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**A STUDY ON SOME
LINEAR EVOLUTION EQUATIONS
WITH TIME DELAY**

**Goong CHEN
Ronald GRIMMER**

Mars 1980

A STUDY ON
SOME LINEAR EVOLUTION EQUATIONS
WITH TIME DELAY

Goong Chen* - Ronald Grimmer**

Résumé : On présente différentes techniques de semi-groupe pour étudier l'équation d'évolution linéaire avec mémoire

$$(VE) \quad \begin{cases} \frac{dx(t)}{dt} = A x(t) + \int_0^t B(t-s) x(s) ds + f(t) \\ x(0) = x_0 \in X \end{cases}$$

dans un espace de Banach X.

On généralise l'approche de R.K. Miller de manière à pouvoir traiter une classe plus large d'équations. Les conditions d'existence de ces semi-groupes sont données. L'existence, l'unicité, le bien posé et l'approximation des solutions de l'équation (VE) sont alors déduites des équations différentielles et semi-groupes qui lui sont associées.

Abstract : We present here various semigroup techniques for studying the linear evolution equation with memory

$$(VE) \quad \begin{cases} \frac{dx(t)}{dt} = A x(t) + \int_0^t B(t-s) x(s) ds + f(t) \\ x(0) = x_0 \in X \end{cases}$$

in a Banach space X.

A generalization of R.K. Miller's semigroup approach is made so that a broader class of equations can be investigated by

his méthode. We determine conditions which ensure the existence of those semigroups. The existence, uniqueness, well-posedness and approximation of the equation (VE) are then derived from the associated differential equations and semigroups.

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1. INTRODUCTION

In this paper, we shall be concerned with the integrodifferential equation (VE)

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s) x(s) ds + f(t), & t \geq 0 \\ x(0) = x_0 \in X \end{cases}$$

in a Banach space X . Throughout this paper we always assume that A is the infinitesimal generator of C_0 semigroup on X and the Hille-Yosida-Phillips conditions

$$\|R^n(\lambda; A)\| \leq M/(\operatorname{Re} \lambda - \omega)^n, \quad n \geq 1$$

are satisfied for the resolvent $R(\lambda; A)$ for some $\omega \geq 0$. Also we assume that f is an element of a Banach space $\mathcal{F}$ of X -valued functions which are defined for $t \geq 0$, and that $B(t)$ is a linear operator on $D(A)$ for each $t \geq 0$ such that $B(\cdot)x \in \mathcal{F}$ for each fixed $x \in D(A)$.

Equation (VE) appears, e.g., in the modelling of heat conduction problems in materials with memory, where $A = \Delta$ is the Laplacian and the kernel $B(t)$ is basically of the form $k(t)A$ with $k \in L^1(0, \infty)$. Linear partial differential integral equations of Volterra type associated with heat conduction problems have been studied by Miller [7], see also the references therein.

In this paper, we are mostly interested in the problems of existence, uniqueness and continuity of solutions with respect to x_0 and f . We are also concerned with approximating the solutions $x(t)$ of (VE) by solutions $x_n(t)$ of

$$(VE)^n \quad \begin{cases} x'_n(t) = A_n x_n(t) + \int_0^t B_n(t-s) x_n(s) ds + f(t), & t \geq 0 \\ x_n(0) = x_0 \end{cases}$$

given that $B_n \rightarrow B$ and $A_n \rightarrow A$ in some sense.

Questions concerning existence uniqueness and well-posedness of solutions of linear Volterra integrodifferential equations in a Banach space have been examined by Miller [6], Miller and Wheeler [8], Chen and Grimmer [1], etc. In the nonlinear case, Crandall and Nohel [2]. Related work concerning integral equations appears in Grimmer and Miller [3], [4].

The approach we are following is similar to that in Miller [6], where he studied (VE) by means of the differential equation

$$(DE)' \quad \begin{cases} \frac{d}{dt} \begin{bmatrix} x(t) \\ F(t, \cdot) \end{bmatrix} = C \begin{bmatrix} x(t) \\ F(t, \cdot) \end{bmatrix} \equiv \begin{bmatrix} A & \delta_0 \\ B(\cdot) & D_s \end{bmatrix} \begin{bmatrix} x(t) \\ F(t, \cdot) \end{bmatrix}, & t \geq 0 \\ \begin{bmatrix} x(0) \\ F(0, \cdot) \end{bmatrix} = \begin{bmatrix} x_0 \\ f \end{bmatrix} \in X \times \mathcal{F} \end{cases}$$

which is a Cauchy problem on $X \times \mathcal{F}$. Here $\mathcal{F} = BU(\mathbb{R}^+; X)$ is the space of bounded uniformly continuous functions from $\mathbb{R}^+ = [0, \infty]$ into X , δ_0 is the Dirac delta function and D_s is the differentiation operator on $\mathcal{F}$. Miller proved that solution of (DE)' give solutions to (VE) ([6, Theorem 3.5]). The choice of $\mathcal{F} = BU$ is necessary ([6, §2]) since $\mathcal{F}$ must at least contain those bounded uniformly continuous X -valued functions. The proof of his theorem will not go through if the function space $\mathcal{F}$ is chosen differently, e.g., say $\mathcal{F} \equiv B^2(\mathbb{R}^+; X)$ (the space of Bochner square integrable functions on $\mathbb{R}^+$).

In §2, we first generalize Miller's scheme so that (VE) can be studied for a broader class of function spaces $\mathcal{F}$. We show that, under appropriate assumptions on B , we have the equivalence of (VE) with a new (DE).

In §3, we compute the resolvent operator $R(\lambda; c)$ and discuss some properties of the spectrum of C .

The main theorems of this paper are given in §4-6.

In §4, we study the case when $\mathcal{F} = BU(\mathbb{R}^+; X)$. We are particularly interested in the case when $B(t)$ is of the form $k(t)A$.

Existence, uniqueness and well-posedness are proved for such kernels B by using perturbation and decomposition techniques for infinitesimal generators. We also point out an error in a recent paper by J. Zabezyk.

In §5, we study the case $\mathcal{F} = B^2(\mathbb{R}^+; X)$. On this $\mathcal{F}$ the Dirac delta function δ_0 is no longer a bounded linear operator, so it becomes more difficult to derive existence theorems for the associated semigroups. We have proved an existence theorem under some assumptions on the Kernel B , by Lumer-Phillips' theorem.

We study approximations by Trotter's theory in §6.

2. THE EQUIVALENCE BETWEEN A VOLTERRA INTEGRODIFFERENTIAL EQUATION (VE) AND AN ASSOCIATED DIFFERENTIAL EQUATION (DE)

We consider the following differential equation

$$(DE) \quad \begin{cases} \frac{d}{dt} z = Cz \\ z(0) = z_0 \in D(C) \in X \times X \times \mathcal{F} \end{cases}$$

which is a Cauchy problem in the Banach space $X \times X \times \mathcal{F}$. Here

$$(2.1) \quad \begin{aligned} z &= \begin{bmatrix} w \\ x \\ y \end{bmatrix} \in X \times X \times \mathcal{F} \quad \text{with} \quad ||z||^2 = ||w||_X^2 + ||x||_X^2 + ||y||_{\mathcal{F}}^2 \\ C &= \begin{bmatrix} 0 & A_0 & 0 \\ 0 & A & \delta_0 \\ 0 & B & D_S \end{bmatrix} \end{aligned}$$

and B is the linear transformation given by $(Bx)(s) = B(s)x$, A_0 is a closed operator in X with domain $D(A_0) \subseteq D(A)$ and with resolvent $R(\lambda; A_0) = (\lambda I - A_0)^{-1}$. In our treatment, A_0 is usually a multiple of either A or perhaps some positive fractional power of A if it exists. D_s is the differentiation operator on $\mathfrak{F}$ defined by $D_s f = f'$ on a domain $D(D_s) \subseteq \mathfrak{F}$ where $f \in D(D_s)$ implies

$$f(s) = \alpha + \int_0^s e(u) du$$

for some $e \in \mathfrak{F}$ and D_s generates the translation semigroup $T(t)$ on $\mathfrak{F}$ given by $T(t) f(s) = f(t+s)$. The domain of C , $D(C)$, is $X \times D(A) \times D(D_s)$. It is routine to verify that C is a closed operator on $X \times X \times \mathfrak{F}$.

