I does not contain accumulation points of Z

One studies here the case where $I \cap \mathrm{Acc}Z = \emptyset$, but $I \cap (\mathrm{Acc}Z + h) \neq \emptyset$. The same notations as in Sections 4.3 and 4.4 are used for $U, \tilde{U}, U_j, b_j, \tilde{X}, X'$.

Since $I - h$ contains accumulation points of Z, the connected components $\tilde{U}_i$, $i \in \mathcal{I} \subset \mathbb{N}$ of $\tilde{U}$ cannot be ordered in increasing order; to recall this fact, one uses a different notation of the indexes, namely $i \in \mathcal{I}$ instead of $j \in \mathcal{J}$. They still correspond to the maximal intervals on which $\tilde{X}$ is monotonous.

Let $\varepsilon > 0$. There exists a finite subset $\mathcal{I}_f$ of $\mathcal{I}$ such that

$$\sum_{i \in \mathcal{I} \setminus \mathcal{I}_f} \mu_\varphi(\tilde{U}_i) \leq \varphi \left(\mathrm{meas}\, \tilde{U} - \sum_{i \in \mathcal{I}_f} \mathrm{meas}\, \tilde{U}_i \right) \leq \varepsilon \ . \tag{27}$$

Let us approximate $\tilde{X}$ by another solution $\tilde{X}^\varepsilon$ of (1)-(5), defined as follows:

$$\tilde{X}^\varepsilon = \tilde{X} \text{ on } \bigcup_{i \in \mathcal{I}_f} \tilde{U}_i, \quad \tilde{X}^\varepsilon = X' \text{ on } U_j \text{ such that } U_j \cap \left(\bigcup_{i \in \mathcal{I}_f} \tilde{U}_i \right) = \emptyset \ ,$$

and, to complete the definition, on any connected component of $U_j \setminus \bigcup_{i \in \mathcal{I}_f} \tilde{U}_i$, one chooses $\tilde{X}^\varepsilon$ increasing and then decreasing if j is even, decreasing and then increasing if j is odd. The function $\tilde{X}^\varepsilon$ verifies (18) for a new, *ordered*, sequence $\tilde{U}_j^\varepsilon$, of lengths $\tilde{b}_j^\varepsilon$. For this sequence, the result of Proposition 9 applies, due to Section 4.4, and gives:

$$\mu_\varphi(\tilde{b}^\varepsilon) \leq \mu_\varphi(b) \text{ for any } \varphi, \quad \mu_{\gamma+1}(\tilde{b}^\varepsilon) \leq \mu_{\gamma+1}(b) - \theta \frac{\|b\|_p^{\frac{3}{p-1}+2+\gamma}}{\|b\|_1^{\frac{3}{p-1}+1}} \ .$$

In order to get the result of Proposition 9 for I, it suffices to prove that for any φ,

$$\mu_\varphi(I - h) \leq \limsup_{\varepsilon \to 0^+} \mu_\varphi(\tilde{b}^\varepsilon) \ .$$

One has

$$\mu_\varphi(I - h) = \sum_{i \in \mathcal{I}} \mu_\varphi(\tilde{U}_i) = \sum_{i \in \mathcal{I}_f} \mu_\varphi(\tilde{U}_i) + \sum_{i \in \mathcal{I} \setminus \mathcal{I}_f} \mu_\varphi(\tilde{U}_i) \ .$$

Now, for any $\varepsilon > 0$,

$$\sum_{i \in \mathcal{I}_f} \mu_\varphi(\tilde{U}_i) \leq \mu_\varphi(\tilde{b}^\varepsilon) \ ,$$

since any $\tilde{U}_i$ with $i \in \mathcal{I}_f$ is included in a set $\tilde{U}_j^\varepsilon$, and any $\tilde{U}_j^\varepsilon$ contains at most one $\tilde{U}_i$ with $i \in \mathcal{I}_f$. Formula (27) permits to conclude.

