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**PERFECT  
ZERO-ONE MATRICES**

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# PERFECT ZERO-ONE MATRICES

by

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## ABSTRACT

A zero-one matrix is called perfect if the polytope of the associated set packing problem has integral vertices only. By this definition, all totally unimodular zero-one matrices are perfect. In this paper we give a characterization of perfect zero-one matrices in terms of forbidden submatrices. The notion of a perfect zero-one matrix is closely related to that one of a perfect graph as well as that one of a "balanced" matrix as introduced by BERGE. Furthermore, the results obtained here bear on an unsolved problem in graph theory, the strong perfect graph conjecture due to C. BERGE.

## RESUME

Nous disons que la matrice  $A$  aux éléments zéro-un est parfaite si le polytope  $\{x \in \mathbb{R}^n \mid Ax \leq e, x \geq 0\}$  n'a que des sommets entiers, où le vecteur  $e$  a toutes ses composantes égales à 1. Toutes les matrices totalement unimodulaires sont donc parfaites. Dans cet article nous donnons une caractérisation des matrices parfaites en termes de sous-matrices interdites. Le concept de matrice parfaite est relié au concept de graphe parfait et aussi à celui de BERGE d'une matrice équilibrée. En effet, les matrices parfaites sont fournies par les matrices d'incidence d'un hypergraphe normal. Les résultats obtenus ici sont reliés à la conjecture forte de BERGE concernant les graphes parfaits.

## 1. INTRODUCTION

In this paper, we consider the polytope defined by the constraints of the following set packing problem :

$$\begin{aligned} & \max c x \\ (P) \quad & A x \leq e \\ & x_j = 0 \text{ or } 1 \quad \forall j \in N = \{1, \dots, n\} \end{aligned}$$

where  $A$  is a  $m \times n$  matrix of zeroes and ones having no zero columns,  $e^T = (1, \dots, 1)$  is the vector having all  $m$  components equal to one, and  $c$  is an arbitrary vector of reals. This problem has recently obtained much attention, see e.g. [1], [2], [6], [14], [18]. By (LP) we denote the linear programming problem obtained from (P) by dropping the integrality requirement on  $x$ .

If the matrix  $A$  involved in problem (P) is totally unimodular [10], then all basic feasible solutions to (LP) are integral, i.e. for any vector  $c$  the integer programming problem (P) can be solved as an ordinary linear programming problem. Generally, the matrix  $A$  encountered in (P) is not totally unimodular. Nevertheless, for certain matrices  $A$  the property that all basic feasible solutions to (LP) are integral, remains true (see Section 2 for relevant examples, also [5]). We call such matrices perfect zero-one matrices. Using some results from graph theory, we give a complete characterization of perfect matrices  $A$  in terms of forbidden submatrices. We give examples that show that -like in the case of balanced matrices, see [5]- it is not possible to characterize perfect zero-one matrices by means of forbidden determinantal values (as is the case for totally unimodular matrices). Indeed, given any natural number  $k$ , there exists a perfect zero-one matrix  $A$  such that has a minor with determinant  $k$ .

This paper in an abbreviated version of the original paper [15] ; see also [16] where the main result of this paper is derived using a different proof techniques.

## 2. PERFECT ZERO-ONE MATRICES

Let  $A$  be any  $m \times n$  matrix of zeroes and ones having no zero column, and define the polytopes  $P$  and  $P_I$  as follows :

$$(2.1) \quad P = \{x \in \mathbb{R}^n \mid Ax \leq e, x_j \geq 0, j = 1, \dots, n\}$$

$$P_I = \text{conv} \{x \in \mathbb{R}^n \mid Ax \leq e, x_j = 0 \text{ or } 1, j = 1, \dots, n\}$$

