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Amel Ben Abda, Slim Chaabane, Fadi El Dabaghi, Mohamed Jaoua. On a Non Linear Geometrical Inverse Problem of Signorini type: Identifiability and Stability. [Research Report] RR-3175, INRIA. 1997. inria-00077198

**HAL Id: inria-00077198**

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Submitted on 29 May 2006

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**N° 3175**

May 1997

————— THÈME 4 —————



*Rapport  
de recherche*



## On a non linear geometrical inverse problem of Signorini type : identifiability and stability

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Thème 4 — Simulation et optimisation  
de systèmes complexes  
Projets Estime , Gamma , Mostra , M3N

Rapport de recherche n° 3175 — May 1997 — 18 pages

**Abstract:** This report deals with a non linear inverse problem : identification of unknown boundaries, on which the prescribed conditions are of Signorini type. We first prove an identifiability result, in both frameworks of steady state thermal and elastostatics testing. Local Lipschitz stability of the solutions, with respect to the boundary measurements, is also established under the assumption that the unknown boundary is part of a  $C^{1,\beta}$  Jordan curve, with  $\beta > 0$ .

**Key-words:** geometrical inverse problems, identification, Signorini type boundary conditions, unknown boundaries, identifiability, Lipschitz local stability, domain derivatives, optimal shape design.

(Résumé : *tsvp*)

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This work was partially achieved during a visit by ABA at Ecole Polytechnique (LMS and CMAP), partially supported by CNRS, and visits by SC at INRIA, partially supported by the french-tunisian CMCU/96-F-1508 research programme entitled *Méthodes de simulations numérique*. MJ's research is partially supported, during his sabbatical leave at INRIA, by a french government's MENESR-DRIC grant.

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# Sur un problème inverse géométrique non linéaire de type Signorini

**Résumé :** On s'intéresse dans ce rapport à un problème inverse non linéaire d'identification de frontières inconnues par des mesures de surface, les conditions aux limites étant de type Signorini. On montre d'abord un résultat d'identifiabilité, valable pour les mesures thermiques dans un cadre stationnaire, comme pour les mesures élastiques, dans un cadre linéaire, homogène et isotrope. La stabilité locale lipschitzienne des solutions vis à vis des mesures est ensuite prouvée, sous l'hypothèse que la frontière inconnue est une partie de courbe de Jordan de classe  $\mathcal{C}^{1,\beta}$ , avec  $\beta > 0$ .

**Mots-clé :** problèmes inverses géométriques, identification, conditions aux limites de Signorini, détection de frontières, identifiabilité, stabilité locale lipschitzienne, dérivation par rapport au domaine, optimisation de forme.

## 1 Introduction

This work is devoted to the study of an inverse geometrical problem, which consists in finding the shape of an unknown part  $\gamma$  of the boundary  $\partial\Omega$  of a two-dimensional body  $\Omega$ . The two extremal points of the unknown boundary  $\gamma$  are supposed to be known, and boundary conditions of Signorini type are prescribed on  $\gamma$ . In the elasticity framework, the direct problem modelizes states of equilibrium of a linear elastic body, the part  $\gamma$  of its boundary being supported by a non deformable friction-free surface.

The practical motivation of this work is related to non destructive control processes. Using steady thermal, electrical, or elastic measurements, the governing state equation (or system) is elliptic (Laplace equation, or Lamé system). Our interest is focused on uniqueness and stability questions. Uniqueness is a crucial point in this kind of problems, since it informs us if a single measurement (or a finite number of them) is enough to insure the identifiability.

Many theoretical studies have been performed for the similar problem of conductivities identification. Kohn & Vogelius [14] established first in 1985 the uniqueness, with infinitely many measurements (that is the whole Neumann-Dirichlet operator), for inclusion domains with analytical boundaries, while Isakov [10, 1988] proved later the same result for Lipschitz boundaries. But the most interesting results, for practical purposes, will come later on, when uniqueness is proved for a single measurement, or at least for a finite number of them. Bellout & Friedman [4, 1988], Alessandrini ([1, 1988]), Isakov [10, 1988], Friedman & Isakov [8, 1989], Isakov & Powell [11, 1990], as well as Bellout, Friedman & Isakov [5, 1992] proved uniqueness results, dropping or weakening paper after paper some of the restrictive assumptions (regularity, monotonicity,...) made previously on the admissible boundaries. As for the papers involving Bellout & Friedman ([4], [5]), the identifiability results are obtained as consequences of Lipschitz local stability ones. Such a question is also present in all Alessandrini's papers. For linear and non linear boundary conditions, uniqueness local Lipschitz stability (in the linear case) results have also been established in [2, 1993] and [3, 1997].

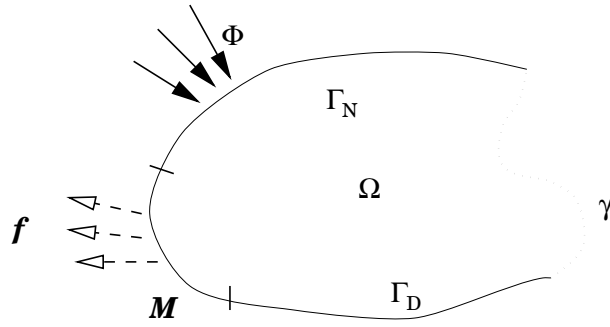
Section 2 is devoted to uniqueness (*identifiability*) questions, in the thermal framework, as well as in the elasticity framework. In the third section, we deal with stability questions. A local Lipschitz stability result is proved, under the assumption that the boundary  $\gamma$  be part of some  $\mathcal{C}^{1,\beta}$  Jordan curve for some  $\beta > 0$ , by using domain derivative techniques, as well as arguments related to analytical functions theory.

## 2 Identifiability

Let  $\Omega$  denote a 2D or 3D domain occupied by the body, and  $\partial\Omega$  its boundary, that we shall divide in three parts as shown in Figure 1 :

$$\partial\Omega = \gamma \cup \Gamma_D \cup \Gamma_N$$

where  $\gamma$  is the unknown part,  $\Gamma_N$  the part where the fluxes used for the measurements are prescribed, and  $\Gamma_D$  the part where an homogeneous Dirichlet condition is prescribed in order to get a well-posed direct problem.

Figure 1: *The domain and its boundary*

## 2.1 Case of steady state thermal testing

### 2.1.1 The direct problem.

Let us denote by  $\Omega_\gamma$  the domain  $\Omega$  with unknown boundary  $\gamma$ . The direct problem is therefore given by :

$$\begin{cases} \Delta u_\gamma & = & 0 & \text{in } \Omega_\gamma \\ u_\gamma & = & 0 & \text{on } \Gamma_D \\ \frac{\partial u_\gamma}{\partial n} & = & \phi & \text{on } \Gamma_N \\ u_\gamma \geq 0 & \frac{\partial u_\gamma}{\partial n} \geq 0 & u_\gamma \frac{\partial u_\gamma}{\partial n} = 0 & \text{on } \gamma \end{cases} \quad (1)$$

where  $\phi$  is a prescribed heat flux on  $\Gamma_N$  ;  $\phi \in H^{-\frac{1}{2}}(\Gamma_N)$  and  $\phi \not\equiv 0$  on  $\Gamma_N$ .

