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QUALITATIVE PROPERTIES OF A NONLINEAR SYSTEM FOR LAMINAR FLAMES WITHOUT IGNITION TEMPERATURE

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Qualitative properties of a nonlinear system for laminar flames without ignition temperature

Martine MARION *

Abstract : We are interested in a system of two nonlinear differential equations $-u'' + c\,u' = f_\varepsilon(u)v$, $-\lambda v'' + c\,v' = -f_\varepsilon(u)v$ on $\mathbb{R}$, with the boundary conditions : $u(-\infty) = 0$, $v(-\infty) = 1$, $u(+\infty) = 1$, $v(+\infty) = 0$, where the unknowns are the functions $u,v : \mathbb{R} \to [0,1]$ and the scalar c. This system arises in the combustion of laminar plane flames. f_ε is $\mathscr{C}^1$ with $f_\varepsilon > 0$ on $]0,1]$ and $f_\varepsilon(0) = 0$. Different qualitative properties of the model are studied here : existence of solutions (we prove that there is a continuum of solutions for $c \geq c_{min}$), asymptotic behaviour of these solutions for an infinite activation energy, relationship with the model of ignition temperature, and lastly convergence of the bounded interval approximation.

Etude mathématique d'un modèle de flamme laminaire sans température d'ignition : II - Cas du système

Martine MARION *

Résumé : On s'intéresse à un système de deux équations différentielles non linéaires $-u'' + c\,u' = f_\varepsilon(u)v$, $-\lambda v'' + c\,v' = -f_\varepsilon(u)v$ sur $\mathbb{R}$, avec les conditions aux limites : $u(-\infty) = 0$, $v(-\infty) = 1$, $u(+\infty) = 1$, $v(+\infty) = 0$, où les inconnues sont les deux applications $u,v : \mathbb{R} \to [0,1]$, et le scalaire c. Ces équations interviennent dans la combustion de flammes laminaires planes. f_ε est $\mathscr{C}^1$ avec $f_\varepsilon > 0$ sur $]0,1]$ et $f_\varepsilon(0) = 0$. Différentes propriétés qualitatives du modèle sont étudiées ici : existence de solutions (on démontre qu'il existe un continuum de solutions pour $c \geq c_{min}$), comportement asymptotique de ces solutions pour une énergie d'activation infinie, lien avec le modèle de la température d'ignition, et enfin, convergence de l'approximation sur un intervalle borné.

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PAPIER RECUPERE ET RECYCLE

0. INTRODUCTION.

One of the simplest problems of combustion is the planar premixed flame, that is the one-dimensional deflagration wave. In a case of a single-step reaction involving one reactant, it reduces to the following system of two reaction-diffusion equations (see Buckmaster and Ludford [2], Williams [10]), where the unknowns are the functions $u,v : \mathbb{R} \to [0,1]$ and the scalar $c \in \mathbb{R}$:

$$(0.1)\quad \begin{cases} -u''+c\,u' = f(u)v, \\ -\lambda v''+c\,v' = -f(u)v, \\ u(-\infty) = 0, \quad u(+\infty) = 1, \\ v(-\infty) = 1, \quad v(+\infty) = 0. \end{cases}$$

u is the (reduced) temperature of the mixture, v is the (reduced) concentration of the reactant and c is an (unknown) massflux. The parameter $\lambda > 0$ is the inverse of the Lewis number. f is a renormalized reaction term depending on a parameter $\varepsilon > 0$ which is the inverse of the activation energy.

The exact term f arising from Arrhenius Law satisfies $f > 0$ on $[0,1]$; in particular, since $f(0) > 0$, it is easily seen that (0.1) has no solution ; this is the classical "cold boundary difficulty" well known to the chemists. Therefore, one modifies the reaction term. A first approach is to introduce an ignition temperature θ : one assumes that there exists some $\theta \in\ ]0,1[$ such that

(0.2) $f \equiv 0$ on $[0,\theta)$ and $f > 0$ on $(\theta,1]$.

Another way is to alter the reaction term so that f satisfies :

(0.3) $f > 0$ on $]0,1]$ and $f(0) = 0$.

