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**A SOUND AND COMPLETE  
AXIOMATIZATION OF EMBEDDED  
CROSS DEPENDENCIES**

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A SOUND AND COMPLETE  
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Abstract

A sound and complete axiomatization of embedded cross dependencies is given. It is proven to be complete through the study of the dual structure of non decomposable sets and the exhibition of an Armstrong Relation for a family of cross dependencies.

Résumé

Un système d'axiomes valide et complet pour les dépendances produit des relations n-aires est donné. La preuve de la complétude est obtenue par l'étude de la structure duale des ensembles d'attributs non décomposables. On exhibe une "Relation d'Armstrong" pour toute famille de dépendances produit.

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## 1. - INTRODUCTION

In the context of the relational model (with which we will assume the reader familiar) many types of data dependencies have been defined : functional dependency (FD's) multivalued (MVD's), join dependencies (JD's). One of the best characterization one can give to such dependencies is a sound and complete set of axioms : A set A of axioms is sound, if given a set F of dependencies every new dependency f that can be derived from F using A is implied by F; it is complete if every f that is implied by F is derivable from F using A.

Complete and sound sets of axioms have been given for FD's, [Arms 74] for MVD's, for FD's and MVD's [Beer 77]. It was also shown that no complete and sound axiomatization could be found for embedded MVD's.

In this paper we study a special case of MVD's, cross dependencies and we give a sound and complete axiomatization for embedded cross dependencies.

This is done through the study of the dual structure of non decomposable sets, and the exhibition of an Armstrong Relation (a relation that satisfies exactly a set of cross dependencies).

## 2. - CROSS DEPENDENCIES

### Definition 1.

Let U be a set of attributes.

A cross dependency Y over U is a set of sets such that

$$(i) \quad X \in Y \Rightarrow X \subseteq U \text{ and } X \neq \emptyset$$

$$(ii) \quad X \in Y, X' \in Y \text{ and } X \neq X' \Rightarrow X \cap X' = \emptyset$$

□

Let  $Y$  be a cross dependency (CD) over  $U$ , we denote  $\text{SCOPE}(Y)$  the set

$$\text{SCOPE}(Y) = \bigcup_{X \in Y} X$$

and we say that the CD is total if  $\text{SCOPE}(Y) = U$  and partial otherwise.

**Definition 2.** Let  $Y$  be a CD over  $U$  and  $R$  be a relation over  $U$ , we say that  $R$  satisfies  $Y$  ( $R \models Y$ ) iff

$$R(\text{SCOPE}(Y)) = \prod_{X \in Y} R(X)$$

where  $\prod$  denotes the cartesian product. Let  $F$  be a family of CD's, we say that  $R$  satisfies  $F$  if it satisfies each  $Y$  in  $F$ . Finally, given  $R$  we denote  $\text{CDF}(R)$  the set of CD's satisfied by  $R$ . We are concerned here with the characterization of the class of cross dependency families (CDF) that are satisfied by relations i.e.

$$\mathcal{C} = \{F \mid \exists R \text{ such that } \text{CDF}(R) = F\}$$

such a characterization is usually done through a set of axioms that must be both sound and complete.

Axioms are sound if  $\forall F \in \mathcal{C}$ ,  $F$  verifies the axioms and they are complete if

$$\forall F \text{ satisfying the axioms, } F \in \mathcal{C} .$$

**Theorem 1.** The following set of axioms is sound for CD's

C1 :  $\forall X \subseteq U$ ,  $\{X\}$  is a CD (trivial CD)

C2 : Let  $Y$  be a CD

Let  $Z \subset U$

Then  $Y_Z = \{X \cap Z \mid X \in Y \text{ and } X \cap Z \neq \emptyset\}$  is a CD  
(Projection)

C3 : Let  $Y = \{X_1, X_2, X_3, \dots, X_n\}$  be a CD then  $Y = \{X_1 X_2, X_3, \dots, X_n\}$  is also a CD (Branch Clustering)

C4 : Let  $Y = \{X_1 X_2, X_3, \dots, X_n\}$  and  $Y' = \{X_1, X_2\}$  be two CD's then  $Y'' = \{X_1, X_2, X_3, \dots, X_n\}$  is also a CD (Transitivity)

□

**Proof :** These axioms are just special cases of the FOHD's axioms given in [Delo 78] .

The main result of this paper is to show that  $C_1, C_2, C_3, C_4$  are indeed complete. To show this we must at first study the notion of non decomposition.

