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G. Chen, R. Grimmer

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**A STUDY ON SOME  
LINEAR EVOLUTION EQUATIONS  
WITH TIME DELAY**

**Goong CHEN  
Ronald GRIMMER**

**Mars 1980**

A STUDY ON  
SOME LINEAR EVOLUTION EQUATIONS  
WITH TIME DELAY

Goong Chen\* - Ronald Grimmer\*\*

Résumé : On présente différentes techniques de semi-groupe pour étudier l'équation d'évolution linéaire avec mémoire

$$(VE) \quad \begin{cases} \frac{dx(t)}{dt} = A x(t) + \int_0^t B(t-s) x(s) ds + f(t) \\ x(0) = x_0 \in X \end{cases}$$

dans un espace de Banach X.

On généralise l'approche de R.K. Miller de manière à pouvoir traiter une classe plus large d'équations. Les conditions d'existence de ces semi-groupes sont données. L'existence, l'unicité, le bien posé et l'approximation des solutions de l'équation (VE) sont alors déduites des équations différentielles et semi-groupes qui lui sont associées.

Abstract : We present here various semigroup techniques for studying the linear evolution equation with memory

$$(VE) \quad \begin{cases} \frac{dx(t)}{dt} = A x(t) + \int_0^t B(t-s) x(s) ds + f(t) \\ x(0) = x_0 \in X \end{cases}$$

in a Banach space X.

A generalization of R.K. Miller's semigroup approach is made so that a broader class of equations can be investigated by

his method. We determine conditions which ensure the existence of those semigroups. The existence, uniqueness, well-posedness and approximation of the equation (VE) are then derived from the associated differential equations and semigroups.

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\* INRIA, Domaine de Voluceau, Rocquencourt, 78150 LE CHESNAY, France.  
and Department of Mathematics, Pennsylvania States University, University Park, PA 16802, U.S.A.

\*\* Department of Mathematics, Southern Illinois University, Carbondale, IL 62901, U.S.A.

1. INTRODUCTION

In this paper, we shall be concerned with the integrodifferential equation (VE)

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s) x(s) ds + f(t), t \geq 0 \\ x(0) = x_0 \in X \end{cases}$$

in a Banach space  $X$ . Throughout this paper we always assume that  $A$  is the infinitesimal generator of  $C_0$  semigroup on  $X$  and the Hille-Yosida-Phillips conditions

$$\|R^n(\lambda ; A)\| \leq M/(\text{Re}\lambda - \omega)^n, \quad n \geq 1$$

are satisfied for the resolvent  $R(\lambda ; A)$  for some  $\omega \geq 0$ . Also we assume that  $f$  is an element of a Banach space  $\mathcal{F}$  of  $X$ -valued functions which are defined for  $t \geq 0$ , and that  $B(t)$  is a linear operator on  $D(A)$  for each  $t \geq 0$  such that  $B(\cdot) x \in \mathcal{F}$  for each fixed  $x \in D(A)$ .

Equation (VE) appears, e.g., in the modelling of heat conduction problems in materials with memory, where  $A = \Delta$  is the Laplacian and the kernel  $B(t)$  is basically of the form  $k(t)A$  with  $k \in L^1(0, \infty)$ . Linear partial differential integral equations of Volterra type associated with heat conduction problems have been studied by Miller [7], see also the references therein.

In this paper, we are mostly interested in the problems of existence, uniqueness and continuity of solutions with respect to  $x_0$  and  $f$ . We are also concerned with approximating the solutions  $x(t)$  of (VE) by solutions  $x_n(t)$  of

$$(VE)^n \quad \begin{cases} x'_n(t) = A_n x_n(t) + \int_0^t B_n(t-s) x_n(s) ds + f(t), t \geq 0 \\ x_n(0) = x_0 \end{cases}$$

given that  $B_n \rightarrow B$  and  $A_n \rightarrow A$  in some sense.

Questions concerning existence uniqueness and well-posedness of solutions of linear Volterra integrodifferential equations in a Banach space have been examined by Miller [6], Miller and Wheeler [8], Chen and Grimmer [1], etc. In the nonlinear case, Crandall and Nohel [2]. Related work concerning integral equations appears in Grimmer and Miller [3], [4].

The approach we are following is similar to that in Miller [6], where he studied (VE) by means of the differential equation

$$(DE)' \quad \begin{cases} \frac{d}{dt} \begin{bmatrix} x(t) \\ F(t, \cdot) \end{bmatrix} = C \begin{bmatrix} x(t) \\ F(t, \cdot) \end{bmatrix} \equiv \begin{bmatrix} A & \delta_0 \\ B(\cdot) & D_s \end{bmatrix} \begin{bmatrix} x(t) \\ F(t, \cdot) \end{bmatrix} & , t \geq 0 \\ \begin{bmatrix} x(0) \\ F(0, \cdot) \end{bmatrix} = \begin{bmatrix} x_0 \\ f \end{bmatrix} \in X \times \mathcal{F} \end{cases}$$

which is a Cauchy problem on  $X \times \mathcal{F}$ . Here  $\mathcal{F} = BU(\mathbb{R}^+ ; X)$  is the space of bounded uniformly continuous functions from  $\mathbb{R}^+ = [0, \infty]$  into  $X$ ,  $\delta_0$  is the Dirac delta function and  $D_s$  is the differentiation operator on  $\mathcal{F}$ . Miller proved that solution of (DE)' give solutions to (VE) ([6, Theorem 3.5]). The choice of  $\mathcal{F} = BU$  is necessary ([6, §2]) since  $\mathcal{F}$  must at least contain those bounded uniformly continuous  $X$ -valued functions. The proof of his theorem will not go through if the function space  $\mathcal{F}$  is chosen differently, e.g., say  $\mathcal{F} \equiv B^2(\mathbb{R}^+ ; X)$  (the space of Bochner square integrable functions on  $\mathbb{R}^+$ ).

In §2, we first generalize Miller's scheme so that (VE) can be studied for a broader class of function spaces  $\mathcal{F}$ . We show that, under appropriate assumptions on  $B$ , we have the equivalence of (VE) with a new (DE).

In §3, we compute the resolvent operator  $R(\lambda ; c)$  and discuss some properties of the spectrum of  $C$ .

The main theorems of this paper are given in §4-6.

In §4, we study the case when  $\mathcal{F} = BU(\mathbb{R}^+ ; X)$ . We are particularly interested in the case when  $B(t)$  is of the form  $k(t)A$ .

Existence, uniqueness and well-posedness are proved for such kernels  $B$  by using perturbation and decomposition techniques for infinitesimal generators. We also point out an error in a recent paper by J. Zabezyk.

In §5, we study the case  $\mathcal{F} = B^2(\mathbb{R}^+ ; X)$ . On this  $\mathcal{F}$  the Dirac delta function  $\delta_0$  is no longer a bounded linear operator, so it becomes more difficult to derive existence theorems for the associated semigroups. We have proved an existence theorem under some assumptions on the Kernel  $B$ , by Lumer-Phillips' theorem.

We study approximations by Trotter's theory in §6.