Our purpose here is the study of this latter model. The first model has been considered in many works and more recently by Berestycki, Nicolaenko and Scheurer (see [1],and the references given theiren). Under condition (0.2) with f Lipschitz continuous except possibly at θ, these authors proved, by a topological degree argument, that (0.1) has a solution (whose uniqueness they conjectured) ; moreover, they derived the asymptotic behaviour of this solution, as $\varepsilon \to 0$.

The second model was only studied rigorously in the case $\lambda = 1$, which is very particular, because it reduces to a scalar equation (indeed, one then has $v = 1 - u$). For this

equation, Fife [4], Johnson [5], Uchiyama [9] showed that there exists a continuum of solutions (for $c \geq c_0 > 0$). Their approach relies on a phase plane argument. In [7], we derived the same result by a shooting method, and we obtained many further qualitative properties ; in particular, we investigated the asymptotic behaviour of the different solutions, as $\varepsilon \to 0$.

The present paper is a sequel of [7] and deals with the model with no ignition temperature in the case of any Lewis number (that is any $\lambda > 0$). Following our study of the scalar case in [7], we consider the model from a triple point of view. - Determination of the set of the solutions. - Asymptotic analysis of these solutions for an infinite activation energy (this type of asymptotics is very much used in practice; see [2,10]). - Relationship with the model with ignition temperature. Furthermore, in view of numerical calculations, we investigate the approximation of Problem (0.1) by an analogous one on a bounded interval $I_a = [-a,+a]$.

For a technical reason (monotony lemma), we obtain much more detailed answers in the case $\lambda < 1$. The paper is divided into two parts : in Section 1, we consider the general case $\lambda > 0$; then, Section 2 is devoted to the case $\lambda < 1$. More precisely, the plan of the paper is as follows :

0. Introduction.
1. General results.
   - 1.1 The solutions of (0.1).
   - 1.2 The role of the ignition temperature.
     - 1.2.1 Construction of a solution of (0.1).
     - 1.2.2 Asymptotic analysis.
   - 1.3 The problem on a bounded interval.
3. Case of the Lewis number larger than 1.
   - 2.1 Monotony lemma.
   - 2.2 Applications.
     - 2.2.1 Determination of the set of solutions of (0.1).
     - 2.2.2 Relationship with the model with ignition temperature.
   - 2.3 Further remarks on the asymptotic analysis.

While the proofs for the scalar case ([7]) are elementary, the study of the system asks for more elaborated techniques, namely, here, the topological degree and the (local) theory of dynamical systems.
The principal open questions are the asymptotic analysis (essentially the determination of $\lim_{\varepsilon\to 0} c_{o,\varepsilon}$, see below) and the case $\lambda > 1$.

Main results.

We assume that f satisfies :

$$(0.4)\quad \begin{cases} f \text{ is Lipschitz continuous on } [0,1], \\ f>0 \text{ on }]0,1] \text{ and } f(0) = 0. \end{cases}$$

The main existence results that we obtain (Theorems 1.1 and 2.7) can be summarized in the following :

Theorem 0.1 : *Under condition* (0.4) :

(a) *Let any* $\lambda > 0$ *be given. There exist two constants* $0 < \underline{c} \le \bar{c}$ *such that* $\forall c \le \underline{c}$ (0.1) *has no solution of the form* (u,v,c) *and* $\forall c \ge \bar{c}$ (0.1) *has a solution of the form* (u,v,c). *Moreover, under the additional condition that* f *is of class* $\mathscr{C}^1$ *in a neighbourhood of* 1, $\forall c > 0$, *there exists at most a couple* (u,v) *(up to a translation of the origin) such that* (u,v,c) *is a solution of* (0.1).

(b) *Suppose that* f *is of class* $\mathscr{C}^1$ *in a neighbourhood of* 1 *and let* $0< \lambda < 1$. *Then there exists* $c_o > 0$ *such that* (0.1) *has a solution of the form* (u,v,c) *if and only if* $c \ge c_o$.

