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PENALIZATION AND REGULARIZATION OF QUADRATIC COST CONTROLLABILITY PROBLEMS IN A FINITE DIMENSIONAL SPACE

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PENALIZATION AND REGULARIZATION OF QUADRATIC COST CONTROLLABILITY PROBLEMS IN A FINITE DIMENSIONAL SPACE

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Résumé : Nous étudions les problèmes de contrôlabilité avec un coût quadratique dans un espace de dimension finis par les méthodes de pénalisation et de régularisation. Les feedback-synthèses associé à ces problèmes sont aussi étudiés. Nous calculons des approximations par la méthode des éléments finis du contrôle optimale sur un exemple numérique pour illustrer notre méthode.

Abstract : We discuss controllability problems with a quadratic cost in a finite dimensional space by the methods of penalization and regularization. Feedback synthesis derived from these problems is also studied. Finite element approximations to the optimal control are computed with a numerical example illustrating our methods.

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§ 0. INTRODUCTION

Given a control system

$$(LC)\begin{cases} \dfrac{dx(t;u)}{dt} = A(t)\,x(t;u) + f(t) + B(t)\,u(t), & o \le t \le T \\ x(o;u) = x_o \end{cases}$$

where

$x(t)$: the state of the system at time t

x_o : initial state ($\in \mathbb{R}^n$)

$u \in$ Uad is an admissible control

$A(t) \equiv (a_{ij}(t))_{nxn} \in C([o,T].\ \mathbb{R}^{nxm})$

$B(t) \equiv (b_{pq}(t))_{nxm} \in L^2([o,T];\mathbb{R}^{nxm}),\quad m \le n$

$f(t) \equiv (f_i(t)) \in L^2([o,T];\ \mathbb{R}^n)$

the controllability problem is : for some prescribed final state x_1, find an admissible control u such that the state of the system at $t=T$ is x_1, i.e.,

$$(0.1) \qquad x(T;u) \equiv x_1$$

Examples of such controllability problems are easy to find. Projectiles being used by military forces are one such instance. The control of a projectiles flight-its pitching, rolling and yawing- is achieved through maneuvering the three principal control surfaces : elevators, ailerons and rudder. We want to achieve a certain target x_1. This becomes a controllability problem, with $u(t)$ steering the system from x_o to x_1.

Of course, the above is just a simplified model because we have not taken many other factors (guidance, gust distrubances and physical constraints) into account. Also, the target may be moving.

Controllability problems have been undergoing extensive study by mathematicians and engineers. For example, the classical rank condition criterion : rank $(B, AB, \ldots, A^{n-1}B) = n$ for controllability of an autonomous system

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + Bu(t) \\ \\ A, B \text{ are constant } n \times n, n \times m \text{ matrices} \end{cases}$$

is well-known, and there are some generalizations of this to partial differential equations (e.g. [9]). Basically, this approach differs very much from that of optimal control in the sense that no optimization is needed in the procedures.

The controllability terminal condition (0.1) actually can be regarded as a contraint thereby enabling one to apply the optimal control technique to study the controllability problem. This is clearly indicated in Lee and Markus' far reaching paper [4], where optimal control problems for systems governed by nonlinear ordinary differential equations are studied in great length and under a very general setting. In particular, the target set can be moving.

An important way to handle physical constraints is by the method of penalization. In [2], [3], [6], this method was used to attack the optimal control problem with contraints : instead of attempting to solve directly the contrained phase optimal control problem, one considers an unconstrained problem wherein the original cost functional is augmented by a nonnegative penalty term. This sharply increases the cost associated with trajectories which violate the phase constraints. In many cases the constrained phase solution of the original optimization problem may be approximated to any desired degree of accuracy by optimal solutions to a sequence of augmented-cost unconstrained problems. In [5], the penalization method is also applied to tackle the optimal control problem, where the governing equation itself is considered as a constraint.

In comparison with the existing literature, our treatment in this paper has the following features :

(i) The problem we are considering is a controllability problem associated with a quadratic cost functional.

(ii) The methods of penalization and regularization are used. In addition to the convergence results, optimization procedures by feedback synthesis are concerned.

(iii) Finite element approximations -a constructive aspect- are included.

In this paper, we will only study a simple model of unconstrained linear (time-varying) systems governed by ordinary differential equations. Systems with constraints and those governed by partial differential equations will be treated in a forthcoming paper.

§I. CONTROLLABILITY AND PENALTY

Throughout this paper, we assume that

$$Uad \equiv L^2(o,T;R^m)$$

Suppose that $\Phi(t,s)$ is the fundamental matrix solution ofr $A(t)$, i.e., $\Phi(t,s)$ satisfies

$$(1.1) \quad \begin{cases} \frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s) & o \le s \le t \le T \\ \Phi(s,s) = I_{nxn} \end{cases}$$

In case $s = o$, we simply write $\Phi(t)$ instead of $\Phi(t,o)$. Then we have

$$\Phi(t,s) = \Phi(t)\Phi(s)^{-1}$$

The solution $x(t;u)$ to (LC) becomes

$$(1.2) \qquad x(t;u) = \Phi(t)\, x_o + \int_o^t \Phi(t,s)\,[f(s) + B(s)\, u(s)]\,ds$$

$$= \Phi(t)\, x_o + F(t) + \int_o^t \Phi(t,s)B(s)u(s)\, ds, \quad F(t) \equiv \int_o^t \Phi(t,s)f(s)ds$$

For any $x(t;u)$ satisfying (LC), suppose we have a quadratic functional

$$J(x,u) \equiv \int_o^T [\,||C\, x(t;u) - z(t)\,||^2 + \langle Nu(t),\, u(t)\rangle]dt$$

where

$$C : L^2(o,T;R^n) \to L^2(o,T;R^p) \text{ is linear}$$
$$N : L^2(o,T;R^m) \to L^2(o,T;R^m) \text{ is linear such that}$$

$$\langle Nu,u\rangle_{L^2_m(o,T)} \geq \nu\, ||u||^2_{L^2_m(o,T)} \qquad \nu > o$$

and $N = N^*$

$$z \in L^2(o,T;R^p)$$

We are now in a position to pose the Optimal Controllability Problem (OCP)

Given some $x_1 \in R^n$, find a control $u \in U_{ad}$ such that the solution $x(t;u)$ of (LC) satisfies the terminal condition $x(T;u) = x_1$ and the control cost $J(x,u)$ is minimal.