2. THE EQUIVALENCE BETWEEN A VOLTERRA INTEGRODIFFERENTIAL EQUATION (VE) AND AN ASSOCIATED DIFFERENTIAL EQUATION (DE)

We consider the following differential equation

$$(DE) \quad \begin{cases} \frac{d}{dt} z = Cz \\ z(0) = z_0 \in D(C) \in X \times X \times \mathcal{F} \end{cases}$$

which is a Cauchy problem in the Banach space  $X \times X \times \mathcal{F}$ . Here

$$(2.1) \quad z = \begin{bmatrix} w \\ x \\ y \end{bmatrix} \in X \times X \times \mathcal{F} \quad \text{with} \quad \|z\|^2 = \|w\|_X^2 + \|x\|_X^2 + \|y\|_{\mathcal{F}}^2$$

$$C = \begin{bmatrix} 0 & A_0 & 0 \\ 0 & A & \delta_0 \\ 0 & B & D_S \end{bmatrix}$$

and  $B$  is the linear transformation given by  $(Bx)(s) = B(s)x$ ,  $A_0$  is a closed operator in  $X$  with domain  $D(A_0) \equiv D(A)$  and with resolvent  $R(\lambda ; A_0) = (\lambda I - A_0)^{-1}$ . In our treatment,  $A_0$  is usually a multiple of either  $A$  or perhaps some positive fractional power of  $A$  if it exists.  $D_s$  is the differentiation operator on  $\mathfrak{F}$  defined by  $D_s f = f'$  on a domain  $D(D_s) \equiv \mathfrak{F}$  where  $f \in D(D_s)$  implies

$$f(s) = \alpha + \int_0^s e(u) du$$

for some  $e \in \mathfrak{F}$  and  $D_s$  generates the translation semigroup  $T(t)$  on  $\mathfrak{F}$  given by  $T(t) f(s) = f(t+s)$ . The domain of  $C$ ,  $D(C)$ , is  $X \times D(A) \times D(D_s)$ . It is routine to verify that  $C$  is a closed operator on  $X \times X \times \mathfrak{F}$ .

We first give some definitions and notations.

Definition 2.1. By a solution  $z(t)$  of (DE) satisfying an initial condition  $z(0) = z_0$  we mean a function  $z : \mathbb{R}^+ \rightarrow D(C)$  with  $z, z'$  and  $Cz$  continuous and  $z'(t) = Cz(t)$  for all  $t \in \mathbb{R}^+$ .

Definition 2.2. A solution  $x(t)$  of (VE) satisfying  $x(0) = x_0$  is a function  $x : \mathbb{R}^+ \rightarrow D(A)$  such that  $x, x'$  and  $Ax$  are continuous and (VE) is satisfied for all  $t \in \mathbb{R}^+$ .

Definition 2.3. The equation (DE) is uniformly well-posed if for each  $z_0 \in D(C)$  the initial value problem  $z(0) = z_0$  has a unique solution  $z(t, z_0)$  and for any  $T > 0$  there is a  $K > 0$  such that :

$$\|z(t, z_0)\| \leq K \|z_0\| \text{ for all } z_0 \in D(C) \text{ for all } t \in [0, T].$$

Definition 2.4. The equation (VE) is uniformly well-posed is for each pair  $(x_0, f)$  with  $(0, x_0, f) \in D(C)$  there is a unique solution  $x(t, x_0, f)$  of (VE) and for any  $T > 0$  there is an  $M > 0$  such that  $\|x(t, x_0, f)\|_X \leq M (\|x_0\|_X + \|f\|_{\mathfrak{F}})$  for all  $t \in [0, T]$ .



Notations : From now on,  $T(t)$  always denote the translation semigroup. We use  $h_s$  to denote the translated function  $T(s)h$ , i.e.,  $h_s(u) = h(s+u)$ .  $\mathcal{L}_\lambda$  denotes the Laplace transform, i.e.,  $\mathcal{L}_\lambda h =$

$$\int_0^\infty e^{-\lambda s} h(s) ds.$$

We use  $*$  to denote the transpose of an element in  $X \times X \times \mathcal{F}$ .

In order to obtain an equivalence relation between solutions of (VE) and those of (DE), we require the following assumptions dealing with  $B$  :

(H1)  $B(\cdot)x(t)$  is continuous as a function of  $t$  on  $\mathbb{R}^+$  into  $\mathcal{F}$  whenever  $x(t)$  and  $A_0 x(t)$  are continuous on  $\mathbb{R}^+$  into  $X$ .  
 In addition,  $B(\cdot)R(\lambda, A_0)$  is a bounded operator from  $X$  into  $\mathcal{F}$ .

(H2)  $B(s)x$  is in  $D(D_s)x$  for each fixed  $x$  in  $D(A_0)$ .

(H3)  $D_s B(s)x(t)$  is locally integrable as a function of  $t$  whenever  $x(t)$  and  $A_0 x(t)$  are continuous.

(H4)  $A_0 x(t)$  is continuous as a function of  $t$  whenever  $Ax(t)$  is continuous as a function of  $t$ .

An example of a function  $B(t)$  which may satisfy (H1)-(H3) is  $B(t) = a(t)A_0$ , where  $a(t)$  is a scalar valued function. Consider  $\mathcal{F} = B^2(\mathbb{R}^+; X)$ .

If  $a \in L^2(\mathbb{R}^+)$ , (H1) is satisfied. If  $a$  is absolutely continuous with  $a' \in L^2(0, \infty)$ , then  $B(s)x = a(s)A_0 x$  is in  $D(D_s)$  for each fixed  $x \in D(A_0)$  and  $a'(s)A_0 x(t)$  is a continuous function of  $t$  into  $\mathcal{F}$  when  $A_0 x(t)$  is continuous. So (H2) and (H3) are satisfied. (H4) is satisfied if  $A = -A_0^2$  and  $A_0$  is invertible, or if  $A = A_0$ .

The following theorem generalizes [6, Theorem 3.5].

Theorem 2.5. Assume (H1) is valid. If  $z = (w,x,y)^*$  is a solution of (DE), then  $x(t)$  is a solution of (VE) with  $f(t) = y(o)(t)$  and  $x_o = x(o)$ . Conversely, if  $f \in D(D_S)$  and if (H1)-(H4) are valid and if  $x(t)$  is a solution of (VE), then  $(w,x,y)^*$  is a solution of (DE) with  $w(t) = w_o + \int_0^t A_o x(s) ds$

$$\text{and } y(t)(s) = f(t+s) + \int_0^t B(t-\tau+s) x(\tau) d\tau$$

Proof : First assume that (H1) is valid and  $z = (w,x,y)^*$  is a solution of (DE). Then  $w, w', x, x', y$  and  $y'$  are all continuous as functions of  $t$  from  $\mathbb{R}^+$  into either  $X$  or  $\mathcal{Y}$ . As the equation

$$y'(t) = D_S y(t) + B(s) x(t) ; y(o) = y_o, \quad t \geq 0$$

has a solution, it follows from [9, p. 110] or [5, p. 488] that the solution is given by

$$y(t) = T(t) y_o + \int_0^t T(t-\tau) B(.) x(\tau) d\tau$$

where  $T(t)$  is the semigroup generated by  $D_S$ , i.e.,  $T(t)$  is the translation semigroup. Hence if  $y(o) = f$ , we see that

$$y(t)(s) = f(t+s) + \int_0^t B(t-\tau+s) x(\tau) d\tau$$

As  $y \in D(D_S)$ ,  $y(t,.)$  is absolutely continuous and so

$$y(t)(o) = f(t) + \int_0^t B(t-\tau) x(\tau) d\tau$$

is continuous in  $t$  by (H1). Now  $x'(t) = Ax(t) + y(t)(o)$  so  $Ax(t)$  is continuous. Therefore

$$x'(t) = Ax(t) + \int_0^t B(t-\tau) x(\tau) d\tau + f(t)$$

is a solution of (VE).

Conversely, if (H1)-(H4) are valid and  $x(t)$  is a solution of (VE) with  $f \in D(D_S)$ , then  $x(t)$  and  $Ax(t)$  are continuous so that  $B(\cdot)x(t)$ , which is in  $D(D_S)$  for each  $t$ , is continuous in  $t$  and  $D_S B(s)x(t)$  is locally integrable as a function of  $t$ . Thus, the equation

$$y'(t) = D_S y(t) + B(s)x(t)$$

has as its solution [9, p. 112]

$$y(t)(s) = f(t+s) + \int_0^t B(t-\tau+s) x(\tau) d\tau$$

and in particular,

$$y(t)(0) = f(t) + \int_0^t B(t-\tau) x(\tau) d\tau$$

so

$$x'(t) = Ax(t) + \delta_0 y(t)$$

Thus, if  $w(t) = w_0 + \int_0^t A_0 x(s) ds$ ,  $z \equiv (w, x, y)^*$  is a solution of  $z' = Cz$  with  $z(0) = (w_0, x_0, y_0)$ . Q.E.D.

