



**HAL**  
open science

# The fluid-dynamic limit of a model Boltzmann equation in the presence of a shock

R. Caflish

► **To cite this version:**

R. Caflish. The fluid-dynamic limit of a model Boltzmann equation in the presence of a shock. RR-0081, INRIA. 1981. inria-00076480

**HAL Id: inria-00076480**

**<https://inria.hal.science/inria-00076480>**

Submitted on 24 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The logo for IRIA (Institut National de Recherche en Informatique et en Automatique) is displayed in a large, bold, white font against a dark, textured background. The letters are stylized and interconnected.

CENTRE DE ROCQUENCOURT

Institut National  
de Recherche  
en Informatique  
et en Automatique

Domaine de Voluceau  
Rocquencourt  
B.P.105  
78153 Le Chesnay Cedex  
France  
Tél. 954 90 20

Rapports de Recherche

N° 81

**THE FLUID-DYNAMIC LIMIT  
OF A MODEL  
BOLTZMANN EQUATION  
IN THE PRESENCE OF A SHOCK**

**Russel CAFLISCH**

Juin 1981

THE FLUID-DYNAMIC LIMIT OF A MODEL BOLTZMANN

EQUATION IN THE PRESENCE OF A SHOCK

Russel CAFLISCH \*

I N R I A

\* Permanent Address : Mathematics Department, Stanford University,  
Stanford , CA 94305, U.S.A.

RESUME :

L'équation de Broadwell est le modèle le plus simple de l'équation de Boltzmann pour lequel les équations de la dynamique des fluides sont non triviales.

Et il a une théorie complète pour l'existence de solution de cette équation avec valeur initiale donnée. Ici nous montrons formellement que si les équations de la dynamique des fluides a une solution . Alors la solution de Broadwell converge vers celle-ci quand le libre parcours moyen tend vers 0. Cette limit est valable même s'il y a un choc dans le flux, bien que dans une fine couche limite le long du choc la convergence n'a pas lieu. Une donnée régulière conduit à l'existence d'une couche initiale qui n'entre cependant pas en interaction avec la couche limite de choc.

ABSTRACT :

The Broadwell equation is the simplest model of the Boltzmann equation of kinetic theory for which the corresponding model fluid dynamic equations are non trivial. For this equation there is a complete existence theory for the initial value problem. Here we show formally that if the model fluid dynamic equations can be solved, the the Broadwell solution asymptotically converges to the fluid dynamic solution as the mean free path goes to zero. This limit is valid even if there is a shock in the fluid flow, although there is a thin shock layer in which the convergence does not hold. Arbitrary smooth initial data is allowed, which leads to a short initial layer of non-convergence, but the initial and shock layers do not interact due to the assumed initial smoothness.

## I. INTRODUCTION

---

The Broadwell model of gas kinetics [1] describes a gas as composed of particles of only six speeds with a binary collision law and spatial variation in only one direction. It leads to a relatively simple model Boltzmann equation (called the Broadwell equation) and to non-trivial fluid dynamic (Euler) equations. There is an asymptotic equivalence between these two equations at small mean free path, which was proved by Caflisch and Papanicolaou [2] when the fluid flow is smooth. The purpose of this paper is to demonstrate formally that even if the fluid flow includes a shock this fluid-dynamic approximation is still valid, although there is a thin shock layer in which the convergence is non-uniform.

From the Broadwell equation a smooth profile for a steady plane shock was found explicitly by Gatignol [6], Broadwell [1], and Caflisch [3]. The key to the present analysis of a time dependent shock is that to leading order the time dependent shock profile has exactly the shape of a steady shock profile with parameters that change in time.

There is a complete existence theory for the initial value problem for the Broadwell equation developed by Inoue and Nishida [9] and Tartar [15]. The classical Boltzmann equation and its fluid-dynamic limit has been investigated by Grad [7, 8], Ellis and Pinsky [5], Nishida [14], Kawashima, Matsumura and Nishida [10] and Caflisch [4] and the corresponding steady plane shock solution was found by Nicolaenko [13]. The fluid-dynamic limit of the Carleman model was analyzed by Kurtz [11] and McKean [12].

After the Broadwell and model Euler equations are introduced in section 2 and 3, the fluid dynamic limit is described in section 4. The Hilbert expansion (or outer expansion) is developed in section 5 and the shock layer expansion in section 6, but the proof that the corresponding equations can be

solved is in section 7. The initial layer expansion is presented in section 8.

This work was performed while visiting INRIA. Their support and the hospitality of Professor Bensoussan is greatly appreciated.

## II. THE BROADWELL EQUATION

---

Broadwell [1] proposed a simplified Boltzmann equation which describes the evolution of a gas composed of particles all of speed  $c$  and each moving in one of the six directions parallel to one of the three spatial axes. If the gas is assumed to have spatial variation in only the  $x$  direction and to be constant in the  $y$  and  $z$  directions, particles moving in the latter two directions have effective velocity 0 and the model describes a gas consisting of 3 species of particles with velocities  $+c, 0$  and  $-c$ . The density function for each species is written respectively as  $f_+, f_0, f_-$ , and the equations for the evolution of the system are :

$$(2.1) \quad \left\{ \begin{array}{l} \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) f_+ = f_0^2 - f_+ f_- \\ \frac{\partial}{\partial t} f_0 = -\frac{1}{2} (f_0^2 - f_+ f_-) \\ \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) f_- = f_0^2 - f_+ f_- \end{array} \right.$$

The terms on the left side of this system represent the streaming of particles ; those on the right represent binary collisions between particles.

We introduce the following vector notation :

$$(2.2) \quad \left\{ \begin{array}{l} f = \begin{pmatrix} f_+ \\ f_0 \\ f_- \end{pmatrix} \quad V = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c \end{pmatrix} \\ \bar{Q}(f, g) = f_0 g_0 - \frac{1}{2} (f_+ g_- + f_- g_+) \\ Q(f, g) = \bar{Q}(f, g) \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix} \\ D = \frac{\partial}{\partial t} + V \frac{\partial}{\partial x}, \end{array} \right.$$

and rewrite (2.1) as :

$$(2.3) \quad Df = Q(f, f).$$

We will now describe the basic properties of (2.3) which can be found in [2] and all of which are analogous to properties of the classical Boltzmann equation.

A density function  $f$  describes a gas only if  $f_i \geq 0$ , for  $i = +, 0, -$ . In fact we will assume throughout that each  $f_i > 0$ . Equation (2.3) preserves positivity, i.e. if  $f_i(t = 0, x) > 0$  for all  $i$  and  $x$  and if  $f(t, x)$  solves (2.3), then  $f_i(t, x) > 0$  for all  $i$  and  $x$ .