The associated variational formulation of such an elliptic inequality, as well as the existence and uniqueness of the solution, are well known (see for example [9, 1976]). Let us briefly recall two equivalent formulations of problem (1) :

$$\begin{cases} u_\gamma \in K \\ a(u_\gamma, v - u_\gamma) \geq L(v - u_\gamma) \quad \forall v \in K \end{cases} \quad (2)$$

or

$$\begin{cases} u_\gamma \in K \\ J(u_\gamma) \leq J(v) \quad \forall v \in K \end{cases} \quad (3)$$

where  $K$  is the closed convex set of  $H^1(\Omega_\gamma)$  defined by :

$$K = \{v \in H^1(\Omega_\gamma); v = 0 \text{ on } \Gamma_D \text{ and } v \geq 0 \text{ on } \gamma\}$$

and where, for  $u$  and  $v$  in  $H^1(\Omega_\gamma)$  :

$$\begin{cases} a(u, v) & = & \int_{\Omega_\gamma} \nabla u \cdot \nabla v \\ L(v) & = & \int_{\Gamma_N} \phi v \\ J(v) & = & \frac{1}{2} a(v, v) - L(v) \end{cases}$$

### 2.1.2 Uniqueness for the inverse problem.

Let  $M$  be an open subset, with positive measure, of  $\Gamma_N$ . We are going to prove the identifiability result, which is that two different admissible boundaries  $\gamma_1$  and  $\gamma_2$  cannot produce the same measured temperature  $f$  on  $M$  for the given flux  $\phi$ .

**Theorem 1 (identifiability)** *Let  $\gamma_1$  and  $\gamma_2$  be two piecewise  $C^{1,1}$  boundaries having the same extremity points. Assume that the corresponding domains  $\Omega_i = (\Omega_{\gamma_i})$  are connected, and let  $u_i = u_{\gamma_i}$  be the solution of problem (1) in domain  $\Omega_{\gamma_i} = \Omega_i$  ( $i = 1, 2$ ). Then, if  $u_1|_M = u_2|_M$ , the boundaries  $\gamma_1$  and  $\gamma_2$  coincide.*

**Proof :** Let us denote by  $\Omega_{12}$  the intersection  $\Omega_{\gamma_1} \cap \Omega_{\gamma_2}$  (see figure2), and let  $\omega = u_1 - u_2$ .  $\omega$  is then solution of the Cauchy problem :

$$\begin{cases} \Delta\omega = 0 & \text{in } \Omega_{12} \\ \omega = 0 & \text{on } M \\ \frac{\partial}{\partial n}\omega = 0 & \text{on } M \end{cases}$$

Since  $\Omega_{12}$  is connected,  $\omega$  vanishes on the whole domain  $\Omega_{12}$  by the Holmgren's unique continuation theorem. Thus :

$$u_1 = u_2 \quad \text{on } \partial(\Omega_1 \cap \Omega_2) \quad (4)$$

and also :

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \quad \text{on } \partial(\Omega_1 \cap \Omega_2) \quad (5)$$

Let  $\mathcal{O} = (\Omega_1 \cup \Omega_2) \setminus \overline{(\Omega_1 \cap \Omega_2)}$ . Assume that  $\mathcal{O} \neq \emptyset$ , and let  $\mathcal{O}_1$  be one of its connected components. Assume for instance that  $\mathcal{O}_1 \subset \Omega_1 \setminus \Omega_2$ .

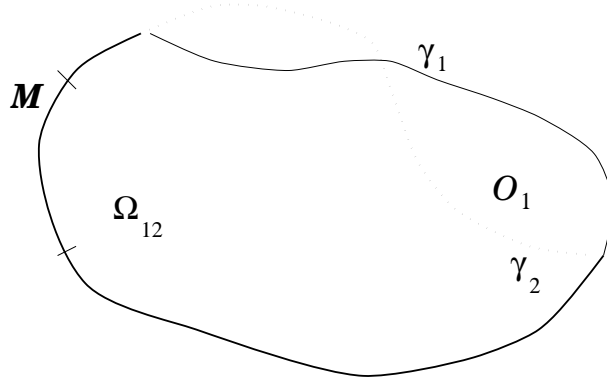


Figure 2: Two possible boundaries

Then, in the open set  $\mathcal{O}_1$ ,  $u_1$  is solution of :

$$\begin{cases} \Delta u_1 = 0 & \text{in } \mathcal{O}_1 \\ u_1 \frac{\partial u_1}{\partial n} = 0 & \text{on } \partial\mathcal{O}_1 \cap \gamma_1 \\ u_1 \frac{\partial u_1}{\partial n} = 0 & \text{on } \partial\mathcal{O}_1 \cap \gamma_2 \end{cases}$$

The boundary conditions on  $\partial\mathcal{O}_1 \cap \gamma_1$  and  $\partial\mathcal{O}_1 \cap \gamma_2$  come from the Signorini boundary condition, and from (4) and (5). It comes then that :

$$\int_{\mathcal{O}_1} |\nabla u_1|^2 = \int_{\partial\mathcal{O}_1} u_1 \frac{\partial u_1}{\partial n} = 0$$



and thus  $u_1$  is constant in  $\mathcal{O}_1$ . By analyticity,  $u_1$  is constant in the whole domain  $\Omega_1$ , and thus  $\frac{\partial u_1}{\partial n} = 0$  on  $\partial\Omega_1$ , which is in contradiction with  $\phi \not\equiv 0$  on  $\Gamma_N$ . ■

**Remark 1 :** This proof drops the interior sphere assumption on the domain  $\Omega_\gamma$ , which was needed in [2] in order to use the Hopf maximum principle.

**Remark 2 :** This result extends to any analytical hypo-elliptic operator, for example for any elliptic operator with constant coefficients.

## 2.2 Case of elastostatics testing

Let us denote by  $u$  the displacement field,  $\varepsilon$  the associated linearized strain tensor, and  $\sigma$  the stress tensor. The material is supposed to be isotropic and homogeneous, and the constitutive law is linear. The stiffness tensor  $R$  then fulfills the classical symmetry and ellipticity conditions, that is for some real positive number  $\rho$  :

$$\left\{ \begin{array}{l} R_{ijkl} = R_{jikl} = R_{klij} \quad i, j, k, l = 1, 2 \\ \sum_{k,l=1,2} R_{ijkl} \xi_{ij} \xi_{kl} \geq \rho \sum_{i,j=1}^2 (\xi_{ij})^2 \quad \forall \xi \in \mathbf{R}^4 \end{array} \right. \quad (6)$$

The direct problem is then the following :

$$\left\{ \begin{array}{lll} \sigma_{ij,i} & = & 0 \quad \text{in } \Omega_\gamma \\ \sigma_{ij} & = & R_{ijkl} (\varepsilon_{kl}(u)) \quad \text{in } \Omega_\gamma \\ u & = & 0 \quad \text{on } \Gamma_D \\ \sigma_{ij} n_j & = & g_i \quad \text{on } \Gamma_N \\ \sigma_{ij} t_j & = & 0 \quad \text{on } \gamma \\ (\sigma_{ij} n_i) n_j \leq 0 \quad u \cdot n \leq 0 \quad ((\sigma \cdot n) \cdot n) (u \cdot n) = 0 & & \text{on } \gamma \end{array} \right. \quad (7)$$

where  $g$  is a prescribed load on  $\Gamma_N$  ( $g \in (H^{-\frac{1}{2}}(\Gamma_N))^2$  and  $g \not\equiv 0$  on  $\Gamma_N$ ).