4.6 Proof of Proposition 9 for general I

Let now I be a general open interval with bounds in $\mathrm{Acc}Z$ and $\mathrm{meas}\ I \leq h$, and let $U = I \setminus \mathrm{Acc}Z$. The set U is open, and is then the (finite or countable) union of its connected components U_i, $i \in \mathcal{I} \subset \mathbb{N}$, which form a partition of U. However in general, the sets U_i cannot be ordered. Since $\mathrm{meas}\ Z = 0$, one has:

$$\sum_{i\in\mathcal{I}} \mathrm{meas}\ U_i = \mathrm{meas}\ U = \mathrm{meas}\ I\ .$$

Moreover, $I \setminus Z = U \setminus Z = \bigcup_{i\in\mathcal{I}}(U_i \setminus Z)$ (disjoint union) and, by the definition of μ_φ,

$$\mu_\varphi(I) = \sum_{i\in\mathcal{I}} \mu_\varphi(U_i)\ .$$

Since $t \in \mathrm{Acc}Z$ implies $t - h \in \mathrm{Acc}Z$ (by Lemma 7), then $(I-h) \setminus \mathrm{Acc}Z \subset U - h$ and $(I-h) \setminus Z = (U-h) \setminus Z = \bigcup_{i\in\mathcal{I}}((U_i - h) \setminus Z)$ (disjoint union). Therefore,

$$\mu_\varphi(I-h) = \sum_{i\in\mathcal{I}} \mu_\varphi(U_i - h)\ .$$

Since Proposition 9 has already been proved for open intervals with bounds in $\mathrm{Acc}Z$ and no accumulation points of Z inside, one has, for any $i \in \mathcal{I}$:

$$\mu_\varphi(U_i - h) \leq \mu_\varphi(U_i)\ ,$$

$$\mu_{\gamma+1}(U_i - h) \leq \mu_{\gamma+1}(U_i) - \theta \frac{\|U_i\|_p^{\frac{3}{p-1}+2+\gamma}}{(\mathrm{meas}\ U_i)^{\frac{3}{p-1}+1}}\ .$$

Summing the previous inequalities, one obtains (16), and:

$$\mu_\varphi(I-h) \leq \mu_\varphi(I) - \theta \sum_{i\in\mathcal{I}} \frac{\|U_i\|_p^{\frac{3}{p-1}+2+\gamma}}{(\mathrm{meas}\ U_i)^{\frac{3}{p-1}+1}}\ .$$

Now, let us prove that

$$\sum_{i\in\mathcal{I}} \frac{\|U_i\|_p^{\frac{3}{p-1}+2+\gamma}}{(\mathrm{meas}\ U_i)^{\frac{3}{p-1}+1}} \geq \frac{\|I\|_p^{\frac{3}{p-1}+2+\gamma}}{(\mathrm{meas}\ I)^{\frac{3}{p-1}+1}}\ .$$

Indeed, let $q \stackrel{\Delta}{=} (\frac{3}{p-1}+1)p/(\frac{3}{p-1}+2+\gamma)$, then $q \geq 1$ since $\gamma \leq 1$ and $p \geq 1$. Therefore

$$\begin{aligned}
\|I\|_p^{\frac{3}{p-1}+2+\gamma} &= (\|I\|_p^p)^{\frac{1}{q}(\frac{3}{p-1}+1)} = \left(\sum_{i\in\mathcal{I}} \|U_i\|_p^p\right)^{\frac{1}{q}(\frac{3}{p-1}+1)} \\
&\leq \max_{i\in\mathcal{I}} \left(\frac{\|U_i\|_p^p}{(\text{meas } U_i)^q}\right)^{\frac{1}{q}(\frac{3}{p-1}+1)} \left(\sum_{i\in\mathcal{I}} (\text{meas } U_i)^q\right)^{\frac{1}{q}(\frac{3}{p-1}+1)} \\
&\leq \sum_{i\in\mathcal{I}} \frac{\|U_i\|_p^{\frac{3}{p-1}+2+\gamma}}{(\text{meas } U_i)^{\frac{3}{p-1}+1}} (\text{meas } I)^{\frac{3}{p-1}+1} \ .
\end{aligned}$$

This ends the proof of (17) for general I with bounds in Acc$Z$, and achieves the proof of Proposition 9.