The equilibrium states  $\omega = \begin{pmatrix} \omega_+ \\ \omega_0 \\ \omega_- \end{pmatrix}$  for the collision process satisfy  $Q(\omega, \omega) = 0$ , i.e.

$$(2.4) \quad \omega_0^2 = \omega_+ \omega_-.$$

They are called local Maxwellians and will play a basic role in the following theory. Each local Maxwellian can be represented in a unique way as

$$(2.5) \quad \begin{cases} \omega_+ = \frac{\rho}{2c^2} (F(u) + cu) \\ \omega_0 = \frac{\rho}{4c^2} (c^2 - F(u)) \\ \omega_- = \frac{\rho}{2c^2} (F(u) - cu) \end{cases}$$

in which  $\rho$  and  $u$  are any functions of  $x$  and  $t$  satisfying  $\rho > 0$ ,  $|u| < c$  (in order to insure positivity) and

$$(2.6) \quad F(u) = \frac{1}{3} c^2 \{2(1 + 3u^2/c^2)^{1/2} - 1\}.$$

During a collision the number and total momentum of the particles is conserved. If we denote



$$(2.7) \quad \begin{cases} \psi_1 = (1, 4, 1) \\ \psi_2 = (c, 0, -c) \end{cases}$$

and use  $\langle, \rangle$  for the usual vector inner product, these conservation law are written respectively as

$$(2.8) \quad \langle \psi_i, Q(f,h) \rangle = 0, \text{ for } i = 1, 2,$$

for any density functions  $f, h$ . Define corresponding moments of  $f$  by

$$(2.9) \quad \begin{cases} \rho_f = f_+ + 4f_0 + f_- = \langle \psi_1, f \rangle \\ \rho_f u_f = m_f = c(f_+ - f_-) = \langle \psi_2, f \rangle . \end{cases}$$

If (2.8) is used in (2.3) we see that a solution of (2.3) must satisfy

$$(2.10) \quad \begin{cases} \frac{\partial}{\partial t} \rho_f + \frac{\partial}{\partial x} m_f = 0 \\ \frac{\partial}{\partial t} m_f + \frac{\partial}{\partial x} c^2 (f_+ + f_-) = 0. \end{cases}$$

Now let  $f$  be a fixed positive distribution function. The linearized collision operator  $L_f$  is defined by

$$(2.11) \quad L_f = 2 Q(f, \cdot) ,$$

and has the matrix representation

$$(2.12) \quad L_f = \begin{pmatrix} -f_- & 2f_0 & -f_+ \\ \frac{1}{2}f_- & -f_0 & \frac{1}{2}f_+ \\ -f_- & 2f_0 & -f_+ \end{pmatrix}$$

Zero is a double eigenvalue of  $L_f$  and  $-(f_+ + f_o + f_-)$  is the third one. Corresponding to (2.8) we choose for left eigenvectors  $\psi_1, \psi_2$  and

$$(2.13) \quad \psi_3 = \psi_3^f = (f_+ + f_o + f_-)^{-1} (f_-, -2f_o, f_+).$$

The dual basis of right eigenvectors is

$$(2.14) \quad \left\{ \begin{array}{l} \phi_1^f = (f_+ + f_o + f_-)^{-1} \begin{pmatrix} \frac{1}{2}f_o \\ \frac{1}{4}(f_+ + f_-) \\ \frac{1}{2}f_o \end{pmatrix} \\ \phi_2^f = (f_+ + f_o + f_-)^{-1} \begin{pmatrix} \frac{1}{2c}(2f_+ + f_o) \\ \frac{1}{4c}(f_- - f_+) \\ -\frac{1}{2c}(2f_- + f_o) \end{pmatrix} \\ \phi_3 = \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix} \end{array} \right.$$

We denote by  $P_f$  the projection operator into the null space of  $L_f$ . It has the explicit form

$$(2.15) \quad P_f = \phi_1^f \psi_1 + \phi_2^f \psi_2,$$

in which the products on the right are tensor products between vectors resulting in  $3 \times 3$  matrices. From (2.9) we see that

$$(2.16) \quad P_f h = \rho_h \phi_1^f + m_h \phi_2^f$$

Given a vector  $h$ , consider the linear system

$$(2.17) \quad L_f u = h.$$

From the Fredholm alternative, (2.17) can be solved if and only if  $P_f h = 0$ , and then the solution is unique up to addition of an element in the nullspace. We denote the inverse of  $L_f$  in the range of  $I - P_f$  by  $K_f$ , such that  $P_f K_f = 0$ . Explicitly if

$$(2.18) \quad h = (\psi_3^f, h) \phi_3 = \tilde{h} \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix}$$

it follows that

$$(2.19) \quad K_f h = L_f^{-1} (I - P_f) h = \frac{\tilde{h}}{(f_+ + f_0 + f_-)} \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix}.$$

### III. THE MODEL EULER EQUATIONS

---

The Euler equations of fluid dynamics for the model gas are derived by assuming that the gas is in a state of equilibrium for the collision process, i.e. that  $f$  is a local Maxwellian. Then  $c^2(f_+ + f_-) = \rho_f F(u_f)$  and the conservation laws (2.10) become

$$(3.1) \quad \begin{cases} \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} m = 0 \\ \frac{\partial}{\partial t} m + \frac{\partial}{\partial x} \rho F\left(\frac{m}{\rho}\right) = 0 \end{cases}$$

These model Euler equations give an alternate description of the gas in terms of its macroscopic density  $\rho$  and momentum  $m$ .

These equations have been studied in [3] and [6], and we will only list some of their properties. Let  $\rho, m$  be a solution of (3.1) with  $\rho(t=0) > 0$  and  $|u(t=0)| = |m/\rho(t=0)| < c$ , to be consistent with (2.9), then

$$(3.2) \quad \rho(t) > 0, \quad |u(t)| < c, \quad \text{for } t \geq 0.$$

The system (3.1) is strictly hyperbolic and genuinely nonlinear with characteristic speeds  $\lambda$  and  $\nu$  given by

$$(3.3) \quad \lambda = 2 \frac{u + F(u)^{1/2}}{\frac{3}{c^2} F(u) + 1} \quad \nu = 2 \frac{u - F(u)^{1/2}}{\frac{3}{c^2} F(u) + 1}$$

which satisfy the bounds

$$(3.4) \quad -c < \nu < 0 < \lambda < c, \quad \text{if } |u| < c.$$

Shock solutions of these equations satisfy the Rankine-Hugoniot jump conditions

$$(3.5) \quad \begin{cases} \rho^{\ell}(u^{\ell} - s) = \rho^{\text{r}}(u^{\text{r}} - s) = a \\ \rho^{\ell}(F(u^{\ell}) - u^{\ell} s) = \rho^{\text{r}}(F(u^{\text{r}}) - u^{\text{r}} s) = b \end{cases}$$

in which  $(\rho^{\ell}, u^{\ell})$  and  $(\rho^{\text{r}}, u^{\text{r}})$  denote the limits from the left and right respectively of  $(\rho, u)$  at the shock and  $s$  is the shock speed. There is also an entropy condition

$$(3.6) \quad u^{\text{r}} < u^{\ell} \quad \text{or} \quad s(\rho^{\ell} - \rho^{\text{r}}) > 0$$

which is equivalent to the condition that either

$$(3.7) \quad \lambda^{\text{r}} < s < \lambda^{\ell} \quad \text{or} \quad v^{\text{r}} < s < v^{\ell}$$

We will assume that (3.1) has a solution  $\rho_0, m_0$  satisfying the following :