We first give some definitions and notations.

Definition 2.1. By a solution $z(t)$ of (DE) satisfying an initial condition $z(0) = z_0$ we mean a function $z : \mathbb{R}^+ \rightarrow D(C)$ with z, z' and Cz continuous and $z'(t) = Cz(t)$ for all $t \in \mathbb{R}^+$.

Definition 2.2. A solution $x(t)$ of (VE) satisfying $x(0) = x_0$ is a function $x : \mathbb{R}^+ \rightarrow D(A)$ such that x, x' and Ax are continuous and (VE) is satisfied for all $t \in \mathbb{R}^+$.

Definition 2.3. The equation (DE) is uniformly well-posed if for each $z_0 \in D(C)$ the initial value problem $z(0) = z_0$ has a unique solution $z(t, z_0)$ and for any $T > 0$ there is a $K > 0$ such that :

$$\|z(t, z_0)\| \leq K \|z_0\| \text{ for all } z_0 \in D(C) \text{ for all } t \in [0, T].$$

Definition 2.4. The equation (VE) is uniformly well-posed is for each pair (x_0, f) with $(0, x_0, f) \in D(C)$ there is a unique solution $x(t, x_0, f)$ of (VE) and for any $T > 0$ there is an $M > 0$ such that $\|x(t, x_0, f)\|_X \leq M(\|x_0\|_X + \|f\|_{\mathfrak{F}})$ for all $t \in [0, T]$.

Notations : From now on, $T(t)$ always denote the translation semigroup. We use h_s to denote the translated function $T(s)h$, i.e., $h_s(u) = h(s+u)$. $\mathcal{L}_\lambda$ denotes the Laplace transform, i.e., $\mathcal{L}_\lambda h =$

$\int_0^\infty e^{-\lambda s} h(s) ds$. We use $*$ to denote the transpose of an element in $X \times X \times \mathfrak{F}$.

In order to obtain an equivalence relation between solutions of (VE) and those of (DE), we require the following assumptions dealing with B :

(H1) $B(\cdot)x(t)$ is continuous as a function of t on $\mathbb{R}^+$ into $\mathfrak{F}$ whenever $x(t)$ and $A_0 x(t)$ are continuous on $\mathbb{R}^+$ into X .
In addition, $B(\cdot)R(\lambda, A_0)$ is a bounded operator from X into $\mathfrak{F}$.

(H2) $B(s)x$ is in $D(D_S)x$ for each fixed x in $D(A_0)$.

(H3) $D_S B(s)x(t)$ is locally integrable as a function of t whenever $x(t)$ and $A_0 x(t)$ are continuous.

(H4) $A_0 x(t)$ is continuous as a function of t whenever $Ax(t)$ is continuous as a function of t .

An example of a function $B(t)$ which may satisfy (H1)-(H3) is $B(t) = a(t)A_0$, where $a(t)$ is a scalar valued function. Consider $\mathfrak{F} = B^2(\mathbb{R}^+; X)$.

If $a \in L^2(\mathbb{R}^+)$, (H1) is satisfied. If a is absolutely continuous with $a' \in L^2(0, \infty)$, then $B(s)x = a(s)A_0 x$ is in $D(D_S)$ for each fixed $x \in D(A_0)$ and $a'(s)A_0 x(t)$ is a continuous function of t into $\mathfrak{F}$ when $A_0 x(t)$ is continuous. So (H2) and (H3) are satisfied. (H4) is satisfied if $A = -A_0^2$ and A_0 is invertible, or if $A = A_0$.

The following theorem generalizes [6, Theorem 3.5].

Theorem 2.5. Assume (H1) is valid. If $z = (w, x, y)^*$ is a solution of (DE), then $x(t)$ is a solution of (VE) with $f(t) = y(o)(t)$ and $x_o = x(o)$. Conversely, if $f \in D(D_S)$ and if (H1)-(H4) are valid and if $x(t)$ is a solution of (VE), then $(w, x, y)^*$ is a solution of (DE) with $w(t) = w_o + \int_0^t A_o x(s) ds$

$$\text{and } y(t)(s) = f(t+s) + \int_0^t B(t-\tau+s) x(\tau) d\tau$$

Proof : First assume that (H1) is valid and $z = (w, x, y)^*$ is a solution of (DE). Then w, w', x, x', y and y' are all continuous as functions of t from $\mathbb{R}^+$ into either X or $\mathcal{Y}$. As the equation

$$y'(t) = D_S y(t) + B(s) x(t) ; y(o) = y_o, \quad t \geq 0$$

has a solution, it follows from [9, p. 110] or [5, p. 488] that the solution is given by

$$y(t) = T(t) y_o + \int_0^t T(t-\tau) B(.) x(\tau) d\tau$$

where $T(t)$ is the semigroup generated by D_S , i.e., $T(t)$ is the translation semigroup. Hence if $y(o) = f$, we see that

$$y(t)(s) = f(t+s) + \int_0^t B(t-\tau+s) x(\tau) d\tau$$

As $y \in D(D_S)$, $y(t, .)$ is absolutely continuous and so

$$y(t)(o) = f(t) + \int_0^t B(t-\tau) x(\tau) d\tau$$

is continuous in t by (H1). Now $x'(t) = Ax(t) + y(t)(o)$ so $Ax(t)$ is continuous. Therefore

$$x'(t) = Ax(t) + \int_0^t B(t-\tau) x(\tau) d\tau + f(t)$$

is a solution of (VE).

Conversely, if (H1)-(H4) are valid and $x(t)$ is a solution of (VE) with $f \in D(D_S)$, then $x(t)$ and $Ax(t)$ are continuous so that $B(\cdot)x(t)$, which is in $D(D_S)$ for each t , is continuous in t and $D_S B(s)x(t)$ is locally integrable as a function of t . Thus, the equation

$$y'(t) = D_S y(t) + B(s)x(t)$$

has as its solution [9, p. 112]

$$y(t)(s) = f(t+s) + \int_0^t B(t-\tau+s) x(\tau) d\tau$$

and in particular,

$$y(t)(0) = f(t) + \int_0^t B(t-\tau) x(\tau) d\tau$$

so

$$x'(t) = Ax(t) + \delta_0 y(t)$$

Thus, if $w(t) = w_0 + \int_0^t A_0 x(s) ds$, $z \equiv (w, x, y)^*$ is a solution of $z' = Cz$ with $z(0) = (w_0, x_0, y_0)$. Q.E.D.