4.7 A proof using Schur convexity when F is odd

When F is odd, then

$$\mathcal{G}_\pm^P = \mathcal{G}_\pm^N \ , \quad \varphi^P = \varphi^N \ .$$

In this section, one proves that $\mu_\varphi(I-h) \leq \mu_\varphi(b')$ for all I such that $I \cap \text{Acc}Z = \emptyset$, which implies Proposition 9, due to Section 4.3. This proof replaces that of Sections 4.4 and 4.5.

Let us first study, as in Section 4.4, the case where $I \cap \text{Acc}Z = I \cap (\text{Acc}Z + h) = \emptyset$ and $\tilde{b} \neq b'$.

Let $U, U_j, b_j, \tilde{U}, \tilde{U}_j, \tilde{b}_j, X'$ be defined as in Section 4.3, and $\tilde{X}, \tilde{\jmath} = \tilde{\jmath}(j), \bar{l} = \bar{l}(j)$ as in Section 4.4. For any $j \in \mathbb{Z}$, let T_j be the following transformation on the function $\tilde{X}$ (see Figure 3):

$$T_j\tilde{X} = \tilde{X} \text{ on } I \setminus U_j, \quad T_j\tilde{X} = X' \text{ on } U_j \ . \tag{28}$$

Let us partition U_j into two subintervals U_j^- and U_j^+, with $U_j^- \leq U_j^+$ and

$$\text{meas } U_j^- = b_j^- \stackrel{\Delta}{=} \mathcal{G}_-^P(b_j), \quad \text{meas } U_j^+ = b_j^+ \stackrel{\Delta}{=} b_j - b_j^- = \mathcal{G}_+^P(b_j) \ .$$

If $\tilde{X}$ is equal to X' on U_j, then $\bar{l} = 0$ and

$$\tilde{U}_{\tilde{\jmath}-1} \cap U_j = U_j^-, \quad \tilde{U}_{\tilde{\jmath}} \cap U_j = U_j^+ \ .$$

The transformation T_j generates the following transformation, also denoted T_j, on $\tilde{U}$ and $\tilde{b}$. The sets $\tilde{U}_{\tilde{\jmath}-2\bar{l}-1}, \ldots, \tilde{U}_{\tilde{\jmath}}$ are replaced by $(T_j\tilde{U})_{\tilde{\jmath}-1} = \tilde{U}_{\tilde{\jmath}-2\bar{l}-1} \cup U_j^-$ and $(T_j\tilde{U})_{\tilde{\jmath}} =$

$\tilde{U}_{\tilde{\jmath}} \cup U_j^+$. Therefore, the numbers $\tilde{b}_{\tilde{\jmath}-2\bar{l}-1}, \ldots, \tilde{b}_{\tilde{\jmath}}$ are replaced by $(T_j\tilde{b})_{\tilde{\jmath}-1} = b^- + b_j^-$ and $(T_j\tilde{b})_{\tilde{\jmath}} = b^+ + b_j^+$, with

$$b^- \stackrel{\Delta}{=} \text{meas}\, \tilde{U}_{\tilde{\jmath}-2\bar{l}-1} \setminus U_j, \quad b^+ \stackrel{\Delta}{=} \text{meas}\, \tilde{U}_{\tilde{\jmath}} \setminus U_j \ .$$

The other components remain unchanged, up to a reindexation.