The methods that we employ here enable us to show a new uniqueness result relative to the model with ignition temperature. Namely, we derive :

Theorem 0.2 : *Suppose that condition*(0.2) *holds with* f *Lipschitz continuous on* [0,1], *except possibly at* θ, *and of class* $\mathscr{C}^1$ *in a neighbourhood of* 1. *For any* $0 < \lambda < 1$, (0.1) *has a unique solution (up to a translation of the origin).*

Hence, as it is the case in the scalar equation, the existence results for the two models are quite different.

Next, we consider the relationship between the solutions of the two models. Let f satisfy (0.4) and, for any $\theta \in]0,1[$, let us introduce the following problem :

$$(0.5)\quad \begin{cases} -u'' + c\,u' = f(u)\,\chi_{[\theta,1]}(u)v, \\ -\lambda v'' + c\,v' = -f(u)\,\chi_{[\theta,1]}(u)v, \\ u(-\infty) = 0, \quad u(0) = z_o, \quad u(+\infty) = 1, \\ v(-\infty) = 1, \quad v(+\infty) = 0, \end{cases}$$

where $\chi_{[\theta,1]}$ denotes the characteristic function of $[\theta,1]$ and z_o is some fixed real $\in]0,1[$ (the condition $u(0)=z_o$ is a normalization related to the translational invariance). From [1], we know that, for any $\lambda > 0$, (0.5) has a solution $(u_\theta, v_\theta, c_\theta)$; in the case $\lambda < 1$, this solution is unique (Theorem 0.2 above). We will show here that there exists a sequence $(\theta_n)_n$ with $\lim \theta_n = 0$ such that $(u_{\theta_n}, v_{\theta_n}, c_{\theta_n})$ converges to a solution of (0.1) ; moreover, in the case $\lambda < 1$, the whole family $(u_\theta, v_\theta, c_\theta)$ converges, as $\theta \to 0$, and the limit is the solution (u_o, v_o, c_o) of (0.1) corresponding to the lower c, c_o, defined at Theorem 0.1 (b).

Lastly, we consider the question of large activation energy asymptotics. We assume that $f = f_\varepsilon$ behaves like $f_\varepsilon(x) = \frac{1}{\varepsilon^2} \exp\left(\frac{1}{\varepsilon}\frac{x-1}{x}\right)$ (see Section 1.2.2. for the precise conditions ; let us recall that ε is the inverse of the activation energy). In general, we only determine the asymptotic behaviour as $\varepsilon \to 0$ of the particular solution obtained above from the limiting model with ignition temperature. Indeed, for any $\varepsilon > 0$, let us denote by $(u_{o,\varepsilon}, v_{o,\varepsilon}, c_{o,\varepsilon})$ this solution. We show that $c_{o,\varepsilon}$ is bounded independently of $\varepsilon > 0$; moreover, if (ε_n) is a sequence with $\lim \varepsilon_n = 0$ and $\lim c_{o,\varepsilon_n} = c_o$, then $(u_{o,\varepsilon_n}, v_{o,\varepsilon_n})$ converges to the unique solution (u_o, v_o) of the following system :

$$\begin{cases} -u_o'' + c_o\,u_o' = c_o\,\delta_{x=x_o}, \\ -\lambda v_o'' + c_o\,v_o' = -c_o\,\delta_{x=x_o}, \\ u_o(-\infty) = 0, \qquad u_o(+\infty) = 1, \\ v_o(-\infty) = 1, \qquad v_o(+\infty) = 0, \end{cases}$$

where x_o is uniquely determined by the condition $u_o(0) = z_o$. One of the main steps in the proof of this result is to remark that the function $h_{o,\varepsilon} = u'_{o,\varepsilon}(u^{-1}_{o,\varepsilon})$ is differentiable at 0 and $h'_{o,\varepsilon}(0) = 0$.

In the case $\lambda < 1$, we obtain informations on the asymptotic behaviour of all the other solutions of (0.1) ; if c is large enough, the corresponding solution of (0.1) (defined for $\varepsilon \leq \varepsilon_o$) or, at least, a subsequence converges as $\varepsilon \to 0$ to a couple (u,v) which does not satisfy any more the boundary conditions at $-\infty$. This illustrates quite well the cold boundary difficulty. The absence of any ignition temperature leads to infinitely many solutions. But, here, we show rigorously that, at the limit $\varepsilon = 0$, only one solution satisfies the correct boundary conditions.