### 3. - NON DECOMPOSITION

**Definition 3.** Let  $U$  be an attribute set.

A non decomposition (ND) over  $U$  is a subset  $X \subseteq U$ .

We say that  $R(U)$  satisfies the ND  $X$  ( $R \models X$ ) iff it does not satisfy any non trivial CD with scope  $X$ .

Finally we denote  $ND(R)$  the set of ND's satisfied by  $R$ .

**Theorem 2.** The following set of axioms is sound for the class of ND's :

N1 :  $\forall A \in U, \{A\}$  is an ND (Trivial ND's)

N2 : If  $Y_1$  and  $Y_2$  are ND's and  $Y_1 \cap Y_2 \neq \emptyset$  then  $Y_1 \cup Y_2$  is an ND. (Transitivity).

□

**Proof :**

- (1)  $N_1$  is trivial
- (2) assume  $Y_1, Y_2$  are ND's with  $Y_1 \cap Y_2 \neq \emptyset$  and assume  $Y_1 \cup Y_2$  is not an ND.

Then there is some partition of  $Y_1 \cup Y_2 \neq \emptyset$  which is a cross decomposition.

Projecting this partition on  $Y_1$  and  $Y_2$  we will generate a non trivial CD with scope  $Y_1$  or  $Y_2$ , hence a contradiction.

Q.E.D.

**4. - RELATIONSHIP BETWEEN CD's AND ND's**

We first associate with each CDF an NDF :

**Definition 4.**

Let  $F$  be a family of CD's. The associated family of ND's  $\bar{F}$  is defined by :

$$\bar{F} = \{Y \mid Y \subseteq U \text{ and } (\forall Z \in F, \text{SCOPE}(Z) = Y \Rightarrow Z \text{ is trivial})\}$$

□

Intuitively it is the greatest family of ND's that can be satisfied knowing that  $F$  is satisfied.

**Definition 5.** Let  $F$  be a family of ND's over  $U$  let  $Y \subseteq U$  we define

$$\text{MAX}_F(Y) = \{Z \mid Z \in F, Z \subseteq Y \text{ and } (\forall Z' \in F, Z \subseteq Z' \subseteq Y \Rightarrow Z' = Z)\}$$

**Theorem 3.** Let  $F$  satisfy  $N_1$  and  $N_2$  then

$\text{MAX}_F(Y)$  is a partition of  $Y$  for all  $Y \subseteq U$ .



Proof :

- (1) Let  $Z_1$  and  $Z_2 \in \text{MAX}_F(Y)$  and assume  $Z_1 \cap Z_2 \neq \emptyset$ .  
Then  $Z_1 \in F$  and  $Z_2 \in F$  and  $F$  satisfies  $N_2 \Rightarrow$   
 $Z_1 \cup Z_2 \in F$  and we have

$$Z_1 \subseteq Z_1 \cup Z_2 \subseteq Y \Rightarrow Z_1 \cup Z_2 = Z_1$$

symmetrically  $Z_1 \cup Z_2 = Z_2 = Z_1$

- (2) To see that  $\text{MAX}_F(Y)$  is a covering of  $Y$  we just have to remember that each  $A \in Y$  is in  $F$  i.e. each element  $A$  will be covered by some element of  $F$ .

Q.E.D.

We can now associate with each NDF a CDF :

Definition 6. Let  $F$  be a family of ND's satisfying  $N_1$  and  $N_2$ . We define the associated family of CD's  $\bar{F}$  by

$$\bar{F} = \{ \text{MAX}_F(Y) \mid Y \subseteq U \}$$

□

Let us now prove a set of results concerning the relationship between  $F$  and  $\bar{F}$ .