We note that, under the assumptions (H1)-(H4), if the solutions of (DE) are unique, then the solutions of (VE) with  $(0, x_0, f) \in D(C)$  are unique when they exist. Similarly, if the solutions of (VE) are unique for  $(0, x_0, f) \in D(C)$  then the solutions of (DE) must also be unique. It follows that if  $C$  generates a  $C_0$  semigroup, then (VE) is uniformly well-posed. Thus we will be concerned with conditions which ensure that  $C$  generates a  $C_0$  semigroup in the subsequent sections.

### 3. RESOLVENTS AND SPECTRUM OF THE OPERATOR C

Let  $C$  be given as in (2.1). The following computation of the resolvent

in a generalization of [6, Theorem 4.1].

Theorem 3.1 For any  $\lambda$  with  $\text{Re}\lambda > 0$ ,  $R(\lambda; C)$  exists if and if only  $R(x, A + \mathcal{L}_\lambda B)$  exists. If  $R(\lambda; C)$  exists, it is given by

$$(3.1) \quad R(\lambda; C) = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} A_0 R(\lambda; A + \mathcal{L}_\lambda B) & \frac{1}{\lambda} A_0 R(\lambda; A + \mathcal{L}_\lambda B) \mathcal{L}_\lambda \\ 0 & R(\lambda; A + \mathcal{L}_\lambda B) & R(\lambda; A + \mathcal{L}_\lambda B) \mathcal{L}_\lambda \\ 0 & R(\lambda; D_s) B R(\lambda; A + \mathcal{L}_\lambda B) & R(\lambda; D_s) [I + B R(\lambda; A + \mathcal{L}_\lambda B)] \mathcal{L}_\lambda \end{bmatrix}$$

Proof : Assume  $R(\lambda; C)$  exists. Then for any  $(f, g, h) \in X \times X \times \mathcal{F}$ , the equation

$$\begin{bmatrix} \lambda I & -A_0 & 0 \\ 0 & \lambda I - A & -\delta_0 \\ 0 & -B & \lambda I - D_s \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$$

is always solvable with solution  $(w, x, y)$ . Thus

$$\begin{cases} \lambda w - A_0 x = f \\ (\lambda I - A)x - \delta_0 y = g \\ -Bx + (\lambda I - D_s)y = h \end{cases}$$

so

$$\begin{aligned} y &= R(\lambda; D_s) (h + Bx) \\ \text{i.e. } y(s) &= \int_0^\infty e^{-\lambda u} T(u) [h(s) + B(s)x] du \\ &= \int_0^\infty e^{-\lambda u} [h_u(s) + B_u(s)x] du \\ &= \int_0^\infty e^{-\lambda u} [h_s(u) + B_s(u)x] du \end{aligned}$$

Therefore

$$(3.2) \quad \delta_0 y = \int_0^\infty e^{-\lambda u} [h(u) + B(u)x] du = \mathcal{L}_\lambda^{-1} (h + Bx)$$

$$\begin{aligned} \text{Also, } X &= R(\lambda; A) [g + \delta_0 y] \\ &= R(\lambda; A) [g + \mathcal{L}_\lambda^{-1} (h+Bx)] \quad (\text{from (3.2.)}) \end{aligned}$$

$$\text{so} \quad [I - R(\lambda; A) \mathcal{L}_\lambda^{-1} B] x = R(\lambda; A) (g + \mathcal{L}_\lambda^{-1} h)$$

the above relation is always invertible with solution

$$x = [I - R(\lambda; A) \mathcal{L}_\lambda^{-1} B]^{-1} R(\lambda; A) (g + \mathcal{L}_\lambda^{-1} h)$$

$$\text{But} \quad [I - R(\lambda; A) \mathcal{L}_\lambda^{-1} B]^{-1} R(\lambda; A) = (\lambda I - A - \mathcal{L}_\lambda^{-1} B)^{-1} = R(\lambda; A + \mathcal{L}_\lambda^{-1} B)$$

Hence  $R(\lambda; A + \mathcal{L}_\lambda^{-1} B)$  exists.

The first component w is given by

$$\begin{aligned} w &= \frac{1}{\lambda} (f + A_0 x) \\ &= \frac{1}{\lambda} [f + A_0 R(\lambda; A + \mathcal{L}_\lambda^{-1} B) (g + \mathcal{L}_\lambda^{-1} h)] \end{aligned}$$

We note that  $A_0 R(\lambda; A + \mathcal{L}_\lambda^{-1} B)$  is a bounded operator by the closed graph theorem.

Conversely, if  $R(\lambda; A + \mathcal{L}_\lambda^{-1} B)$  exists. Because each of the above steps is reversible,  $R(\lambda; C)$  must also exist and is given by (3.1). Q.E.D.

We note, in particular, that  $R(\lambda; A + \mathcal{L}_\lambda^{-1} B)$  exists provided that  $\mathcal{L}_\lambda^{-1} B$  is a bounded operator on  $X$  and  $\text{Re } \lambda > \omega + M \|\mathcal{L}_\lambda^{-1} B\|$ . Having found  $R(\lambda; C)$ , one can proceed with matrix multiplications to find the iterated resolvent  $R^n(\lambda; C)$ . Because  $R^n(\lambda; C)$  does not have a simple representation, it is in

general very difficult to verify whether the Hille-Yosida-Phillips criterion is satisfied. This makes any attempt impractical to directly prove that  $C$  is infinitesimal generator.

From the appearance of  $C$ , we know that its spectrum depends on the spectrum of  $A$  and  $D_S$  as well as the behavior of  $B$ . We refer the readers to [10] for the definitions of resolvent and point, continuous and redi-due spectrum, which we denote by  $\rho$ ,  $p\sigma$ ,  $c\sigma$  and  $r\sigma$ , respectively.

Different  $\mathcal{F}$ s give different spectrum for  $D_S$ . For example, if  $\mathcal{F} \equiv BU(\mathbb{R}^+; X)$ , then

$$\begin{aligned} \rho(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re}\lambda > 0\} \\ p\sigma(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re}\lambda \leq 0\} \\ c\sigma(D_S) &= r\sigma(D_S) = \emptyset \end{aligned}$$

but if  $\mathcal{F} \equiv B^2(\mathbb{R}^+; X)$ , then

$$\begin{aligned} \rho(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re}\lambda > 0\} \\ p\sigma(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re}\lambda < 0\} \\ c\sigma(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re}\lambda = 0\} \\ r\sigma(D_S) &= \emptyset \end{aligned}$$

The spectrum of the operator  $C$  can usually be classified by a careful computation:

An important subset of the spectrum of  $C$ , called the essential spectrum  $e\sigma(C)$ , merits special attention. There are many non-equivalent definitions of  $e\sigma$ . We will use the one given in [5].

The following theorem indicates the invariance of the essential spectrum under the perturbation by  $B$  when  $B$  and  $A$  have certain properties.

Theorem 3.2. Let  $C_1$  denote the operator 
$$\begin{bmatrix} 0 & A_0 & 0 \\ 0 & A & \delta_0 \\ 0 & 0 & D_S \end{bmatrix}$$

on  $X \times X \times \mathcal{Y}$ . Assume that for some  $\lambda$ ,  $A$  has a compact resolvent  $R(\lambda;A)$ . If the operator  $\mathcal{B}$  defined by  $\mathcal{B} x \equiv B(\cdot)x$  is a bounded linear operator from  $X$  into  $\mathcal{Y}$ , then for (i)  $\mathcal{Y} \equiv BU(\mathbb{R}^+;X)$  or (ii)  $\mathcal{Y} \equiv B^2(\mathbb{R}^+;X)$  and  $X$  = a Hilbert space,  $C$  has the same essential spectrum as  $C_1$ .

Proof : We want to show that the bounded operator  $\bar{B}$  defined by

$$\bar{B}(w,x,y)^* \equiv (0,0,\mathcal{B}x)^* \in X \times X \times \mathcal{Y}$$

is  $C_1$ -compact ([5]).