If there is a u which steers the system from x_o to x_1, (with a minimal cost $J(x,u)$), we say that (LC) is controllable from x_o to x_1. If (LC) is controllable for arbitrary x_o and x_1, we say that (LC) is (globally) controllable.

Without requiring the minimality of the control cost, the concept of controllability is dual to that of observality. The following duality theorem can be found in [7].

Theorem 1.1 The system

$$(1.3) \qquad \frac{dx(t;u)}{dt} = A(t)\, x(t;u) + B(t)\, u(t)$$

is controllable for any initial and final states x_o, x_1 if and only if the system

$$(LO) \qquad \begin{cases} \dfrac{dy(t)}{dt} = -A^*(t)\, y(t) \\ w(t) = B(t)^* y(t) \end{cases}$$

is observable. (LO) is observable if and only if

$$(1.4) \qquad Z(T) \equiv \Phi(T) \left[\int_0^T \Phi(s)^{-1} B(s)\, B^*(s)\, (\Phi(s)^{-1})^* ds \right] \Phi(T)^*$$

is a positive definitive matrix. Furthermore, for any x_o, x_1, the control

$$\hat{u}(t) = B^*(t)\, \Phi^*(T,t)\, Z(T)^{-1} [x_1 - \Phi(T)\, x_o] \qquad o \le t \le T$$

steers (1.3) from x_o to x_1 such that $||\hat{u}||_{L_m^2(u,T)}$ is minimal.

Here we prove a slightly generalized version of the above theorem.

Theorem 1.2. The system (LC) is controllable for any initial states x_o and x_1 if and only if the system (LO) is observable. (LO) is observable if and only if

$$(1.5) \qquad Z_N(T) \equiv \int_o^T \Phi(T,s)\, B(s)\, N(s)^{-1}\, B^*(s) \Phi^*(T,s) ds$$

is a positive definite matrix. For any x_o, x_1 the control

$$(1.6) \qquad \hat{u}(t) \equiv N(t)^{-1}B^*(t)\Phi^*(T,t)Z_N^{-1}(T)[x_1 - \Phi(T)x_o - F(T)], \quad o \le t \le T$$

steers (LC) from x_o to x_1 such that $< N\hat{u}, \hat{u} >_{L_m^2(o,T)}$ is minimal.

Proof : We first note that Theorem 1.1 remains true with the presence of $f(t) \not\equiv o$ in (1.3) because only a translation $F(T)$ is resulted from f at $t = T$.

Now, we show that $\hat{u}$ steers (LC) from x_o to x_1.

$$x(T) = \Phi(T)x_o + \int_o^T \Phi(T,s)\,[B(s)\hat{u}(s) + f(s)]ds$$

$$= \Phi(T)x_o + F(T) + \int_o^T \Phi(T,s)B(s)N^{-1}(s)B^*(s)\Phi^*(T,s)ds\{Z_N^{-1}(T)[x_1 - \Phi(T)x_o - F(T)]\}$$

$$= \Phi(T)x_o + F(T) + Z_N(T)\,\{Z_N^{-1}(T)\,[\,x_1 - \Phi(T)x_o - F(T)]\}$$

$$= x_1$$

Note that there is no question about the existence of $Z_N^{-1}(T)$ because $Z(T)$ in (1.4) is positive and N is invertible.

Next we prove that $\hat{u}$ has the minimal cost $< Nu,u >$. Let u also steer x_o to x_1. Then

$$x_1 - \Phi(T)x_o - F(T) = \int_o^T \Phi(T,s)\,B(s)\,u(s)\,ds = \int_o^T \Phi(T,s)\,B(s)\,\hat{u}(s)\,ds$$

so

$$\int_o^T \Phi(T,s)\,B(s)\,[u(s) - \hat{u}(s)]\,ds = o$$

$$\int_0^T [x_1 - \Phi(T)x_0 - F(T)]^* \, (Z_N^{-1}(T))^* \, \Phi(T,s) \, B(s) \, [u(s) - \hat{u}(s)] \, ds = 0 \quad \text{(as a scalar)}$$

Hence
$$\int_0^T \hat{u}^*(s) \, N(s) \, [u(s) - \hat{u}(s)] ds = 0 \quad \text{i.e., } < \hat{u}, N(u-\hat{u}) >_{L_m^2(0,T)} = 0$$

with $N^* = N$, we derive

$$< Nu, u > - < N\hat{u}, \hat{u} > = < N(u - \hat{u}), u - \hat{u} > \geq \nu || u - \hat{u}||^2_{L_m^2(0,T)} \geq 0$$

The above is equal to 0 if and only if $u = \hat{u}$. Q.E.D.

We have some more to say about this theorem in §III.

Now return to the (OCP). Instead of directly minimizing $J(x, u)$, we consider

$$(1.7) \qquad J_\varepsilon(x, u) \equiv J(x, u) + \psi(\varepsilon)|x(T ; u) - x_1|^2 \qquad \varepsilon > 0$$

where $\psi(\varepsilon)$ is a positive continuous function from R^+ into R^+, strictly decreasing on $(0, \varepsilon_0)$ for some $\varepsilon_0 > 0$ and such that

$$(1.8) \qquad \psi(\varepsilon) \to +\infty \qquad \text{as} \qquad \varepsilon \downarrow 0$$

The simplest such $\psi(\varepsilon) = 1/\varepsilon$. In general, $\psi(\varepsilon)$ is a function of the mesh size ε in finite element approximations. Here in (1.7), we regard the controllability terminal condition $x(T ; u) = x_1$ as a constraint. If $x(T;u) = x_1$ is not satisfied, the penalty $\psi(\varepsilon)|x(T;u) - x_1|^2$ is imposed.

Theorem 1.3 For every $\varepsilon > 0$, there is a unique $\hat{u}_\varepsilon \in L^2(0,T ; R^m)$ wich minimizes $J_\varepsilon(x, u)$.

The proof is standard : find the weak limit of a subsequence of a mini-

zing sequence and use the lower semicontinuity of $J_\varepsilon(x,u)$ to show minimalty. The uniqueness follows from strict convexity of J_ε. Q.E.D.