Theorem 4. Let  $F$  be a family of CD's satisfying  $C_1, C_2, C_3, C_4$ . Then  $\bar{F}$  satisfies  $N_1$  and  $N_2$ .

Proof :

$$\bar{F} = \{ Y \mid \forall Z \in F, \text{SCOPE}(Z)=Y \Rightarrow Z \text{ trivial} \}$$

N1 :  $\{A\} \in \bar{F}$  for all  $A \in A$   
 $\forall A \in U, \forall Z \in F$  if  $\text{SCOPE}(Z) = \{A\}$ . Then  $Z$  is trivial.

Q.E.D.

N2 : Assume  $X_1 \in \bar{F}$  and  $X_2 \in \bar{F}$  and  $X_1 \cap X_2 = \emptyset$  then

$$\forall Z_1 \in F \text{ s.t. } \text{SCOPE}(Z_1) = X_1 \quad Z_1 = \{X_1\}$$

$$\forall Z_2 \in F \text{ s.t. } \text{SCOPE}(Z_2) = X_2 \quad Z_2 = \{X_2\}$$

Let  $Z$  be such that  $\text{SCOPE}(Z) = X_1 \cup X_2$  by the projection axioms  $Z$  can be projected on  $X_1$  and  $X_2$ . It is clear that if  $Z$  is non trivial at least one of these projections will be non trivial, hence a contradiction, therefore  $Z$  is trivial and  $X_1 \cup X_2 \in F$ .

Q.E.D.

**Lemma 1.** If  $R$  satisfies a family of ND's  $F$  and  $\bar{F}$  then  $\text{NDF}(R) = F$ .

**Proof :**  $X \in \text{NDF}(R)$   
 Assume  $X \notin F$   
 then  $\text{MAX}_F(X)$  has at least two elements i.e. there is  $(X_1, X_2)$  in  $\bar{F}$  non trivial and with scope  $X$ , i.e.  $X$  is decomposable hence a contradiction.

Q.E.D.

**Theorem 5.** If  $R$  satisfies a family of ND's  $F$  and  $\bar{F}$  then

$$\text{CDF}(R) = (\bar{F})^*$$

**Proof :** Let  $Y = (X_1, X_2, \dots, X_n) \in \text{CDF}(R)$  and consider

$$Y' = \bigcup_i \text{MAX}_F(X_i)$$

(1) from transitivity axiom :  $Y' \Rightarrow Y$

(2)  $Y' \in \bar{F}$  because

$NDF(R) = F \Rightarrow \forall X \in F, X \subseteq SCOPE(Y)$   
 $X \in \text{MAX}_{\bar{F}}(X_i)$  otherwise  $X$  would be decomposable

therefore  $Y' = \text{MAX}_{\bar{F}}(\bigcup_i X_i)$

i.e.  $\forall Y \in \bar{F} \Rightarrow Y$

Q.E.D.

**Theorem 6.** Let  $F$  be a family of CD's satisfying  $C_1 C_2 C_3 C_4$  then  $(\bar{F})^{\times} = F$ .

Proof :  $F \subseteq (\bar{F})^{\times}$

Let  $Y = \{X_1, X_2, \dots, X_n\} \in F$  and let  $Y' = \text{MAX}_{\bar{F}}(SCOPE(Y))$

claim  $\forall X \in Y', \exists X_i \in Y$  s.t.  $X \subseteq X_i$

proof : assume it is not true. Then there exists some  $X \in Y'$  not included in any  $X_i$ . It is therefore covered by more than one  $X_i$ , assume by  $X_1$  and  $X_2$  (the proof generalises easily)

$(X_1, X_2) \in F$  and by the projection axiom  $(X_1 \cap X_2, X_1 \cap X) \in F$  therefore  $X \notin \bar{F}$  and  $X$  cannot be in  $\text{MAX}_{\bar{F}}(SCOPE(Y))$ .

Q.E.D.

Therefore  $Y'$  is a refinement of  $Y$ . By the clustering axiom  $Y' \Rightarrow Y$  and  $Y' \in \bar{F} \Rightarrow Y \in (\bar{F})^{\times}$

Q.E.D.