1 - GENERAL RESULTS

This section is devoted to the study of the laminar plane flame for any Lewis number. Section 1.1 deals with existence results. In Section 1.2, we investigate the relationship between the considered model and the one with ignition temperature and we derive the asymptotic behaviour for large activation energy of a particular solution of (0.1). In Section 1.3 we study the problem analogous to (0.1) on a bounded interval $[-a,+a]$ and the convergence of solutions as $a \to +\infty$. Through out this section, we assume condition (0.4) and we denote :

$$(1.1) \qquad \sigma = \sup_{x \in]0,1[} \frac{f(x)(1-x)}{x} ,$$

$$(1.2) \quad \begin{cases} \Lambda = \max \left(\dfrac{1}{\lambda+|\lambda-1|} , \dfrac{1}{1+|\lambda-1|} \right) , \\ \Lambda' = \min \left(\dfrac{\lambda+|\lambda-1|}{\lambda} , \dfrac{|\lambda-1|+1}{\lambda} \right) . \end{cases}$$

1.1. The solutions of (0.1).

Our main results can be summarized in the following :

Theorem 1.1 : Under condition (0.4) :

(i) $\forall c \leq \sqrt{2\Lambda \int_0^1 f(s)\,(1-s)\,ds}$, (0.1) *has no solution of the form* (u,v,c).

(ii) $\forall z_o \in]0,1[$, $\forall c \geq 2\sqrt{\sigma\Lambda'}$, (0.1) *has got a solution* (u,v,c) *satisfying* $u(0) = z_o$.

(ii) *If we assume, in addition, that* f *is of class* $\mathscr{C}^1$ *in a neighbourhood of* 1, *then,* $\forall c > 0$, *there exists at most one solution of the form* (u,v,c) *(up to a translation of the origin).*

The proof of this theorem will be given in three steps.

a) Preliminary results ; proof of (i).

Let us extend f to $\mathbb{R}$ by setting; :

$f(x) = 0 \quad \forall x \leq 0 , \qquad f(x) = f(1) \quad \forall x \geq 1.$

The first lemma states a classical and very useful result ; let us assume that, for some $c > 0$, there exist $x_0, x_1 \in]-\infty,+\infty]$ and two functions $u,v \in \mathscr{C}^2([x_0, x_1[, \mathbb{R})$ satisfying the following conditions :

(1.3) $\quad -u''+cu' = f(u)v$ and $-\lambda v'' + cv' = - f(u)v$ on $[x_0,x_1[$.

$$(1.4)\quad \begin{cases} u(x_1) = \lim\limits_{x \to x_1} u(x) = 1 , \quad v(x_1) = \lim\limits_{x \to x_1} v(x) = 0, \\ u'(x_1) = \lim\limits_{x \to x_1} u'(x) \geq 0 , \quad v'(x_1) = \lim\limits_{x \to x_1} v'(x) \leq 0 \text{ (one assumes that those} \\ \text{limits exist).} \end{cases}$$

(1.5) $\quad v \geq 0$ on $]x_0, x_1[$.

Lemma 1.2 : *Under conditions* (1.3), (1.4) *and* (1.5), *we have* :

(i) $u' \geq 0$ *and* $v' \leq 0$ *on* $[x_0, x_1[$.

(ii) *Moreover, under the additional assumption* :

(1.6) *There exists* $x_2 \in [x_0, x_1]$ *such that* $-u'(x_2) + c\,u(x_2) - \lambda\, v'(x_2) + c\,v(x_2) = c$

we have, $\forall x \in [x_0, x_1]$,

$$(1.7)\quad \begin{cases} |u(x) + v(x)-1| \leq |\lambda-1|\,v(x) \\ |u(x) + v(x)-1| \leq \left|\frac{\lambda-1}{\lambda}\right|(1-u(x)) \end{cases}$$

$$(1.8)\quad \begin{cases} |u(x) + \lambda v(x)-1| \leq |\lambda-1|\,v(x) \\ |u(x) + \lambda v(x)-1| \leq |\lambda-1|\,(1-u(x)) \end{cases}$$

The proof, wich is easy, will be omitted (see for example Berestycki, Nicolaenko and Scheurer [1]).