Sor for every $\varepsilon > o$, let $\hat{u}_\varepsilon$ be the unique control which minimizes $J_\varepsilon(x,u)$. For $o < \varepsilon_2 < \varepsilon_1$ $(<\varepsilon_o)$,

$$J_{\varepsilon_2}(x, \hat{u}_{\varepsilon_2}) = J(x, \hat{u}_{\varepsilon_2}) + \psi(\varepsilon_2)\left| x(T ; \hat{u}_{\varepsilon_2}) - x_1\right|^2$$

$$= J(x, \hat{u}_{\varepsilon_2}) + \psi(\varepsilon_1)\left| x(T ; \hat{u}_{\varepsilon_2}) - x_1\right|^2 + (\psi(\varepsilon_2) - \psi(\varepsilon_1))\left| x(T ; \hat{u}_{\varepsilon_2}) - x_1\right|^2$$

$$\geq J_{\varepsilon_1}(x, \hat{u}_{\varepsilon_1}) + (\psi(\varepsilon_2) - \psi(\varepsilon_1))\left| x(T ; \hat{u}_{\varepsilon_2}) - x_1\right|^2$$

$$\geq J_{\varepsilon_1}(x, \hat{u}_{\varepsilon_1})$$

Accordingly, $J_\varepsilon(x,\hat{u}_\varepsilon)$ is increasing as ε is decreasing to o. This means that cost gets higher as more controllability is achieved.

The following is the fundamental theorem.

Theorem 1.4 The system (LC) is controllable from x_o to x_1, if and only if $J_\varepsilon(x,\hat{u}_\varepsilon)$ is bounded from above by some $M > o$ (depending on x_o, x_1). If (LC) is controllable from x_o to x_1, then $\hat{u}_\varepsilon$ converges strongly to a control $\hat{u}$ (as $\varepsilon \to o$) which solves the (OPC).

Proof : If (LC) is controllable from x_o to x_1, then there is a control $\bar{u}$ which steers (LC) from x_o to x_1, so

$$(1.9) \qquad J(x,\bar{u}) = J(x,\bar{u}) + \psi(\varepsilon)\left|x(T;\bar{u}) - x_1\right|^2 \geq \min_u J_\varepsilon(x,u) = J_\varepsilon(x,\hat{u}_\varepsilon)$$

Thus $J_\varepsilon(x, \hat{u}_\varepsilon)$ is bounded from above for all $o<\varepsilon<\varepsilon_o$ by $M \equiv J(x,\bar{u})$.

Conversely, assume that all $J_\varepsilon(x,\bar{u}_\varepsilon)$ is bounded from above by some $M>o$. Then

$$J(x, \hat{u}_\varepsilon) + \psi(\varepsilon) \left| x(T;\hat{u}_\varepsilon) - x_1 \right|^2 \leq M$$

Hence

$$(1.10) \qquad \left| x(T;u_\varepsilon) - x_1 \right|^2 \leq M/\psi(\varepsilon) \to o \text{ as } \varepsilon \to o$$

Because $\{\hat{u}_\varepsilon\}$ is bounded in $L^2_m(o,T)$, it contains a subsequence $\hat{u}_{\varepsilon_n} \to \hat{u}$ weakly convergent. From (1.2) and (1.10), we see that

$$x(T ; \hat{u}) = \lim_{\varepsilon_n \downarrow o} \{\Phi(T)\, x_o + F(T) + \int_o^t \Phi(T, s)\, B(s)\, \hat{u}_{\varepsilon_n}(s)\, ds\}$$

$$= \lim_{\varepsilon_n \downarrow o} x(T ; \hat{u}_{\varepsilon_n}) = x_1$$

So $\hat{u}$ accomplishes controllability. (LC) is controllable from x_o to x_1. From (1.9) we have

$$(1.11) \qquad J(x, \hat{u}) \geq J_\varepsilon(x, \hat{u}_\varepsilon) \geq J(x, \hat{u}_\varepsilon).$$

On the other hand, weak consequence $\hat{u}_{\varepsilon_n} \to \hat{u}$ implies the strong convergence of $x(t ; \hat{u}_{\varepsilon_n})$ to $x(t ; \hat{u})$. We have

$$(1.12) \qquad J(x, \hat{u}) \leq \liminf_{\varepsilon_n \downarrow o} J(x, \hat{u}_{\varepsilon_n})$$

Combining (1.11) and (1.12) and the remark before Theorem 1.4., we conclude

$$J(x, \hat{u}) = \lim_{\varepsilon_n \downarrow o} J(x, \hat{u}_{\varepsilon_n})$$

This implies that $\hat{u}_{\varepsilon_n}$ converges strongly to $\hat{u}$ in $L^2_m(o,T)$ since $\langle Nv,v\rangle^{1/2}$ is an equivalent norm in $L^2_m(o,T)$. We also remark that the whole sequence $\{\hat{u}_\varepsilon\}$ converges strongly to $\hat{u}$, since every subsequence $\{\hat{u}_{\varepsilon_n}\}$ contains a subsequence converging strongly to $\hat{u}$.

Now for any $\bar{u}$ steering x_o to x_1, we have

$$J(x, \bar{u}) \leq \liminf J_{\varepsilon_n}(\hat{u}_{\varepsilon_n})$$

for a subsequence $\hat{u}_{\varepsilon_n}$. From (.19) we deduce that

$$J(\bar{u}) \geq J(\hat{u})$$

Hence $\hat{u}$ solves the (OCP).

The rate of convergence (1.10) can be improved to

$$\psi(\varepsilon)^{1/2} | x(T ; \hat{u}_\varepsilon) - x_1|^2 \to o \qquad \text{as} \quad \varepsilon \downarrow o$$

because $J(x, \hat{u}_\varepsilon)$ converges to $J(x, \hat{u})$. Q.E.D.

Theorem 1.4 formulates an equivalent condition for controllability depending on initial and terminal conditions only. In the next theorem, we will formulate an equivalent condition for global controllability.