Proof :  $(\bar{F})^{\times} \subseteq F$

$F$  being closed under  $C_1 C_2 C_3 C_4$  it is sufficient to show  $\bar{F} \subseteq F$

Let  $Y \in \bar{F}$  then there is some  $Z$  s.t.

$$Y = \text{MAX}_{\bar{F}}(Z) = (X_1 X_2 \dots X_n)$$

assume  $Y \in F$  then

$$\exists X_i \text{ s.t. } (X_i, \bigcup_{j \neq i} X_j) \notin F$$

(otherwise by projection and transitivity  $Y \in F$ )  
therefore there exists some  $T \in \bar{F}$  that intersects  
both  $X_i$  and  $\bigcup_{j \neq i} X_j$  and  $X_i$  is not a maximal  
undecomposable hence a contradiction.

Q.E.D.

### 5. - COMPLETENESS OF ND's

**Theorem 7.** Given a family of ND's  $F$  there exists a relation  $R_F$   
such that

$$R_F \models F \text{ and } R_F \not\models \bar{F}$$

**Proof :** We build the relation as follows :

(1) define some coding  $c : F \rightarrow N$

(2) For each  $A \in U$  define

$$D(A) = \{c(Y) \mid Y \in F, A \subseteq Y\}$$

(3) Define

$$R_0(A_1 A_2 \dots A_n) = D(A_1) \times D(A_2) \times \dots \times D(A_n)$$

(4) For each  $Y \in F, |Y| > 1$  delete from  $R_0$  all tuples  $x$   
such that

$$x(Y) = (c(Y), c(Y), \dots, c(Y))$$

Thus obtaining relation  $R_F$ .

Before proceeding to the proof let us look at an example

$$U = A B C$$

$$F = \{\{A\}, \{B\}, \{C\}, \{AB\}, \{ABC\}\}$$

define  $c$  as follows :

$$c(\{A\})=1 \quad ; \quad c(\{B\})=2 \quad ; \quad c(\{C\})=3 \quad ; \quad c(\{ABC\})=4 \quad ; \\ c(\{AB\})=5 \quad ;$$

Then

$$\begin{array}{ll} D(A) = \{1,4,5\} & (A \in \{A\}, \{AB\}, \{ABC\}) \\ D(B) = \{2,4,5\} & (B \in \{B\}, \{AB\}, \{ABC\}) \\ D(C) = \{3,4\} & (C \in \{C\}, \{ABC\}) \end{array}$$

$$R_o(ABC) = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

The final relation is

$$R_o(ABC) = \begin{bmatrix} 4 & 4 & 4 \\ 5 & 5 & 3 \\ 5 & 5 & 4 \end{bmatrix}$$

$[4 \ 4 \ 4]$  is deleted because  $\{ABC\} \in F$

$[5 \ 5 \ 3]$  and  $[5 \ 5 \ 4]$  are deleted because  $\{AB\} \in F$ .

It is easy to check that by removing these 3 tuples we have forbidden any decomposition of ABC on AB, while all others are satisfied. (A,C) and (B,C) are satisfied cross decompositions.

We now proceed to proving that  $R_F$  is indeed the good candidate.

Claim 1.

$Y \in F$  and  $X \not\subseteq Y$

Then  $(c(Y), c(Y), \dots, c(Y)) \in R_F(X)$

□

Proof :

$\forall A \in X \quad c(Y) \in R_0(A) \quad (\text{because } A \subseteq X \subseteq Y \in F)$

$\forall A \notin X \quad c(A) \in R_0(A) \quad (\text{because } \{A\} \in F)$

Denote  $U-X = B_1 B_2 \dots B_p$  (single attributes).

Let  $x_0$  be the following tuple :

R	X	$B_1 B_2 \dots B_p$
	$c(Y) \dots c(Y)$	$c(B_1) \dots c(B_p)$

$x_0 \in R_0(U)$

more over  $\forall Y' \in F, |Y'| > 1$

$$x_0(Y') \neq (c(Y') \dots c(Y'))$$

therefore  $x_0 \in R_F$  and  $x_0(X) \in R_F(X)$ .