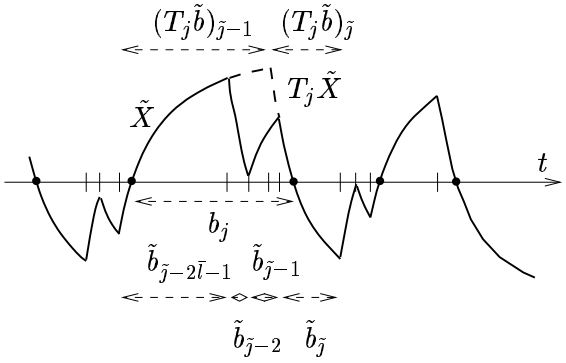


Figure 3: Suboptimal predecessors and their transformation (j is even, $\bar{l}(j) = 1$)

Lemma 12. *For all predecessor $\tilde{b}$ of b and for all $j \in \mathbb{Z}$, one has*

$$\mu_\varphi(\tilde{b}) \leq \mu_\varphi(T_j\tilde{b}) \ .$$

Proof. Let us consider the two $(2\bar{l}+2)$-dimensional vectors

$$u \stackrel{\Delta}{=} (\tilde{b}_{\tilde{\jmath}-2\bar{l}-1}, \ldots, \tilde{b}_{\tilde{\jmath}}), \quad v \stackrel{\Delta}{=} (0, \ldots, 0, (T_j\tilde{b})_{\tilde{\jmath}-1}, (T_j\tilde{b})_{\tilde{\jmath}}) \ . \tag{29}$$

Since $T_j\tilde{b}$ and $\tilde{b}$ differ only (up to a reindexation) by the coordinates present in u and v, one has

$$\mu_\varphi(T_j\tilde{b}) - \mu_\varphi(\tilde{b}) = \sum_{l=1}^{2\bar{l}+2} \varphi^P(v_l) - \sum_{l=1}^{2\bar{l}+2} \varphi^P(u_l) \ .$$

The proof of Lemma 12 then reduces to show $g(u) \leq g(v)$ for $g : \mathbb{R}^{2\bar{l}+2} \to \mathbb{R}, u \mapsto g(u) = \sum_{l=1}^{2\bar{l}+2} \varphi^P(u_l)$. Since φ^P is a convex function, g is Schur-convex (see [16], [9], and [12, Proposition 3.C.1, p. 64], where a general presentation of the subject may be found). In other words, $g(u) \leq g(v)$ for any $u, v \in \mathbb{R}^{2\bar{l}+2}$ such that u is *majorized* by v, that is fulfilling

the two following conditions:

$$\sum_{l=1}^{2\bar{l}+2} u_l = \sum_{l=1}^{2\bar{l}+2} v_l \ , \tag{30}$$

$$\max_{\substack{\text{card } \mathcal{L} = L \\ \mathcal{L} \subset \{1,\dots,2\bar{l}+2\}}} \sum_{l\in\mathcal{L}} u_l \leq \max_{\substack{\text{card } \mathcal{L} = L \\ \mathcal{L} \subset \{1,\dots,2\bar{l}+2\}}} \sum_{l\in\mathcal{L}} v_l, \quad \forall L = 1,\dots,2\bar{l}+2 \ . \tag{31}$$

It then suffices to prove that v majorizes u, for u, v given by (29).

Since

$$\sum_{j'=-\infty}^{+\infty} \tilde{b}_{j'} = \sum_{j'=-\infty}^{+\infty} (T_j\tilde{b})_{j'} = \sum_{j=-\infty}^{+\infty} b_j \ ,$$

one has $\sum_{l=1}^{2\bar{l}+2} u_l = \sum_{l=1}^{2\bar{l}+2} v_l$, that is (30). Since v has only two nonzero coordinates, (31) is fulfilled for $L \geq 2$. The case $L = 1$ is equivalent to

$$\max_{l=0,\dots,2\bar{l}+1} \tilde{b}_{\tilde{\jmath}-l} \leq \max\{(T_j\tilde{b})_{\tilde{\jmath}-1}, (T_j\tilde{b})_{\tilde{\jmath}}\} \ .$$