Theorem 1.5. (Global uniform bounds)

The system (LC) is controllable from x_o to x_1 for arbitrarity given x_o, x_1 if and only if there exists a positive constants M independent of x_o, x_1 such that

$$J_\varepsilon(x, \hat{u}_\varepsilon) \le M\,|x_o - \Phi(T)x_1 - F(T)|^2 + 4\,||C||^2 \|\Phi(.)x_o + F(.)\|^2_{L_n^2(o,T)} + 2\,\|z\|^2_{L_p^2(o,T)}$$

Proof: By theorem 1.2, the control

$$\bar{u}(t) \equiv N(t)^{-1} B^*(t)\Phi^*(T,t)\, Z_N^{-1}\,[x_1 - \Phi(T)x_o - F(T)]$$

steers (LC) from x_o to x_1. Therefore

$$J(x, \bar{u}) \ge J_\varepsilon(x, \hat{u}_\varepsilon) \qquad \text{for all} \quad o<\varepsilon<\varepsilon.$$

But

$$J(x,\bar{u}) = \int_o^T (|Cx(t\,;\,\bar{u}) - z(t)|^2 + \langle N(t)\,\bar{u}(t), \bar{u}(t)\rangle)\,dt$$

$$\le \int_o^T [2|Cx(t\,;\,\bar{u})|^2 + 2|z(t)|^2 + \langle N(t)\,\bar{u}(t), \bar{u}(t)\rangle]\,dt$$

It is clear that

$$\langle N\bar{u}, \bar{u}\rangle \le K_1 |\,x_o - \Phi(T)\,x_o - F(T)|^2$$

with

$$K_1 \equiv \Big|\Big| \int_o^T Z_N^{-1*}\,\Phi(T,t)\,B(t)\,B^*(t)\,\Phi(T,t)^*\,Z_N^{-1}\,dt\Big|\Big|$$

Also from (1.2)

$$2\int_o^T |C\,x(t,\bar{u})|^2\,dt \le 4\Big\{\int_o^T [|C(\Phi(t)\,x_o + F(t)|^2 + |\,C\int_o^T \Phi(t,s)\,B(s)$$

$$N^{-1}(s)\,B^*(s)\,\Phi^*(t,s)\,.\,Z_N^{-1}\,(x_1 - \Phi(T)\,x_o - F(T))|^2\,ds]\,dt\Big\}$$

$$\leq 4\,||C||^2\,||\Phi(\cdot)\,x_o + F(\cdot)||^2_{L^2_n(o,T)} + K_2\,||x_o - \Phi(T)x_o - F(T)||^2$$

where K_2 can be similarly defined.

Let $M \equiv K_1 + K_2$, the proof is complete. Q.E.D.

§ II. REGULARIZATION AND PENALIZATION

In this section we adopt an approach [5] which provides us with a sequence of regularized controls approximating optimal control. Let

$$U_1 \equiv H^1_m(O, T) = \{u \mid u,\ u' \in L^2_m(o, T)\}$$

and consider minimizing

$$(2.1)\qquad J_{\varepsilon_1,\varepsilon_2}(x,u) \equiv J(x,u) + \psi(\epsilon_1)\,|\,x(T;u) - x_1\,|^2 + \varepsilon_2 \int_o^T \langle u'(t), u'(t)\rangle\, dt$$

for all $u \in H^1_m(o,T)$. Using the same argument as in Theorem 1.3, one easily proves that there is a unique element $\hat{u}_{\varepsilon_1,\varepsilon_2}$ in U_1 which minimizes $J_{\varepsilon_1,\varepsilon_2}$.

Theorem 2.1. Assume the system is controllable from x_o to x_1. As $\varepsilon_1,\varepsilon_2 \to o$ $\hat{u}_{\varepsilon_1,\varepsilon_2}$ converges strougly in $L^2_m(o,T)$ to some $\hat{u}$ which is the unique solution for the (OCP). For each fixed $\varepsilon_1 > o$, $\hat{u}_{\varepsilon_1,\varepsilon_2}$ converges strongly in $L^2_m(o,T)$ to $\hat{u}_{\varepsilon_1}$ which is the unique solution in Theorem 1.4.

Proof : Let $\hat{u}_{\varepsilon_1,\varepsilon_2}$, $\hat{u}_{\varepsilon_1}$ minimize $J_{\varepsilon_1,\varepsilon_2}(x,u)$ and $J_{\varepsilon_1}(x,u)$, respectively. Since $H^1_m(o,T)$ is dense in $L^2_m(o,T)$, there exists an H^1_m - sequence $w^{\epsilon_1}_n \to \hat{u}_{\epsilon_1}$ strongly in $L^2_m(o,T)$ such that

$$J(x, w^{\epsilon 1}_n) + \psi(\epsilon_1)\,|x(T\,;\,w^{\epsilon 1}_n) - x_1|^2 \leq J(x,\hat{u}_{\epsilon_1}) + \psi(\epsilon_1)\,|x(T;\hat{u}_{\epsilon_1}) - x_1|^2 + \eta/2$$

for any $\eta > 0$ provided n is large enough.

On the other hand, we have

$$J_{\varepsilon_1,\varepsilon_2}(x, \hat{u}_{\varepsilon_1,\varepsilon_2}) \leq J_{\varepsilon_1,\varepsilon_2}(x, w_n^{\epsilon_1}) \leq J_{\varepsilon_1}(x, \hat{u}_{\epsilon_1}) + \eta/2 + \varepsilon_2 ||w_n^{\epsilon_1}||_{L^2}$$

We choose ε_2 so small that $\varepsilon_2 || w_n^{\epsilon_1'} || \leq \eta/2$. Thus for any $\eta > 0$, there exists $\varepsilon(\eta)$ such that for $\varepsilon_2 \leq \varepsilon(\eta)$,

$$J_{\varepsilon_1,\varepsilon_2}(x, \hat{u}_{\varepsilon_1,\varepsilon_2}) \leq J_{\varepsilon_1}(\hat{u}_{\varepsilon_1}) + \eta$$

so

$$\overline{\lim_{\varepsilon_2 \downarrow 0}} J_{\varepsilon_1,\varepsilon_2}(x, \hat{u}_{\varepsilon_1,\varepsilon_2}) \leq J_{\varepsilon_1}(\hat{u}_{\varepsilon_1})$$

Now as $\varepsilon_1, \varepsilon_2 \to 0$, $\hat{u}_{\varepsilon_1,\varepsilon_2}$ is bounded in $L^2_m(0,T)$, so it contains a subsequence converging weakly to $\hat{u}$ in L^2_m for some $\hat{u}$. According to the proof of Theorem 1.4., this $\hat{u}$ steers (LC) from x_0 to x_1. Also,

$$J(x,u) \leq \lim_{\varepsilon_1,\varepsilon_2 \downarrow 0} J(x, \hat{u}_{\varepsilon_1,\varepsilon_2}) \leq \lim_{\varepsilon_1 \downarrow 0} J_{\varepsilon_1}(x, \hat{u}_{\varepsilon_1}) = J(x, \hat{u})$$

Hence for a subsequence $\varepsilon_1^{(n)}$, $\varepsilon_2^{(n)}$, we have

$$\lim_{\varepsilon_1^{(n)}, \varepsilon_2^{(n)} \downarrow 0} < N\hat{u}_{\varepsilon_1^{(n)}, \varepsilon_2^{(n)}}, \hat{u}_{\varepsilon_1^{(n)}, \varepsilon_2^{(n)}} > = < N\hat{u}, \hat{u} >$$

This implies the strong convergence of $\hat{u}_{\varepsilon_1,\varepsilon_2} \to \hat{u}$ in L^2_m.