Let  $\{(w_n, x_n, y_n)\}$  be a sequence in  $D(C_1) (\equiv D(C))$  bounded in  $X \times X \times \mathcal{Y}$  such that  $\{C_1(w_n, x_n, y_n)^* = (A_0 x_n, Ax_n + \delta_0 y_n, y'_n)\}$  is bounded. (i)  $\mathcal{Y} \equiv BU(\mathbb{R}^+;X)$  : this implies that  $\{Ax_n\}$  is bounded in  $X$ , so  $\{(\lambda I - A)x_n\}$  is also bounded in  $X$ . Hence

$$x_n = (\lambda I - A)^{-1} (\lambda I - A)x_n$$

has a convergent subsequence, which we still denote by  $\{x_n\}$ . Thus

$$\begin{aligned} & \| \bar{B}(w_n, x_n, y_n) - \bar{B}(w_m, x_m, y_m) \|_{X \times X \times \mathcal{Y}} \\ &= \sup_{t \in \mathbb{R}^+} \| B(t)x_n - B(t)x_m \|_X \leq \| \mathcal{B} \|_{\mathcal{L}(X, \mathcal{Y})} \| x_n - x_m \|_X \rightarrow 0 \text{ as} \\ & n, m \rightarrow \infty \end{aligned}$$

Therefore  $\bar{B}$  is  $C_1$ -compact.

(ii)  $\mathcal{Y} \equiv B^2(\mathbb{R}^+;X)$  : since both  $\{y_n\}$  and  $\{y'_n\}$  are bounded, so is

$$\frac{1}{2} |\delta_0 y_n|_X^2 = | \langle y'_n, y_n \rangle_{\mathcal{Y} \times \mathcal{Y}} |$$

Thus  $\{Ax_n\} (= \{(Ax_n + \delta_0 y_n) - \delta_0 y_n\})$  as a sum of two bounded sequences is also bounded. So is  $\{(\lambda I - A)x_n\}$ . Hence

$$x_n = (\lambda I - A)^{-1} (\lambda I - A)x_n$$

has a convergent subsequence which we still denote by  $\{x_n\}$  as before. Now

$$\begin{aligned} & \|\bar{B}(w_n, x_n, y_n) - \bar{B}(w_m, x_m, y_m)\|_{X \times X \times \mathcal{F}}^2 \\ &= \int_0^\infty \|B(t)(x_n - x_m)\|_X^2 dt \leq \|B\|_{\mathcal{L}(X, \mathcal{F})}^2 \|x_n - x_m\|^2 \rightarrow 0 \text{ as } n, m \rightarrow \infty \end{aligned}$$

Therefore  $\bar{B}$  is also  $C_1$ -compact.

Since  $C = C_1 + \bar{B}$ , by [5, Theorem 5.35] the proof is complete. Q.E.D.

Remark : For any bounded invertible operator,  $P$ ,  $P^{-1}C_1P$  has the same spectrum classification as  $C_1$ , as  $R(\lambda; P^{-1}C_1P) = P^{-1}R(\lambda; C_1)P$  is true for  $\lambda \in \rho(C_1)$ . This observation will be useful in the subsequent proof of theorem 4.1.

4. THEOREMS ON EXISTENCE, UNIQUENESS AND CONTINUITY(I) :  $\mathcal{F} \equiv BU(\mathbb{R}^+; X)$

Throughout this section we assume that  $\mathcal{F} \equiv BU(\mathbb{R}^+; X)$  and  $A_0 = A$ . For this  $\mathcal{F}$ , (DE) may also be treated in the simpler setting  $X \times \mathcal{F}$  such as in [1], [6]. It is easy to see in the subsequent treatment that any results valid in the setting  $X \times \mathcal{F}$  are also valid in  $X \times X \times \mathcal{F}$ , and vice versa.

Theorem 4.1. Suppose  $B$  can be written as  $B = FA + K$  where  $F : X \rightarrow \mathcal{F}$  with range  $F \equiv D(D_S)$  and  $K : X \rightarrow \mathcal{F}$  are bounded linear operators. Then  $C$  generates a  $C_0$  semigroup on  $X \times X \times \mathcal{F}$ .

Remark : The above theorem holds for any  $\mathcal{F}$  such that  $\delta_0$  is a bounded operator from  $\mathcal{F}$  into  $X$ .



Proof : We first note that the operator on  $X \times X \times \mathcal{F}$  given by

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & D_S \end{bmatrix}$$

generates a  $C_0$  semigroup on  $X \times X \times \mathcal{F}$ . The operator on  $X \times X \times \mathcal{F}$  given by

$$P = \begin{bmatrix} I_X & -I_X & 0 \\ 0 & I_X & 0 \\ 0 & -F & I_{\mathcal{F}} \end{bmatrix}$$

is invertible with inverse

$$P^{-1} = \begin{bmatrix} I_X & I_X & 0 \\ 0 & I_X & 0 \\ 0 & F & I_{\mathcal{F}} \end{bmatrix}$$

and for any  $(w, x, y) \in P^{-1}(D(C_2)) = P^{-1}(X \times D(A) \times D(D_S))$ ,

$$\begin{aligned} P^{-1}C_2P(w, x, y)^* &= P^{-1}C_2(w, x, y, -Fx + y) \\ &= P^{-1}(0, Ax, D_S(-Fx + y)) \\ &= (Ax, Ax, FAx + D_S(-Fx+y)) = (Ax, Ax, FAx - D_S Fx + D_S y) \end{aligned}$$

Since  $F$  maps  $X$  into  $D(D_S)$ , we have  $P^{-1}(X \times D(A) \times D(D_S)) = X \times D(A) \times D(D_S)$ . Also,  $D_S F$  is a closed operator from  $X$  into  $\mathcal{F}$  with domain  $X$ . By the closed graph theorem,  $D_S F$  is a bounded operator. Thus

$$P^{-1}C_2P = \begin{bmatrix} 0 & A & 0 \\ 0 & A & 0 \\ 0 & FA - D_S F & D_S \end{bmatrix}$$

we must remark that in general the above is not true if  $F$  does not map  $X$  into  $D(D_S)$ . Now

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_0 \\ 0 & D_S F + K & 0 \end{bmatrix}$$

is a bounded operator, so  $P^{-1}C_2P + E = C$  is an infinitesimal generator of a  $C_0$ -semigroup [9, P. 50] or [5, P. 497] with  $D(C) = X \times D(A) \times D(D_S)$  Q.E.D.

Corollary 4.2. (Hille-Yosida-Phillips' conditions). Let A satisfy the Hille-Yosida-Phillips conditions  $\|R^n(\lambda; A)\| \leq M(\operatorname{Re}\lambda - \omega)^{-n}$  for some  $\omega \geq 0$  and let  $\alpha_1, \alpha_2, \alpha_3$  denote

$$\alpha_1 = \|F\|_{\mathcal{L}(X, \mathfrak{Y})}, \alpha_2 = \|D_S F\|_{\mathcal{L}(X, \mathfrak{Y})}, \alpha_3 = \|K\|_{\mathcal{L}(X, \mathfrak{Y})}$$

Then the iterated resolvent  $R^n(\lambda; C)$  is bounded by

$$\|R^n(\lambda; C)\|_{\mathcal{L}(X \times X \times \mathfrak{Y})} \leq \frac{M(2 + \alpha_1)^2}{[\operatorname{Re}\lambda - \omega - M(2 + \alpha_1)^2(1 + \alpha_2 + \alpha_3)]^n} \quad \text{all } n \in \mathbb{Z}^+$$

if  $\operatorname{Re}\lambda$  is large enough.