We note that, under the assumptions (H1)-(H4), if the solution of (DE) are unique, then the solutions of (VE) with $(0, x_0, f) \in D(C)$ are unique when they exist. Similarly, if the solutions of (VE) are unique for $(0, x_0, f) \in D(C)$ then the solutions of (DE) must also be unique. It follows that if C generates a C_0 semigroup, then (VE) is uniformly well-posed. Thus we will be concerned with conditions which ensure that C generates a C_0 semigroup in the subsequent sections.

3. RESOLVENTS AND SPECTRUM OF THE OPERATOR C

Let C be given as in (2.1). The following computation of the resolvent

in a generalization of [6, Theorem 4.1].

Theorem 3.1 For any λ with $\text{Re } \lambda > 0$, $R(\lambda; C)$ exists if and if only $R(x, A + \mathcal{L}_\lambda B)$ exists. If $R(\lambda; C)$ exists, it is given by

$$(3.1) \quad R(\lambda; C) = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} A_0 R(\lambda; A + \mathcal{L}_\lambda B) & \frac{1}{\lambda} A_0 R(\lambda; A + \mathcal{L}_\lambda B) \mathcal{L}_\lambda \\ 0 & R(\lambda; A + \mathcal{L}_\lambda B) & R(\lambda; A + \mathcal{L}_\lambda B) \mathcal{L}_\lambda \\ 0 & R(\lambda; D_s) B R(\lambda; A + \mathcal{L}_\lambda B) & R(\lambda; D_s) [I + B R(\lambda; A + \mathcal{L}_\lambda B)] \mathcal{L}_\lambda \end{bmatrix}$$

Proof : Assume $R(\lambda; C)$ exists. Then for any $(f, g, h) \in X \times X \times \mathcal{F}$, the equation

$$\begin{bmatrix} \lambda I & -A_0 & 0 \\ 0 & \lambda I - A & -\delta_0 \\ 0 & -B & \lambda I - D_s \end{bmatrix} \begin{bmatrix} w \\ x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$$

is always solvable with solution (w, x, y) . Thus

$$\begin{cases} \lambda w - A_0 x = f \\ (\lambda I - A)x - \delta_0 y = g \\ -Bx + (\lambda I - D_s)y = h \end{cases}$$

so

$$\begin{aligned} y &= R(\lambda; D_s) (h + Bx) \\ \text{i.e.} \quad y(s) &= \int_0^\infty e^{-\lambda u} T(u) [h(s) + B(s)x] du \\ &= \int_0^\infty e^{-\lambda u} [h_u(s) + B_u(s)x] du \\ &= \int_0^\infty e^{-\lambda u} [h_s(u) + B_s(u)x] du \end{aligned}$$

Therefore

$$(3.2) \quad \delta_o y = \int_0^{\infty} e^{-\lambda u} [h(u) + B(u)x] du = \mathcal{L}_{\lambda}^{-1} (h + Bx)$$

$$\begin{aligned} \text{Also, } X &= R(\lambda; A) [g + \delta_o y] \\ &= R(\lambda; A) [g + \mathcal{L}_{\lambda}^{-1} (h + Bx)] \quad (\text{from (3.2.)}) \end{aligned}$$

$$\text{so} \quad [I - R(\lambda; A) \mathcal{L}_{\lambda}^{-1} B] x = R(\lambda; A) (g + \mathcal{L}_{\lambda}^{-1} h)$$

the above relation is always invertible with solution

$$x = [I - R(\lambda; A) \mathcal{L}_{\lambda}^{-1} B]^{-1} R(\lambda; A) (g + \mathcal{L}_{\lambda}^{-1} h)$$

$$\text{But} \quad [I - R(\lambda; A) \mathcal{L}_{\lambda}^{-1} B]^{-1} R(\lambda; A) = (\lambda I - A - \mathcal{L}_{\lambda}^{-1} B)^{-1} = R(\lambda; A + \mathcal{L}_{\lambda}^{-1} B)$$

Hence $R(\lambda; A + \mathcal{L}_{\lambda}^{-1} B)$ exists.

The first component w is given by

$$\begin{aligned} w &= \frac{1}{\lambda} (f + A_o x) \\ &= \frac{1}{\lambda} [f + A_o R(\lambda; A + \mathcal{L}_{\lambda}^{-1} B) (g + \mathcal{L}_{\lambda}^{-1} h)] \end{aligned}$$

We note that $A_o R(\lambda; A + \mathcal{L}_{\lambda}^{-1} B)$ is a bounded operator by the closed graph theorem.

Conversely, if $R(\lambda; A + \mathcal{L}_{\lambda}^{-1} B)$ exists. Because each of the above steps is reversible, $R(\lambda; C)$ must also exist and is given by (3.1). Q.E.D.

We note, in particular, that $R(\lambda; A + \mathcal{L}_{\lambda}^{-1} B)$ exists provided that $\mathcal{L}_{\lambda}^{-1} B$ is a bounded operator on X and $\text{Re } \lambda > \omega + M \|\mathcal{L}_{\lambda}^{-1} B\|$. Having found $R(\lambda; C)$, one can proceed with matrix multiplications to find the iterated resolvent $R^n(\lambda; C)$. Because $R^n(\lambda; C)$ does not have a simple representation, it is in

general very difficult to verify whether the Hille-Yosida-Phillips criterion is satisfied. This makes any attempt impractical to directly prove that C is infinitesimal generator.

From the appearance of C , we know that its spectrum depends on the spectrum of A and D_S as well as the behavior of B . We refer the readers to [10] for the definitions of resolvent and point, continuous and residue spectrum, which we denote by ρ , $p\sigma$, $c\sigma$ and $r\sigma$, respectively.

Different $\mathcal{F}$ s give different spectrum for D_S . For example, if $\mathcal{F} \equiv BU(\mathbb{R}^+; X)$, then

$$\begin{aligned}\rho(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > 0\} \\ p\sigma(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda \leq 0\} \\ c\sigma(D_S) &= r\sigma(D_S) = \emptyset\end{aligned}$$

but if $\mathcal{F} \equiv B^2(\mathbb{R}^+; X)$, then

$$\begin{aligned}\rho(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda > 0\} \\ p\sigma(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda < 0\} \\ c\sigma(D_S) &= \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda = 0\} \\ r\sigma(D_S) &= \emptyset\end{aligned}$$

The spectrum of the operator C can usually be classified by a careful computation:

An important subset of the spectrum of C , called the essential spectrum $e\sigma(C)$, merits special attention. There are many non-equivalent definitions of $e\sigma$. We will use the one given in [5].

The following theorem indicates the invariance of the essential spectrum under the perturbation by B when B and A have certain properties.

Theorem 3.2. Let C_1 denote the operator

$$\begin{bmatrix} 0 & A_0 & 0 \\ 0 & A & \delta_0 \\ 0 & 0 & D_S \end{bmatrix}$$

on $X \times X \times \mathcal{Y}$. Assume that for some λ , A has a compact resolvent $R(\lambda; A)$. If the operator $\mathcal{B}$ defined by $\mathcal{B}x \equiv B(\cdot)x$ is a bounded linear operator from X into $\mathcal{Y}$, then for (i) $\mathcal{Y} \equiv BU(\mathbb{R}^+; X)$ or (ii) $\mathcal{Y} \equiv B^2(\mathbb{R}^+; X)$ and X = a Hilbert space, C has the same essential spectrum as C_1 .

Proof : We want to show that the bounded operator $\bar{B}$ defined by

$$\bar{B}(w, x, y)^* \equiv (0, 0, \mathcal{B}x)^* \in X \times X \times \mathcal{Y}$$

is C_1 -compact ([5]).