It is well known that the solution of (7) is unique (see for example [13, 1988]), and the associated variational formulations are similar to (2) and (3), where the convex set  $K$  is defined as follows :

$$K = \{v \in (H^1(\Omega_\gamma))^2 ; v = 0 \text{ on } \Gamma_D \text{ and } v \cdot n \leq 0 \text{ on } \gamma\}$$

We can then settle the identifiability result exactly in the same way than for thermal testing (theorem 1). Its proof works also the same way, except it uses Almansi's lemma, which generalises Holmgren's theorem to elliptic systems [17].

**Remark :** Although they were formulated in 2D situations, these identifiability results extend without difficulty to 3D.

## 3 Stability.

In this section, problem (1) is again considered. The overspecified data on the open set  $M$  of the boundary  $\partial\Omega$  have been obtained by measurements, and are thus subject to errors. The stability means, roughly speaking, that *small* errors on the measurements lead to *small* perturbations on the unknown geometry. To formalize this idea, let us consider a set  $\Gamma_{ad}$  of admissible geometries, and the mapping  $\eta$  defined, the *identifying* flux  $\phi$  of the previous section being given, by :

$$\begin{aligned} \eta &: \Gamma_{ad} \longmapsto L^2(M) \\ \gamma &\longmapsto f = u_\gamma|_M \end{aligned}$$

The identifiability result proved in the previous section means that this mapping is one-to-one, and therefore, that the mapping :

$$\begin{aligned} \eta &: \Gamma_{ad} \longmapsto \eta(\Gamma_{ad}) \\ \gamma &\longmapsto f = u_\gamma|_M \end{aligned}$$

is invertible. The stability will be established if one proves, after having equipped  $\Gamma_{ad}$  with an appropriate topology, that  $\eta^{-1}$  is continuous. But this might be not sufficient for numerical purposes. This is the reason why we shall be focusing our attention on Lipschitz stability, even if the results expected hold only locally. We shall be using for that the derivatives with respect to the domain as a basic tool.

### 3.1 Derivatives of the solution with respect to the domain

To prove local stability results, we need to map an admissible boundary onto another one, close to it. Following Murat-Simon [16], we shall use mappings from the whole domain  $\Omega$  onto  $\Omega_h$ , defined as follows :

$$F_h = Id + h\theta$$

where  $\theta$  is a  $(\mathcal{C}^1(\mathcal{B}))^2$  vector-field defined on some bounded ball  $\mathcal{B}$  containing  $\overline{\Omega_\gamma}$ , verifying :

$$\left\{ \begin{array}{l} \theta \equiv 0 \quad \text{on } \Gamma_D \cup \Gamma_N \\ \theta \equiv 0 \quad \text{in some neighbourhood of } \Gamma_D \\ \theta \cdot \tau = 0 \quad \text{on } \gamma \\ \theta_n = \frac{\partial \theta_n}{\partial \tau} = 0 \quad \text{on } \partial\gamma \end{array} \right. \quad (8)$$

There exists some constant  $h_0$  such that  $F_h$  be a diffeomorphism for any  $h$  such that  $|h| < h_0$ . Let us denote by  $\Omega_{\gamma_h}$ , or by  $\Omega_h$  the set :

$$\Omega_h = (Id + h\theta)(\Omega) \quad (9)$$

the boundary of which is  $\partial\Omega_h = \gamma_h \cup \Gamma_D \cup \Gamma_N$ ,  $\gamma_h$  being the image by  $F_h$  of  $\gamma$ . Let us now denote by  $u_h$  the solution of the non linear boundary problem on  $\Omega_h$  :

$$\left\{ \begin{array}{l} \Delta u_h = 0 \quad \text{in } \Omega_h \\ u_h = 0 \quad \text{on } \Gamma_D \\ \frac{\partial u_h}{\partial n} = \phi \quad \text{on } \Gamma_N \\ u_h \geq 0 \quad \frac{\partial u_h}{\partial n} \geq 0 \quad u_h \frac{\partial u_h}{\partial n} = 0 \quad \text{on } \gamma_h \end{array} \right. \quad (10)$$

and let  $u^h$  be its “transported” on the original domain  $\Omega_\gamma$ , also denoted  $\Omega$  :

$$u^h = u_h \circ F_h \quad (11)$$

Denoting by  $u^0$  the solution  $u_\gamma$  of problem (1), we can define a partition of the unknown boundary  $\gamma$  into two parts : a “Dirichlet” part  $\gamma_D$  on which the boundary condition  $u^0 = 0$  is fulfilled, and a “Neumann” part  $\gamma_N = \gamma \setminus \gamma_D$ . Then :

$$\gamma_D = \{x \in \gamma; u^0(x) = 0\} \quad (12)$$

and therefore :

$$\gamma_N = \gamma \setminus \gamma_D = \{x \in \gamma; u^0(x) > 0\} \quad (13)$$

Let us then define the following convex set :

$$\mathcal{S} = \left\{ v \in H^1(\Omega_\gamma); \int_\Omega \nabla u^0 \nabla v = \int_{\Gamma_N} \phi v; v|_{\Gamma_D} = 0 \text{ and } v \geq 0 \text{ a.e. on } \gamma_D \right\} \quad (14)$$

The following expansion result is due to J. Sokolowski & J.P. Zolesio [20, 1982].

**Theorem 2 (Sokolowski-Zolesio)** *The scalar field  $u^h$  can be expanded as follows :*

$$u^h = u^0 + h u^1 + h O(h) \quad (15)$$

where  $u^1$  and  $O(h)$  are elements of  $H^1(\Omega)$  verifying :

- $\lim_{h \rightarrow 0} O(h) = 0$  in  $H^1(\Omega)$
- $u^1$  is the unique solution of the following variational inequality in  $\mathcal{S}$  :

$$\int_\Omega \nabla u^1 \nabla v \geq \int_\Omega \left( \frac{\partial \theta}{\partial M}{}^t + \frac{\partial \theta}{\partial M} \right) \nabla u^0 \nabla v - \int_\Omega (\nabla u^0 \nabla v) \operatorname{div} \theta \quad (16)$$

for any  $v \in \mathcal{S}$

### 3.2 Local Lipschitz stability

From now on, we shall denote by a subscript  $N$  the solutions of problems with a prescribed Neumann boundary condition ( i.e. the given flux  $\frac{\partial u}{\partial n} = \phi$  on  $\Gamma_N$ ), and by the subscript  $D$  the solutions of problems with the measured temperature as prescribed Dirichlet boundary value (i.e.  $u|_M = f$ ). The solution of the Signorini direct problem (1), with the prescribed flux  $\phi$ , will then be denoted  $u_N$ , or  $u_N^0$ , and its derivative with respect to the domain  $u_N^1$ . The solution of the Signorini problem with prescribed flux on the perturbed domain  $\Omega_h$  will be denoted  $u_{hN}$ , while its “transported” on the original domain  $\Omega$  is  $u_N^h$ .