Proof. Since  $R(\lambda; P^{-1}C_2P) = P^{-1}R(\lambda; C_2)P$ , therefore

$$R^n(\lambda; P^{-1}C_2P) = P^{-1}R^n(\lambda; C_2)P$$

$$\text{and } \|R^n(\lambda; P^{-1}C_2P)\| \leq \|P^{-1}\| \|R^n(\lambda; C_2)\| \|P\|$$

$$\leq \frac{M(2 + \alpha_1)^2}{[\operatorname{Re}\lambda - \omega]^n} \quad (\because \|P\| \leq 2 + \alpha_1, \|P^{-1}\| \leq 2 + \alpha_1)$$

$$\text{Now } C = P^{-1}C_2P + E$$

From The proof of [9, Theorem 3.1.1.], one easily sees that

$$\|R^n(\lambda; C)\| \leq \frac{M(2 + \alpha_1)^2}{[\operatorname{Re}\lambda - \omega - M(2 + \alpha_1)^2(1 + \alpha_2 + \alpha_3)]^n}$$

$$\text{because } \|E\|_{\mathcal{L}(X \times X \times \mathfrak{Y})} \leq 1 + \alpha_2 + \alpha_3.$$

Q.E.D.

Corollary 4.3. Suppose (H1)-(H4) are valid with  $B(t) = a_1(t)A + a_2(t)I$ , where  $a_1, a'_1, a_2$  and  $a'_2$  are bounded uniformly continuous scalar functions on  $\mathbb{R}^+$ . The integral equation (VE) is uniformly well-posed for  $(x_0, f)$  with  $(0, x_0, f)$  in  $D(C)$ .

We remark that this corollary is essentially Miller's Theorem 7.3. [6] which was obtained via greatly different techniques. Other results, similar to those in [6, §7] follow in a similar manner.

Actually, a much more general result follows from Theorem 4.1. As  $A$  generates a  $C_0$  semi-group,  $A$  has a resolvent  $R(\lambda, A) = (\lambda I - A)^{-1}$  for  $\lambda$  with  $\text{Re} \lambda$  sufficiently large. As in [6, p. 181], Miller noted that if  $x(t)$  satisfies (VE), then  $\gamma(t)$  defined by  $\gamma(t) = x(t) \exp(-\lambda_0 t)$  ( $\lambda_0 > 0$ ) satisfies the equation

$$(VE)_\lambda \gamma'(t) = (A - \lambda_0 I) \gamma(t) + \int_0^t \exp(-\lambda_0(t-s)) B(t-s) \gamma(s) ds + \exp(-\lambda_0 t) f(t)$$

We may then, without any essential loss of generality, assume  
 (WLOG<sub>1</sub>)  $A$  has a bounded inverse  $A^{-1}$ , or  
 (WLOG<sub>2</sub>)  $A$  generates a uniformly bounded semigroup. (By changing the norm on  $X$ , we can actually assume that  $A$  generates a contraction semigroup).

Corollary 4.4. Under the convention of (WLOG<sub>1</sub>), assume (H1)-(H4) are valid for  $\mathcal{F} = BU(\mathbb{R}^+; X)$  with  $A_0 = A$  and assume furthermore

$$\|B(t)x\| \leq \beta (\|x\| + \|Ax\|), \quad \text{for } x \in D(A), \quad \beta > 0$$

Then the integral equation (VE) is uniformly well-posed for  $(x_0, f)$  with  $(0, x_0, f) \in D(C)$ .

Proof : Define  $F : X \rightarrow D(D_S)$  by  $F = BA^{-1}$ . Then  $B = FA$  and

$$\|Fx\|_{BU(\mathbb{R}^+; X)} = \sup_{t \in \mathbb{R}^+} \|B(t)A^{-1}x\| \leq \beta (\|A^{-1}x\| + \|x\|)$$

So F is a bounded operator. The proof follows from theorem 4.1. Q.E.D.

The techniques used in Theorem 4.1. were motivated by results obtained recently by Zabczyk [11]. In fact, it appears at first that a slight modification of Zabczyk's theorem 1 part 2 would allow us to obtain our result even if F does not map X into the domain of D<sub>S</sub>. This is not the case as Zabczyk's result is not correct. First of all the computations at lines 18 and 19 on [11, p. 525] gives  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  rather than  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Secondly, we have a counterexample which shows that :

$$\begin{bmatrix} I & F \\ 0 & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

need not generate a semigroup if F does not map into the domain of A. Consider the operator

$$\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \quad , \quad \text{i.e., } A \equiv B \equiv D$$

which generates a semigroup on X x X if D generates a semigroup S(t) on X. Now consider the operator

$$\begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \quad \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix} \equiv A$$

If Zabczyk's result were correct, A would generate a C<sub>0</sub> semigroup on X x X. Now choose x<sub>0</sub> ∈ D(D) which has the property that S(t)Dx<sub>0</sub> ∉ D(D) for any t ≥ 0. Then (0, x<sub>0</sub>)<sup>\*</sup> ∈ D(A) and there must be a classical solution of w' = Aw, w = (x, y)<sup>\*</sup> through this point if A generates a C<sub>0</sub> semigroup. This solution must have second coordinate S(t)x<sub>0</sub>. The first coordinate must satisfy

$$\begin{cases} y' = Dy + DS(t)x_0 \\ y(0) = y \end{cases}$$

However, as shown in Pazy [9, p.111] this problem has no solution for if y where a solution then y(t) must satisfy

$$y(t) = \int_0^t S(t-s) DS(s)x_0 ds = t S(t) Dx_0$$

This is impossible, however, as  $S(t) Dx_0$  differentiable would mean  $S(t) Dx_0 \in D(D)$ . Interestingly, the operator  $D$  which yields the easiest such example is  $D = D_S$  on  $X \equiv BU(\mathbb{R}^+; \mathbb{R})$ . Choosing  $x_0$  to be a function with only one derivative in  $BU(\mathbb{R}^+; \mathbb{R})$ , we are clear that translating the function will not smooth the function in general.

We restate a corrected version of Zabczyk's result in the following, which can be proven as above or will follow from his Theorem 1 a(1). All of his subsequent related results must also be modified accordingly.

Theorem 4.5. Let  $X$  and  $Y$  be Banach spaces and  $A$  and  $B$  generate  $C_0$  semigroups on  $X$  and  $Y$  respectively. If  $F : Y \rightarrow D(A)$  then

$$\begin{bmatrix} I & F \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

Also generates a semigroup.

An additional existence theorem will be given in §6 (Theorem 6.5.').

5. THEOREMS ON EXISTENCE, UNIQUENESS AND CONTINUITY (II) :  $\mathcal{F} = B^2(\mathbb{R}^+; X)$

If the Dirac delta function is not a bounded operator on  $\mathcal{F}$  into  $X$ , different techniques must be used. This is the case when  $\mathcal{F} = B^2(\mathbb{R}^+; X)$  of course. Our results in this direction are not as general because the unbounded operator  $\delta_0$  must also be dealt with. What we have obtained here are similar to Theorems 3 and 4 of our earlier work [1] for the case  $\mathcal{F} = BU(\mathbb{R}^+; X)$ .

. Throughout this section, we will follow the convention (WLOG<sub>2</sub>) in §4.

Theorem 5.1. Let  $X$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ . Let  $A$  be the generator of a contraction semigroup on  $X$  and let  $\mathcal{F} = B^2(\mathbb{R}^+; X)$  with the usual norm. Suppose  $B(\cdot) : X \rightarrow \mathcal{F}$  is defined on the domain of  $A$  at least and  $\|B(\cdot)x\|^2 \leq -M_1 \langle Ax, x \rangle$  and  $\|A_0 x\|_X^2 \leq -M_2 \langle Ax, x \rangle$  for all  $x \in D(A)$  for some positive constants  $M_1, M_2$ . Assume that  $R(\lambda + \alpha; A + \mathcal{L}_\lambda B)$  exists as a bounded operator on  $X$  for some  $\lambda > 0$  and some  $\alpha$  with  $2\alpha > M_1 + M_2$ . Then  $C$  generates a  $C_0$  semigroup  $\|S(t)\|$  with  $\|S(t)\| \leq e^{\alpha t}$ .