where  $e^T = (1, \dots, 1)$  has  $m$  components all equal to one. The matrix  $A$  is called perfect if  $P = P_I$ , i.e. if the polytope  $P$  defined in (2.1) has only integral vertices. Note that  $\dim P = \dim P_I = n$  holds. Denote  $G$  the (intersection) graph associated with the matrix  $A$ , i.e. the nodes of  $G$  correspond to the columns of  $A$  and two nodes of  $G$  are linked by an edge if the associated columns  $a^i$  and  $a^j$  of  $A$  have at least one + 1 entry in common, see e.g. [14]. A clique in  $G$  is a maximal complete subgraph of  $G$ . Let  $C$  denote the node set of any clique in  $G$ . Then by Theorem 2.1. of [14], the inequality

$$(2.2) \quad \sum_{j=1}^n a_j^c x_j \leq 1, a_j^c = \begin{cases} 1 & \text{if } j \in C \\ 0 & \text{if not} \end{cases}$$

yields a facet of  $P_I$ , i.e. a face of dimension  $n-1$  of  $P_I$ . Clearly, every facet of the polytope  $P_I$  is essential in defining  $P_I$ . Hence it is a necessary condition for  $A$  to be perfect that  $A$  contain the incidence (row-) vectors of all cliques of the associated graph  $G$ . In order to characterize perfect matrices we can thus restrict ourselves to considering "clique"-matrices, i.e. matrices  $A$  which contain the incidence vectors of all cliques of the associated graph  $G$ .

Let  $A$  be any clique-matrix of size  $m \times n$  and let  $G$  be the associated graph. Let  $\bar{G}$  denote the complement of  $G$  and denote  $B$  a clique matrix of  $\bar{G}$ . Similarly to (2.1) define the polytopes  $Q$  and  $Q_I$ , respectively as follows :

$$(2.3) \quad Q = \{x \in \mathbb{R}^n \mid Bx \leq \hat{e}, x_j \geq 0, j = 1, \dots, n\}$$

$$Q_I = \text{conv} \{x \in \mathbb{R}^n \mid Bx \leq \hat{e}, x_j = 0 \text{ or } 1, j = 1, \dots, n\}$$

where  $\hat{e}^T = (1, \dots, 1)$  has all components equal to one and is dimensioned compatibly with  $B$ . Note that  $B$  has no zero column and that  $\dim Q = \dim Q_I = n$  holds. The vertices of  $Q_I$  correspond to complete subgraphs of  $G$  and vice versa. Furthermore, every maximal independent node set in  $G$  defines a clique of  $\bar{G}$  (and vice versa). Consequently, there exists a (incomplete) "duality" relation between the vertices of  $P_I$  and the facets of  $Q$  (and hence, between the vertices of  $Q_I$  and the facets of  $P$ ), see e.g. [11]. In the terminology of Fulkerson [7],  $Q(P, \text{resp.})$  is the anti-blocker of  $P_I(Q_I, \text{resp.})$ .

Let  $A$  be any clique-matrix of size  $m \times n$ . By Theorem 1 of [6],  $A$  is perfect if and only if the associated graph  $G$  is perfect. Consequently,  $A$  is imperfect if and only if the graph  $G$  contains an induced subgraph of  $G'$  which is almost perfect, i.e.  $G'$  is imperfect, but every proper induced subgraph of  $G'$  is perfect, see [13], [15], [17]. Since induced subgraphs of  $G$  having  $k$  nodes correspond uniquely to  $m \times k$  submatrices of  $A$  (and vice versa) we can make, without loss of generality, the assumption that  $G$  is an almost perfect graph. Denote  $\alpha(G)$  the maximum cardinality of a stable (independent) node set in  $G$  and define  $\alpha(\bar{G})$  likewise for  $\bar{G}$ .

Lemma 1 : Let  $A$  be a clique-matrix of an almost perfect graph  $G, \alpha = \alpha(G)$  and  $\bar{\alpha} = \alpha(\bar{G})$ . Then  $\sum_{j=1}^n x_j \leq \bar{\alpha}$  provides a facet of  $Q_I$  and  $x = (1/\bar{\alpha})e$  is a fractional vertex of  $P$ .