Corollary 1.3 : *Under conditions* (1.3), (1.4), (1.5) *and* (1.6), *we have*

(1.9) $\forall x \in [x_0, x_1] \qquad \Lambda(1-u(x)) \le v(x) \le \Lambda'(1-u(x))$.

Lemma 1.4 : *Suppose* u,v,c *is a solution of* (0.1). *Then* :

$$c > 0, \qquad 0 < u' \le c, \qquad \frac{-c}{\lambda} \le v' < 0,$$

u *and* v *satisfy the inequalities* (1.7), (1.8) *and* (1.9)

Proof : Let $a < b$ and let us integrate the equation for u on [a,b] to obtain the relation :

$$(1.10) \qquad -u'(b) + u'(a) + c\,(u(b)-u(a)) = \int_a^b f(u)v\,dx$$

From (1.10), it is easily seen that :

$$\lim_{x \to \pm\infty} u'(x) = 0 \quad \text{and} \quad c = \int_{-\infty}^{+\infty} f(u)v\,dx > 0$$

So we can apply lemma 1.2 (and Corollary 1.3) to (u,v).

Therefore it remains to show the inequalities : $0 < u' \le c$ and $-\frac{c}{\lambda} < v' \le 0$.

Suppose that $\exists\, x_0 \in \mathbb{R}$ with $u'(x_0) = 0$. If $f(u(x_0))v(x_0) > 0$, from the equation satisfied by u, we have $u''(x_0) < 0$, which is impossible (Lemma 1.2). On the other hand, if $f(u(x_0))\,v(x_0) = 0$, then $u(x_0) = 0$ or $v(x_0) = 0$, hence $u \equiv 0$ or $v \equiv 0$ which is again a contradiction.

Lastly $e^{-cx}\,u'(x) = \int_x^{+\infty} e^{-cs} f(u(s))v(s)\,ds \le e^{-cx} \int_x^{+\infty} f(u)v\,ds = e^{-cx}c$

Thus we have proved that $0 < u' \le c$. In the same way, it can be shown that $-\frac{c}{\lambda} \le v' < 0$. □

Lemma 1.5 : *Suppose* u, v, c *is a solution of* (0.1). *Then* :

$$c > \sqrt{2\Lambda \int_0^1 f(s)(1-s)\,ds}$$

Proof : Multiplying the equation for u successively by u and u' and integrating over $[a,b] \subset \mathbb{R}$, we obtain :

$$(1.11) \quad -u(b)\,u'(b) + u(a)u'(a) + \int_a^b u'^2 dx + \frac{c}{2}\,(u^2(b) - u^2(a)) = \int_a^b f(u)\,v\,u\,dx$$

$$(1.12) \quad -\frac{1}{2}\,u'^2(b) + \frac{1}{2}\,u'^2(a) + c\int_a^b u'^2\,dx = \int_a^b f(u)v\,u'\,dx$$

We can let $a \to -\infty$ and $b \to +\infty$ in (1.10), (1.11) and (1.12).

Since $\int_{-\infty}^{+\infty} f(u)\,v\,u\,dx < \int_{-\infty}^{+\infty} f(u)\,v\,dx$, (1.10) and (1.11) imply :

$$\int_{-\infty}^{+\infty} u'^2 dx < \frac{c}{2}\,.$$

Thus combining this last inequality with (1.2) and using the fact that $c > 0$ (lemma 1.4), we deduce

$$\int_{-\infty}^{+\infty} f(u)\,v\,u'\,dx < \frac{c^2}{2}$$

It is now easy to conclude the proof, as $v \geq \Lambda(1-u)$ and $u' > 0$. □

(b) Proof of Theorem 1.1 (ii)

The first (and main) step will consist in proving the following Proposition :

Proposition 1.6 : $\forall z_o \in]0,1[$ *and* $\forall\, c > 0$, *there exist* $u,v \in \mathscr{C}^2(\mathbb{R},\mathbb{R})$ *such that* :

$$(1.13)\begin{cases} -u'' + cu' = f(u)\,v \text{ and } -\lambda v'' + c\,v' = -f(u)\,v \quad \text{on } \mathbb{R}, \\ u(0) = z_o, \quad u(+\infty) = 1, \quad v(+\infty) = 0, \\ u' > 0 \text{ and } v' < 0 \quad \text{on } \mathbb{R}, \\ u \text{ and } v \text{ satisfy the relations (1.7), (1.8) and (1.9).} \end{cases}$$

We will then show that, if c is large enough, the couple defined by Proposition 1.6, satisfies $u(-\infty) = 0$ and $v(-\infty) = 1$ and hence is a solution of (0.1). This will complete the proof of Theorem 1.1 (ii).