Let $\{(w_n, x_n, y_n)\}$ be a sequence in $D(C_1)$ ($\equiv D(C)$) bounded in $X \times X \times \mathcal{Y}$ such that $\{C_1(w_n, x_n, y_n)^* = (A_0 x_n, Ax_n + \delta_0 y_n, y'_n)\}$ is bounded. (i) $\mathcal{Y} \equiv BU(\mathbb{R}^+; X)$: this implies that $\{Ax_n\}$ is bounded in X , so $\{(\lambda I - A)x_n\}$ is also bounded in X . Hence

$$x_n = (\lambda I - A)^{-1} (\lambda I - A)x_n$$

has a convergent subsequence, which we still denote by $\{x_n\}$. Thus

$$\|\bar{B}(w_n, x_n, y_n) - \bar{B}(w_m, x_m, y_m)\|_{X \times X \times \mathcal{Y}}$$

$$= \sup_{t \in \mathbb{R}^+} \|B(t)x_n - B(t)x_m\|_X \leq \|\mathcal{B}\|_{\mathcal{L}(X, \mathcal{Y})} \|x_n - x_m\|_X \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

Therefore $\bar{B}$ is C_1 -compact.

(ii) $\mathcal{Y} \equiv B^2(\mathbb{R}^+; X)$: since both $\{y_n\}$ and $\{y'_n\}$ are bounded, so is

$$\frac{1}{2} \|\delta_0 y_n\|_X^2 = |\langle y'_n, y_n \rangle_{\mathcal{Y} \times \mathcal{Y}}|$$

Thus $\{Ax_n\} (= \{(Ax_n + \delta_0 y_n) - \delta_0 y_n\})$ as a sum of two bounded sequences is also bounded. So is $\{(\lambda I - A)x_n\}$. Hence

$$x_n = (\lambda I - A)^{-1} (\lambda I - A)x_n$$

has a convergent subsequence which we still denote by $\{x_n\}$ as before. Now

$$\begin{aligned} & ||\bar{B}(w_n, x_n, y_n) - \bar{B}(w_m, x_m, y_m)||^2_{X \times X \times \mathcal{F}} \\ &= \int_0^\infty ||B(t)(x_n - x_m)||^2_X dt \leq ||\mathcal{B}||^2_{\mathcal{L}(X, \mathcal{F})} ||x_n - x_m||^2 \rightarrow 0 \text{ as } n, m \rightarrow \infty \end{aligned}$$

Therefore $\bar{B}$ is also C_1 -compact.

Since $C = C_1 + \bar{B}$, by [5, Theorem 5.35] the proof is complete. Q.E.D.

Remark : For any bounded invertible operator, P , $P^{-1}C_1P$ has the same spectrum classification as C_1 , as $R(\lambda; P^{-1}C_1P) = P^{-1}R(\lambda; C_1)P$ is true for $\lambda \in \rho(C_1)$. This observation will be useful in the subsequent proof of theorem 4.1.

4. THEOREMS ON EXISTENCE, UNIQUENESS AND CONTINUITY(I) : $\mathcal{F} \equiv BU(R^+; X)$

Throughout this section we assume that $\mathcal{F} \equiv BU(R^+; X)$ and $A_0 = A$. For this $\mathcal{F}$, (DE) may also be treated in the simpler setting $X \times \mathcal{F}$ such as in [1], [6]. It is easy to see in the subsequent treatment that any results valid in the setting $X \times \mathcal{F}$ are also valid in $X \times X \times \mathcal{F}$, and vice versa.

Theorem 4.1. Suppose B can be written as $B = FA + K$ where $F : X \rightarrow \mathcal{F}$ with range $F \equiv D(D_S)$ and $K : X \rightarrow \mathcal{F}$ are bounded linear operators. Then C generates a C_0 semigroup on $X \times X \times \mathcal{F}$.

Remark : The above theorem holds for any $\mathcal{F}$ such that δ_0 is a bounded operator from $\mathcal{F}$ into X .

Proof : We first note that the operator on $X \times X \times \mathcal{F}$ given by

$$C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & D_S \end{bmatrix}$$

generates a C_0 semigroup on $X \times X \times \mathcal{F}$. The operator on $X \times X \times \mathcal{F}$ given by

$$P = \begin{bmatrix} I_X & -I_X & 0 \\ 0 & I_X & 0 \\ 0 & -F & I_{\mathcal{F}} \end{bmatrix}$$

is invertible with inverse

$$P^{-1} = \begin{bmatrix} I_X & I_X & 0 \\ 0 & I_X & 0 \\ 0 & F & I_{\mathcal{F}} \end{bmatrix}$$

and for any $(w, x, y) \in P^{-1}(D(C_2)) = P^{-1}(X \times D(A) \times D(D_S))$,

$$\begin{aligned} P^{-1}C_2P(w, x, y)^* &= P^{-1}C_2(w, x, y, -Fx + y) \\ &= P^{-1}(0, Ax, D_S(-Fx + y)) \\ &= (Ax, Ax, FAx + D_S(-Fx + y)) = (Ax, Ax, FAx - D_SFx + D_Sy) \end{aligned}$$

Since F maps X into $D(D_S)$, we have $P^{-1}(X \times D(A) \times D(D_S)) = X \times D(A) \times D(D_S)$. Also, D_SF is a closed operator from X into $\mathcal{F}$ with domain X . By the closed graph theorem, D_SF is a bounded operator. Thus

$$P^{-1}C_2P = \begin{bmatrix} 0 & A & 0 \\ 0 & A & 0 \\ 0 & FA - D_SF & D_S \end{bmatrix}$$

we must remark that in general the above is not true if F does not map X into $D(D_S)$. Now

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_0 \\ 0 & D_S F + K & 0 \end{bmatrix}$$

is a bounded operator, so $P^{-1}C_2P + E = C$ is an infinitesimal generator of a C_0 -semigroup [9, P. 50] or [5, P. 497] with $D(C) = X \times D(A) \times D(D_S)$ Q.E.D.

Corollary 4.2. (Hille-Yosida-Phillips' conditions). Let A satisfy the Hille-Yosida-Phillips conditions $\|R^n(\lambda; A)\| \leq M(\operatorname{Re} \lambda - \omega)^{-n}$ for some $\omega \geq 0$ and let $\alpha_1, \alpha_2, \alpha_3$ denote

$$\alpha_1 = \|F\|_{\mathcal{L}(X, \mathfrak{F})}, \alpha_2 = \|D_S F\|_{\mathcal{L}(X, \mathfrak{F})}, \alpha_3 = \|K\|_{\mathcal{L}(X, \mathfrak{F})}$$

Then the iterated resolvent $R^n(\lambda; C)$ is bounded by

$$\|R^n(\lambda; C)\|_{\mathcal{L}(X \times X \times \mathfrak{F})} \leq \frac{M(2 + \alpha_1)^2}{[\operatorname{Re} \lambda - \omega - M(2 + \alpha_1)^2(1 + \alpha_2 + \alpha_3)]^n} \quad \text{all } n \in \mathbb{Z}^+$$

if $\operatorname{Re} \lambda$ is large enough.