At this point, we have also to make it clear that we denote by  $\gamma$  the unknown boundary, *without its extreme points*, so that it is an open subset of  $\partial\Omega$ .

#### 3.2.1 Some preliminary technical results

The proof of the local Lipschitz stability result is somewhat technical, and needs some additional light to be thrown on the topological features of the partition  $(\gamma_D, \gamma_N)$  of the unknown boundary  $\gamma$ . The desired result would be that the sets  $\gamma_D$  and  $\gamma_N$  defined by (12) and (13), be also characterized - up to neglectible sets - as follows :

$$\gamma_D = \left\{ x \in \gamma; \frac{\partial u_N^0}{\partial n}(x) > 0 \right\} \text{ and thus } \gamma_N = \left\{ x \in \gamma; \frac{\partial u_N^0}{\partial n}(x) = 0 \right\}$$

Actually, such a result could not be proved. The reason is that the set  $(\gamma_D \setminus \overset{\circ}{\gamma}_D)$ , where both  $u_N^0$  and  $\frac{\partial u_N^0}{\partial n}$  could vanish, might be some closed subset of  $\gamma_D$  of positive measure and void interior, such as Cantor p-adic sets. As far as such possibility is not excluded, the best we can expect in characterizing these sets is the following.

**Theorem 3** *Assume the unknown boundary  $\gamma$  be part of a  $\mathcal{C}^{1,\beta}$  Jordan curve, for some  $\beta > 0$ . Then,  $\gamma_N$  is an open subset of  $\gamma$ , and  $\gamma_D$  a closed one, the interior of which is - up to a neglectible set - the following :*

$$\gamma_D^\circ = \left\{ x \in \gamma; \frac{\partial u_N^0}{\partial n}(x) > 0 \right\} \quad (17)$$

**Proof :** As a matter of fact, it happens that the solution  $u_N^0$  of the Signorini problem is locally (i.e. in the vicinity of any open subset of  $\gamma$ ) smoother than the solution of the associated mixed Dirichlet-Neumann problem (Dirichlet boundary condition on  $\gamma_D$ , Neumann boundary condition on  $\gamma_N$ ) : singularities having a  $\rho^{\frac{1}{2}}$ -behaviour at the vicinity of the switch-points ( $\rho$  being the distance to these points) from one boundary to another occur for this latter, while the positiveness condition eliminates them from the Signorini solution. More precisely, according to Lions [15, 1969] and to Khodja-Moussaoui [12, 1992], we have :

$$u_N^0 \in H^2(\mathcal{O}) \cap \mathcal{C}^{1,\beta}(\overline{\mathcal{O}}), \text{ with } 0 < \beta < \frac{1}{2} \quad (18)$$

$\mathcal{O}$  being any open subset of  $\Omega$  such that  $\overline{\mathcal{O}} \cap \partial\Omega \subset \gamma$ .

It follows that  $u_N^0$  is continuous on  $\gamma$ , so that  $\gamma_N$  is an open subset, and  $\gamma_D$  a closed one of  $\gamma$ .  $\frac{\partial u_N^0}{\partial n}(x)$  is also continuous on  $\gamma$ , so that by the Signorini condition,  $\frac{\partial u_N^0}{\partial n}(x) = 0$  on  $\gamma_N$ . The set :

$$\mathcal{A} := \left\{ x \in \gamma; \frac{\partial u_N^0}{\partial n}(x) > 0 \right\}$$

is therefore an open subset of  $\gamma_D$ , and accordingly of its interior  $\gamma_D^\circ$ .

To prove that  $\mathcal{A} = \gamma_D^\circ$ , it is sufficient to establish that the set  $(\gamma_D^\circ \setminus \mathcal{A})$  has no accumulation points in  $\gamma_D^\circ$ , which would insure that its measure is zero. The forthcoming lemma is dealing with this issue. ■

**Lemma 1** *The set  $(\gamma_D^\circ \setminus \mathcal{A}) = \{x \in \gamma_D^\circ; \nabla u(x) = 0\}$  has no accumulation points in  $\gamma_D^\circ$ , which means that all its points are isolated.*

**Proof of the lemma :** Let  $x_0$  be a non-isolated point of  $(\gamma_D^\circ \setminus \mathcal{A})$ . Let  $\vartheta_{x_0}$  be some open neighbourhood of  $x_0$ , included in  $\gamma_D^\circ$ . We can find some sequence  $x_n \in (\gamma_D^\circ \setminus \mathcal{A}) \cap \vartheta_{x_0}$  verifying :

$$\begin{cases} x_n \neq x_0 \forall n \in \mathbf{N} \\ \lim_{n \rightarrow +\infty} x_n = x_0 \end{cases} \quad (19)$$

Choose now some open subset  $\mathcal{O}$  of  $\Omega$  such that  $\overline{\mathcal{O}} \cap \partial\Omega \subset \vartheta_{x_0}$ .  $\partial\mathcal{O}$  is part of some  $\mathcal{C}^{1,\beta}$  Jordan curve, and it is then possible to find two conformal mappings  $\varphi$  and  $\psi$ , the first one mapping the unit disc  $\mathcal{D}$  on  $\mathcal{O}$  and the second one mapping  $\mathcal{D}$  on some simply connected domain  $\Theta$  of the half plane  $\{(x, y) \in \mathbf{R}^2; y \geq 0\}$ . The boundary of  $\Theta$  is some  $\mathcal{C}^{1,\beta}$  Jordan curve, and we can suppose a part of it is a segment  $[a, b]$  included in the  $x$ -axis.