From the previous computations and as $b^+, b^- \geq 0$, it is sufficient to prove

$$\max\{\tilde{b}_{\tilde{\jmath}-2\bar{l}-1} - b^-,\ \tilde{b}_{\tilde{\jmath}-2\bar{l}+1},\dots,\ \tilde{b}_{\tilde{\jmath}-1}\} \leq b_j^- \ , \tag{32}$$

$$\max\{\tilde{b}_{\tilde{\jmath}-2\bar{l}},\dots,\ \tilde{b}_{\tilde{\jmath}-2},\ \tilde{b}_{\tilde{\jmath}} - b^+\} \leq b_j^+ \ , \tag{33}$$

that is

$$\begin{cases} \max\limits_{\substack{l\,=\,1,\dots\,2\bar{l}\,+\,1 \\ l \text{ odd}}} \text{meas } \tilde{U}_{\tilde{\jmath}-l} \cap U_j \leq \text{meas } U_j^- \ , \\ \max\limits_{\substack{l\,=\,0,\dots\,2\bar{l} \\ l \text{ even}}} \text{meas } \tilde{U}_{\tilde{\jmath}-l} \cap U_j \leq \text{meas } U_j^+ \ . \end{cases}$$

Since the intervals $\tilde{U}_l$ are the maximal intervals on which $\tilde{X}$ is monotonous, the previous inequalities mean that the maximum length of a subinterval of U_j on which $\tilde{X}$ increases (resp. decreases) is less than the length of the (unique) subinterval of U_j on which X' increases (resp. decreases), that is b_j^- (resp. b_j^+) if j is even, b_j^+ (resp. b_j^-) if j is odd. It is easy to see from Figure 3 that this holds. This is indeed a consequence of the stationnarity of the equation $\dot{X} = \varepsilon + F(X)$ ($\varepsilon = \pm 1$).

For instance, let us suppose that j is even, and consider an interval $(t', t'') \subset U_j = (t, t + b_j)$ where $\tilde{X}$ increases. Then $\tilde{X}$ satisfies $\dot{\tilde{X}} = 1 + F(\tilde{X})$ on (t', t''), and $\tilde{X}(t') \geq 0$. But X' satisfies the same equation on $(t, t + b_j^-)$, and $X'(t) = 0$. If $t'' - t' > b_j^-$, then $\tilde{X}(t'') > X'(t + b_j^-)$ (by the stationnarity of $\dot{X} = 1 + F(X)$ and the fact that X' and

$\tilde{X}$ increase), and $t'' > t + b_j^-$. Using all the properties of X' and $\tilde{X}$, that is: X' verifies $\dot{X}' = -1 + F(X')$ on $(t+b_j^-, t+b_j)$, $\tilde{X}$ either increases or is a solution of the same equation on $(t'', t+b_j)$, $\tilde{X}(t'') > X'(t+b_j^-)$ and $\tilde{X}(t+b_j) = X'(t+b_j) = 0$, one obtains that the trajectory of $\tilde{X}$ on $[t'', t+b_j]$ contains all the trajectory of X' on $[t+b_j^-, t+b_j]$ but at different instants. This is impossible, since there is not enough time: $t + b_j - t'' < (t+b_j) - (t+b_j^-)$.

From this, one deduces (32). Analogously, one may show (33). This achieves the proof of Lemma 12. □

By applying the transformations T_j successively for $j \in \{-m, \ldots, m\}$, one obtains a function $\tilde{X}^m \stackrel{\Delta}{=} (T_{-m} \circ \cdots \circ T_m)\tilde{X}$ (the composition is commutative), equal to X' on the closure $\mathrm{cl}\,(\bigcup_{j=-m}^{m} U_j)$. Since the limit of the latter increasing sequence of intervals is equal to I, $\tilde{X}^m$ tends to X' (pointwise). Applying the following lemma to $\tilde{b}^m = (T_{-m} \circ \cdots \circ T_m)\tilde{b}$, one obtains $\mu_\varphi(\tilde{b}^m) \to \mu_\varphi(b')$ when $m \to +\infty$, and finally deduces, with the help of Lemma 12, that $\mu_\varphi(\tilde{b}) \leq \mu_\varphi(b')$. Therefore, b' is an *optimal* predecessor, in the sense that

$$\mu_\varphi(b') = \max_{\tilde{b} \text{ predecessor of } b} \mu_\varphi(\tilde{b}) \ ,$$

and (17) is proved for general I such that I and $I - h$ do not contain accumulation points of Z.