Proof. Since $R(\lambda; P^{-1}C_2P) = P^{-1}R(\lambda; C_2)P$, therefore

$$\begin{aligned} R^n(\lambda; P^{-1}C_2P) &= P^{-1}R^n(\lambda; C_2)P \\ \text{and } \|R^n(\lambda; P^{-1}C_2P)\| &\leq \|P^{-1}\| \|R^n(\lambda; C_2)\| \|P\| \\ &\leq \frac{M(2 + \alpha_1)^2}{[\operatorname{Re} \lambda - \omega]^n} \quad (\because \|P\| \leq 2 + \alpha_1, \|P^{-1}\| \leq 2 + \alpha_1) \end{aligned}$$

$$\text{Now } C = P^{-1}C_2P + E$$

From The proof of [9, Theorem 3.1.1.], one easily sees that

$$\|R^n(\lambda; C)\| \leq \frac{M(2 + \alpha_1)^2}{[\operatorname{Re} \lambda - \omega - M(2 + \alpha_1)^2(1 + \alpha_2 + \alpha_3)]^n}$$

$$\text{because } \|E\|_{\mathcal{L}(X \times X \times \mathfrak{F})} \leq 1 + \alpha_2 + \alpha_3.$$

Q.E.D.

Corollary 4.3. Suppose (H1)-(H4) are valid with $B(t) = a_1(t)A + a_2(t)I$, where a_1, a'_1, a_2 and a'_2 are bounded uniformly continuous scalar functions on $\mathbb{R}^+$. The integral equation (VE) is uniformly well-posed for (x_0, f) with $(0, x_0, f)$ in $D(C)$.

We remark that this corollary is essentially Miller's Theorem 7.3. [6] which was obtained via greatly different techniques. Other results, similar to those in [6, §7] follow in a similar manner.

Actually, a much more general result follows from Theorem 4.1. As A generates a C_0 semi-group, A has a resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ for λ with $\operatorname{Re} \lambda$ sufficiently large. As in [6, p. 181], Miller noted that if $x(t)$ satisfies (VE), then $\gamma(t)$ defined by $\gamma(t) = x(t) \exp(-\lambda_0 t)$ ($\lambda_0 > 0$) satisfies the equation

$$(VE)_\lambda \gamma'(t) = (A - \lambda_0 I) \gamma(t) + \int_0^t \exp(-\lambda_0(t-s)) B(t-s) \gamma(s) ds + \exp(-\lambda_0 t) f(t)$$

We may then, without any essential loss of generality, assume
(WLOG₁) A has a bounded inverse A^{-1} , or
(WLOG₂) A generates a uniformly bounded semigroup. (By changing the norm on X , we can actually assume that A generates a contraction semigroup).

Corollary 4.4. Under the convention of (WLOG₁), assume (H1)-(H4) are valid for $\mathcal{F} = BU(\mathbb{R}^+; X)$ with $A_0 = A$ and assume furthermore

$$\|B(t)x\| \leq \beta (\|x\| + \|Ax\|), \quad \text{for } x \in D(A), \quad \beta > 0$$

Then the integral equation (VE) is uniformly well-posed for (x_0, f) with $(0, x_0, f) \in D(C)$.

Proof : Define $F : X \rightarrow D(D_S)$ by $F = BA^{-1}$. Then $B = FA$ and

$$\|Fx\|_{BU(\mathbb{R}^+; X)} = \sup_{t \in \mathbb{R}^+} \|B(t)A^{-1}x\| \leq \beta (\|A^{-1}x\| + \|x\|)$$

So F is a bounded operator. The proof follows from theorem 4.1. Q.E.D.

The techniques used in Theorem 4.1. were motivated by results obtained recently by Zabczyk [11]. In fact, it appears at first that a slight modification of Zabczyk's theorem 1 part 2 would allow us to obtain our result even if F does not map X into the domain of D_S . This is not the case as Zabczyk's result is not correct. First of all the computations at lines 18 and 19 on [11, p. 525] gives $\frac{1}{t} \frac{d}{dt} S(t)x_0$ rather than $\frac{d}{dt} S(t)x_0$. Secondly, we have a counterexample which shows that :

$$\begin{bmatrix} I & F \\ 0 & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

need not generate a semigroup if F does not map into the domain of A . Consider the operator

$$\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}, \quad \text{i.e., } A \equiv B \equiv D$$

which generates a semigroup on $X \times X$ if D generates a semigroup $S(t)$ on X . Now consider the operator

$$\begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \quad \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix} \equiv A$$

If Zabczyk's result were correct, A would generate a C_0 semigroup on $X \times X$. Now choose $x_0 \in D(D)$ which has the property that $S(t)Dx_0 \notin D(D)$ for any $t > 0$. Then $(0, x_0)^* \in D(A)$ and there must be a classical solution of $w' = Aw$, $w = (x, y)^*$ through this point if A generates a C_0 semigroup. This solution must have second coordinate $S(t)x_0$. The first coordinate must satisfy

$$\begin{cases} y' = Dy + DS(t)x_0 \\ y(0) = y \end{cases}$$

However, as shown in Pazy [9, p.111] this problem has no solution for if y where a solution then $y(t)$ must satisfy

$$y(t) = \int_0^t S(t-s) DS(s)x_0 ds = t S(t) Dx_0$$

This is impossible, however, as $S(t) D x_0$ differentiable would mean $S(t) D x_0 \in D(D)$. Interestingly, the operator D which yields the easiest such example is $D = D_s$ on $X \equiv BU(\mathbb{R}^+; R)$. Choosing x_0 to be a function with only one derivative in $BU(\mathbb{R}^+; R)$, we are clear that translating the function will not smooth the function in general.

We restate a corrected version of Zabczyk's result in the following, which can be proven as above or will follow from his Theorem 1 a(1). All of his subsequent related results must also be modified accordingly.

Theorem 4.5. Let X and Y be Banach spaces and A and B generate C_0 semigroups on X and Y respectively. If $F : Y \rightarrow D(A)$ then

$$\begin{bmatrix} I & F \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

Also generates a semigroup.

An additional existence theorem will be given in §6 (Theorem 6.5.').

5. THEOREMS ON EXISTENCE, UNIQUENESS AND CONTINUITY (II) : $\mathcal{F} = B^2(\mathbb{R}^+; X)$

If the Dirac delta function is not a bounded operator on $\mathcal{F}$ into X , different techniques must be used. This is the case when $\mathcal{F} = B^2(\mathbb{R}^+; X)$ of course. Our results in this direction are not as general because the unbounded operator δ_0 must also be dealt with. What we have obtained here are similar to Theorems 3 and 4 of our earlier work [1] for the case $\mathcal{F} = BU(\mathbb{R}^+; X)$.

. Throughout this section, we will follow the convention (WLOG₂) in §4.

Theorem 5.1. Let X be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let A be the generator of a contraction semigroup on X and let $\mathcal{F} = B^2(\mathbb{R}^+; X)$ with the usual norm. Suppose $B(\cdot) : X \rightarrow \mathcal{F}$ is defined on the domain of A at least and $\|B(\cdot)x\|^2 \leq -M_1 \langle Ax, x \rangle$ and $\|A_0 x\|_X^2 \leq -M_2 \langle Ax, x \rangle$ for all $x \in D(A)$ for some positive constants M_1, M_2 . Assume that $R(\lambda + \alpha; A + \mathcal{L}_\lambda B)$ exists as a bounded operator on X for some $\lambda > 0$ and some α with $2\alpha > M_1 + M_2$. Then C generates a C_0 semigroup $\|S(t)\|$ with $\|S(t)\| \leq e^{\alpha t}$.