Moreover, by the Kellogg-Warschawski theorem (see Pommerenke [18]), we have :

$$\begin{cases} \varphi \text{ ( and } \psi \text{ ) are diffeomorphisms from } \overline{\mathcal{D}} \text{ on } \overline{\Omega} \text{ ( and on } \overline{\Theta} \text{ )} \\ \varphi \text{ and } \psi \text{ are differentiable on } \overline{\mathcal{D}} \\ \varphi'(z) \neq 0 \text{ and } \psi'(z) \neq 0 \quad \forall z \in \overline{\mathcal{D}} \end{cases} \quad (20)$$

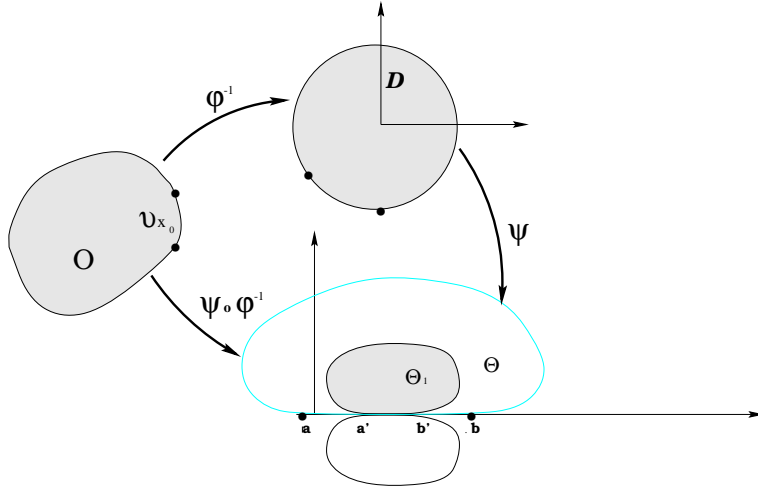


Figure 3: The conformal mappings

The function  $w = u_N^0 \circ (\varphi \circ \psi^{-1})$  is then harmonic in  $\Theta$ .  $x_0 \in \gamma_D^\circ$  and, up to a rotation, we can assume that  $(\psi \circ \varphi^{-1})(x_0) \in ]a, b[ \times \{0\}$ . Since  $u_N^0$  vanishes in some neighbourhood of  $x_0$ ,  $w$  vanishes in the associated neighbourhood of  $(\psi \circ \varphi^{-1})(x_0)$ , and we can then find two real numbers  $a'$  and  $b'$ ,  $a < a' < b' < b$  such that  $w(t, 0) = 0 \quad \forall t \in ]a', b'['$ .

Let now  $\Theta_1$  be an open subset of  $\Theta$  with  $C^1$  boundary, such that  $\partial\Theta_1 \cap (x'ox) = [a', b'] \times \{0\}$ . Let us denote by  $\Theta_1^s$  the symmetrized set, with respect to the  $x$ -axis, of  $\Theta_1$ . This set includes  $]a', b'[ \times ]0, \infty[$ , and we can define on it the harmonic function  $\tilde{w}$ , by the Schwarz reflexion principle :

$$\begin{cases} \tilde{w}(x, y) = w(x, y) & \text{if } y \geq 0 \\ \tilde{w}(x, y) = -w(x, -y) & \text{if } y < 0 \end{cases} \quad (21)$$

There exists some integer  $N$  such that  $\forall n \geq N$ ,  $\psi \circ \varphi^{-1}(x_n) \in ]a', b'[ \times \{0\}$ . Let  $t_n$  be the point of  $]a', b'[$  such that :

$$\psi \circ \varphi^{-1}(x_n) = (t_n, 0) \quad \forall n \geq N$$

Then :

$$\begin{cases} \lim_{n \rightarrow +\infty} (t_n, 0) = \psi \circ \varphi^{-1}(x_0) \\ \nabla w(t_n, 0) = 0 \quad \forall n \geq N \\ (t_n, 0) \neq \psi \circ \varphi^{-1}(x_0) \end{cases} \quad (22)$$

$\tilde{w}$  is the real part of an holomorphic function  $h$  in  $\Theta_1^s$ . According to the Cauchy-Riemann conditions, the imaginary part's gradient of such a function will also vanish in  $(t_n, 0)$  for all  $n \geq N$ . This means that, inside the domain  $\Theta_1^s$ , the zeros of  $h'(z)$  are not isolated, which is not possible since  $h'$  is holomorphic in this domain. ■

The forthcoming lemmas are technical points needed for the proof of the final stability result. Proofs of lemmas 2 and 5 are not really different from the linear case. Their statements are recalled mostly for the reader's convenience.

Let us define the  $L^2(\Omega)$  function  $u'_N = u_N^1 - \langle \nabla u_N^0, \theta \rangle$ . This function is known as the *Eulerian* derivative of the solution with respect to the domain, while  $u_N^1$  is the *Lagrangian* one. The first result is that  $\Delta u'_N = 0$  in  $\Omega$ .

**Lemma 2** Define the  $L^2$  function  $u'_N = u_N^1 - \langle \nabla u_N^0, \theta \rangle$ . Then :

$$\Delta u'_N = 0 \text{ in } \Omega$$

**Proof** : It works exactly as for the linear case [19, Simon]. ■

**Lemma 3** Suppose  $u_N^1$  vanishes on  $M$ . Then,  $u_N^1 = \langle \nabla u_N^0, \theta \rangle$  in  $\Omega$ , and moreover :

$$\beta := \left\langle \frac{\partial u_N^0}{\partial n}, \theta_n \frac{\partial u_N^0}{\partial n} \right\rangle_{H^{-\frac{1}{2}}(\gamma_D^\circ) \times H^{\frac{1}{2}}(\gamma_D^\circ)} = 0 \quad (23)$$

**Proof** : Let us choose some  $v \in H^1(\Omega)$  such that  $v|_{\gamma \cup \Gamma_D} = 0$ . Therefore,  $v \in \mathcal{S}$  and also  $-v \in \mathcal{S}$ , so that inequation (16) leads to the following equation :

$$\int_{\Omega} \nabla u_N^1 \nabla v = \int_{\Omega} \left( \frac{\partial \theta}{\partial M}{}^t + \frac{\partial \theta}{\partial M} \right) \nabla u_N^0 \nabla v - \int_{\Omega} (\nabla u_N^0 \nabla v) \operatorname{div} \theta \quad (24)$$

which gives, by Green formula :

$$\frac{\partial u_N^1}{\partial n} = \left\langle \left( \frac{\partial \theta}{\partial M} + \frac{\partial \theta}{\partial M}{}^t \right) \nabla u_N^0, \vec{n} \right\rangle - \operatorname{div} \theta \frac{\partial u_N^0}{\partial n} \text{ on } M$$

$\theta$  vanishes in a neighbourhood of  $M$ , and thus  $\frac{\partial u_N^1}{\partial n} = 0$  on  $M$ .  $u'_N$  is hence solution of :

$$\begin{cases} \Delta u'_N = 0 & \text{in } \Omega \\ u'_N = 0 & \text{on } M \\ \frac{\partial u'_N}{\partial n} = 0 & \text{on } M \end{cases} \quad (25)$$

By the Holmgren's unique continuation theorem,  $u'_N = 0$  in  $\Omega$ , and it follows that :

$$u_N^1 = \langle \nabla u_N^0, \theta \rangle \quad (26)$$

By the above identity, the trace of  $u_N^1$  on  $\partial\Omega$  is  $\theta_n \frac{\partial u_N^0}{\partial n}$ , which is therefore an element of  $H^{\frac{1}{2}}(\partial\Omega)$ . On the other hand,  $u_N^0 \in \{v \in H^1(\Omega); \Delta v \in L^2(\Omega)\}$ , so that we can define its normal derivative as an element of  $H^{-\frac{1}{2}}(\partial\Omega)$ , and accordingly as an element of  $H^{-\frac{1}{2}}(\gamma_D^\circ)$ . The duality product in (23) makes then sense.