Proof : Consider the operator

$$\begin{bmatrix} -\alpha I_X & A_0 & 0 \\ 0 & A - \alpha I_X & \delta_0 \\ 0 & B & D_S - \alpha I_{\mathfrak{Y}} \end{bmatrix}$$

on  $X \times D(A) \times D(D_S)$ . We wish to employ the theorem of Lumer-Phillips [9] for the operator  $C_\alpha$ . We first show that  $C_\alpha$  is dissipative. Using  $\langle \cdot, \cdot \rangle$  as inner product in each space  $X, \mathfrak{Y}$  and  $X \times X \times \mathfrak{Y}$ , we see that if  $z = (w, x, y) \in D(C_\alpha)$ ,

$$\begin{aligned} \langle C_\alpha z, z \rangle &= \langle -\alpha w + A_0 x, w \rangle_X + \langle (A - \alpha I)x + \delta_0 y, x \rangle_X + \langle Bx + (D_S - \alpha I)y, y \rangle_{\mathfrak{Y}} \\ &= -\alpha (||w||^2 + ||x||^2 + ||y||^2) + \langle A_0 x, w \rangle + \langle Ax, x \rangle + \langle \delta_0 y, x \rangle + \langle Bx, y \rangle + \langle D_S y, y \rangle \\ &\leq -\alpha (||w||^2 + ||x||^2 + ||y||^2) + ||A_0 x|| ||w|| + \langle Ax, x \rangle + ||\delta_0 y|| ||x|| + ||B(\cdot)x||_{\mathfrak{Y}} ||y||_{\mathfrak{Y}} + \langle D_S y, y \rangle \end{aligned}$$

Now  $\langle D_S y, y \rangle = -||\delta_0 y||_X^2 / 2$  and

$$||Bx||_{\mathfrak{Y}} ||y||_{\mathfrak{Y}} \leq (\epsilon^2 ||Bx||_{\mathfrak{Y}}^2 + \epsilon^{-2} ||y||_{\mathfrak{Y}}^2) / 2$$

Similarly ,

$$||A_0 x|| ||w|| \leq (\epsilon^2 ||A_0 x||^2 + \epsilon^{-2} ||w||^2) / 2$$

Choosing  $\epsilon$  sufficiently small and  $\alpha = \max ((\epsilon^{-2} + 1) / 2, (M_1 + M_2 + 1) / 2)$ , we will obtain  $\langle C_\alpha z, z \rangle \leq 0$ . So  $C_\alpha$  is a dissipative operator.

Now consider  $\lambda I - C_\alpha$ . We want to show that the range of  $\lambda I - C_\alpha$  is  $X \times X \times \mathfrak{Y}$  for some  $\lambda > 0$ . But

$$(\lambda I - C_\alpha)^{-1} = ((\lambda + \alpha)I - C^{-1}) = R(\lambda + \alpha; C)$$

exists provided that  $R(\lambda + \alpha; A + \mathcal{L}_\lambda B)$  exists by theorem 3.1.

Therefore  $C_\alpha$  generates a contraction semigroup and, hence,  $C$  generates a semigroup  $S(t)$  with  $\|S(t)\| \leq e^{\alpha t}$ . Q.E.D.

Applying this to our original problem, we have

Corollary 5.2. Let  $X, A$  and  $\mathcal{F}$  be as above. Suppose (H1)-(H4) are valid with  $\|B(t)x\|_X^2 \leq -b(t) \langle Ax, x \rangle$  a.e.  $\mathbb{R}^+$  for some  $b \in L^1(\mathbb{R}^+)$  and  $\|A_0 x\|^2 \leq -M \langle Ax, x \rangle$  for some  $M > 0$ . Assume  $R(\lambda + \alpha; A + \mathcal{L}_\lambda B)$  exists as a bounded operator on  $X$  for some  $\lambda > 0$  and some  $\alpha > 1/2(M \int_0^\infty b(t) dt)$ . Then (VE) is uniformly well posed for  $(x_0, f)$  where  $(0, x_0, f) \in D(C)$ .

## 6. APPROXIMATIONS

In this section we will be concerned with approximating the solutions of (VE) by those of  $(VE)_n$ . A result of this kind has been obtained in our earlier work [1, Theorem 5]. Here we will study this problem under the general setting of this paper. Our results are motivated by a close examination of the proofs of Theorems 4.1 and 5.1.

First, we consider the differential equations

$$(DE)_n \quad z'_n = C^n z_n, \quad z_n(0) = z(0) \in X \times X \times \mathcal{F}$$

where

$$C^n \equiv \begin{bmatrix} 0 & A_n & 0 \\ 0 & A_n & \delta_0 \\ 0 & F_n A_n & D_s \end{bmatrix}$$

$$\mathcal{F} \equiv BU(\mathbb{R}^+; X)$$

We shall assume that  $A_n$  generates a  $C_0$  semigroup on  $W$  and  $F_n : X \rightarrow D(D_S)$  for all  $n$ . Theorem 4.1. implies that each operator  $C^n$  generates a  $C_0$  semigroup.

Theorem 6.1. Suppose  $\{A_n\}$  and  $A$  are infinitesimal generators of  $C_0$  semigroups  $\{S_n(t)\}$  and  $\{S(t)\}$  such that  $\{A_n\}$  and  $A$  are defined on a common domain  $D(A)$  and  $A_n x \rightarrow Ax$  for every  $x \in D(A)$ . Suppose there are constants  $M > 0$ ,  $\omega \geq 0$  such that  $\|S_n(t)\| \leq Me^{\omega t}$  and  $\|S(t)\| \leq Me^{\omega t}$ . Suppose further that  $F_n$  and  $F$  are bounded linear operators mapping  $X$  into  $D(D_S)$  such that  $F_n x \rightarrow Fx$  and  $D_S F_n x \rightarrow D_S Fx$  in  $\mathfrak{Y}$  for all  $x \in X$ . Then with  $B \equiv FA$ , if  $z_n(0) = z(0) = z_0$  is in  $D(C) = D(C^n)$ , we have  $z_n(t) \rightarrow z(t)$  as  $n \rightarrow \infty$  for all  $t \geq 0$  and the convergence is uniform on bounded  $t$  intervals.

Proof: We first note that  $C = P^{-1}C_2P + Q$  and  $C^n = P_n^{-1}C_2^n P_n + Q_n$  where

$$C_2^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_n & 0 \\ 0 & 0 & D_S \end{bmatrix}$$

$$P = \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & -F & I \end{bmatrix}, \quad P_n = \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & -F_n & I \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_0 \\ 0 & D_S F & 0 \end{bmatrix}, \quad Q_n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_0 \\ 0 & D_S F_n & 0 \end{bmatrix}$$

Our assumptions have the effect of making  $P$ ,  $P_n$ ,  $Q$  and  $Q_n$  bounded operators. In fact, by the uniform boundedness principle,  $P_n \rightarrow P$ ,  $P_n^{-1} \rightarrow P^{-1}$  and  $Q_n \rightarrow Q$  in the uniform operator topology and there is a uniform bound for  $\|P_n^{-1}\|$ ,  $\|P^{-1}\|$ ,  $\|P_n\|$ ,  $\|P\|$ ,  $\|Q_n\|$  and  $\|Q\|$ ; call it  $M_1$ .



We choose  $\lambda$  with  $\text{Re}\lambda > \omega + MM_1^3$ . From [9. p. 50], we see that  $\{C_n\}$  and  $\{C\}$  generates semigroups  $\{S_n(t)\}$  and  $\{S(t)\}$  satisfying

$$\|S_n(t)\| \leq MM_1^2 e^{(\omega+MM_1^3)t}, \quad \|S(t)\| \leq MM_1^2 e^{(\omega+MM_1^3)t}$$

with the resolvent conditions

$$\|R^n(\lambda; C_n)\| \leq \frac{MM_1^2}{(\text{Re}\lambda - \omega - MM_1^3)^n}$$

$$\|R^n(\lambda; C)\| \leq \frac{MM_1^2}{(\text{Re}\lambda - \omega - MM_1^3)^n}$$

Now we want to show that for all  $z \in X \times X \times \mathcal{F}$ ,  $R(\lambda; C_n) z \rightarrow R(\lambda; C)z$  as  $n \rightarrow \infty$ . Since  $R(\lambda; C_n) = P^{-1}R(\lambda; C_2^n + P_n Q_n P_n^{-1}) P_n$  and  $R(\lambda; C) = P^{-1}R(\lambda; C_2 + PQP^{-1})P$ , this is equivalent to showing that  $R(\lambda; C_2^n + P_n Q_n P_n^{-1})z \rightarrow R(\lambda; C_2 + PQP^{-1})z$  for all  $z$ . Let

$$k = R(\lambda; C_2 + PQP^{-1})z, \quad k \in D(C)$$

and let 
$$z_n \equiv [\lambda I - (C_2^n + P_n Q_n P_n^{-1})] k$$

From the given assumptions, we see immediately that  $z_n$  tends to  $z$  as  $n \rightarrow \infty$ .