Proof : Denote by  $e^j$  the row vector with  $n-1$  components equal to +1 and having a zero in the  $j$ -th component. Since  $G$  is almost perfect it follows Theorem 1 of [13] that :

$$\max_{x \in P} e^j x = \alpha \text{ for } j = 1, \dots, n.$$

Define  $H_j$  to be the halfspace given by

$$H_j = \{x \in R^n \mid e^j x \leq \alpha\} \text{ for } j = 1, \dots, n.$$

Consequently  $P \subseteq H_j$  for  $j = 1, \dots, n$  or equivalently,

$$P \subseteq H = \bigcap_{j=1}^n H_j$$

By definition of  $e^j$ ,  $j = 1, \dots, n$ ,  $H$  is a pointed polyhedral convex cone with its apex at  $\bar{x} = \frac{\alpha}{n-1} e$ . By Theorem 1 of [13], we have that  $\alpha(G) = \alpha(\bar{G}) = n - 1$  holds, since  $G$  is almost perfect. Hence it follows that :

$$A\bar{x} \leq \frac{\bar{\alpha} \alpha}{n-1} e = e.$$

Consequently, since  $H$  is pointed,  $\bar{x} = \frac{\alpha}{n-1} e = \frac{1}{n} \frac{\alpha}{\bar{\alpha}} e$  is a vertex of  $P$ , satisfying  $0 < \bar{x} < e$ . Consequently, the inequality  $\sum_{j=1}^n x_j \leq \bar{\alpha}$  provides a facet of  $Q_I$ . Q.E.D.

Remark 1 : By Theorem 1 of [12] it follows that  $G$  is almost perfect if and only if  $\bar{G}$  is almost perfect. Consequently, we have that  $\bar{x} = \frac{1}{\bar{\alpha}} e$  is a fractional vertex of  $Q$  and  $\sum_{j=1}^n x_j \leq \bar{\alpha}$  provides a facet for  $P_I$ .

Remark 2 : Since  $\bar{x} = \frac{1}{\bar{\alpha}} e$  is a vertex of  $P$ , we can reorder the rows of the matrix  $A$  as follows :

$$A = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix}$$

where the row sums of  $\tilde{A}_1$  all equal  $\bar{\alpha}$  and the row sums of  $\tilde{A}_2$  are all (strictly) less than  $\bar{\alpha}$ . Furthermore,  $\tilde{A}_1$  is of size  $\hat{m} \times n$  with  $m \geq \hat{m} \geq n$  and  $\tilde{A}_1$  contains a (at least one) nonsingular submatrix of size  $n \times n$ . By Remark 1, we have a similar partitioning of  $B$  into  $\tilde{B}_1$  and  $\tilde{B}_2$ , where  $\tilde{B}_1$  has  $\tilde{m}$  rows,  $\tilde{m} > n$ , having row sums equal to  $\alpha$  and  $\tilde{B}_1$  contains a  $n \times n$  nonsingular submatrix.

Remark 3 : In graphical notation, Remark 2 implies that every almost perfect graph  $G$  contains at least  $|G|$  maximum cliques of the cardinality  $\bar{\alpha} = \alpha(\bar{G})$ . This has been observed earlier by Sachs and can be found, though without proof, in [17].

For  $j = 1, \dots, n$  define :

$$(2.4) \quad P_j = P \cap \{x \in \mathbb{R}^n \mid x_j = 0\}$$

and let  $Q_j$  be defined analogously with  $P$  replaced by  $Q$ . Intersecting  $P$  with  $x_j = 0$  corresponds to the operation of deleting from the graph  $G$  node  $j$  and all edges incident to it. Hence we have

Remark 4 : If  $A$  is a clique-matrix of an almost perfect graph  $G$ , then  $P_j \subseteq P_I$  ( $Q_j \subseteq Q_I$ , resp.) holds for  $j = 1, \dots, n$ .

Lemma 2 : Let  $A$  be a  $m \times n$  clique-matrix of an almost perfect graph  $G$ , then  $A$  contains a  $n \times n$  nonsingular submatrix  $A_1$  whose column and row sums are all equal to  $\bar{\alpha} = \alpha(\bar{G})$ . Furthermore, any row of  $A$  which is not in  $A_1$  is either component-wise identical to some row of  $A_1$ , or has a row sum strictly less than  $\bar{\alpha}$ .