#### Condition S

For  $0 \leq t \leq T$ , the fluid state  $(\rho_0(x, t), m_0(x, t))$  has a single shock which starts at  $t = t_1 > 0$  and moves along the curve  $\dot{x} = p(t)$ , with  $p'(t) = s(t)$ . The fluid state is smooth away from the shock and tangentially to the shock ; across the shock it satisfies (3.5). The shock curve  $x = p(t)$  is smooth and starts off in a characteristic direction; to be specific we assume that

$$(3.8) \quad s(t_1) = \lambda(t_1)$$

It follows that the shock has zero strength at  $t = t_1$ , i.e.  $u_0^{\text{r}}(t_1) = u_0^{\ell}(t_1)$  and  $\rho_0^{\text{r}}(t_1) = \rho_0^{\ell}(t_1)$ , and that

$$(3.9) \quad 0 < \lambda^{\text{r}}(t) \leq s(t) \leq \lambda^{\ell}(t) \quad \text{for} \quad t > t_1.$$

Next we study the linearized inhomogeneous Euler equations :

$$(3.10) \quad \begin{cases} \frac{\partial}{\partial t} \rho_1 + \frac{\partial}{\partial x} m_1 = 0 \\ \frac{\partial}{\partial t} m_1 + \frac{\partial}{\partial x} ((F(u_0) - u_0 F'(u_0)) \rho_1 + F'(u_0) m_1) = \frac{\partial}{\partial x} w_1 \end{cases}$$

in which the inhomogeneity  $w_1$  will be a function of  $\rho_0$ ,  $u_0 = m_0 / \rho_0$  and their derivatives. The functions  $\rho_1$  and  $u_1$  can be discontinuous along  $x = p(t)$  and must satisfy the following linearized inhomogeneous Rankin-Hugoniot conditions

$$(3.11) \quad \begin{cases} [-s \rho_1 + m_1] = \alpha_1 \\ [-s m_1 + (F(u_0) - u_0 F'(u_0)) \rho_1 + F'(u_0) m_1 + w_1] = \beta_1 \end{cases}$$

in which the brackets [ ] denote the jump across the shock, and  $\alpha_1$  and  $\beta_1$  will come from the inner expansion at the shock.

The system (3.10) can be diagonalized as

$$(3.12) \quad \frac{\partial}{\partial t} \begin{pmatrix} \theta_1 \\ \eta_1 \end{pmatrix} + N \frac{\partial}{\partial x} \begin{pmatrix} \theta_1 \\ \eta_1 \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ w_1 \end{pmatrix} - A^{-1} \left( \frac{\partial}{\partial t} A + \frac{\partial}{\partial x} (A N) \right) \begin{pmatrix} \theta_1 \\ \eta_1 \end{pmatrix}$$

in which

$$(3.13) \quad N = \begin{pmatrix} \lambda & 0 \\ 0 & \nu \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 \\ \lambda & \nu \end{pmatrix}$$

$$\begin{pmatrix} \theta_1 \\ \eta_1 \end{pmatrix} = A \begin{pmatrix} \rho_1 \\ m_1 \end{pmatrix}$$

The jump condition (3.11) becomes

$$(3.14) \quad (-s A + AN) \begin{pmatrix} \theta_1 \\ \eta_1 \end{pmatrix} + \begin{bmatrix} 0 \\ w_1 \end{bmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$$

The system (3.12) has the same characteristics as (3.1) with slopes  $\lambda$  and  $\nu$ . It follows from (3.9) that the  $\lambda^r$ ,  $\lambda^l$ , and  $\nu^r$  characteristics are incoming at the shock while the  $\nu^l$  characteristic is outgoing (cf. Figure 1).

Thus  $\theta_1^r, \theta_1^\ell$  and  $\eta_1^r$  will be determined by integrating along the characteristics and  $\eta_1^\ell$  can be chosen to solve (3.14) (here  $\theta_1^r, \theta_1^\ell, \eta_1^r, \eta_1^\ell$  denote the limits of  $\theta_1$  and  $\eta_1$  from each side of the shock as before). Since there are two equations in (3.14) this leads to a solvability condition on  $\alpha_1$  and  $\beta_1$  which is found after eliminating  $\eta_1^\ell$  to be

$$(3.15) \quad \begin{cases} v^\ell \alpha_1 - \beta_1 = (v^\ell - \lambda^r)(s - \lambda^r) \theta_1^r + (v^\ell - v^r)(s - v^r) \eta_1^r - \\ \quad - (v^\ell - \lambda^\ell)(s - \lambda^\ell) \theta_1^\ell + [w_1]. \end{cases}$$

The remaining equation for  $\eta_1^\ell$  can be rewritten as

$$(3.16) \quad \begin{cases} \eta_1^\ell = (s - v^\ell)^{-2} \{ (s \alpha_1 - \beta_1) - (s - \lambda^r)^2 \theta_1^r - (s - v^r)^2 \eta_1^r + \\ \quad + (s - \lambda^\ell)^2 \theta_1^\ell + [w_1] \} \end{cases}$$

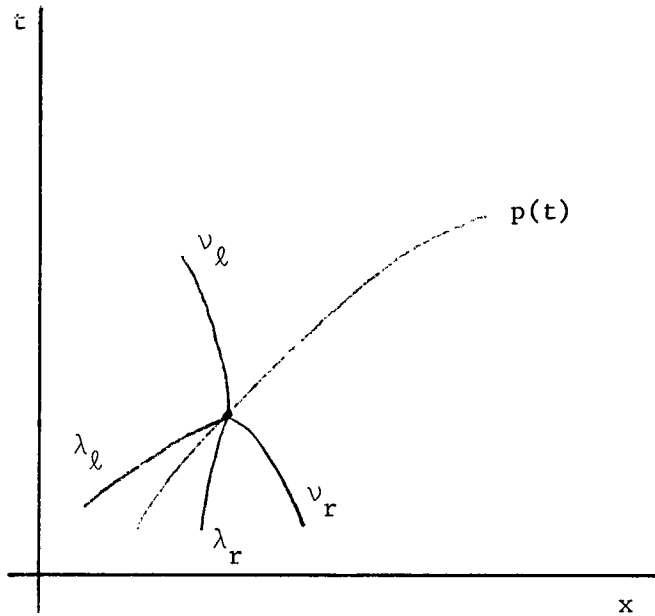


Figure 1 : Shock curve with characteristics :

$$v_\ell < 0, \quad v_r < 0, \quad 0 < \lambda_r < s < \lambda_\ell$$

#### IV. THE FLUID-DYNAMIC LIMIT

---

The model Boltzmann equation (2.3) can be rewritten in dimensionless form as

$$(4.1) \quad Df^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon)$$

$$(4.2) \quad f^\varepsilon(t=0, x) = \bar{f}(x)$$

in which the one parameter  $\varepsilon$  is the mean free time between collisions. The fluid dynamic limit which relates (4.1) and (3.1) is the limit  $\varepsilon \rightarrow 0$ .