For fixed ε_1, one easily sees from the proof of [5, Theorem 1.5 , p. 361] that $\hat{u}_{\varepsilon_1,\varepsilon_2}$ converges strongly to $\hat{u}_{\varepsilon_1}$ as $\varepsilon_2 \to 0$. Q.E.D.

§ III. FEEDBACK SYNTHESIS

The optimal control $\hat{u}_\varepsilon$ minimizing J_ε can be characterized [5] by the variational equation

$$\langle Cx(o\,;\hat{u}) - z,\ Cx(o\,;v) - Cx(t\,;\hat{u})\rangle_{L^2_p(o,T)} + \langle N\hat{u},\ v-\hat{u}\rangle_{L^2_m(o,T)} +$$

$$+ \psi(\varepsilon)\ \langle x(T\,;\hat{u}) - x_1,\ x(T\,;v) - x(T\,;\hat{u})\rangle_{R^n} = o$$

Introducing the adjoint state $p_\varepsilon(t)$

$$\begin{cases} \dfrac{dP_\varepsilon(t)}{dt} = -A^*\, p_\varepsilon(t) - C^*Cx(t\,;\hat{u}_\varepsilon) + C^*z(t) \\ p_\varepsilon(T) = \psi(\varepsilon)\ [x(T\,;\hat{u}_\varepsilon) - x_1] \end{cases}$$

we obtain

$$\int_o^T \langle N(t)\ \hat{u}_\varepsilon(t) + B^*(t)\ p_\varepsilon(t),\ v(t) - \hat{u}_\varepsilon(t)\rangle_{R^m} = o \qquad \forall\ v \in U_{ad}$$

Hence

$$(3.1) \qquad \hat{u}_\varepsilon = -N^{-1} B^* p_\varepsilon$$

Thus $x(t\,;\hat{u}_\varepsilon)$ and $p_\varepsilon(t)$ are coupled through

$$(3.2) \qquad \begin{cases} \dfrac{dx}{dt} - Ax + BN^{-1} B^* p_\varepsilon = f \\ x(o\,;\hat{u}_\varepsilon) = x_o \\ \dfrac{dp_\varepsilon}{dt} + A^* p_\varepsilon + C^*Cx = C^*z \\ p_\varepsilon(T) = \psi(\varepsilon)\ [x(T\,;\hat{u}_\varepsilon) - x_1] \end{cases}$$

on [o,T].

Consider the above (3.2) begining at some intermediate time $s \in [o,T]$. We write

$$(3.3)\quad \begin{cases} \dfrac{dx}{dt} - Ax + BN^{-1} B^* p_\varepsilon = f \\ \dfrac{dp_\varepsilon}{dt} + A^* p_\varepsilon + C^* C x = C^* z \\ x(s\,;\,\hat{u}_\varepsilon) = h \\ p_\varepsilon(T) = \psi(\varepsilon)\,[x(T\,;\,\hat{u}_\varepsilon) - x_1] \end{cases}$$

We decompose x, p_ε into $x = y_1 + y_2$, $p_\varepsilon = P_1 + P_2$, where y_1, P_1 satisfy

$$(3.4)\quad \begin{cases} \dfrac{dy_1}{dt} - Ay_1 + BN^{-1} B^* P_1 = o \\ -\dfrac{dP_1}{dt} + A^* P_1 + C^* C y_1 = o \\ y_1(s) = h \\ P_1(T) = \psi(\varepsilon)\, y_1(T) \end{cases}$$

and y_2, P_2 satisfy

$$(3.5)\quad \begin{cases} \dfrac{dy_2}{dt} - Ay_2 + BN^{-1} B^* P_2 = f \\ -\dfrac{dP_2}{dt} + A^* P_2 + C^* C y_2 = C^* z \\ y_2(s) = o \\ P_2(s) = \psi(\varepsilon)\, y_2(T) - \psi(\varepsilon)\, x_1 \end{cases}$$

From (3.3) we see that $P_1(t)$ $(s \le t \le T)$ is a linear function of h, in particular, at $t = s$. So we write $P_1(s) = \mathbb{P}(s)h$. $P_2(t)$ is independent of h, so we denote it by $\gamma_\varepsilon(t)$. Thus we have the affine relation

$$(3.6) \qquad p_\varepsilon(t) = \mathbb{P}_\varepsilon(t)h + \gamma_\varepsilon(t) = \mathbb{P}_\varepsilon(t)\, x(t\,;\, \hat{u}_\varepsilon) + \gamma_\varepsilon(t)$$

thereby achieving the decoupling

$$(3.7) \qquad \mathbb{P}'_\varepsilon + \mathbb{P}_\varepsilon A + A^*\mathbb{P}_\varepsilon - \mathbb{P}_\varepsilon B N^{-1} B^*\mathbb{P}_\varepsilon = -C^*C$$

$$(3.8) \qquad \mathbb{P}_\varepsilon(T) = \psi(\varepsilon)\ \ \mathrm{Inxn}$$

and $\gamma_\varepsilon(t)$ satisfies

$$(3.9) \qquad \gamma'_\varepsilon + A^*\gamma_\varepsilon - \mathbb{P}_\varepsilon B N^{-1} B^*\gamma_\varepsilon = -\mathbb{P}_\varepsilon f + C^*z$$

$$(3.10) \qquad \gamma_\varepsilon(T) = -\psi(\varepsilon)\, x_1$$

One easily verifies that P has the following properties ([8]) :