Now

$$\begin{aligned} R(\lambda; C_2^n + P_n Q_n P_n^{-1})z &= R(\lambda; C_2^n + P_n Q_n P_n^{-1})z_n \\ &\quad + R(\lambda; C_2^n + P_n Q_n P_n^{-1})(z - z_n) \end{aligned}$$

and

$$\begin{aligned} &\lim_{n \rightarrow \infty} |R(\lambda; C_2^n + P_n Q_n P_n^{-1})(z - z_n)| \\ &\leq \lim_{n \rightarrow \infty} \frac{M}{(\text{Re}\lambda - \omega - MM_1^3)} |z - z_n| = 0 \end{aligned}$$

Hence 
$$\lim_{n \rightarrow \infty} R(\lambda; C_2^n + P_n Q_n P_n^{-1})z = \lim_{n \rightarrow \infty} R(\lambda; C_2^n + P_n Q_n P_n^{-1})z_n$$

$$= k = R(\lambda; C_2 + P Q P^{-1})z$$

It now follows from Trotter's approximation theorem [9, p. 57] or [5, p. 504] that the proof is complete. Q.E.D.

Corresponding to the case  $\mathcal{F} = B^2(\mathbb{R}^+; X)$  and Theorem 5.1, we consider

(DE)<sub>n</sub> 
$$z'_n = C^n z_n, \quad z_n(0) = z(0)$$

$$C^n = \begin{bmatrix} 0 & A_0^{(n)} & 0 \\ 0 & A_n & \delta_0 \\ 0 & B_n & D_s \end{bmatrix}$$

and obtain the following similar result.

Theorem 6.2. Let  $X$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ . Let  $\mathcal{F} = B^2(\mathbb{R}^+; X)$  be equipped with the usual norm. Assume that all the assumptions of Theorem 5.1. are satisfied for each triple  $(A_n, B_n, A_0^{(n)})$  and  $(A, B, A_0)$  with the same constants  $M_1, M_2$  and  $\alpha$ . Assume furthermore  $D(A_n) = D(A)$ ,  $D(A_0^{(n)}) = D(A_0)$  and for all  $x \in D(A)$ ,  $A_n x \rightarrow A x$ ,  $A_0^{(n)} x \rightarrow A_0 x$  in  $X$  and  $B_n x \rightarrow B x$  in  $\mathcal{F}$ . If  $z_n(0) = z(0) = z_0$  is in  $D(C) = D(C^n)$ , we have  $z_n(t) \rightarrow z(t)$  as  $n \rightarrow \infty$  for all  $t \geq 0$  and the convergence is uniform on bounded  $t$  intervals.

Proof. Note first that each of the operators  $C^n$  generates a semigroup  $S_n(t)$  with  $\|S_n(t)\| \leq e^{\alpha t}$ . We are thus able to argue as in the previous theorem that because  $C^n z \rightarrow C z$  for  $z \in D(C)$  that  $R(\lambda; C^n)z \rightarrow R(\lambda; C)z$  for all  $z \in X \times X \times \mathcal{F}$ . The proof again follows from the Trotter approximation theorem.

Theorems 6.1 and 6.2 have immediate application to our stated objective of obtaining results which ensure that the solution  $x_n(t)$  of (VE)<sub>n</sub> tends to the solution  $x(t)$  of (VE). Corresponding to theorem 6.1. we are able to

obtain the following result.

Theorem 6.3. Let (H1) - (H4) be valid for  $\mathcal{F}=BU(\mathbb{R}^+, X)$  and  $A_0=A$ . Suppose  $A_n$  and  $A$  are the generators of  $C_0$  semigroups  $\{S_n(t)\}$  and  $\{S(t)\}$  respectively and that  $\|S_n(t)\| \leq Me^{\omega t}$ ,  $\|S(t)\| \leq Me^{\omega t}$  for some constants  $M>0$ ,  $\omega>0$ . Suppose that the operators  $A_n$  and  $A$  have common domain  $D(A)$  and that  $A_n x \rightarrow Ax$  for every  $x \in D(A)$ . Also, suppose that  $B_n(\cdot)x \rightarrow B(\cdot)x$  in  $BU(\mathbb{R}^+; X)$  for every  $x \in D(A)$  and that  $\|B_n(\cdot)x\|_{BU} \leq \beta(\|x\| + \|A_n x\|)$ ,  $\|B(\cdot)x\| \leq \beta(\|x\| + \|Ax\|)$  for all  $x \in D(A)$  for some positive constant  $\beta$  for all  $n$ . Then for  $(0, x_0, f) \in D(C)$ , we have  $x_n(t) \xrightarrow{n \rightarrow \infty} x(t)$  pointwise in  $t$  for all  $t \geq 0$ . The convergence is uniform on bounded  $t$  intervals.

Proof : We first argue in a similar fashion as in the proof of Theorem 6.1., we obtain

$$(6.1) \quad R(\lambda; A_n)y \rightarrow R(\lambda; A)y \quad n \rightarrow \infty \quad y \in X, \quad \forall \lambda > \omega$$

Now, instead of considering  $(VE)_n$  and  $(VE)$ , we consider  $(VE)_{n,\lambda}$  and  $(VE)_n$ . We make the factorization

$$\begin{aligned} B_{n,\lambda} &\equiv \exp(-\lambda s)B_n = F_n(A_n - \lambda I), \quad \text{with } F_n \equiv -B_{n,\lambda} R(\lambda; A_n) \\ B_\lambda &\equiv \exp(-\lambda s)B = F(A - \lambda I), \quad \text{with } F \equiv -B_\lambda R(\lambda; A) \end{aligned}$$

For  $x \in X$ , we have  $R(\lambda; A)x \in D(A)$ ,  $R(\lambda; A_n)x \in D(A)$  and

$$(6.2) \quad \begin{aligned} \|F_n x - Fx\|_{BU} &\leq \|B_\lambda R(\lambda; A)x - B_{n,\lambda} R(\lambda; A)x\| + \\ &\quad + \|B_{n,\lambda} R(\lambda; A)x - B_{n,\lambda} R(\lambda; A_n)x\| \end{aligned}$$

The first term on the right of (6.2) vanishes as  $n \rightarrow \infty$ . Consider the second term

$$\begin{aligned} \left\| B_{n,\lambda} R(\lambda;A)x - B_{n,\lambda} R(\lambda;A_n)x \right\| &\leq \beta \left( \left\| R(\lambda;A)x - R(\lambda;A_n)x \right\| + \right. \\ &\quad \left. + \left\| A_n R(\lambda;A)x - A_n R(\lambda;A_n)x \right\| \right) \end{aligned}$$

Using (6.1), we see the vanishing of the first term on the right as  $n \rightarrow \infty$ . And

$$\begin{aligned} \left\| A_n R(\lambda;A)x - A_n R(\lambda;A_n)x \right\| &\leq \left\| A_n R(\lambda;A)x - AR(\lambda;A)x \right\| + \left\| AR(\lambda;A)x - A_n R(\lambda;A_n)x \right\| \\ &= \left\| A_n R(\lambda;A)x - AR(\lambda;A)x \right\| + \left\| \lambda R(\lambda;A)x - \lambda R(\lambda;A_n)x \right\| \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

We thus obtain  $F_n x \rightarrow Fx$  for every  $x \in X$ .

Similarly, we can prove that  $D_S F_n \rightarrow D_S Fx \quad \forall x \in X$ .