Let $z_o \in]0,1[$ and $c > 0$. The couple (u,v) of Proposition 1.6 will be obtained as the limit as $a \to +\infty$ of the solutions of an auxiliary problem posed on a bounded interval $[0,a]$. More precisely, $\forall a > 0$, let us consider the following problem :

$$(1.14)\begin{cases} \text{To find } u,\ v \in \mathscr{C}^2([0,a],\ \mathbb{R}) \text{ and } \mu \geq 0 \text{ such that :} \\ -u''+cu' = f(u)v \text{ and } -\lambda v'' + cv' = -f(u)\,v \text{ on } [0,a], \\ u(0)=z_o,\quad u(a)=1,\quad u'(0)=\mu, \\ -\lambda v'(0) + c\,v(0) = c(1-z_o)+\mu,\quad v(a) = 0. \end{cases}$$

Our purpose is to show that, $\forall a > 0$, (1.14) has got a solution (u_a, v_a, μ_a) and that (u_a, v_a) converges, as $a \to +\infty$, to (u,v) satisfying the conclusions of Proposition 1.6.

We solve (1.14) by using a topological degree argument. More precisely, $\forall \tau \in [0,1]$ let us denote by (1.15) the problem analogous to (1.14), where f is replaced by τf. Let $E = \mathscr{C}^1([0,a],\ \mathbb{R}) \times \mathscr{C}^1([0,a],\ \mathbb{R}) \times \mathbb{R}$. $\forall \tau \in [0,1]$, let K_τ : $E \to E$ be defined by :

$$K_\tau(u,v,\mu) = (U(u,v,\mu),\ V(u,v,\mu),\ \mu - u\,(a) + 1)$$

where (U,V) is the unique solution of the following linear system :

$$\begin{cases} -U''+cU' = \tau f(u)v, \qquad -\lambda V'' + cV' = -\tau f(u)v, \\ U(0) = z_o, \qquad U'(0) = \mu, \\ -\lambda V'(0) + c\,V(0) = c\,(1-z_o) + \mu, \qquad V(a) = 0. \end{cases}$$

Then the solutions of (1.15) are precisely the fixed points of K_τ in E. It is easily seen that, $\forall \tau$, K_τ is a compact operator and that the application $\tau \to K_\tau$ is continuous. In the next two lemmae, we derive the a-priori estimates which will enable us to use a topological degree argument.

Lemma 1.7 : *Suppose* u_a, v_a, μ_a *is a solution of* (1.15). *Then* :

$v_a \geq 0$ *on* $[0,a]$, $\quad u'_a > 0$ *and* $v'_a < 0$ *on* $[0,a]$,

u_a *and* v_a *satisfy the relations* (1.7), (1.8) *and* (1.9) *on* $[0,a]$.

Proof : Let us argue by contradiction and suppose that $\min_{[0,a]} v_a(x) < 0$.

Then either there exists $x \in [0,a]$ with $v_a(x) < 0$ and $v'_a(x) = 0$, or $\forall x \in [0,a]$ such that $v_a(x) < 0$, one has $v'_a(x) > 0$. In the first case, the maximum principle leads to a contradiction. In the second case, we would have $-\lambda v'_a(0) + cv_a(0) < 0$, which is impossible since $-\lambda v'_a(0) + c v_a(0) = c(1-z_o) + \mu_a$. Now, since $v_a \geq 0$, one has $v'_a(a) < 0$ (thus $v'_a < 0$ on $[0,a]$). Hence, from $-u'_a + c u_a - \lambda v'_a + c v_a = c$ on $[0,a]$, we can deduce $u''_a(a) > 0$ (thus $u'_a > 0$ on $[0,a]$). Lastly , according to lemma 1.2 and Corollary 1.3, u_a, v_a satisfy the relations (1.7)-(1.9) on $[0,a]$. □