1. $\mathbb{P}_\varepsilon^*(t) = \mathbb{P}_\varepsilon(t)$ for all $t \in [o,T]$
2. $\mathbb{P}_\varepsilon(t) \ge o$, i.e., $\langle \mathbb{P}_\varepsilon(t)y, y \rangle_{\mathbb{R}^n} \ge o$ for $y \in \mathbb{R}^n$
3. $\mathbb{P}_\varepsilon(t_1) \le \mathbb{P}_\varepsilon(t_2)$ if $t_1 \ge t_2$
4. $\mathbb{P}_{\varepsilon_2}(t) \ge \mathbb{P}_{\varepsilon_1}(t)$ if $o < \varepsilon_2 < \varepsilon_1 < \varepsilon_o$

The terminal conditions (3.8) and (3.10) indicate that

$$\|\mathbb{P}_\varepsilon(T)\| \to \infty\ ,\ |\gamma_\varepsilon(T)| \to \infty \quad \text{as} \quad \varepsilon \downarrow o$$

This seems to be a great disadvantage for Riccati equations synthesis.

Now consider the special case of $C \equiv o$. Then

$$J(x,u) = \int_o^T \left(|z(t)|^2 + \langle N(t)\, u(t), u(t) \rangle \right) dt$$

so $J(x,u)$ is minimal if and only if $< Nu, u >$ is minimal. The optimal control $\hat{u}$ on $[o,T]$ is known in Theorem 1.2. By Bellman's dynamic programming, the optimal control $\hat{u}_s(t)$ which minimizes $<Nu,u>_{L_m^2(s,T)}$ with initial and terminal states $x(s\,;\,\hat{u}_\varepsilon)$ and x_1 (resp.) is

$$(3.11)\qquad \hat{u}(s,t) = N^{-1}(t)\,B^*(t)\,\Phi^*(T,t)\,Z_N^{-1}(T,s)\,[x_1 - \Phi(T,s)\,x(s\,;\,\hat{u}_\varepsilon) - F(T,s)] \quad (s \le t \le T)$$

where

$$Z_N(T,s) \equiv \int_s^T \Phi(T,\zeta)\,B(\zeta)\,N(\zeta)^{-1}\,B^*(\zeta)\,\Phi^*(T,\zeta)\,d\zeta$$

$$F(T,s) = \int_s^T \Phi(T,\zeta)\,f(\zeta)\,d\zeta$$

In the context of (3.3), $x(s\,;\,\hat{u}_\varepsilon)$ is h. Therefore, comparing (3.1), (3.6), (3.11) and letting $t = s$, we get

Theorem 3.1. Assume that $C \equiv o$ and B is nonsingular (a priori, $m = n$). Then

$$P_\varepsilon(.) \to \Phi^*(T,.)\,Z_N^{-1}(T,.)\,\Phi(T,.)$$

$$\gamma_\varepsilon(.) \to \Phi^*(T,.)\,Z_N^{-1}(T,.)\,[x_1 - F(T,.)]$$

as $\varepsilon \to o$.

The above theorem explores to some extent the structure of solutions to Riccati equations. One can see that as $s \to T$, the (operator) norm of $\Phi^*(T,s)\,Z_N^{-1}(T,s)\,\Phi(T,s)$ becomes unbounded. This coincides with our earlier observation that $||P_\varepsilon(s)|| \to \infty$ as $s \to T$. Similar observation also holds for $\gamma_\varepsilon(s)$ as $s \to T$. Nevertheless, we do not knows what the rate of convergence is as related to ε.

In general $m < n$ so B can not be nonsingular, we tend to think that theorem 3.1. remains true. We have yet no proof to confirm this.

As for the cost functional $J_{\varepsilon_1,\varepsilon_2}$ in (2.1), the optimal control $\hat{u}_{\varepsilon_1,\varepsilon_2}$ is determined by

$$\begin{cases} x' = Ax + Bu + f \\ x(o) = x_o \\ p' = A^*p - C^*Cx + C^*z \\ p(T) = \psi(\varepsilon_1)\ [x(T) - x_1] \\ -\varepsilon_2\, u'' + Nu + B^*p = o \\ u'(o) = u'(T) = o \\ o \le t \le T \end{cases}$$

This is a coupled system of initial, terminal and boundary value problems.

§ IV. NUMERICAL APPROXIMATIONS AND EXAMPLE

We choose S^h to be the space of continuous precewise cubics on $[o,T]$ on a mesh of length h and let

$$\hat{S}^h \equiv \prod_{i=1}^{m} S^h$$

Let $\{\phi_i\}$ be a basis for S^h. If $[o,T]$ is divided into k intervals, then the number of elements in $\{\phi_i\}$ is $3k+1$. Consequently, the set of elements

$$\hat{\phi}_{ij} \equiv \phi_i\, \hat{e}_j, \quad \hat{e}_j : \text{the j-th unit vector, } 1 \le j \le m$$

becomes a basis for $\hat{S}^h$ consisting of $NS \equiv (3k+1).m$ elements.

We want to approximate the optimal control $\hat{u}$ by elements in the space $\hat{S}^h$. According to the theory we have established in § II, $\hat{u}$ can be approximated to any desired accuracy by $\hat{u}_\varepsilon$. Therefore, we consider the minimization problem

$$\underset{v\in\hat{S}^h}{\text{Min}} \quad J(x,v) = \{ \int_0^T [\,||C\,\underline{x}(t\,;\,v) - \underline{z}(t)||^2 + \langle \underline{N}(t)\,v(t);\,v(t)\rangle]\,dt +$$

$$+ \psi(h)\,|x(T\,;\,v) - x_1|^2\}$$

where $\underline{x}$, $\underline{z}$, $\underline{N}$ are finite element representations of these functions. The variational formulation of this problem is :

$$(4.1)\quad \begin{cases} \text{Find } \hat{u}_h \in \hat{S}^h \text{ such that} \\ \int_0^T [\langle C\,\underline{x}(t\,;\,v) - \underline{z}(t),\, C\,\underline{x}(t\,;\,v) - C\,\underline{x}(t\,;\,\hat{u}_h)\rangle + \langle \underline{N}(t)\,v(t), \\ v(t) - \hat{u}_h(t)\rangle]\,dt + \psi(h)\,\langle \underline{x}(T\,;v) - x_1,\, \underline{x}(T\,;\,v) - \underline{x}(T\,;\,\hat{u}_h)\rangle = \\ = 0 \qquad\qquad v \in \hat{S}^h \end{cases}$$

We state the following fundamental theorem here. Its proof will be given in detail in a forthcoming paper.