Let

$$(4.3) \quad \bar{\rho}(x) = \langle \psi_1, \bar{f}(x) \rangle, \quad \bar{m}(x) = \langle \psi_2, \bar{f}(x) \rangle$$

and suppose that  $(\rho_0(x,t), m_0(x,t))$  is a solution of (3.1) satisfying condition S with

$$(4.4) \quad \rho_0(t=0) = \bar{\rho}, \quad m_0(t=0) = \bar{m}$$

Let  $\omega(x,t)$  be the local Maxwellian density associated with  $\rho_0$  and  $m_0$  as in (2.5). Then we will construct an expansion for  $f^\varepsilon$  in powers of  $\varepsilon$ , so that  $f^\varepsilon$  is a formal solution of (4.1) and formally satisfies

$$(4.5) \quad \sup_{\substack{0 < \delta \leq t \leq T \\ |p(t)-x| \geq \delta > 0}} |f^\varepsilon(x,t) - \omega(x,t)| \leq \kappa \varepsilon,$$

in which the constant  $\kappa$  will depend on  $\delta$ .

The solution  $f^\varepsilon$  will be the sum of three terms -an outer expansion (the Hilbert expansion)  $f^H$ , a shock layer expansion  $f^S$ , and an initial layer expansion  $f^I$ - written as



$$(4.6) \quad f^\varepsilon(x, t) = f^H(x, t) + f^S\left(y = \frac{x - p(t)}{\varepsilon}, t\right) + f^I\left(x, \tau = \frac{t}{\varepsilon}\right)$$

in which  $\lim_{|y| \rightarrow \infty} f^S = \lim_{\tau \rightarrow \infty} f^I = 0$ , i.e.  $f^S$  is confined to a neighborhood of the

shock and  $f^I$  to a neighborhood of  $t = 0$ .

The restrictions to  $\delta \leq t$  and  $|p(t) - x| \geq \delta$  in (4.5) are due to the initial and shock layers.

We shall obtain asymptotic expansions for  $f^H$ ,  $f^S$  and  $f^I$ , although we are unable to show that these expansions are asymptotic to an actual solution.

## V. HILBERT EXPANSION WITH A SHOCK

---

The Hilbert expansion is

$$(5.1) \quad f^H = \omega + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$$

Substituting this into (4.1) and equating the coefficients of power of  $\varepsilon$  leads to

$$(5.2) \quad 0 = Q(\omega, \omega)$$

$$(5.3) \quad D\omega = L_\omega f_1$$

$$(5.4) \quad Df_1 = L_\omega f_2 + Q(f_1, f_1)$$

From (5.2) we conclude that  $\omega$  is a local Maxwellian as suggested by the notation. A solution  $f_1$  of (5.3) exists if and only if

$$(5.5) \quad \langle \psi_1, D\omega \rangle = \langle \psi_2, D\omega \rangle = 0$$

These are exactly the model Euler equations (3.1) for

$$(5.6) \quad \rho_0 = \langle \psi_1, \omega \rangle, \quad m_0 = \langle \psi_2, \omega \rangle$$

and their solution is taken to be the fluid state described in section 4, with a single shock at  $x = p(t)$  and satisfying condition S.

Now from (2.19) we have

$$(5.7) \quad f_1 = \tilde{f}_1 + K_g D\omega$$

with

$$(5.8) \quad P_\omega \tilde{f}_1 = \tilde{f}_1,$$

i.e.  $\tilde{f}_1$  is a purely fluid-dynamic vector and is determined by its moments, which are

$$(5.9) \quad \left\{ \begin{array}{l} \rho_1 = \langle \psi_1, \tilde{f}_1 \rangle = \langle \psi_1, f_1 \rangle \\ m_1 = \langle \psi_2, \tilde{f}_1 \rangle = \langle \psi_2, f_1 \rangle. \end{array} \right.$$

These two quantities are determined by the solvability condition for (5.4), which is

$$(5.10) \quad p_\omega D f_1 = p_\omega D \tilde{f}_1 + p D K_\omega (I - p_\omega) D\omega = 0$$

It was shown in [2] that

$$(5.11) \quad c^2(f_{1+} + f_{1-}) = (F(u_0) - u_0 F'(u_0)) \rho_1 + F'(u_0) m_1 - \mu(u_0) \frac{\partial}{\partial x} u_0$$

and that (5.10) are exactly the following inhomogeneous linearized Euler equations

$$(5.12) \quad \left\{ \begin{array}{l} \frac{\partial}{\partial t} \rho_1 + \frac{\partial}{\partial x} m_1 = 0 \\ \frac{\partial}{\partial t} m_1 + \frac{\partial}{\partial x} \{ (F(u_0) - u_0 F'(u_0)) \rho_1 + F'(u_0) m_1 \} = \frac{\partial}{\partial x} ( \mu(u_0) \frac{\partial}{\partial x} u_0 ), \end{array} \right.$$

in which

$$(5.13) \quad \mu(u) = 2 \frac{c^2 - F(u)}{(1 + 3u^2/c^2)^{3/2}}.$$

The jump conditions for  $\rho_1$  and  $m_1$  at  $x = p(t)$  are taken to be (3.11) with  $w_1 = \mu(u_0) \frac{\partial}{\partial x} u_0$ ;  $\alpha_1$  and  $\beta_1$  will be determined later. Using (5.11) the jump conditions can be rewritten as

$$(5.14) \quad \begin{cases} [-s(f_{1+} + 4f_{10} + f_{1-}) + c(f_{1+} - f_{1-})] = \alpha_1 \\ [-sc(f_{1+} - f_{1-}) + c^2(f_{1+} + f_{1-})] = \beta_1 \end{cases}$$

This procedure is continued to find  $f_i$  for  $i \geq 2$  in the obvious way.

## VI. THE SHOCK EXPANSION

---

The shock expansion is

$$(6.1) \quad f^S(x, t) = g_0(y, t) + \epsilon g_1(y, t) + \epsilon^2 g_2(y, t) + \dots$$

with

$$(6.2) \quad y = \frac{x - p(t)}{\epsilon}$$

The equation for  $g_i$  are found by supposing that  $f^H + f^S$  is a continuous solution of (4.1). After substituting (5.1) and (6.1) into (4.1) and matching powers of  $\epsilon$ , the terms containing only  $\omega$  or  $f_i$  can be eliminated using the Hilbert expansion equations. To simplify the calculations we replace  $\omega$  and  $f_i$  by their Taylor series expansion in  $x$  around  $x = p(t)$  which must be done separately for  $x < p(t)$  and  $x > p(t)$ , and we denote

$$(6.3) \quad \left\{ \begin{array}{ll} \omega^r = \omega(x = p(t) + ) & \omega^l = \omega(x = p(t) - ) \\ \frac{\partial \omega^r}{\partial x} = \frac{\partial \omega}{\partial x} (x = p(t) + ) & \frac{\partial \omega^l}{\partial x} = \frac{\partial \omega}{\partial x} (x = p(t) - ) \\ \text{etc. for } \frac{\partial^n \omega}{\partial x^n} \text{ and } f_i. \end{array} \right.$$

The resulting equations are

$$(6.4) \quad (V - sI) \frac{\partial}{\partial y} g_0 = \begin{cases} Q(g_0 + 2\omega^l, g_0), & y < 0 \\ Q(g_0 + 2\omega^r, g_0), & y > 0 \end{cases}$$

$$(6.5) \quad (V - sI) \frac{\partial}{\partial y} g_1 = \begin{cases} 2Q(g_0 + \omega^l, g_1) + 2Q(y \frac{\partial \omega^r}{\partial x} + f_1^l, g_0) - \frac{\partial}{\partial t} g_0, & y < 0 \\ 2Q(g_0 + \omega^r, g_1) + 2Q(y \frac{\partial \omega^r}{\partial x} + f_1^r, g_0) - \frac{\partial}{\partial t} g_0, & y > 0 \end{cases}$$

We look for a solution satisfying :

(6.6.)<sub>a</sub> Each sum  $\omega + g_0, f_1 + g_1, f_2 + g_2$  is continuous

(6.6.)<sub>b</sub> For  $t \leq t_1, g_0 = g_1 = \dots = 0$

(6.6.)<sub>c</sub> As  $y \rightarrow \pm\infty, g_0 \rightarrow 0, g_1 \rightarrow 0, \dots$

The first condition implies that each  $g_i$  is continuous except at  $x = p(t)$  ; the last condition is to insure that the shock layer is confined to a small region around the shock.