Lemma 1.8 : $\forall a_o > 0$, *there exists a constant* k $(=k(a_o))$ *such that*,

$\forall a \geq a_o$, $\forall \tau \in [0,1]$, *one has* $\|u_a\|_{\mathscr{C}^1([0,a],\mathbb{R})} \leq k$ *and*

$\|v_a\|_{\mathscr{C}^1([0,a],\mathbb{R})} \leq k$, *where* (u_a, v_a, μ_a) *is any solution of* (1.15).

Proof : Let $a \geq a_o > 0$. From (1.12),we have :

$$\forall x \in [0,a] \quad u'^2_a(x) = u'^2_a(0) + 2c \int_0^x u'^2_a \, ds - 2 \int_0^x \tau f(u_a) \, v_a u'_a \, ds$$

We can now deduce,using the lemma 1.7 (and (1.9)), that :

$$\forall x \in [0,a] \quad u'^2_a(x) \geq u'^2_a(0) - 2\Lambda' \tau \int_{z_o}^1 f(s)(1-s) \, ds$$

Thus, if $u'_a(0) > \sqrt{2\Lambda' \tau \int_{z_o}^1 f(s)(1-s) \, ds} = c_1$, we obtain

$$1-z_o = u_a(a) - u_a(0) \geq a \sqrt{u'^2_a(0) - c_1^2} \geq a_o \sqrt{u'^2_a(0) - c_1^2}$$

that is : $u_a'(0) \leq \sqrt{c_1^2 + \left(\frac{1-z_o}{a_o}\right)^2}$

Therefore, in any case, we have $u_a'(0) \leq \sqrt{c_1^2 + \left(\frac{1-z_o}{a_o}\right)^2}$

hence : $u_a'(0) \leq \sqrt{2\Lambda' \int_{z_o}^{1} f(s)(1-s)ds + \left(\frac{1-z_o}{a_o}\right)^2} = \alpha(a_o)$

Notice that $\alpha(a_o)$ does not depend on τ. Besides, from Lemma 1.7, we know that $z_o \leq u_a \leq 1$, $0 \leq v_a \leq \Lambda'(1-z_o)$ and $u_a' > 0$ on $[0,a]$. Hence, since $-u_a' + c\,u_a$ is an increasing function, we obtain $\forall x \in [0,a]$ $u_a'(x) \leq \alpha(a_o) + c(1-z_o)$. It is now easy to conclude the proof, as $v_a' = \frac{1}{\lambda}(-u_a' + c\,u_a + c\,v_a - c)$ on $[0,a]$. □

We can now state an existence result for the problem (1.14) :

<u>*Lemma 1.9*</u> : *$\forall a > 0$, there exists a solution (u_a, v_a, μ_a) of the problem (1.14). Moreover, $\forall a_o > 0$, u_a and v_a are bounded independently of $a \geq a_o$ in $\mathscr{C}^2([0,a],\mathbb{R})$.*

<u>Proof</u> : Let $a \geq a_o > 0$. According to the lemmae 1.7 and 1.8, we have : $\forall \tau \in [0,1]$, $(0,0,0) \notin (I-K_\tau)(\partial\Omega)$, where

$$\Omega = \{(u,v,\mu) \in E \mid \|u\|_{\mathscr{C}^1([0,a])}, \|v\|_{\mathscr{C}^1([0,a])} < 2\,k(a_o),\ 0 < \mu < 2\,k(a_o)\}$$

Thus the Leray-Schauder degree of $(I-K_\tau)$ with respect to Ω and 0 is well-defined and further is independent of $\tau \in [0,1]$ (see [8] for the definition and the main properties of the Leray-Schauder degree).