Theorem 4.1. As sume that (LC) is controllable from x_o to x_1 and let $\hat{u}$ be the solution of the (OCP). Let $\hat{u}_h$ denote the unique solution of (4.1). Then

$$\hat{u}_h \to \hat{u} \qquad \text{in } L^2(o,T) \qquad \text{as } h \to o$$

Setting

$$\hat{u}_h = \sum_{i=1}^{NS} \alpha_i\,\hat{\Psi}_i$$

and letting v run through all the basis elements $\{\hat{\Psi}_j\}_1^{NS}$, the variational equation (4.1) is reduced to the following NS x 1 quadratic system :

$$(4.2)\qquad (I_{NSxNS} - \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}_{NSx1} \vec{\alpha}^{*}\ )\ \{[\ \psi(h)^{-1}\ (G+D)\ +\ M\ ]\vec{\alpha}\ +\ [\psi(h)^{-1}\beta\ +\ \xi]\ \} = o$$

where

$$\vec{\alpha}^{*} = [\alpha_1,\ \alpha_2,\ \ldots,\alpha_{NS}]$$

$$G = (\gamma_{ij})_{NSxNS},\quad \gamma_{ij} \equiv \int_o^T \langle C \int_o^t \Phi(t,s)B(s)\ \psi_i(s)ds,\ C \int_o^t \Phi(t,s)B(s)\psi_j(s)ds \rangle dt$$

$$D = (\delta_{ij})_{NSxNS},\quad \delta_{ij} \equiv \int_o^T \langle N(s)\ \psi_i(s),\ \psi_j(s)\rangle\ ds$$

$$M = (\mu_{ij})_{NSxNS}\quad \mu_{ij} \equiv \langle \int_o^T \Phi(T,s)B(s)\ \psi_i(s)ds,\ \int_o^T \Phi(T,s)B(s)\psi_j(s)ds \rangle$$

$$\vec{\beta}^{*} = [\beta_1,\ \beta_2,\ \ldots,\ \beta_{NS}]\ ,\ \beta_j \equiv \int_o^T \langle C(\Phi(t)x_o + F(t)) - z(t),\ C \int_o^t \Phi(t,s)B(s)\psi_j(s)$$

$$ds \rangle\ dt$$

$$\vec{\zeta}^{*} = [\zeta_1,\zeta_2,\ \ldots,\ \zeta_{NS}]\ ,\ \zeta_j \equiv \langle\ \Phi(T)x_o + F(T) - x_1,\ \int_o^t \Phi(T,s)B(s)\psi_j(s)ds \rangle\ .$$

System 4.2. is very ill-conditioned. The standard non-linear system solvers will not work for (4.2) with cubics. This is typical of known penalization methods for linear partial differential equations [1]. So, instead, we solve the linearized system

$$(4.3)\qquad [\psi(h)^{-1}\ (G+D)+M\ \vec{\alpha}]+\ [\psi(h)^{-1}\ \beta+\zeta\ ]\ = o$$

The matrix $\psi(h)^{-1}\ (G+D)+M$ is also very ill-conditioned for large $\psi(h)$.

But we can use some standard stiff linear system solvers and compute

$$\alpha = - \left[\psi(h)^{-1}(G + D) + M\right]^{-1} \left[\psi(h)^{-1} \beta+\zeta \right]$$

It is clear that if α is a solution of (4.3), then α is also a solution of (4.2). In fact, (4.2) corresponds to the linear variational problem

$$(4.4)\quad \begin{cases} \text{Find } \hat{u}_h \in \hat{S}^h \text{ such that} \\ \int_0^T \left[\langle C\underline{x}(t\,;\,\hat{u}_h) - \underline{z}(t),\ C(\underline{x}(t;v) - \underline{\Phi}(t)x_o - \underline{F}(t))\rangle + \langle N(t)\hat{u}_h(t), \right. \\ \left. v(t)\rangle\right]dt + \psi(h)\ \langle \underline{x}(T\,;\,\hat{u}_h) - x_1,\ \underline{x}(T\,;\,v) - \Phi(T)\,x_o - F(T)\rangle = o \end{cases}$$

Thus a solution of (4.4) is a solution of (4.1).

We end this paper with a simple example.

Example : $n = m = 2$, $T = \pi/2$

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} o & 1 \\ -1 & o \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & o \\ o & 1 \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \begin{bmatrix} x_1(o) \\ x_2(o) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$$

$$\Phi(t,\ s) = \begin{bmatrix} \cos\,(t - s) & \sin\,(t - s) \\ -\sin\,(t - s) & \cos\,(t - s) \end{bmatrix}$$