Denote

$$(6.7) \quad k = \begin{cases} g_0 + \omega^l, & y < 0 \\ g_0 + \omega^r, & y > 0 \end{cases}$$

Then  $k$  is continuous and solve

$$(6.8) \quad (V - sI) \frac{\partial}{\partial y} k = Q(k, k)$$

$$(6.9) \quad \begin{cases} k(y = \infty) = \omega^r \\ k(y = -\infty) = \omega^l \end{cases}$$

This system was analyzed in [3] and [6]. Applying (2.8) and (3.5) to (6.8) and (6.9) yields the following identities for  $\rho_k = \langle \psi_1, k \rangle$  and  $m_k = \langle \psi_2, k \rangle$  :

$$(6.10) \quad \begin{cases} -s f_k + m_k = a \\ -s m_k + c^2(k_+ + k_-) = b, \end{cases}$$

from which it follows that

$$(6.11) \quad k = \begin{pmatrix} k_+ \\ k_o \\ k_- \end{pmatrix} = \Lambda k_o + A_o$$

with

$$(6.12) \quad \Lambda = \begin{pmatrix} \frac{2s}{c-s} \\ 1 \\ \frac{-2s}{c+s} \end{pmatrix} \quad A_o = \begin{pmatrix} \frac{a+b/c}{2(c-s)} \\ 0 \\ \frac{-a+b/c}{2(c+s)} \end{pmatrix}$$

Now we have

$$(6.13) \quad Q(k,k) = \{ \bar{Q}(\Lambda, \Lambda) k_o^2 + 2\bar{Q}(\Lambda, A_o) k_o + \bar{Q}(A_o, A_o) \} \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix} \\ = q(k_o) \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix}$$

Since  $Q(\omega^r, \omega^r) = Q(\omega^l, \omega^l) = 0$ , it must be that

$$(6.14) \quad q(k_o) = \bar{Q}(\Lambda, \Lambda) (k_o - \omega_o^r) (k_o - \omega_o^l)$$

Therefore  $k_o$  solves

$$(6.15) \quad -s \frac{\partial}{\partial y} k_o = -\frac{1}{2} q(k_o),$$

which has the solution

$$(6.16) \quad k_o(y) = -\frac{1}{2}(\omega_o^l - \omega_o^r) \tanh \left( \frac{1}{4s} \bar{Q}(\Lambda, \Lambda) (\omega_o^l - \omega_o^r) (y + y_o) \right) + \frac{1}{2}(\omega_o^l + \omega_o^r)$$

Since

$$(6.17) \quad \bar{Q}(\Lambda, \Lambda) = \frac{c^2 + 3s^2}{c^2 - s^2} \quad \omega_o^l - \omega_o^r = \frac{1}{4c^2} (c^2 - s^2) (\rho_o^l - \rho_o^r),$$

we see that  $k$  satisfies (6.9) if and only if the entropy condition (3.6) is true, or equivalently if and only if

$$(6.18) \quad s(\omega_o^l - \omega_o^r) > 0$$

The shift parameter  $y_o = y_o(t)$  may depend on time and will be chosen later.

Apply (2.8) to (6.5) and integrate once using (6.6)<sub>c</sub> to obtain

$$(6.19) \quad \begin{aligned} & \begin{pmatrix} -s(g_{1+} + 4g_{1o} + g_{1-}) + c(g_{1+} - g_{1-}) \\ -sc(g_{1+} - g_{1-}) + c^2(g_{1+} + g_{1-}) \end{pmatrix} = \\ & \left\{ \begin{array}{l} - \int_{-\infty}^y \frac{\partial}{\partial t} \begin{pmatrix} g_{0+} + 4g_{0o} + g_{0-} \\ c(g_{0+} - g_{0-}) \end{pmatrix} (z, t) dz, \quad y < 0 \\ \int_y^{\infty} \frac{\partial}{\partial t} \begin{pmatrix} g_{0+} + 4g_{0o} + g_{0-} \\ c(g_{0+} - g_{0-}) \end{pmatrix} (z, t) dz, \quad y > 0 \end{array} \right. \end{aligned}$$

This can be rewritten as in (6.11) as

$$(6.20) \quad g_1 = \Lambda g_{1o} + A_1$$

with



$$A_1 = \begin{cases} \int_{-\infty}^y b_1(z) dz, & y < 0 \\ \int_y^{\infty} b_1(z) dz, & y > 0 \end{cases}$$

(6.21)

$$b_1(z) = \begin{pmatrix} \frac{2c}{(c-s)^2} \\ 0 \\ -\frac{2c}{(c-s)^2} \end{pmatrix} \frac{\partial}{\partial t} g_0(t, z)$$

Then (6.5) can be reduced to a single linear first order equation for  $g_{10}$ . There is a complicated inhomogeneity which we abbreviate as  $\bar{Q}_1$ , and then

$$(6.22) \quad -s \frac{\partial}{\partial y} g_{10} = -\bar{Q}(k, g_1) + \bar{Q}_1$$

Now

$$(6.23) \quad \bar{Q}(k, g_1) = \bar{Q}(\Lambda k_0 + A_0, \Lambda g_{10} + A_1) = \frac{1}{2} q'(k_0) g_{10} + \bar{Q}(A_0, A_1),$$

and (6.22) can be rewritten as

$$(6.24) \quad \frac{\partial}{\partial y} g_{10} = \frac{1}{2s} q'(k_0) g_{10} + \bar{Q}_2$$

From (6.18), (6.9) and the quadratic nature of  $q$ , it follows that for some positive number  $\bar{q}$ ,

$$(6.25) \quad \begin{cases} \frac{1}{2s} q'(k_0) < -\bar{q} & \text{for } y \text{ large enough} \\ \frac{1}{2s} q'(k_0) > \bar{q} & \text{for } y \text{ large enough.} \end{cases}$$

Also the inhomogeneous term  $Q_2$  is a linear functional of  $g_0$  and  $\frac{\partial}{\partial t} g_0$

which is decaying exponentially to 0 at  $y = \pm\infty$ . Therefore  $g_{10}$  and also  $g_{1+}$  and  $g_{1-}$  are decaying exponentially to 0 at  $y = \pm\infty$ . This is true for any values of  $g_{10}^r$  and  $g_{10}^l$ , the right and left limits of  $g_{10}$  at  $y = 0$ .