Proof : By Remark 1, the point  $\bar{x} = (1/\alpha)e$  is a fractional vertex of  $Q$ . The point  $x^j = (1/\alpha)e^j$  satisfies  $x^j \in Q_j$  for  $j = 1, \dots, n$  where  $e^j$  is the vector defined in the proof of Lemma 1. Since  $Q_j \subseteq Q_I$  holds for  $j = 1, \dots, n$  it follows by convexity that  $\tilde{x} \in Q_I$  where

$$(2.5) \quad \tilde{x} = (1/n) \sum_{j=1}^n x^j = (\bar{\alpha}/n)e$$

Since  $\sum_{j=1}^n \tilde{x}_j = \bar{\alpha}$  holds it follows from Carathéodory's theorem [11] that there exists a  $n \times n$  nonsingular submatrix  $A_1$  of the matrix  $\tilde{A}_1$  defined in Remark 2 such that :

$$(2.6) \quad \tilde{x}^T = \gamma^T A_1, \gamma_i \geq 0 \text{ for } i = 1, \dots, n \text{ and } \sum_{i=1}^n \gamma_i = 1$$

holds. Let  $B_1$  be any  $n \times n$  nonsingular submatrix of the matrix  $\tilde{B}_1$  defined in Remark 2. Then we obtain from (2.6) :

$$\gamma^T A_1 B_1^T = \tilde{x}^T B_1^T = \frac{\bar{\alpha}}{n} e^T B_1^T = \frac{n-1}{n} e^T$$

Define  $D = A_1 B_1^T$ . Then  $D$  is a  $n \times n$  nonsingular matrix of zeroes and ones since the columns of  $B_1^T$  are (a subset of) vertices of  $P_I$  (satisfying  $e^T x \leq \alpha$  with equality). Furthermore,  $D$  cannot contain a row consisting of +1 entries only. For, if there were such a row, then necessarily  $B_1$  would have to be singular since the row sums of  $B_1$  all equal  $\alpha$ . Then we have that :



$$(2.7) \quad \gamma^T D = \frac{n-1}{n} e^T$$

where  $\gamma_i \geq 0$  for  $i = 1, \dots, n$  and  $\sum_{i=1}^n \gamma_i = 1$ . From (2.7) we obtain :

$$\gamma^T D e = n - 1$$

Consequently,  $\gamma_i > 0$  implies that the row sum of row  $i$  of  $D$  equals  $n-1$ , since  $\sum_{i=1}^n \gamma_i = 1$ . Suppose now that  $\gamma_1 \leq \dots \leq \gamma_k$  with  $k < n$  satisfies  $\gamma_1 > 0$ ,  $\gamma_{k+1} = \dots = \gamma_n = 0$ . Since  $D$  is nonsingular we can rearrange the rows and columns of  $D$  such that  $D$  has the form

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

with

$$D_1 = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix} \quad D_2 = \begin{pmatrix} 1 & \dots & 1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & \dots & 1 \end{pmatrix}$$

where  $D_1$  is of size  $k \times k$  and has zeroes only in the main diagonal,  $D_2$  is of size  $(n-k) \times (n-k)$  and consists entirely of ones. If  $k < n$ , obviously, (2.7) cannot have a solution satisfying  $\gamma_i \geq 0$  and  $\sum_{i=1}^n \gamma_i = 1$ . On the other hand  $\sum_{i=1}^n \gamma_i = 1$  implies  $k \geq 1$ . Consequently,  $k = n$  and  $D$  has the general form  $E - R$ , where  $E$  is the  $n \times n$  matrix consisting entirely of ones and  $R$  is a  $n \times n$  permutation matrix. Consequently, from (2.7) we have that

$$\gamma^T = \frac{n-1}{n} e^T D^{-1} = \frac{1}{n} e^T.$$

To complete the proof of Lemma 2, we note that the relation  $A_1 B_1^T = E - R$  implies that :

$$A_1^{-1} = \frac{\alpha}{n-1} E - B_1^T R^T.$$

Suppose now that the matrix A contains a row  $a^T$  satisfying  $a^T e = \bar{\alpha}$  which is not contained in the submatrix  $A_1$ . Then we have :

$$(2.8) \quad 0 \leq a^T A_1^{-1} = e^T - a^T B_1^T R^T = Z \leq e^T.$$

From  $Z e = 1$  and the integrality of Z it follows that such row  $a^T$  is componentwise identical to some row in  $A_1$  and thus Lemma 2 follows. Q.E.D.