Lemma 13. *Let $\tilde{b}$ be a predecessor of b and $\tilde{X}$ satisfy* (18). *If $\tilde{X} = X'$ on the closure* $\mathrm{cl}\,(\bigcup_{j=-m}^{m} U_j)$, *then*

$$|\mu_\varphi(\tilde{b}) - \mu_\varphi(b')| \leq \frac{\varphi^P(h)}{h} \mathrm{meas} \left(I \setminus \bigcup_{j=-m+1}^{m-1} U_j \right) = \frac{\varphi^P(h)}{h} \left(\sum_{j=-\infty}^{-m} b_j + \sum_{j=m}^{+\infty} b_j \right) \ .$$

Proof. The proof proceeds from the following inequalities:

$$\begin{aligned} |\mu_\varphi(\tilde{b}) - \mu_\varphi(b')| &\leq \left| \sum_{j \in \tilde{\mathcal{J}}_m} \varphi^P(\tilde{b}_j) - \sum_{j \in \mathcal{J}'_m} \varphi^P(b'_j) \right| \\ &\quad \text{where } \tilde{\mathcal{J}}_m \stackrel{\Delta}{=} \left\{ j \in \mathbb{Z} \ : \ \tilde{U}_j \not\subset \mathrm{cl} \left(\bigcup_{l=-m}^{m} U_l \right) \right\} \text{ and similarly for } \mathcal{J}'_m \\ &\leq \frac{\varphi^P(h)}{h} \max \left\{ \sum_{j \in \tilde{\mathcal{J}}_m} \tilde{b}_j, \sum_{j \in \mathcal{J}'_m} b'_j \right\} \\ &\leq \frac{\varphi^P(h)}{h} \left(\sum_{j=-\infty}^{-m} b_j + \sum_{j=m}^{+\infty} b_j \right) \ , \end{aligned}$$

using the fact that $\varphi^P(s) \leq s\varphi^P(h)/h$ for $0 \leq s \leq h$ (due to convexity of φ^P and $\varphi^P(0) = 0$) and the definition of $\tilde{\mathcal{J}}_m$, $\mathcal{J}'_m$. □

We now consider the case where $I \cap (\mathrm{Acc}Z + h) \neq \emptyset$. We get from Section 4.5 that

$$\mu_\varphi(I - h) \leq \liminf_{\varepsilon \to 0^+} \mu_\varphi(\tilde{b}^\varepsilon) \ ,$$

where the $\tilde{b}^\varepsilon$ are predecessors of b. From what was proved in the present section, we hence deduce $\mu_\varphi(I - h) \leq \mu_\varphi(b')$.

Remark that one may also generalize Lemma 12 to this case. Indeed, the weaker order relation induced by (31), called *weak majorization*, suffices to prove $g(u) \leq g(v)$, since φ^P is increasing in $\mathbb{R}^+$ and $u_l, v_l \geq 0$; this result is proved in [19, 20] and may be found in [12, 3.C.1.b, p. 64]. The weak majorization is satisfied by the vectors u and v generated by some finite subsets of measures of those $\mathrm{cl}\,(\tilde{U}_i)$ and $(T_j\tilde{U})_i$ which intersect $\mathrm{cl}\,(U_j)$. Then, Lemma 12 is deduced by passing to the limit.