Proof : Consider the operator

$$\begin{bmatrix} -\alpha I_X & A_0 & 0 \\ 0 & A - \alpha I_X & \delta_0 \\ 0 & B & D_S - \alpha I_{\mathcal{Y}} \end{bmatrix}$$

on $X \times D(A) \times D(D_S)$. We wish to employ the theorem of Lumer-Phillips [9] for the operator C_α . We first show that C_α is dissipative. Using $\langle \cdot, \cdot \rangle$ as inner product in each space X , $\mathcal{Y}$ and $X \times X \times \mathcal{Y}$, we see that if $z = (w, x, y) \in D(C_\alpha)$,

$$\begin{aligned} \langle C_\alpha z, z \rangle &= \langle -\alpha w + A_0 x, w \rangle_X + \langle (A - \alpha I)x + \delta_0 y, x \rangle_X + \langle Bx + (D_S - \alpha I)y, y \rangle_{\mathcal{Y}} \\ &= -\alpha (||w||^2 + ||x||^2 + ||y||^2) + \langle A_0 x, w \rangle + \langle Ax, x \rangle + \langle \delta_0 y, x \rangle + \langle Bx, y \rangle + \langle D_S y, y \rangle \\ &\leq -\alpha (||w||^2 + ||x||^2 + ||y||^2) + ||A_0 x|| ||w|| + \langle Ax, x \rangle + ||\delta_0 y|| ||x|| + ||B(\cdot)x||_{\mathcal{Y}} ||y||_{\mathcal{Y}} + \langle D_S y, y \rangle \end{aligned}$$

$$\text{Now } \langle D_S y, y \rangle = -||\delta_0 y||_X^2 / 2 \text{ and}$$

$$||Bx||_{\mathcal{Y}} ||y||_{\mathcal{Y}} \leq (\epsilon^2 ||Bx||_{\mathcal{Y}}^2 + \epsilon^{-2} ||y||_{\mathcal{Y}}^2) / 2$$

Similarly ,

$$||A_0 x|| ||w|| \leq (\epsilon^2 ||A_0 x||^2 + \epsilon^{-2} ||w||^2) / 2$$

Choosing ϵ sufficiently small and $\alpha = \max ((\epsilon^{-2} + 1)/2, (M_1 + M_2 + 1)/2)$, we will obtain $\langle C_\alpha z, z \rangle \leq 0$. So C_α is a dissipative operator.

Now consider $\lambda I - C_\alpha$. We want to show that the range of $\lambda I - C_\alpha$ is $X \times X \times \mathcal{Y}$ for some $\lambda > 0$. But

$$(\lambda I - C_\alpha)^{-1} = ((\lambda + \alpha)I - C^{-1}) = R(\lambda + \alpha; C)$$

exists provided that $R(\lambda + \alpha; A + \mathcal{L}_\lambda B)$ exists by theorem 3.1.

Therefore C_α generates a contraction semigroup and, hence, C generates a semigroup $S(t)$ with $\|S(t)\| \leq e^{\alpha t}$. Q.E.D.

Applying this to our original problem, we have

Corollary 5.2. Let X, A and $\mathfrak{F}$ be as above. Suppose (H1)-(H4) are valid with $\|B(t)x\|_X^2 \leq -b(t) \langle Ax, x \rangle$ a.e. $\mathbb{R}^+$ for some $b \in L^1(\mathbb{R}^+)$ and $\|A_0 x\|^2 \leq -M \langle Ax, x \rangle$ for some $M > 0$. Assume $R(\lambda + \alpha; A + \mathcal{L}_\lambda B)$ exists as a bounded operator on X for some $\lambda > 0$ and some α with $\alpha > 1/2(M + \int_0^\infty b(t)dt)$. Then (VE) is uniformly well posed for (x_0, f) where $(0, x_0, f) \in D(C)$.

6. APPROXIMATIONS

In this section we will be concerned with approximating the solutions of (VE) by those of $(VE)_n$. A result of this kind has been obtained in our earlier work [1, Theorem 5]. Here we will study this problem under the general setting of this paper. Our results are motivated by a close examination of the proofs of Theorems 4.1 and 5.1.

First, we consider the differential equations

$$(DE)_n \quad z'_n = C^n z_n, \quad z_n(0) = z(0) \in X \times X \times \mathfrak{F}$$

where

$$C^n \equiv \begin{bmatrix} 0 & A_n & 0 \\ 0 & A_n & \delta_0 \\ 0 & F_n A_n & D_s \end{bmatrix}$$

$$\mathfrak{F} \equiv BU(\mathbb{R}^+; X)$$

We shall assume that A_n generates a C_0 semigroup on W and $F_n : X \rightarrow D(D_S)$ for all n . Theorem 4.1. implies that each operator C^n generates a C_0 semigroup.

Theorem 6.1. Suppose $\{A_n\}$ and A are infinitesimal generators of C_0 semigroups $\{S_n(t)\}$ and $\{S(t)\}$ such that $\{A_n\}$ and A are defined on a common domain $D(A)$ and $A_n x \rightarrow Ax$ for every $x \in D(A)$. Suppose there are constants $M > 0$, $\omega \geq 0$ such that $\|S_n(t)\| \leq Me^{\omega t}$ and $\|S(t)\| \leq Me^{\omega t}$. Suppose further that F_n and F are bounded linear operators mapping X into $D(D_S)$ such that $F_n x \rightarrow Fx$ and $D_S F_n x \rightarrow D_S Fx$ in $\mathcal{Y}$ for all $x \in X$. Then with $B \equiv FA$, if $z_n(0) = z(0) = z_0$ is in $D(C) = D(C^n)$, we have $z_n(t) \rightarrow z(t)$ as $n \rightarrow \infty$ for all $t \geq 0$ and the convergence is uniform on bounded t intervals.

Proof: We first note that $C = P^{-1}C_2P+Q$ and $C^n = P_n^{-1}C_2^n P_n + Q_n$ where

$$C_2^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_n & 0 \\ 0 & 0 & D_S \end{bmatrix}$$

$$P = \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & -F & I \end{bmatrix}, \quad P_n = \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & -F_n & I \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_0 \\ 0 & D_S F & 0 \end{bmatrix}, \quad Q_n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_0 \\ 0 & D_S F_n & 0 \end{bmatrix}$$

Our assumptions have the effect of making P , P_n , Q and Q_n bounded operators. In fact, by the uniform boundedness principle, $P_n \rightarrow P$, $P_n^{-1} \rightarrow P^{-1}$ and $Q_n \rightarrow Q$ in the uniform operator topology and there is a uniform bound for $\|P_n^{-1}\|, \|P^{-1}\|, \|P_n\|, \|P\|, \|Q_n\|$ and $\|Q\|$; call it M_1 .

We choose λ with $\operatorname{Re} \lambda > \omega + MM_1^3$. From [9. p. 50], we see that $\{C_n\}$ and $\{C\}$ generates semigroups $\{S_n(t)\}$ and $\{S(t)\}$ satisfying

$$||S_n(t)|| \leq MM_1^2 e^{(\omega + MM_1^3)t}, \quad ||S(t)|| \leq MM_1^2 e^{(\omega + MM_1^3)t}$$

with the resolvent conditions

$$||R^n(\lambda; C_n)|| \leq \frac{MM_1^2}{(\operatorname{Re} \lambda - \omega - MM_1^3)^n}$$

$$||R^n(\lambda; C) || \leq \frac{MM_1^2}{(\operatorname{Re} \lambda - \omega - MM_1^3)^n}$$

Now we want to show that for all $z \in X \times X \times \mathcal{F}$, $R(\lambda; C_n) z \rightarrow R(\lambda; C)z$ as $n \rightarrow \infty$. Since $R(\lambda; C_n) = P^{-1}R(\lambda; C_2^n + P_n Q_n P_n^{-1}) P_n$ and $R(\lambda; C) = P^{-1}R(\lambda; C_2 + PQP^{-1})P$, this is equivalent to showing that $R(\lambda; C_2^n + P_n Q_n P_n^{-1})z \rightarrow R(\lambda; C_2 + PQP^{-1})z$ for all z . Let

$$k = R(\lambda; C_2 + PQP^{-1})z, \quad k \in D(C)$$

and let
$$z_n \equiv [\lambda I - (C_2^n + P_n Q_n P_n^{-1})]^{-1} k$$

From the given assumptions, we see immediately that z_n tends to z as $n \rightarrow \infty$.