By theorem 6.1, the proof is complete.

Q.E.D.

Theorem 6.3. has an interesting application to integrodifferential equations arising in the study of heat conduction in materials with memory. In particular, Miller [7] examined the equation

$$\begin{aligned} \text{(HVE)} \quad \theta'(t) &= c \Delta \theta(t) - a(o) \theta(t) + c \int_0^t b(t-\tau) \Delta \theta(\tau) d\tau - \\ &\quad - \int_0^t a'(t-\tau) \theta(\tau) d\tau + f(t) \\ \theta(o) &= \theta_o \end{aligned}$$

where  $\Delta$  is the Laplacian,  $c > 0$ ,  $a'$  and  $b$  are continuously differentiable real valued functions in  $L^1(\mathbb{R}^+)$ . Assumptions made regarding the set  $\Omega$  on which the Laplacian is considered and on the boundary conditions make  $A \equiv c\Delta - a(o)I$  a generator of a  $C_0$  semigroup on  $L^p(\Omega)$ ,  $1 < p < \infty$ . If instead of (HVE) we consider

$$\begin{aligned}
 \text{(HVE)}_n \quad \theta'_n(t) &= c\Delta\theta_n(t) - a_n(o)\theta_n(t) + c \int_0^t b_n(t-\tau)\Delta\theta_n(\tau)d\tau - \\
 &\quad - \int_0^t a'_n(t-\tau)\theta_n(\tau)d\tau + f(t) \\
 \theta_n(o) &= \emptyset
 \end{aligned}$$

we see that Theorem 6.3. can be immediately applied. If the assumptions  $b_n \rightarrow b$ ,  $b'_n \rightarrow b'$ ,  $a_n \rightarrow a'$  and  $a''_n \rightarrow a''$  in  $BU(\mathbb{R}^+)$  are satisfied, we see that  $\theta_n(t)$  converges to  $\theta(t)$  uniformly on bounded intervals. In particular, if  $b \equiv b' \equiv a' \equiv a'' = 0$ , then  $\theta_n(t)$  converges to  $\theta_o(t)$  which is the solution of

$$\begin{aligned}
 \text{(HE)} \quad \theta'_o(t) &= c\Delta\theta_o(t) + f(t) \\
 \theta_o(o) &= \emptyset_o
 \end{aligned}$$

We thus conclude that if  $a$  and  $b$  are small (in the sense of Theorem 6.3), the solution  $\theta(t)$  of (HVE) differs only slightly from  $\theta_o(t)$  because of the memory term.

The above discussion leads us to consider a related problem. If  $x(t, \epsilon)$  is the solution of the equation

$$\text{(VE)}_\epsilon \quad x'(t) = Ax(t) + \epsilon \int_0^t B(t-s) x(s) ds + f(t), \quad x(o) = x_o$$

we would like to compare  $x(t, \epsilon)$  with  $x(t, o)$ , the solution of

$$\text{(VE)}_o \quad x'(t) = Ax(t) + f(t), \quad x(o) = x_o$$

The corresponding differential equations are

$$\text{(DE)}_\epsilon \quad z' = C(\epsilon)z$$

and

$$\text{(DE)}_o \quad z' = C(o)z$$

where

$$C(\epsilon) = \begin{bmatrix} 0 & A & 0 \\ 0 & A & \delta_0 \\ 0 & 0 & D_S \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & B & 0 \end{bmatrix} \text{ and } C(0) = \begin{bmatrix} 0 & A & 0 \\ 0 & A & \delta_0 \\ 0 & 0 & D_S \end{bmatrix}$$

If we assume that  $B = FA$  where  $F : X \rightarrow \mathcal{F}$  with  $\text{Range } F \subseteq D(D_S)$  is a bounded linear operator, we may apply the argument in the proof of Theorem 6.1 to show that for  $\text{Re}\lambda > \omega_1$

$$||R^n(\lambda; C(\epsilon))|| \leq M_2 / (\text{Re}\lambda - \omega_1)^n$$

for some constants  $M_2$  and  $\omega_1$  which are independent of  $\epsilon$  as long as  $|\epsilon| \leq 1$ . As an immediate consequence of these observations and [5, Theorem 2.19, p. 507] we obtain the following result.

Theorem 6.4. Suppose  $B = FA$  where  $F : X \rightarrow D(D_S)$  is a bounded linear operator. Let  $z(t, \epsilon)$  be the solution of  $(DE)_\epsilon$  with  $z(0) = z_0 \in D(C(\epsilon))$ . Then

$$z(t, \epsilon) = z(t, 0) + \epsilon z_1(t) + o(\epsilon)$$

In addition, if  $x(t, \epsilon)$  is the solution of  $(VE)_\epsilon$  where  $(0, x_0, f)^* \in D(C(\epsilon))$ , then

$$x(t, \epsilon) = x(t, 0) + \epsilon x_1(t) + o(\epsilon)$$

Theorem 6.1. can be modified in another way so that we may obtain a more general existence theorem for  $(VE)$  and  $(DE)$ . In particular, it removes some restriction that  $F$  must map  $X$  into  $D(D_S)$ .

Theorem 6.5. Suppose  $\{A_n\}$  and  $\{A\}$  are infinitesimal generators of  $C_0$  semi-groups  $\{S_n(t)\}$  and  $\{S(t)\}$  such that  $\{A_n\}$  and  $\{A\}$  have common domain  $D(A)$  and  $A_n x \rightarrow Ax$  for every  $x \in D(A)$ . Assume there are constants  $M > 0$ ,  $\omega \geq 0$  such that  $||S_n(t)|| \leq Me^{\omega t}$  and  $||S(t)|| \leq Me^{\omega t}$ . Suppose  $R(\lambda; A + \mathcal{L}_\lambda B)$  exists for some  $\lambda$  with  $\text{Re}\lambda > \omega$  and that  $\{F_n\}$  and  $\{F\}$  are bounded linear operators with  $F_n$  mapping  $X$  into  $D(D_S)$  and  $F_n x \rightarrow Fx$  in  $\mathcal{F}$  for all  $x \in X$ . Suppose further that there exists positive constant  $N$  so that  $||F_n|| + ||D_S F_n|| \leq N$

for all  $n$ . If  $B = FA$ , then  $C$  generates a  $C_0$  semigroup.

Proof : It follows as in the proof of Theorem 6.1. that

$$\|R^n(\lambda; C_k)\| \leq M_2 / (\operatorname{Re}\lambda - \omega_1)^n \quad \operatorname{Re}\lambda > \omega_1$$

for some constants  $M_2, \omega_1$  independent of  $n, k$ . Also, as the  $F_n$  are uniformly bounded,  $C^n z \rightarrow Cz$  for every  $z \in D(C^n) = D(C)$ . It follows from Theorem 3.1. that  $R(\lambda; C)$  exists and so  $C$  must generate a semigroup  $\mathcal{J}(t)$  with  $\|\mathcal{J}(t)\| \leq M_2 e^{\omega_1 t}$  [9, p. 90]. Q.E.D.

A special case of theorem 6.5. applied to (VE) yields the following result.

Theorem 6.6. Suppose  $B(t) = a(t)A$  where  $a : \mathbb{R}^+ \rightarrow \mathbb{R}$  is bounded and uniformly Lipschitzian. If the solution of (VE) are unique when they exists, then (VE) is uniformly well-posed.

Proof : As  $B(t) = a(t)A$  with  $a(t)$  bounded,  $R(\lambda; A + \mathcal{L}_\lambda B)$  exists for all  $\lambda$  with  $\operatorname{Re}\lambda$  sufficiently large. Furthermore, as  $a(t)$  is uniformly Lipschitzian, it is the uniform limit of a sequence of functions  $a_n(t)$  where  $a'_n(t)$  is bounded and uniformly continuous. Hence, we take  $B_n(t) = a_n(t)A$  and  $F_n = a_n I$  in Theorem 6.5. to get that (DE) is uniformly well-posed. As the solutions of (VE) are unique, the results follows. Q.E.D.

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