Remark 5 : In graphical notation, Lemma 2 implies that every almost perfect graph G has exactly  $|G|$  maximum cliques of the cardinality  $\bar{\alpha} = \alpha(\bar{G})$  and  $|G|$  maximal independent node sets of the cardinality  $\alpha = \alpha(G)$ .

Lemma 2 suggests the following definition :

Definition : Let A be a zero-one matrix of size  $m \times n$ ,  $m \geq n$ . A is said to have property  $\pi_{\beta, n}$  if the following conditions are met :

- (i) A contains a  $n \times n$  nonsingular submatrix  $A_1$  whose row and column sums are all equal to  $\beta$ .
- (ii) Each row of A which is not a row of  $A_1$  either is componentwise equal to some row of  $A_1$  or has a row sum strictly less than  $\beta$ .

Remark 6 : Let A be a  $m \times n$  matrix of zeroes and ones and G its associated intersection graph. A is a clique-matrix, i.e. A contains as rows vectors the incidence vectors of all cliques in G, if and only if A does not contain any  $m \times k$  submatrix  $A'$  having the property  $\pi_{\beta, k}$  with  $\beta = k-1$  and  $\beta \geq 2$ .

We will sketch the proof of Remark 6 only, since the assertion of Remark 6 is probably a known result. The necessity of the condition is obvious. To prove sufficiency, let C be the node set of a clique in G such that the associated incidence vector  $a^T$  with  $a_j = 1$  if  $j \in C$ ,  $a_j = 0$  otherwise, is not contained in A. Define  $P' = P \cap \{x \in R^n \mid x_j = 0, \forall j \in N - C\}$  and  $P'_j = P' \cap \{x \in R^n \mid x_j = 0\}$  for

all  $j \in C$ . Obviously,  $P'$  must have at least one fractional vertex. We now distinguish two cases : (i)  $P'_j$  has integral vertices only for all  $j \in C$  or (ii) there exists a  $j \in C$  such that  $P'_j$  has a fractional vertex. In case (i), we can show by an argument completely analogous to the one used in the proof of Lemma 1 that  $A$  contains a  $m \times k$  submatrix  $A'$  having property  $\pi_{\beta,k}$  with  $\beta = k - 1$ ,  $\beta \geq 2$  and  $k = |C|$ . In case (ii), we can restrict attention to any  $P'_j$  having a fractional vertex for  $j \in C$ . Abusing slightly the notation, we redefine  $C$  to be  $C - \{j\}$ , redefine the polytopes  $P'$ ,  $P'_j$  with respect to the new set  $C$  and find again the two cases mentioned above. (Note that the new set  $C$  defines a complete subgraph in  $G$ , which is no longer a clique in  $G$ . This, however, does not affect the argument). Clearly, case (ii) can happen only finitely many times and finally, we obtain a set  $C$  having at least three elements and which is such that case (i) prevails. This completes the out-line of the proof of Remark 6.

The following theorem states a necessary and sufficient condition for an arbitrary zero-one matrix to be perfect :

Theorem 1 : Let  $A$  be any zero-one matrix of size  $m \times n$ . The following two conditions are equivalent :

- (i)  $A$  is perfect.
- (ii) For  $\beta \geq 2$  and  $3 \leq k \leq n$ ,  $A$  does not contain any  $m \times k$  submatrix  $A'$  having the property  $\pi_{\beta,k}$ .