$u_N^1$  vanishes wherever  $\theta$  does, and in particular :

$$u_N^1 = 0 \text{ on } \Gamma_N \cup \Gamma_D$$

Since  $u_N^1 \in \mathcal{S}$ , we have  $\int_{\Omega} \nabla u_N^0 \nabla u_N^1 = \int_{\Gamma_N} \phi u_N^1 = 0$  and, by using Green formula and theorem 3, we get (23). ■

**Lemma 4** Suppose  $\gamma_D^\circ = \emptyset$ . Then,  $\text{meas}(\gamma_D) = 0$ .

**Proof :** Suppose  $\gamma_D$  has an accumulation point  $x_0$ , interior to  $\gamma$ . There exists then some open “interval”  $\vartheta_{x_0}$  of  $\gamma$ , containing  $x_0$ , and a sequence of points  $x_n \in \gamma_D \cap \vartheta_{x_0}$ , each one of them being supposed to be different from the other, such that :

$$\begin{cases} x_n & \neq x_0 \quad \forall n \in \mathbf{N} \\ \lim_{n \rightarrow +\infty} (x_n) & = x_0 \\ u_N^0(x_n) & = 0 \quad \forall n \in \mathbf{N} \end{cases} \quad (27)$$

By Khodja-Moussaoui [12], we know that  $u_N^0 \in \mathcal{C}^{1,\beta}(\vartheta_{x_0})$ . It comes out that we can find a sequence  $\xi_n \in ]x_n, x_{n+1}[$ , such that  $\frac{\partial u_N^0}{\partial \tau}(\xi_n) = 0$ .

The sequence  $\xi_n$  is not stationary, and converges to  $x_0$ . Furthermore,  $\frac{\partial u_N^0}{\partial n}(\xi_n) = 0 \quad \forall n$ , since  $\frac{\partial u_N^0}{\partial n} = 0$  on  $\gamma$ , so that all points  $\xi_n$  are in the set  $\{x \in \vartheta_{x_0}; \nabla u_N^0(x) = 0\}$ .

To conclude, we shall apply the result proved in lemma 1 to the conjugate function of  $u_N^0$  : Let  $h = u_N^0 + iv$  be the holomorphic function associated with  $u_N^0$ . By the Cauchy-Riemann conditions,  $\frac{\partial v}{\partial \tau} = 0$  on  $\vartheta_{x_0}$ , so that the function  $w = v - v(x_0)$  is a harmonic function vanishing on  $\vartheta_{x_0}$ . Therefore,  $x_0$  is an accumulation point of the set  $\{x \in \vartheta_{x_0}; \nabla w(x) = 0\}$ , which by lemma 1 cannot have any. This ends the proof of lemma 4. ■

**Lemma 5** Let  $\vartheta_{x_0}$  be some open subset of  $\gamma$ , and  $\mathcal{O}$  some open subset of  $\Omega$  such that  $\overline{\mathcal{O}} \cap \partial\Omega \subset \vartheta_{x_0}$ . Then, for any  $v \in H^2(\mathcal{O})$ , vanishing in some neighbourhood of  $\partial\mathcal{O} \setminus \vartheta_{x_0}$ , we have :

$$\int_{\vartheta_{x_0}} \theta_n \frac{\partial u_N^0}{\partial \tau} \frac{\partial v}{\partial \tau} = 0 \quad (28)$$

**Proof :** Let  $v$  be such a test function.

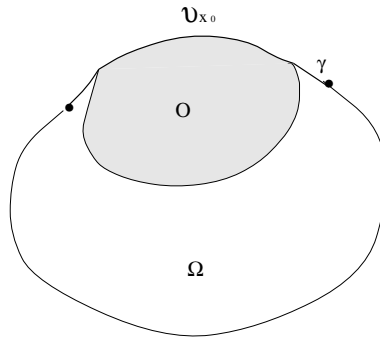


Figure 4: The open set  $\mathbf{O}$

Then,  $v \in \mathcal{S}$  and  $(-v) \in \mathcal{S}$ , so that inequality (16) becomes the equality :

$$\int_{\Omega} \nabla u_N^1 \nabla v = \int_{\Omega} \left( \frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M} \right) \nabla u_N^0 \nabla v - \int_{\Omega} \langle \nabla u_N^0, \nabla v \rangle \text{div} \theta \quad (29)$$

By Green formula, the left handside of the above equation becomes :

$$\int_{\Omega} \nabla u_N^1 \nabla v = - \int_{\mathcal{O}} u_N^1 \Delta v + \int_{\partial \mathcal{O}} u_N^1 \frac{\partial v}{\partial n}$$

The proof ends therefore exactly as for the linear case, by using Green formulae and lemma 3 [2]. ■

### 3.2.2 The final stability result

We are now able to prove the final Lipschitz stability result.

**Theorem 4 (Lipschitz stability)** *Suppose  $\theta$  fulfills (8) and  $\theta_n \not\equiv 0$  on  $\gamma$ . Then, denoting  $u_{hN}|_M$  by  $f_h$ , we have :*

$$\lim_{h \rightarrow 0} \frac{|f - f_h|_{L^2(M)}}{h} > 0 \quad (30)$$

**Proof :** According to the expansion (15), (30) is equivalent to the following :

$$|u_N^1|_{L^2(M)} > 0 \quad (31)$$

Let us suppose that  $u_N^1 = 0$  a.e. on  $M$ . We shall now consider two cases :

• **First case :**  $\gamma_D^\circ = \emptyset$

By theorem 3, we derive that  $\frac{\partial u_N^0}{\partial n} = 0$  on  $\gamma$ , and  $u_N^0$  is then solution of the linear “Neumann” problem :

$$\begin{cases} \Delta u_N^0 = 0 & \text{in } \Omega_\gamma \\ u_N^0 = 0 & \text{on } \Gamma_D \\ \frac{\partial u_N^0}{\partial n} = \phi & \text{on } \Gamma_N \\ \frac{\partial u_N^0}{\partial n} = 0 & \text{on } \gamma \end{cases} \quad (32)$$

On the other hand, we also know that  $u_N^1$  is solution of :

$$\begin{cases} u_N^1 \in \mathcal{S} \\ \int_{\Omega} \nabla u_N^1 \nabla v \geq \int_{\Omega} \left( \frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M} \right) \nabla u^0 \nabla v - \int_{\Omega} (\nabla u^0 \nabla v) \operatorname{div} \theta, \forall v \in \mathcal{S} \end{cases} \quad (33)$$

Since  $\operatorname{meas}(\gamma_D) = 0$  (by lemma 4), it comes that  $\mathcal{S} = V := \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_D\}$ .  $u_N^1$  is therefore solution of the linear problem :

$$\begin{cases} u_N^1 \in V \\ \int_{\Omega} \nabla u_N^1 \nabla v = \int_{\Omega} \left( \frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M} \right) \nabla u_N^0 \nabla v - \int_{\Omega} (\nabla u_N^0 \nabla v) \operatorname{div} \theta, \forall v \in V \end{cases} \quad (34)$$

Actually, this means that  $u_N^1$  is the derivative, with respect to the domain, of the linear “Neumann” problem. Referring to [2], we know that  $|u_N^1|_{L^2(M)} > 0$ , which is in contradiction with our assumption on  $u_N^1|_M = 0$ . This ends the proof for this first case.