Next we check the continuity of  $f_1 + g_1$ . First we require that

$$(6.26) \quad g_{10}^r - g_{10}^l = - (f_{10}^r - f_{10}^l)$$

This still leaves one value, say  $g_{10}^r$  undetermined. The continuity for the other two components is checked by first writing the following jump condition from (6.19)

$$(6.27) \quad \begin{bmatrix} -s(g_{1+} + 4g_{10} + g_{1-}) + c(g_{1+} - g_{1-}) \\ -sc(g_{1+} - g_{1-}) + c^2(g_{1+} + g_{1-}) \end{bmatrix} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \begin{pmatrix} g_{0+} + 4g_{00} + g_{0-} \\ c(g_{0+} - g_{0-}) \end{pmatrix} (y, t) dy$$

Comparison with (5.14) shows that we must take

$$(6.28) \quad \begin{cases} \alpha_1 = - \int_{-\infty}^{\infty} \frac{\partial}{\partial t} (g_{0+} + 4g_{00} + g_{0-}) (y, t) dy \\ \beta_1 = - \int_{-\infty}^{\infty} \frac{\partial}{\partial t} c(g_{0+} - g_{0-}) (y, t) dy \end{cases}$$

From (6.13) these can be calculated explicitly to be

$$(6.29) \quad \begin{cases} \alpha_1(t) = - y_0'(t) (\rho^l(t) - \rho^r(t)) - y_0(t) (\rho^{l'}(t) - \rho^{r'}(t)) \\ \beta_1(t) = s(t) \alpha_1(t) - s'(t) y_0(t) (\rho^l(t) - \rho^r(t)). \end{cases}$$

Finally from (6.26) and (6.24) we see that we can write the dependence of  $g_{10}$  on the (so far) undetermined value  $g_{10}^r$  as

$$(6.30) \quad g_{10}(y) = e^{-\gamma(y)} g_{10}^r + \tilde{g}_{10}(y)$$

in which

$$(6.31) \quad \gamma(y) = \int_0^y \frac{1}{2s} q'(k_0(z)) dz$$

and  $\tilde{g}_{10}$  is known in terms of  $f_1$ ,  $\omega$  and  $g_0$  and is decaying exponentially to 0 at  $y = \pm\infty$ .

A similar procedure is used to solve for  $g_1$ . We will only display the result that the inhomogeneities in the jump condition are

$$(6.32) \quad \begin{aligned} \alpha_2 &= \frac{d}{dt}(\bar{\gamma} g_{10}^r) + \frac{d}{dt} \left( \int_{-\infty}^{\infty} \langle \psi_1, \Lambda g_{10} + A_1 \rangle dy \right) \\ \beta_2 &= \frac{d}{dt}(s \bar{\gamma} g_{10}^r) + \frac{d}{dt} \left( \int_{-\infty}^{\infty} \langle \psi_2, \Lambda g_{10} + A_1 \rangle dy \right) \end{aligned}$$

in which

$$(6.33) \quad \bar{\gamma}(t) = \frac{4c^2}{s^2 - c^2} \int_{-\infty}^{\infty} e^{-\gamma(y)} dy.$$

## VII. SOLUTION OF THE LINEARIZED EULER EQUATION WITH A SHOCK

---

In this section we solve the linearized Euler equations (3.12) with the jump conditions (3.15) and (3.16) with  $\alpha_1$  and  $\beta_1$  given by (6.29). First we define the characteristic curves  $X_1(t, \bar{X})$  and  $X_2(t, \bar{X})$  starting from  $\bar{X}$ , which satisfy

$$(7.1) \quad \begin{cases} \frac{d}{dt} X_1(t, \bar{X}) = \lambda & \frac{d}{dt} X_2(t, \bar{X}) = \nu \\ X_1(0, \bar{X}) = X_2(0, \bar{X}) = \bar{X} \end{cases}$$

Denote by  $\bar{t}(\bar{X})$  the time at which the characteristic curve  $X_2(t, \bar{X})$  crosses the shock, so that  $X_2(\bar{t}(\bar{X}), \bar{X}) = p(\bar{t}(\bar{X}))$ . Using (6.29) the equations (3.12), (3.15), and (3.16) can be written as

$$(7.2) \quad \begin{cases} \frac{d}{dt} \theta_1(X_1, t) = a_1 \theta_1 + a_2 \eta_1 + b_1 \\ \frac{d}{dt} \eta_1(X_2, t) = a_3 \theta_1 + a_4 \eta_1 + b_2 \\ \eta_1^\ell = a_5 \theta_1^r + a_6 \eta_1^r + a_7 \theta_1^\ell + a_8 y_0 + b_3, \text{ at } t = \bar{t}(\bar{X}) \\ \frac{d}{dt} y_0 = a_9 \theta_1^r + a_{10} \eta_1^r + a_{11} \theta_1^\ell + a_{12} \eta_1^\ell + a_{13} y_0 + b_4 \end{cases}$$

in which  $a_i$  and  $b_i$  are functions of  $\rho_0$ ,  $m_0$ , and their derivatives. In particular there are constants  $\lambda_1$  and  $\gamma_1$  so that

$$(7.3) \quad |a_i| \leq \gamma_1 \quad |b_i| \leq \gamma_1 e^{\lambda_1 t}$$

Now we solve these equations by iteration. The  $(K+1)^{\text{st}}$  iterate is chosen to be

$$(7.4) \left\{ \begin{array}{l} \theta_1^{(K+1)}(X_1, t) = \int_0^t (a_1 \theta_1^{(K)} + a_2 \eta_1^{(K)} + b_1)(X_1(s), s) ds + \theta_1(\bar{X}, 0) \\ \eta_1^{(K+1)}(X_1, t) = \int_0^t (a_3 \theta_1^{(K)} + a_4 \eta_1^{(K)} + b_2)(X_2(s), s) ds + \eta_1(\bar{X}, 0) \text{ for} \\ \qquad \qquad \qquad 0 \leq t \leq \bar{t}(\bar{X}) \\ \eta_1^{\ell(K+1)} = a_5 \theta_1^{r(K+1)} + a_6 \eta_1^{r(K+1)} + a_7 \theta_1^{\ell(K+1)} + a_8 y_o^{(K+1)} + b_3 \text{ at} \\ \qquad \qquad \qquad t = \bar{t}(\bar{X}) \\ \eta_1^{(K+1)}(X_1, t) = \int_{\bar{t}}^t (a_3 \theta_1^{(K)} + a_4 \eta_1^{(K)} + b_2)(X_2(s), s) ds + \eta_1^{\ell(K+1)}(X_1(\bar{t}), \bar{t}) \\ \qquad \qquad \qquad \text{for } \bar{t} \leq t \\ y_o^{(K+1)}(t) = \int_0^t (a_9 \theta_1^{r(K)} + a_{10} \eta_1^{r(K)} + a_{11} \theta_1^{\ell(K)} + a_{12} \eta_1^{\ell(K)} + \\ \qquad \qquad \qquad + a_{13} y_o^{(K)} + b_4)(s) ds. \end{array} \right.$$