Proof : Suppose that  $A$  is perfect and that (ii) is violated. Then there exists a  $m \times k$  submatrix  $A'$  of  $A$  having property  $\pi_{\beta,k}$  for some  $\beta \geq 2$  and  $k \geq 3$ . Suppose the columns of  $A$  have been ordered such that  $A'$  coincides with the  $k$  first columns. Then  $\bar{x}$  defined by  $\bar{x}_j = \frac{1}{\beta}$  for  $j = 1, \dots, k$ ,  $\bar{x}_j = 0$  for  $j = k+1, \dots, n$ , is a fractional vertex of the polytope  $P$  defined in (2.1). Hence, by definition,  $A$  cannot be perfect.

On the other hand, suppose that  $A$  is such that (ii) holds. Then  $A$  must be perfect. For if not, then by Remark 6 and Theorem 1 of [6] the intersection graph  $G$  associated with  $A$  must be imperfect.. Consequently  $G$  contains an induced subgraph  $G'$  that is almost perfect. Again by Remark 6, the clique-matrix of  $G'$  is a  $m' \times k$  submatrix  $A'$  of  $A$  where  $k = |G'|$ . Let  $A''$  denote the  $m' \times k$  submatrix of  $A'$  whose rows correspond to the cliques of  $G'$ . By Lemma 2,  $A''$  has the property

$\pi_{\beta,k}$  with  $\beta = \alpha(\bar{G}') \geq 2$ . Since the  $m-m'$  truncated rows of  $A$  not contained in  $A''$  are dominated by some row in  $A''$ , the  $m \times k$  submatrix  $A'$  of  $A$  must also have property  $\pi_{\beta,k}$  with  $\beta = \alpha(\bar{G}')$ . Thus (ii) cannot be satisfied by  $A$ . Q.E.D.

Corollary 1 : Let  $A$  be a zero-one matrix of size  $m \times n$ .  $A$  is almost perfect, i.e.  $A$  is the clique-matrix of an almost perfect graph  $G$ , if and only if the following two conditions are met :

- (i)  $A$  has the property  $\pi_{\beta,k}$  for some  $\beta$  satisfying  $2 \leq \beta \leq \lfloor \frac{n}{2} \rfloor$  and  $k = n$ .
- (ii)  $A$  does not contain any  $m \times k$  submatrix having property  $\pi_{\beta,k}$  for  $\beta \geq 2$  and  $3 \leq k \leq n-1$ .

Remark 7. The strong perfect graph conjecture [2], [3], if true, now is reduced to proving that the only zero-one matrix  $A$  of size  $m \times n$  satisfying the conditions (i) and (ii) of Corollary 1 and  $2 \leq \beta < \lfloor \frac{n}{2} \rfloor$  is the circulant of odd size having exactly two positive entries in every row and column, i.e.  $A$  is the clique-matrix of an odd cycle without chords. A characterization of almost perfect matrices in graphical terms appears advantageous if one wants to check the perfection of a zero-one matrix. For, similarly to the criterion for the total unimodularity of a matrix  $A$ , a direct check of the perfection of a zero-one matrix via the necessary and sufficient criterion of Theorem 1 is -computationally- an impossible task, whereas graphical criteria -at least in the context of total unimodularity- are relatively easily verified or known to be satisfied by the physical conditions of a problem under consideration.

Example 1 : Since a zero-one matrix  $A$  is perfect if it is the clique-matrix of a perfect graph (and vice versa), the clique-matrices of perfect graphs furnish examples of zero-one matrices satisfying the condition (ii) of Theorem 1. Among the graphs known to be perfect are the rigid circuit graphs [7] (or "triangulated" graphs [2]), the comparability graphs and the "i-triangulated" graphs, see e.g. [2]. The example of rigid circuit graphs provides examples of zero-one matrices which are perfect, but not totally unimodular. (I am indebted to D.R. Fulkerson for this example).