• **Second case :**  $\gamma_D^\circ \neq \emptyset$



By theorem 3,  $\frac{\partial u_N^0}{\partial n}$  is a strictly positive distribution on  $\gamma_D^\circ$ , and  $\theta_n \frac{\partial u_N^0}{\partial n} = u_N^1$  is positive on  $\gamma_D^\circ$ , since  $u_N^1 \in \mathcal{S}$ .

- If  $\gamma_D^\circ = \gamma$ , the proof ends here since this is in contradiction with the assumption  $\theta_n \not\equiv 0$  on  $\gamma$ .
- Let us therefore consider the case  $\gamma_D^\circ \neq \gamma$ .

By lemma 3, we know that  $\beta := \left\langle \frac{\partial u_N^0}{\partial n}, \theta_n \frac{\partial u_N^0}{\partial n} \right\rangle_{H^{-\frac{1}{2}}(\gamma_D^\circ) \times H^{\frac{1}{2}}(\gamma_D^\circ)} = 0$ , and we can then conclude that  $\theta_n \frac{\partial u_N^0}{\partial n} = 0$  on  $\gamma_D^\circ$ , which by using the characterization of theorem 3 gives  $\theta_n \equiv 0$  on  $\gamma_D^\circ$ .

There exists some point  $x_0$  of  $\gamma_N$  where  $\theta_n$  does not vanish, and therefore some open connected neighbourhood  $\vartheta_{x_0}$  of  $x_0$  in  $\gamma_N$  where  $\theta_n$  does not change sign. Otherwise, assuming  $\theta_n$  vanishes on  $\gamma_N$ , and since it does not vanish identically on  $\gamma$ , there would exist some point  $x_0 \in \gamma$  such that  $\theta_n(x_0) \neq 0$ .  $\theta_n$  being continuous - it would not vanish in some open neighbourhood  $\vartheta_{x_0}$  of  $x_0$ . Of course,  $\vartheta_{x_0} \subset \gamma \setminus \gamma_N = \gamma_D$ , and  $\vartheta_{x_0}$  being an open set,  $\vartheta_{x_0} \subset \gamma_D^\circ$ . This not possible since  $\theta_n$  vanishes on  $\gamma_D^\circ$ .

The “interval”  $\vartheta_{x_0} = ]a, b[$  (with respect to the curvilinear abscissa) can be chosen maximal, i.e. such that  $\theta_n u_N^0(a) = \theta_n u_N^0(b) = 0$ . Moreover,  $\theta_n$  remains of constant sign (say positive) on  $\vartheta_{x_0}$ . The point is now to prove that  $\nabla u_N^0$  vanishes as well on  $\vartheta_{x_0}$ , which by Holmgren’s theorem would lead to  $u_N^0 \equiv 0$  in  $\Omega$ .

We are now going to construct a special family of functions  $v$ , fulfilling the conditions of lemma 5, in order to achieve the proof of the theorem. Up to a local map, the “interval”  $\vartheta_{x_0}$  can be identified to  $]0, 1[$ . Then, given two positive real numbers  $c$  and  $\varepsilon$ , it is possible to construct a family of functions  $\xi_\varepsilon \in C^2(\bar{\Omega})$  such that :

$$\begin{cases} \xi_\varepsilon \equiv 1 \text{ for } x \in ]\varepsilon, 1 - \varepsilon[ \\ \xi_\varepsilon \equiv 0 \text{ for } x \in ]0, \frac{\varepsilon}{2}[ \cup ]1 - \frac{\varepsilon}{2}, 1[ \\ 0 \leq \xi_\varepsilon \leq 1 \\ |\nabla \xi_\varepsilon(x)| \leq \frac{c}{\varepsilon} \quad \forall x \in \bar{\Omega} \\ |\nabla^2 \xi_\varepsilon(x)| \leq \frac{c}{\varepsilon^2} \quad \forall x \in \bar{\Omega} \end{cases} \quad (35)$$

Let us now denote  $\vartheta_{x_0}$  by  $\vartheta$ , and by  $\vartheta_\varepsilon$  the “interval”  $]\varepsilon, 1 - \varepsilon[$ . Using (28) with  $v = \xi_\varepsilon u_N^0$  gives, for any  $\varepsilon > 0$  :

$$\int_{\vartheta_\varepsilon} \theta_n \left( \frac{\partial u_N^0}{\partial \tau} \right)^2 + \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} + \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n \left( \frac{\partial u_N^0}{\partial \tau} \right)^2 \xi_\varepsilon = 0 \quad (36)$$

Two situations have to be considered.

First situation :  $\bar{\vartheta} \subset \gamma$ .

Referring again to Khodja-Moussaoui [12], we know that  $u_N^0 \in C^{1,\beta}(\bar{\vartheta})$ ,  $\beta < \frac{1}{2}$ . There exist then two positive constants  $c_1$  and  $c_2$  such that :

$$\left| \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} \right| \leq \frac{c_1}{\varepsilon} \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n u_N^0 \quad (37)$$

and that

$$\left| \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n \left( \frac{\partial u_N^0}{\partial \tau} \right)^2 \xi_\varepsilon \right| \leq c_2 \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n \quad (38)$$

It is clear that  $\lim_{\varepsilon \rightarrow 0} \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n = 0$ .

As for the right handside of (37), let us denote by  $\chi$  the local map  $]0, 1[ \mapsto \vartheta = ]a, b[$ . The mean value theorem gives us two real numbers  $\alpha_\varepsilon \in ]0, \varepsilon[$ , and  $\beta_\varepsilon \in ]1 - \varepsilon, 1[$  such that :

$$\int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n u_N^0 = \{ [(\theta_n u_N^0)(\chi(\alpha_\varepsilon))\chi'(\alpha_\varepsilon)] + [(\theta_n u_N^0)(\chi(\beta_\varepsilon))\chi'(\beta_\varepsilon)] \} \varepsilon$$

and accordingly :

$$\left| \frac{1}{\varepsilon} \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n u_N^0 \right| = \{ [(\theta_n u_N^0)(\chi(\alpha_\varepsilon))\chi'(\alpha_\varepsilon)] + [(\theta_n u_N^0)(\chi(\beta_\varepsilon))\chi'(\beta_\varepsilon)] \}$$

It comes out then, since  $(\theta_n u_N^0)$  is continuous on  $\gamma$ . and that  $(\theta_n u_N^0)(\chi(0)) = (\theta_n u_N^0)(\chi(1)) = 0$  :

$$\lim_{\varepsilon \rightarrow 0} \left| \frac{1}{\varepsilon} \int_{\vartheta \setminus \vartheta_\varepsilon} \theta_n u_N^0 \right| = 0$$

On the other hand, according to [12],  $u_N^0 \in \mathcal{C}^{1,\beta}(\overline{\vartheta})$ , and we we get :

$$\lim_{\varepsilon \rightarrow 0} \int_{\vartheta_\varepsilon} \theta_n \left( \frac{\partial u_N^0}{\partial \tau} \right)^2 = 0$$

Making  $\varepsilon \rightarrow 0$  in equation (36) then leads to :

$$\int_{\vartheta} \theta_n \left( \frac{\partial u_N^0}{\partial \tau} \right)^2 = 0$$

But  $\theta_n > 0$  on  $\vartheta$ , so that  $\frac{\partial u_N^0}{\partial \tau} = 0$  on  $\vartheta$ . Since  $\frac{\partial u_N^0}{\partial \tau}$  is also vanishing on  $\vartheta$ , the Holmgren's theorem provides the argument to conclude.