A Appendix - Proof of Lemma 10

In a first step, one uses Gagliardo-Nirenberg result [8, 13] under the following form: for any $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ such that $f' \in L^q(\mathbb{R})$, one has

$$\|f\|_1^{\frac{pq-p+q}{p-1}} \ \|f'\|_q^q \geq \left(\frac{q}{2q-1}\right)^q \|f\|_p^{\frac{p(2q-1)}{p-1}} \ . \tag{34}$$

For sake of completness, one provides the proof of (34). Let us first prove that $\lim_{b \to -\infty} f(b)$ exists — and hence is 0. For any $s', s'' \in \mathbb{R}$, one has (by Hölder inequality):

$$\frac{q}{2q-1} \left| \left[|f(s)|^{\frac{2q-1}{q}} \right]_{s'}^{s''} \right| \leq \int_{s'}^{s''} |f'(s)| \ |f(s)|^{\frac{q-1}{q}} \ ds \leq \left(\int_{s'}^{s''} |f'(s)|^q \ ds \right)^{\frac{1}{q}} \left(\int_{s'}^{s''} |f(s)| \ ds \right)^{\frac{q-1}{q}} \ ,$$

which tends to 0 when $\max\{s', s''\} \to -\infty$. From the following inequalities:

$$\begin{aligned}
\|f\|_p^p &= \int_{\mathbb{R}} |f(s)|^p \, ds = \int_{\mathbb{R}} |f(s)| \, |f(s)|^{p-1} \, ds \\
&= \int_{\mathbb{R}} |f(s)| \left(\int_{-\infty}^{s} (|f|^{\frac{2q-1}{q}})'(s') \, ds' \right)^{\frac{q(p-1)}{2q-1}} \, ds \\
&= \left(\frac{2q-1}{q} \right)^{\frac{q(p-1)}{2q-1}} \int_{\mathbb{R}} |f(s)| \left(\int_{-\infty}^{s} f'(s') \, \mathrm{sgn} f(s') \, |f(s')|^{\frac{q-1}{q}} \, ds' \right)^{\frac{q(p-1)}{2q-1}} \, ds \\
&\leq \left(\frac{2q-1}{q} \right)^{\frac{q(p-1)}{2q-1}} \int_{\mathbb{R}} |f(s)| \, ds \left(\left(\int_{\mathbb{R}} |f'(s)|^q \, ds \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |f(s)| \, ds \right)^{\frac{q-1}{q}} \right)^{\frac{q(p-1)}{2q-1}} \\
&\quad \text{(by Hölder inequality)} \\
&= \left(\frac{2q-1}{q} \right)^{\frac{q(p-1)}{2q-1}} \|f\|_1^{\frac{pq-p+q}{2q-1}} \|f'\|_q^{\frac{q(p-1)}{2q-1}} ,
\end{aligned}$$

one gets (34) when $p < +\infty$. Inequality (34) holds for $p = +\infty$ too, by passing to the limit.

Now, let us define $f \in W^{1,\infty}(\mathbb{R})$ by:

$$f(s) = (b_{j+1} - b_j)s + (j+1)b_j - jb_{j+1} \text{ for } s \in [j, j+1) .$$

One proves easily that $\|f\|_1 \leq \|b\|_1$, $\|f'\|_q = \|b_{j+1} - b_j\|_q$. Moreover,

$$\begin{aligned}
\|f\|_p^p &= \frac{1}{p+1} \sum_{j \in \mathbb{Z}} \frac{b_{j+1}|b_{j+1}|^p - b_j|b_j|^p}{b_{j+1} - b_j} \\
&= \frac{1}{2(p+1)} \sum_{j \in \mathbb{Z}} \left(|b_{j+1}|^p + |b_j|^p + \frac{b_{j+1} + b_j}{b_{j+1} - b_j} (|b_{j+1}|^p - |b_j|^p) \right) \\
&\geq \frac{1}{2(p+1)} \sum_{j \in \mathbb{Z}} (|b_{j+1}|^p + |b_j|^p) = \frac{1}{p+1} \|b_j\|_p^p .
\end{aligned}$$

This achieves the proof of Lemma 10 in the case $p < +\infty$, with

$$C_{p,q} = \left(\frac{q}{2q-1} \right)^q \left(\frac{1}{p+1} \right)^{\frac{2q-1}{p-1}} .$$

Since

$$\lim_{p \to +\infty} C_{p,q} = \left(\frac{q}{2q-1} \right)^q ,$$

Lemma 10 also holds for $p = +\infty$, with $C_{\infty,q}$ equal to this limit.

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