Example 2 : Consider the graph  $G$  of Figure 1 and its associated clique-matrix  $A$  in Figure 2. The submatrix  $A'$  made up of columns 1, 2, 3 and 4 and rows 1, 6, 11 and 16 has a determinant of 3. Consequently,  $A$  is not totally unimodular. Due to the simple structure of  $A$ , we find by inspection that  $A$  is perfect. (Checking Condition (ii) of Theorem 1) amounts to proving that  $G$  does not contain an odd cycle without chords). Furthermore, the matrix  $A$  provides us with an example where  $A^T$ , the transpose of a perfect matrix, is not perfect. This is remarkable since it is different from total unimodularity. The above example can be generalized to prove that given any natural number  $k$  there exists a perfect matrix  $A$  having a minor whose determinant in absolute values equals  $k$ . (Replace  $K_4$  by  $K_{k+1}$  and the  $G_i$ ,  $i = 1, \dots, 4$ , by  $k+1$  copies of  $G_1$ , say, each of which is connected to  $K_{k+1}$  in a similar fashion as done above). This indicates why a characterization of perfect matrices in terms of forbidden matrices is appropriate rather than a characterization in terms of forbidden subdeterminants (which is not possible here, but possible for totally unimodular matrices).

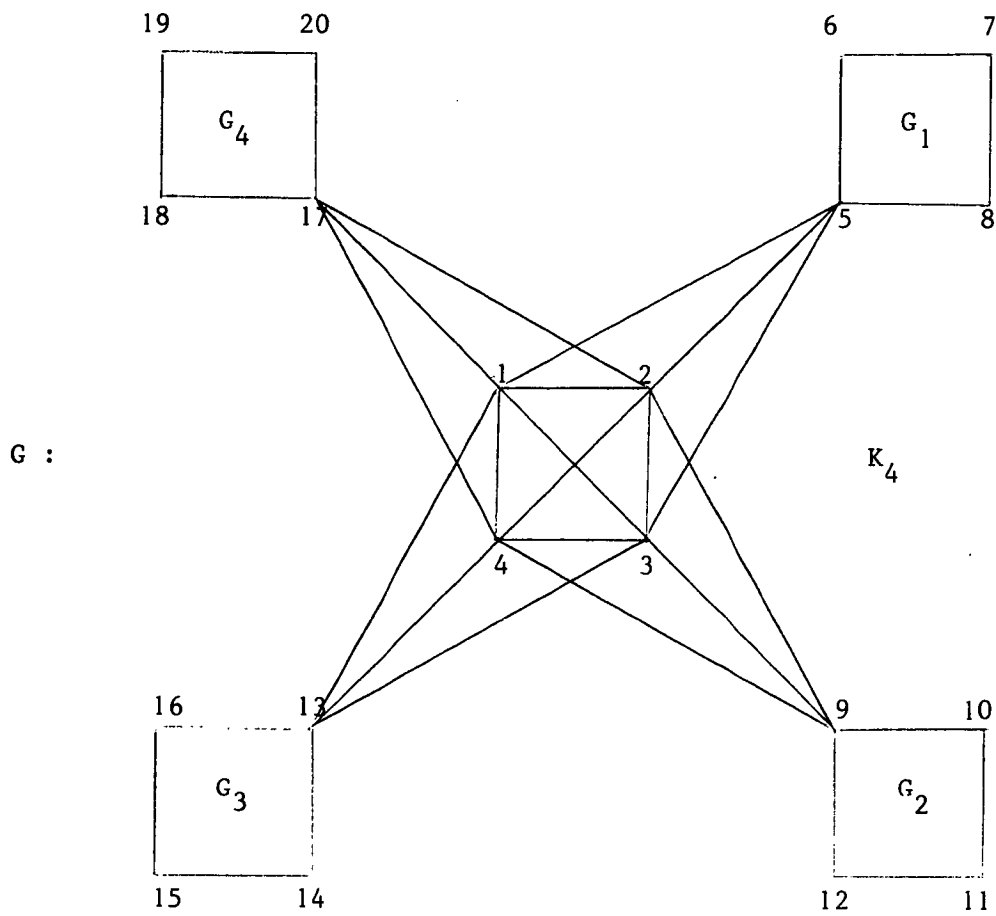


FIGURE 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
A =	1	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 2

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