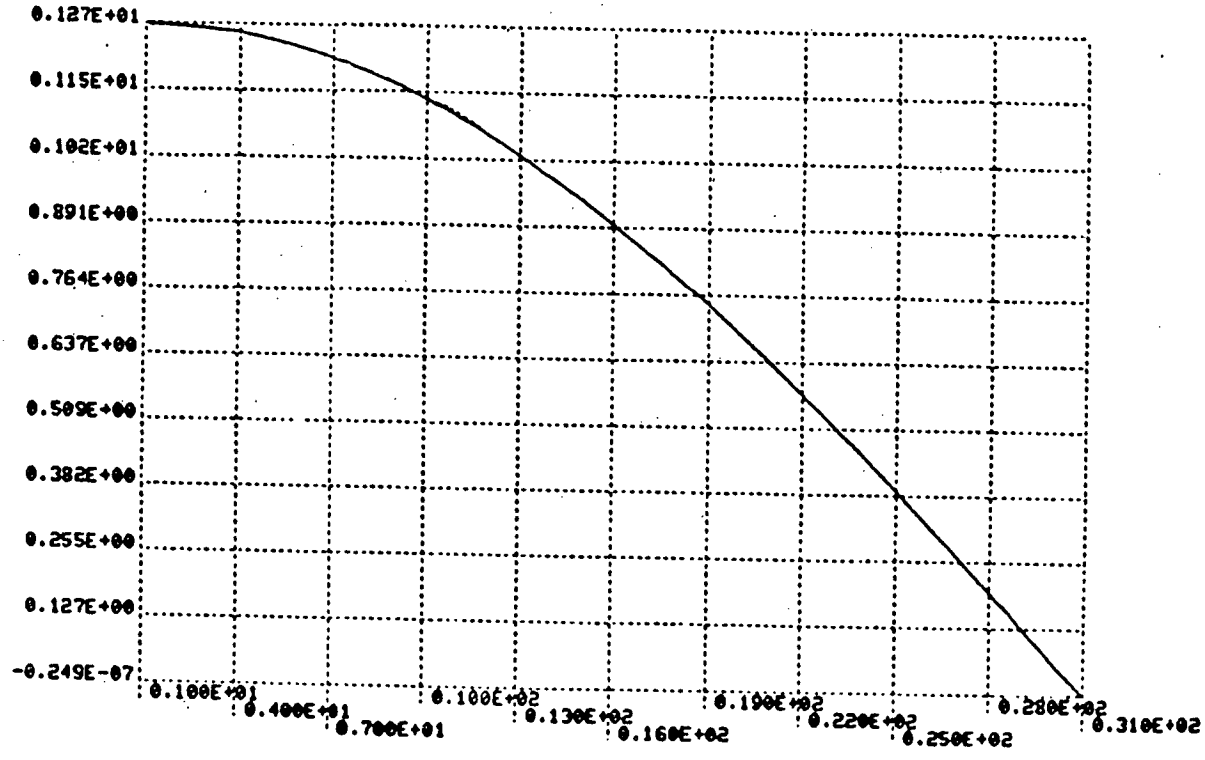
$$C = o_{2x2}\ ,\ z = o_{2x1}\ ,\ N \equiv I_{2x2}$$

The exact solution and the optimal control are, respectively,

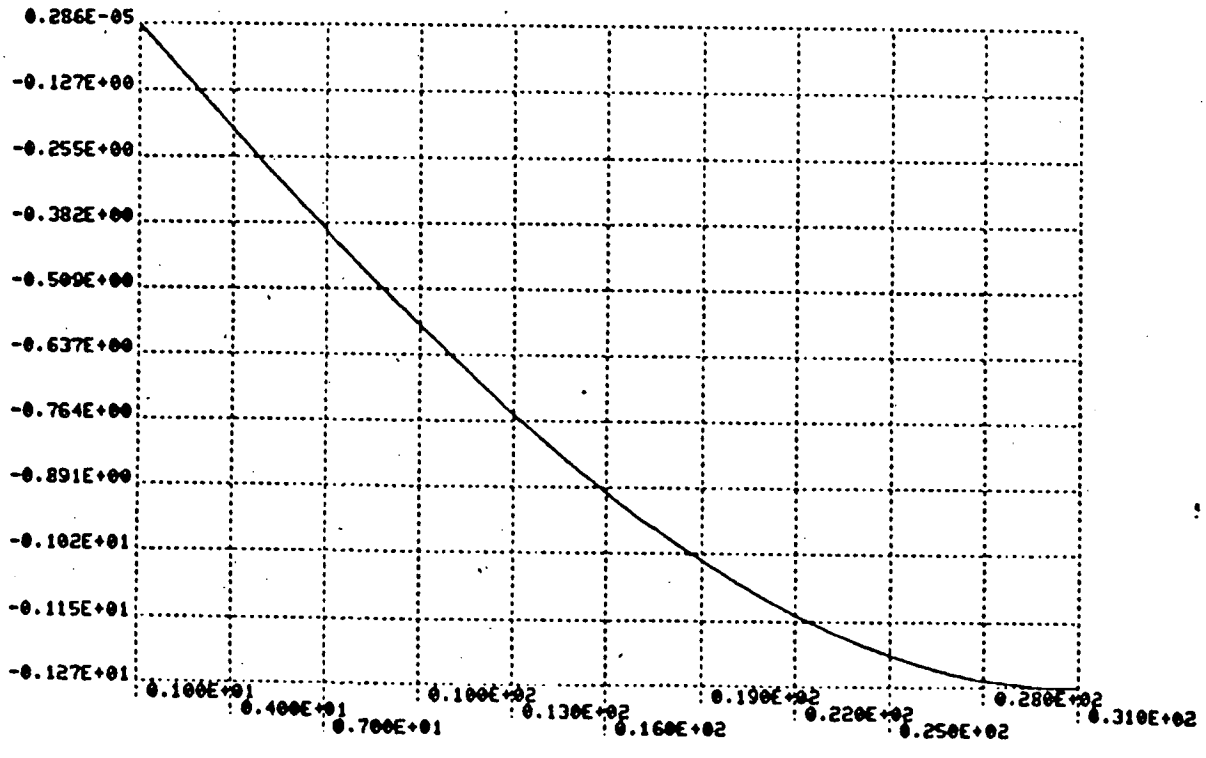
$$x(t) = \begin{bmatrix} (\frac{4}{\pi} t - 1) \cos t + \sin t \\ - (\frac{4}{\pi} t - 1) \sin t + \cos t \end{bmatrix}$$

$$u(t) = \begin{bmatrix} \frac{4}{\pi} \cos t \\ - \frac{4}{\pi} \sin t \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

For covenience, we imput the exact $\Phi(t,s)$ in our computation of (4.3). In the following two graphs, one can see that the broken line (representing the finite element solution coincides almost exactly with the dark line (exact solution)). The error is $\leq 10^{-4}$.


0.127E+01
0.115E+01
0.102E+01
0.891E+00
0.764E+00
0.637E+00
0.509E+00
0.382E+00
0.255E+00
0.127E+00
-0.249E-07
0.100E+01
0.400E+01
0.700E+01
0.100E+02
0.130E+02
0.160E+02
0.190E+02
0.220E+02
0.250E+02
0.280E+02
0.310E+02

(i) $u_1(t)$


0.286E-05
-0.127E+00
-0.255E+00
-0.382E+00
-0.509E+00
-0.637E+00
-0.764E+00
-0.891E+00
-0.102E+01
-0.115E+01
-0.127E+01
0.100E+01
0.400E+01
0.700E+01
0.100E+02
0.130E+02
0.160E+02
0.190E+02
0.220E+02
0.250E+02
0.280E+02
0.310E+02

(ii) $u_2(t)$

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