Q.E.D.

Claim 2.

Let  $X \subseteq U, Y \subseteq U, X \cap Y = \emptyset$

if  $x \in R_F(X), y \in R_F(Y)$  and  $xy \notin R_F(XY)$

then  $\exists Z \subseteq XY, Z \in F$  such that

$$xy(z) = (c(z) \dots c(z)).$$

**Proof :**

$$x \in R_F(X) \Rightarrow x \in R_O(X)$$

$$y \in R_F(Y) \Rightarrow y \in R_O(Y)$$

therefore  $xy \in R_O(XY)$

$$\text{Let } T = U - XY = B_1 B_2 \dots B_p$$

Define tuple  $v$  as follows

R	X	Y	$B_1 \dots B_p$
$v$	$x$	$y$	$c(B_1) \dots c(B_p)$

Then  $v \in R_O$  and  $v(XY) = xy$  and since  $xy \notin R_F$ ,  $v \notin R_F$   
 i.e. it was erased because for some  $Z \in F$ ,  $|Z| > 1$

$$v(z) = (c(z) \dots c(z))$$

since all  $B_i$ 's have size 1,  $Z$  is necessarily included in  $XY$ .

**Claim 3.**

Let  $Y \subseteq U$  and  $MAX_F(Y) = \{X_1, X_2, \dots, X_n\}$

Let  $x_i \in R_F(X_i) \quad \forall i = 1, 2, \dots, n$

then  $x_1 x_2 \dots x_n \in R_F(X_1 X_2 \dots X_n)$ .

**Proof :**

Assume  $x = x_1 x_2 \dots x_n \notin R_F(X_1 \dots X_n)$

then by Claim 2,  $\exists Z \subseteq X_1 X_2 \dots X_n$  such that

$$x(Z) = c(z) c(z) \dots c(z)$$

(1) assume  $Z \subseteq X_i$  for some  $i$

$$\text{then } x_i(z) = (c(z) \dots c(z))$$

and  $x_i \notin R_f(X_i)$  which is a contradiction.

- (2) Then it is necessarily the case that  $Z$  intersects two  $X_i$ 's say  $X_1$  and  $X_2$ , then since  $X_1 \in F$ ,  $X_2 \in F$  and  $Z \in F$ , by rule  $N_2$  :  $X_1 \cup X_2 \in F$   
i.e.  $X_1$  and  $X_2$  are not maximal in  $Y$  which is a contradiction

Q.E.D.

This claim is clearly equivalent to  $R$  satisfies  $\bar{F}$ .

**Claim 4.**

Let  $X \in F$  then  $R_F(X)$  is not decomposable.  
(which is equivalent to say  $R \not\models F$ ).

**Proof :**

Assume  $|X| > 1$  (otherwise the result is trivial)  
then  $X = AX_1$  with  $|X_1| \geq 1$   
 $(c(x)) \in R_F(A)$   
 $(c(x) \dots c(x)) \in R_F(X_1)$   
and  $(c(x) \dots c(x)) \notin R_F(X_1, A)$

Q.E.D.

**6. - COMPLETENESS OF CD**

We can now state the main theorem of this paper :

**Theorem 8**

Axioms  $C_1, C_2, C_3, C_4$  are complete for CD's.

**Proof :**

Let  $F$  satisfy  $C_1, C_2, C_3, C_4$   
By theorem 4,  $\bar{F}$  satisfies  $N_1$  and  $N_2$   
By theorem 7, there exists  $R_{\bar{F}}$  such that

$$R_{\bar{F}} \models \bar{F} \text{ and } R_{\bar{F}} \models \bar{F}^*$$



By theorem 6,  $\overline{F}^* = F$  i.e.

$R_{\overline{F}}$  satisfies  $F$  and  $\overline{F}$

Finally by theorem 5

$$\text{CDF}(R_{\overline{F}}) = F$$

Q.E.D.

The reader should note that  $R_{\overline{F}}$  is indeed an armstrong relation for  $F$ .

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