Suppose that for some positives constants  $\gamma_2$ ,  $\gamma_3$  and  $\lambda_2$

$$(7.5) \left\{ \begin{array}{l} |\theta_1^{(K)}(t)| \leq \gamma_3 e^{\lambda_2 t} \\ |\eta_1^{(K)}(t)| \leq \gamma_3 e^{\lambda_2 t} \quad \text{for } 0 \leq t \leq \bar{t} \\ |\eta_1^{(K)}(t)| \leq \gamma_3 e^{\lambda_2 t} \quad \text{for } \bar{t} \leq t \\ |y_o^{(K)}(t)| \leq \gamma_3 e^{\lambda_2 t} \\ |\theta_1(t=0)| < \frac{1}{2} \gamma_2 \quad |\eta_1(t=0)| < \frac{1}{2} \gamma_2 \end{array} \right.$$

After requiring that

$$(7.6) \quad \begin{cases} \lambda_1 \leq \lambda_2, \quad \gamma_1 \leq \gamma_2 \leq \gamma_3 \\ \lambda_2^{-1} (5 \gamma_2 \gamma_3 + \gamma_2) \leq \frac{1}{2} \gamma_2 \\ 4 \gamma_2^2 + \gamma_2 \leq \frac{1}{2} \gamma_3 \end{cases}$$

we can make the following estimates :

$$(7.7) \quad \begin{cases} |\theta_1^{(K+1)}(t)| \leq \lambda_2^{-1} (2 \gamma_2 \gamma_3 + \gamma_2) e^{\lambda_2 t} + \frac{1}{2} \gamma_2 \leq \gamma_2 e^{\lambda_2 t} \\ |\eta_1^{(K+1)}(t)| \leq \lambda_2^{-1} (2 \gamma_2 \gamma_3 + \gamma_2) e^{\lambda_2 t} + \frac{1}{2} \gamma_2 \leq \gamma_2 e^{\lambda_2 t} \quad \text{for } 0 \leq t \leq \bar{t} \\ |\eta_1^{\ell(K+1)}(t)| \leq (4 \gamma_2^2 + \gamma_2) e^{\lambda_2 \bar{t}} \leq \gamma_3 e^{\lambda_2 t} \\ |\eta_1^{(K+1)}(t)| \leq \lambda_2^{-1} (2 \gamma_2 \gamma_3 + \gamma_2) e^{\lambda_2 t} + (4 \gamma_2^2 + \gamma_2) e^{\lambda_2 \bar{t}} \leq \gamma_3 e^{\lambda_2 t} \quad \text{for } \bar{t} \leq t \\ |y_0^{(K+1)}(t)| \leq \lambda_2^{-1} (5 \gamma_2 \gamma_3 + \gamma_2) e^{\lambda_2 t} \leq \gamma_2 e^{\lambda_2 t} \end{cases}$$

If in addition we require that

$$(7.8) \quad \lambda_2^{-1} \gamma_2 \leq \mu \quad (3 + \gamma_2) \mu < 1,$$

then we find that

$$(7.9) \quad \begin{cases} \sup_{0 \leq t \leq T} e^{-\lambda_2 t} |\Delta^{K+1} \theta_1(t)| \leq \mu \sup_{0 \leq t \leq T} e^{-\lambda_2 t} (|\Delta^K \theta_1(t)| + |\Delta^K \eta_1(t)|) \\ \sup_{0 \leq t \leq \bar{t}} e^{-\lambda_2 t} |\Delta^{K+1} \eta_1(t)| \leq \mu \sup_{0 \leq t \leq \bar{t}} e^{-\lambda_2 t} (|\Delta^K \theta_1(t)| + |\Delta^K \eta_1(t)|) \\ \sup_{0 \leq t \leq T} e^{-\lambda_2 t} |\Delta^{K+1} y_0(t)| \leq \mu \sup_{0 \leq t \leq T} e^{-\lambda_2 t} (|\Delta^K \theta_1(t)| + |\Delta^K \eta_1(t)| + |\Delta^K y_0(t)|) \end{cases}$$

⋮

$$\left. \begin{aligned}
 e^{-\lambda_2 \bar{t}} |\Delta^{K+1} \eta_1^{\ell}(\bar{t})| &\leq \gamma_2 e^{-\lambda_2 \bar{t}} (|\Delta^{K+1} \theta_1^r(\bar{t})| + |\Delta^{K+1} \eta_1^r(\bar{t})| + |\Delta^{K+1} \theta_1^{\ell}(\bar{t})| + \\
 &\quad + |\Delta^{K+1} y_0(\bar{t})|) \\
 \sup_{\bar{t} \leq t \leq T} e^{-\lambda_2 t} |\Delta^{K+1} \eta_1(t)| &\leq \mu \sup_{t \leq t \leq T} e^{-\lambda_2 t} (|\Delta^K \theta_1(t)| + |\Delta^K \eta_1(t)|) + \\
 &\quad + e^{-\lambda_2 \bar{t}} |\Delta^{K+1} \eta_1^{\ell}(\bar{t})|
 \end{aligned} \right\}$$

in which  $\Delta^{K+1} \theta_1 = \theta_1^{(K+1)} - \theta_1^{(K)}$ , etc.. These can be combined to get

$$(7.10) \quad \left\{ \begin{aligned}
 \sup_{0 \leq t \leq T} e^{-\lambda_2 t} (|\Delta^{K+1} \theta_1| + |\Delta^{K+1} \eta_1| + |\Delta^{K+1} y_0|) &\leq \\
 \leq (3 + \gamma_2) \mu \sup_{0 \leq t \leq T} e^{-\lambda_2 t} (|\Delta^K \theta_1| + |\Delta^K \eta_1| + |\Delta^K y_0|). &
 \end{aligned} \right.$$

Using (7.8) it follows that

$$7.11) \quad \theta_1^{(K)} \rightarrow \theta_1 \quad \eta_1^{(K)} \rightarrow \eta_1 \quad y_0^{(K)} \rightarrow y_0$$

which are solutions of (7.2) satisfying the bounds (7.5).

Now that  $\theta_1$ ,  $\eta_1$  and  $y_0$  are known, we can find  $f_1$  and  $g_1$  from (5.7), (6.20), and (6.30) although this does not determine  $g_{10}^r$ . It is found along with  $\theta_2$  and  $\eta_2$  using the linearized Euler equations and their jump conditions exactly as above, in which  $\alpha_2$  and  $\beta_2$  are given by (6.32).

### VIII. INITIAL LAYER EXPANSION

The initial layer expansion is

$$(8.1) \quad f^I(x, t) = h_0(x, \tau) + \epsilon h_1(x, \tau) + \epsilon^2 h_2(x, \tau) + \dots$$

with

$$(8.2) \quad \tau = \frac{t}{\epsilon}$$

As before we ask that  $f^H + f^I$  is a (formal) solution of the Boltzmann equation. Using the Hilbert expansion equations we can omit all terms involving only  $\omega$  or  $f_i$ , and we replace the remaining  $\omega$  or  $f_i$  terms by their Taylor series expansions in  $t$  around  $t = 0$ . We also require that

$$(8.3.) \quad \begin{cases} \omega(x, 0) + h_0(x, 0) = \bar{f}(x) \\ f_i(x, 0) + h_i(x, 0) = 0 \end{cases}$$

and

$$(8.4) \quad h_i \rightarrow 0 \text{ as } \tau \rightarrow \infty$$

The shock does not appear until  $t = t_1$  or  $\tau = t_1/\epsilon$ , by which time  $f^I$  will be quite small. Thus we can ignore the effect of  $f^S$  on  $f^I$  and vice-versa. The equations for the  $h_i$ 's are

$$(8.5) \quad \frac{\partial}{\partial \tau} h_0 = Q(2\omega + h_0, h_0)$$

$$(8.6) \quad \frac{\partial}{\partial \tau} h_1 = 2 Q(\omega + h_0, h_1) + 2 Q(\tau \frac{\partial}{\partial \tau} \omega + f_1, h_0) - v \frac{\partial}{\partial x} h_0$$

$$(8.7) \quad \begin{cases} \frac{\partial}{\partial \tau} h_2 = 2 Q(\omega + h_0, h_2) + 2 Q(\tau \frac{\partial}{\partial \tau} \omega + f_1, h_1) + 2 Q(\frac{1}{2} \tau^2 \frac{\partial^2}{\partial t^2} \omega + \\ + \tau \frac{\partial}{\partial \tau} f_1 + f_2, h_0) - v \frac{\partial}{\partial x} h_1 \end{cases}$$