Second situation :  $\overline{\vartheta} \not\subset \gamma$ . We can suppose as well that  $\vartheta = \gamma$ .

In this situation, the regularity results are helpless, since they do not hold up to  $\partial\gamma$ . To prove the following

$$\lim_{\varepsilon \rightarrow 0} \int_{\vartheta - \vartheta_\varepsilon} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} = 0$$

it is sufficient to prove that the integral on the “interval”  $\chi([0, \varepsilon])$  vanishes. Integrating by parts, we get :

$$\int_{\chi([0, \varepsilon])} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} = \int_{\chi([0, \varepsilon])} \theta_n u_N^0 \frac{\partial}{\partial \tau} (u_N^0)^2 \frac{\partial \xi_\varepsilon}{\partial \tau} = \left[ \theta_n (u_N^0)^2 \frac{\partial \xi_\varepsilon}{\partial \tau} \right]_{\chi(0)}^{\chi(\varepsilon)} - \int_{\chi([0, \varepsilon])} (u_N^0)^2 \frac{\partial}{\partial \tau} \left( \theta_n \frac{\partial \xi_\varepsilon}{\partial \tau} \right) \quad (39)$$

The assumption on  $\theta'_n(\chi(0)) = 0$  is used here to get  $\theta_n(\chi(\varepsilon)) = \varepsilon o(\varepsilon)$ ;  $\lim_{\varepsilon \rightarrow 0} o(\varepsilon) = 0$ . On the other hand,  $u_N^0$  is continuous at  $\chi(0)$ , and  $|\nabla \xi_\varepsilon| \leq \frac{c}{\varepsilon}$ , so that :

$$\lim_{\varepsilon \rightarrow 0} \left( \theta_n(u_N^0)^2 \frac{\partial \xi_\varepsilon}{\partial \tau} \right) (\chi(\varepsilon)) = 0 \quad (40)$$

Making use again - after the change of variables  $\chi$  - of the mean value theorem on the interval  $[0, \varepsilon]$ , we can write down the following, for some  $\alpha_\varepsilon \in ]0, \varepsilon[$  :

$$\int_{\chi([0, \varepsilon])} (u_N^0)^2 \frac{\partial}{\partial \tau} \left( \theta_n \frac{\partial \xi_\varepsilon}{\partial \tau} \right) = u_N^0(\chi(\alpha_\varepsilon))^2 \chi'(\alpha_\varepsilon) \left[ \theta_n(\alpha_\varepsilon) \frac{\partial^2 \xi_\varepsilon}{\partial \tau^2}(\chi(\alpha_\varepsilon)) + \frac{\partial^2 \xi_\varepsilon}{\partial \tau^2}(\chi(\alpha_\varepsilon)) \frac{\partial \theta_n}{\partial \tau}(\chi(\alpha_\varepsilon)) \right] \times \varepsilon$$

Using the following properties,

$$\begin{cases} u_N^0 \circ \chi \text{ and } \theta_n \circ \chi \text{ are continuous at } 0 \\ |\nabla^2 \xi_\varepsilon| \leq \frac{c}{\varepsilon^2} \\ \theta_n(\chi(\varepsilon)) = \varepsilon o_1(\varepsilon); \lim_{\varepsilon \rightarrow 0} o_1(\varepsilon) = 0 \\ \frac{\partial \theta_n}{\partial \tau}(\chi(\varepsilon)) = o_2(\varepsilon); \lim_{\varepsilon \rightarrow 0} o_2(\varepsilon) = 0 \\ |\nabla \xi_\varepsilon| \leq \frac{c}{\varepsilon} \end{cases} \quad (41)$$

we derive :

$$\lim_{\varepsilon \rightarrow 0} \int_{\chi([0, \varepsilon])} u_N^0 \frac{\partial}{\partial \tau} \left( \theta_n \frac{\partial \xi_\varepsilon}{\partial \tau} \right) = 0$$

which gives, by using (39) and (40) :

$$\lim_{\varepsilon \rightarrow 0} \int_{\chi([0, \varepsilon])} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} = 0$$

Of course, we have also  $\lim_{\varepsilon \rightarrow 0} \int_{\chi([1-\varepsilon, 1])} \theta_n u_N^0 \frac{\partial u_N^0}{\partial \tau} \frac{\partial \xi_\varepsilon}{\partial \tau} = 0$ , so that :

$$\lim_{\varepsilon \rightarrow 0} \int_{\vartheta_\varepsilon} \theta_n \left( \frac{\partial u_N^0}{\partial \tau} \right)^2 = 0$$

It follows, since  $\theta_n > 0$  on  $\vartheta$ , that  $\frac{\partial u_N^0}{\partial \tau}$  vanishes on  $\vartheta$  and, according to the fact that  $\frac{\partial u_N^0}{\partial n}$  is also vanishing on  $\vartheta$ , the Holmgren's theorem gives the final argument to conclude, which ends the proof of the theorem. ■

## 4 Conclusion

The inverse problem with unilateral boundary conditions for the Laplace equation is clearly not of great physical interest. However, most the theoretical difficulties expected in more realistic situations (namely the inverse elastic problem, or the coupled thermoelastic one), are as well gathered in the present "model" problem, which makes its study of great interest.

The identifiability uses classical tools : the Holmgren's continuation theorem, and variational arguments. As for Lipschitz stability results, they are also based on the Holmgren's theorem, and use

as a basic tool the derivatives with respect to the domain. However, serious difficulties arise from the possible lack of connectivity of parts  $(\gamma_D, \gamma_N)$  the unknown boundary defined by the Signorini solution :although this latter is smoother than the solution of the related mixed linear problem, the possibility that  $(\gamma_D \setminus \overset{\circ}{\gamma}_D)$  be some closed set of positive measure and void interior, such as a Cantor p-adic set, could not be excluded. In such a situation, the Holmgren's theorem is no more the "magic" straightforward tool we are used to in the linear situations. The conditions for its final use have to be patiently built up, by using sharp informations on the structure of the Signorini solution on the unknown boundary, backed with arguments coming from the analytical functions theory. The Lipschitz stability result proved this way is hence limited to 2D situations, although an extension to 3D might be not excluded.

The development of an appropriate identification algorithm, which is the aim of a forthcoming work currently in progress, will also be facing difficulties similar to those encountered above, particularly when differentiating the cost function. The present work provides useful tools to overcome them.

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Éditeur  
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)  
ISSN 0249-6399