Now

$$R(\lambda; C_2^n + P_n Q_n P_n^{-1})z = R(\lambda; C_2^n + P_n Q_n P_n^{-1})z_n$$

$$+ R(\lambda; C_2^n + P_n Q_n P_n^{-1})(z - z_n)$$

and

$$\lim_{n \rightarrow \infty} |R(\lambda; C_2^n + P_n Q_n P_n^{-1})(z - z_n)|$$

$$\leq \lim_{n \rightarrow \infty} \frac{M}{(\operatorname{Re} \lambda - \omega - MM_1^3)} |z - z_n| = 0$$

$$\begin{aligned} \text{Hence} \quad \lim_{n \rightarrow \infty} R(\lambda; C_2^{n+P} Q_n P_n^{-1}) z &= \lim_{n \rightarrow \infty} R(\lambda; C_2^{n+P} Q_n P_n^{-1}) z_n \\ &= k = R(\lambda; C_2 + P Q P^{-1}) z \end{aligned}$$

It now follows from Trotter's approximation theorem [9, p. 57] or [5, p. 504] that the proof is complete. Q.E.D.

Corresponding to the case $\mathcal{F} = B^2(\mathbb{R}^+; X)$ and Theorem 5.1, we consider

$$(DE)_n \quad z'_n = C^n z_n, \quad z_n(0) = z(0)$$

$$C^n = \begin{bmatrix} 0 & A_o^{(n)} & 0 \\ 0 & A_n & \delta_o \\ 0 & B_n & D_s \end{bmatrix}$$

and obtain the following similar result.

Theorem 6.2. Let X be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{F} = B^2(\mathbb{R}^+; X)$ be equipped with the usual norm. Assume that all the assumptions of Theorem 5.1. are satisfied for each triple $(A_n, B_n, A_o^{(n)})$ and (A, B, A_o) with the same constants M_1, M_2 and α . Assume furthermore $D(A_n) = D(A)$, $D(A_o^{(n)}) = D(A_o)$ and for all $x \in D(A)$, $A_n x \rightarrow A x$, $A_o^{(n)} x \rightarrow A_o x$ in X and $B_n x \rightarrow B_x$ in $\mathcal{F}$. If $z_n(0) = z(0) = z_o$ is in $D(C) = D(C^n)$, we have $z_n(t) \rightarrow z(t)$ as $n \rightarrow \infty$ for all $t \geq 0$ and the convergence is uniform on bounded t intervals.

Proof. Note first that each of the operators C^n generates a semigroup $S_n(t)$ with $\|S_n(t)\| \leq e^{\alpha t}$. We are thus able to argue as in the previous theorem that because $C^n z \rightarrow C z$ for $z \in D(C)$ that $R(\lambda; C^n) z \rightarrow R(\lambda; C) z$ for all $z \in X \times X \times \mathcal{F}$. The proof again follows from the Trotter approximation theorem.

Theorems 6.1 and 6.2 have immediate application to our stated objective of obtaining results which ensure that the solution $x_n(t)$ of $(VE)_n$ tends to the solution $x(t)$ of (VE) . Corresponding to theorem 6.1. we are able to

obtain the following result.

Theorem 6.3. Let (H1) - (H4) be valid for $\mathcal{F} = BU(\mathbb{R}^+, X)$ and $A_0 = A$. Suppose A_n and A are the generators of C_0 semigroups $\{S_n(t)\}$ and $\{S(t)\}$ respectively and that $\|S_n(t)\| \leq Me^{\omega t}$, $\|S(t)\| \leq Me^{\omega t}$ for some constants $M > 0$, $\omega > 0$. Suppose that the operators A_n and A have common domain $D(A)$ and that $A_n x \rightarrow Ax$ for every $x \in D(A)$. Also, suppose that $B_n(\cdot)x \rightarrow B(\cdot)x$ in $BU(\mathbb{R}^+; X)$ for every $x \in D(A)$ and that $\|B_n(\cdot)x\|_{BU} \leq \beta(\|x\| + \|A_n x\|)$, $\|B'_n(\cdot)x\| \leq \beta(\|x\| + \|A_n x\|)$ for all $x \in D(A)$ for some positive constant β for all n . Then for $(0, x_0, f) \in D(C)$, we have $x_n(t) \xrightarrow{n \rightarrow \infty} x(t)$ pointwise in t for all $t \geq 0$. The convergence is uniform on bounded t intervals.

Proof : We first argue in a similar fashion as in the proof of Theorem 6.1., we obtain

$$(6.1) \quad R(\lambda; A_n)y \rightarrow R(\lambda; A)y \quad n \rightarrow \infty \quad y \in X, \quad \forall \lambda > \omega$$

Now, instead of considering $(VE)_n$ and (VE) , we consider $(VE)_{n,\lambda}$ and $(VE)_n$. We make the factorization

$$\begin{aligned} B_{n,\lambda} &\equiv \exp(-\lambda S)B_n = F_n(A_n - \lambda I), \text{ with } F_n \equiv -B_{n,\lambda} R(\lambda; A_n) \\ B_{\lambda} &\equiv \exp(-\lambda S)B = F(A - \lambda I), \text{ with } F \equiv -B_{\lambda} R(\lambda; A) \end{aligned}$$

For $x \in X$, we have $R(\lambda; A)x \in D(A)$, $R(\lambda; A_n)x \in D(A)$ and

$$(6.2) \quad \begin{aligned} \|F_n x - Fx\|_{BU} &\leq \|B_{\lambda} R(\lambda; A)x - B_{n,\lambda} R(\lambda; A)x\| + \\ &\quad + \|B_{n,\lambda} R(\lambda; A)x - B_{n,\lambda} R(\lambda; A_n)x\| \end{aligned}$$

The first term on the right of (6.2) vanishes as $n \rightarrow \infty$. Consider the second term

$$\begin{aligned} \|B_{n,\lambda} R(\lambda;A)x - B_{n,\lambda} R(\lambda;A_n)x\| &\leq \beta (\|R(\lambda;A)x - R(\lambda;A_n)x\| + \\ &+ \|A_n R(\lambda;A)x - A_n R(\lambda;A_n)x\|) \end{aligned}$$

Using (6.1), we see the vanishing of the first term on the right as $n \rightarrow \infty$. And

$$\begin{aligned} \|A_n R(\lambda;A)x - A_n R(\lambda;A_n)x\| &\leq \|A_n R(\lambda;A)x - AR(\lambda;A)x\| + \|AR(\lambda;A)x - A_n R(\lambda;A_n)x\| \\ &= \|A_n R(\lambda;A)x - AR(\lambda;A)x\| + \|\lambda R(\lambda;A)x - \lambda R(\lambda;A_n)x\| \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

We thus obtain $F_n x \rightarrow Fx$ for every $x \in X$.

Similarly, we can prove that $D_S F_n \rightarrow D_S Fx \quad \forall x \in X$.