Here we denote  $\omega = \omega(t=0)$ ,  $\frac{\partial}{\partial t} \omega = \frac{\partial}{\partial \tau} \omega(t=0)$ ,  $f_1 = f_1(t=0)$ , etc...

From (5.6), (4.3) and (4.4) we have that  $\langle \psi_i, \omega(t=0) \rangle = \langle \psi_i, f \rangle$ . It follows from (8.3) and (8.5) that

$$(8.8) \quad \langle \psi_i, h_o(\tau=0) \rangle = \langle \psi_i, h_o(\tau) \rangle = 0 \text{ for } i = 1, 2, \quad 0 \leq \tau \leq T$$

Thus

$$(8.9) \quad h_o = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} h_{0o}$$

so that (8.5) reduces to

$$(8.10) \quad \frac{\partial}{\partial \tau} h_{0o} = -\frac{1}{2} \{ 2(\omega_o + \omega_+ + \omega_-) - 3 h_{0o} \} h_{0o}$$

which has the solution

$$(8.11) \quad \left\{ \begin{array}{l} h_{0o}(x, \tau) = -\frac{1}{3} (\omega_o + \omega_+ + \omega_-) \tanh \left( \frac{1}{2} (\omega_o + \omega_+ + \omega_-) (\tau + \tau_o(x)) \right) + \\ \quad + \frac{1}{3} (\omega_o + \omega_+ + \omega_-) \end{array} \right.$$

This goes exponentially to 0 as  $\tau \rightarrow \infty$ . The shift parameter  $\tau_o(x)$  is chosen at each  $x$  so that  $h_{0o}(x) + \omega_o(x) = f_o(x)$ . Since

$$(8.12) \quad 0 < (h_{0o} + \omega_o) + h_{0+} + \omega_+ + (h_{0-} + \omega_-) = -3h_{0o} + (\omega_o + \omega_+ + \omega_-)$$

then any suitable initial value of  $h_{0o}$  can be attained from (8.10) by a proper choice of  $\tau_o$ .

We only briefly describe the rest of the expansion since it is exactly as in [2] (only the tanh form of  $h_{0o}$  was left out of that account). From (8.6) and (8.4) we see that

$$(8.13) \quad \langle \psi_i, h_1(\tau) \rangle = - \int_{\tau}^{\infty} \langle \psi_i, v \frac{\partial}{\partial x} h_0(s) \rangle ds$$

This gives the initial values for  $\rho_1$  and  $m_1$  in (5.9) and (5.12) since (8.9) implies that

$$(8.14) \quad \left\{ \begin{array}{l} \rho_1(t=0) = - \langle \psi_1, h_1(\tau=0) \rangle \\ m_1(t=0) = - \langle \psi_2, h_2(\tau=0) \rangle \end{array} \right.$$

The two relations in (8.13) allow (8.6) to be reduced to a single linear first order equation for  $h_{10}$  with a negative linear coefficient. Since the inhomogeneities in the equation decrease to 0 exponentially as  $\tau \rightarrow \infty$ , so will  $h_{10}$ . The initial value of  $h_{10}$  is found from the initial values of  $\omega$ ,  $\rho_1$  and  $m_1$  through (8.3.) and (5.7).

REFERENCES

- [1] *BROADWELL G.E.*  
Shock structure in a simple discrete velocity gas.  
Phys. Fluids 7, 1964, pp. 1243-1247.
- [2] *CAFLISCH R.E. - PAPANICOLAOU G.C.*  
The fluid dynamical limit of a nonlinear model Boltzmann equation.  
Comm. Pure Appl. Math 32, 1979, pp. 589-616.
- [3] *CAFLISCH R.E.*  
Navier Stokes and Boltzmann shock profiles for a model of gas dynamics.  
Comm. Pure Appl. Math. 32, 1979, pp. 521-554.
- [4] *CAFLISCH R.E.*  
The fluid dynamic limit of the nonlinear Boltzmann equation.  
Comm. Pure Appl. Math. 33, 1980, pp. 651-666.
- [5] *ELLIS R. - PINSKY M.*  
The first and second fluid approximations to the linearized Boltzmann equation.  
J. Math. Pures. Appl. 54, 1975, pp. 125-156.
- [6] *GATIGNOL R.*  
Contribution à la théorie cinétique des gaz à répartition discrete de vitesses.  
These de doctorat, Paris 1973.
- [7] *GRAD H.*  
Asymptotic equivalence of the Navier Stokes and non-linear Boltzmann equations.  
Proc. Symp. Appl. XVII Appl. Nonlin. PDE in Math. Phys., 1965, Providence, R.I. pp. 154-183.
- [8] *GRAD H.*  
Singular and non-uniform limits of solutions of the Boltzmann equation.  
SIAM-AMS Proceedings, I. Transport Theory, 1969, Providence, R.I., pp. 296-308.

- [9] *INOUE H. - NISHIDA T.*  
On the Broadwell model of the Boltzmann equation for a simple discrete velocity gas.  
Appl. Math. Optim. 3, 1976, pp. 27-49.
- [10] *KAWASHIMA S. - MATSUMURA A. - NISHIDA T.*  
On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier Stokes equation.  
Comm. Math. Phys. 70, 1979, pp. 97-124.
- [11] *KURTZ T.G.*  
Convergence of sequences of semi-groups of nonlinear operators with an application to gas kinetics.  
Trans. AMS 186, 1973, pp. 259-272.
- [12] *McKEAN H.P.*  
The central limit theorem for Carleman's equation.  
Israel J. Math. 21, 1975, pp. 54-92.
- [13] *NICOLAENKO B.*  
Shock wave solutions of the Boltzmann equation as a nonlinear bifurcation problem from the essential spectrum.  
Colloques CNRS, th. Cinétiques et Relativistes.
- [14] *NISHIDA T.*  
Fluid dynamical limit of the nonlinear Boltzmann equation in the level of the compressible Euler equations.  
Comm. Math. Phys. 61, 1978, pp. 119-148.
- [15] *TARTAR L.*  
Existence globale pour un système hyperbolique semilinéaire de la théorie cinétique des gaz.  
Séminaire Goulaouis-Schwarz 1975/76, n° 1, MR 57 # 6865.

Imprimé en France  
par  
l'Institut National de Recherche en Informatique et en Automatique