By theorem 6.1, the proof is complete.

Q.E.D.

Theorem 6.3. has an interesting application to integrodifferential equations arising in the study of heat conduction in materials with memory. In particular, Miller [7] examined the equation

$$\begin{aligned} \text{(HVE)} \quad \theta'(t) &= c \Delta \theta(t) - a(o) \theta(t) + c \int_0^t b(t-\tau) \Delta \theta(\tau) d\tau - \\ &\quad - \int_0^t a'(t-\tau) \theta(\tau) d\tau + f(t) \\ \theta(o) &= \theta_o \end{aligned}$$

where Δ is the Laplacian, $c > 0$, a' and b are continuously differentiable real valued functions in $L^1(\mathbb{R}^+)$. Assumptions made regarding the set Ω on which the Laplacian is considered and on the boundary conditions make $A \equiv c\Delta - a(o)I$ a generator of a C_0 semigroup on $L^p(\Omega)$, $1 < p < \infty$. If instead of (HVE) we consider

$$\begin{aligned}
 \text{(HVE)}_n \quad \theta'_n(t) &= c\Delta\theta_n(t) - a_n(o)\theta_n(t) + c \int_0^t b_n(t-\tau)\Delta\theta_n(\tau)d\tau - \\
 &\quad - \int_0^t a'_n(t-\tau)\theta_n(\tau)d\tau + f(t) \\
 \theta_n(o) &= \emptyset
 \end{aligned}$$

we see that Theorem 6.3. can be immediately applied. If the assumptions $b_n \rightarrow b$, $b'_n \rightarrow b'$, $a_n \rightarrow a'$ and $a''_n \rightarrow a''$ in $BU(R^+)$ are satisfied, we see that $\theta_n(t)$ converges to $\theta(t)$ uniformly on bounded intervals. In particular, if $b \equiv b' \equiv a' \equiv a'' = 0$, then $\theta_n(t)$ converges to $\theta_o(t)$ which is the solution of

$$\begin{aligned}
 \text{(HE)} \quad \theta'_o(t) &= c\Delta\theta_o(t) + f(t) \\
 \theta_o(o) &= \emptyset_o
 \end{aligned}$$

We thus conclude that if a and b are small (in the sense of Theorem 6.3), the solution $\theta(t)$ of (HVE) differs only slightly from $\theta_o(t)$ because of the memory term.

The above discussion leads us to consider a related problem. If $x(t, \epsilon)$ is the solution of the equation

$$\text{(VE)}_\epsilon \quad x'(t) = Ax(t) + \epsilon \int_0^t B(t-s) x(s) ds + f(t), \quad x(o) = x_o$$

we would like to compare $x(t, \epsilon)$ with $x(t, o)$, the solution of

$$\text{(VE)}_o \quad x'(t) = Ax(t) + f(t), \quad x(o) = x_o$$

The corresponding differential equations are

$$\text{(DE)}_\epsilon \quad z' = C(\epsilon)z$$

and

$$\text{(DE)}_o \quad z' = C(o)z$$

where

$$C(\epsilon) = \begin{bmatrix} 0 & A & 0 \\ 0 & A & \delta_0 \\ 0 & 0 & D_S \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & B & 0 \end{bmatrix} \text{ and } C(0) = \begin{bmatrix} 0 & A & 0 \\ 0 & A & \delta_0 \\ 0 & 0 & D_S \end{bmatrix}$$

If we assume that $B = FA$ where $F : X \rightarrow \mathcal{F}$ with $\text{Range } F \subseteq D(D_S)$ is a bounded linear operator, we may apply the argument in the proof of Theorem 6.1 to show that for $\text{Re } \lambda > \omega_1$

$$||R^n(\lambda; C(\epsilon))|| \leq M_2 / (\text{Re } \lambda - \omega_1)^n$$

for some constants M_2 and ω_1 which are independent of ϵ as long as $|\epsilon| \leq 1$. As an immediate consequence of these observations and [5, Theorem 2.19, p. 507] we obtain the following result.

Theorem 6.4. Suppose $B = FA$ where $F : X \rightarrow D(D_S)$ is a bounded linear operator. Let $z(t, \epsilon)$ be the solution of $(DE)_\epsilon$ with $z(0) = z_0 \in D(C(\epsilon))$. Then

$$z(t, \epsilon) = z(t, 0) + \epsilon z_1(t) + o(\epsilon)$$

In addition, if $x(t, \epsilon)$ is the solution of $(VE)_\epsilon$ where $(0, x_0, f)^* \in D(C(\epsilon))$, then

$$x(t, \epsilon) = x(t, 0) + \epsilon x_1(t) + o(\epsilon)$$

Theorem 6.1. can be modified in another way so that we may obtain a more general existence theorem for (VE) and (DE) . In particular, it removes some restriction that F must map X into $D(D_S)$.

Theorem 6.5. Suppose $\{A_n\}$ and $\{A\}$ are infinitesimal generators of C_0 semi-groups $\{S_n(t)\}$ and $\{S(t)\}$ such that $\{A_n\}$ and $\{A\}$ have common domain $D(A)$ and $A_n x \rightarrow Ax$ for every $x \in D(A)$. Assume there are constants $M > 0$, $\omega \geq 0$ such that $||S_n(t)|| \leq Me^{\omega t}$ and $||S(t)|| \leq Me^{\omega t}$. Suppose $R(\lambda; A + \mathcal{L}_\lambda B)$ exists for some λ with $\text{Re } \lambda > \omega$ and that $\{F_n\}$ and $\{F\}$ are bounded linear operators with F_n mapping X into $D(D_S)$ and $F_n x \rightarrow Fx$ in $\mathcal{F}$ for all $x \in X$. Suppose further that there exists positive constant N so that $||F_n|| + ||D_S F_n|| \leq N$

for all n . If $B = FA$, then C generates a C_0 semigroup.

Proof : It follows as in the proof of Theorem 6.1. that

$$||R^n(\lambda; C_k)|| \leq M_2 / (\operatorname{Re} \lambda - \omega_1)^n \quad \operatorname{Re} \lambda > \omega_1$$

for some constants M_2, ω_1 independent of n, k . Also, as the F_n are uniformly bounded, $C^n z \rightarrow Cz$ for every $z \in D(C^n) = D(C)$. It follows from Theorem 3.1. that $R(\lambda; C)$ exists and so C must generate a semigroup $\mathcal{J}(t)$ with $||\mathcal{J}(t)|| \leq M_2 e^{\omega_1 t}$ [9, p. 90]. Q.E.D.

A special case of theorem 6.5. applied to (VE) yields the following result.

Theorem 6.6. Suppose $B(t) = a(t)A$ where $a : \mathbb{R}^+ \rightarrow \mathbb{R}$ is bounded and uniformly Lipschitzian. If the solution of (VE) are unique when they exists, then (VE) is uniformly well-posed.

Proof : As $B(t) = a(t)A$ with $a(t)$ bounded, $R(\lambda; A + \mathcal{L}_\lambda B)$ exists for all λ with $\operatorname{Re} \lambda$ sufficiently large. Furthermore, as $a(t)$ is uniformly Lipschitzian, it is the uniform limit of a sequence of functions $a_n(t)$ where $a'_n(t)$ is bounded and uniformly continuous. Hence, we take $B_n(t) = a_n(t)A$ and $F_n = a_n I$ in Theorem 6.5. to get that (DE) is uniformly well-posed. As the solutions of (VE) are unique, the results follows. Q.E.D.

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