It remains to compute this degree (and to show that it is different from 0). In order to do this, let us explicitly write down $\Phi_o = I-K_o$:
$\Phi_o((u,v,\mu) = (u-U_o(\mu),\ v-V_o(\mu),\ u(a)-1)$
where $U_o(\mu)$ is the function : $x \to \frac{\mu}{c}(e^{cx}-1)$ and $V_o(\mu)$ the function : $x \to (1-z_o+\frac{\mu}{c})(1-e^{\frac{c}{\lambda}(x-a)})$. $\forall t \in [0,1]$, let us introduce the operator $\Phi_t : E \to E$:

$$\Phi_t((u,v,\mu)) = (u-u_o(\mu),\ v-V_o(\mu),\ (1-t)(u(a)-1) + t(\tfrac{\mu}{c}(e^{ca}-1)+z_o))$$

It is easily seen that the map : $t \to \Phi_t$ is continuous and that, $\forall t \in [0,1]$, $(0,0,0) \notin \Phi_t(\partial\Omega)$. Therefore $d(\Phi_o, \Omega, 0) = d(\Phi_1, \Omega, 0) = 1$, since the function : $\mu \to \frac{\mu}{c}(e^{ca}-1)+z_o$ is increasing.

Thus we proved $d(I-K_1, \Omega, 0) \neq 0$. It follows that there exists a solution (u_a, v_a, μ_a) of the problem (1.14) and u_a and v_a are bounded independently of $a \geq a_o$ in $\mathscr{C}^2([0,a], \mathbb{R})$. □

<u>End of the proof of Proposition 1.6</u> : We claim that (u_a, v_a) defined at Lemma 1.9 converges as $a \to +\infty$ to (u,v) satisfying (1.13). Indeed let us extend u_a and v_a to $\mathbb{R}^+$ by setting $\forall x \geq a$ $u_a(x)=1$ and $v_a(x)=0$. Since, by Lemma 1.9, u_a and v_a are bounded independently of $a \geq a_o$ in $H^2_{loc}(\mathbb{R}^+)$, there exists a sequence (a_n) with $\lim a_n = +\infty$ such that $u_{a_n} \to u$ and $v_{a_n} \to v$ in $\mathscr{C}^1_{loc}(\mathbb{R}^+)$.

We remark that (u,v) satisfy;

$$\begin{cases} -u'' + c\,u' = f(u)v, & \text{on } \mathbb{R}^+ \\ -\lambda v'' + c\,v' = -f(u)v, \end{cases}$$

and

$$\begin{cases} -u' + c\,u - \lambda v' + c\,v = c \text{ on } \mathbb{R}^+, \\ u' \geq 0 \text{ and } z_o \leq u < 1 \ , \ v' \leq 0 \text{ and } 0 \leq v \leq \Lambda'(1-z_o), \\ u(0) = z_o \ , \ v(0) \geq \Lambda\,(1-z_o). \end{cases}$$

We can now see that $\ell_1 = u(+\infty) = 1$ and $\ell_2 = v(+\infty) = 0$. Indeed, integrating the equation satisfied by u (resp v), we get that $u'(+\infty) = \lim_{x\to+\infty} u'(x)$ (resp $v'(+\infty)$) exists and hence is equal to 0. Then, the equation satisfied by u yields $u''(+\infty)=0$ and $f(\ell_1)\ell_2=0$. Therefore, as $z_o \leq \ell_1 \leq 1$, we obtain $\ell_2=0$, and thus $\ell_1=1$, since $-u'(+\infty)+c\,u\,(+\infty) - \lambda\, v'(+\infty) + c\,v\,(+\infty) = c$.

(u,v) can be extended in a unique way to a maximal solution, denoted again by (u,v), of :

$$\begin{cases} -u'' + c\,u' = f(u)v, \\ -\lambda v'' + c\,v' = -f(u)v. \end{cases}$$

We know that $v \geq 0$ on $\mathbb{R}^+$ and $v(0) > 0$. On the other hand, according to Lemma 1.2, if $v \geq 0$ on $[x_o, +\infty[$ for some $x_o \in \mathbb{R}$, then $v' \leq 0$ on $[x_o, +\infty[$. Therefore obviously we have $v(x) \geq 0$ $\forall x$. Then it is easily seen that u and v are defined on $\mathbb{R}$ and that Lemma 1.2 and Corollary 1.3 can be applied to (u,v) on $\mathbb{R}$.

Finally it can be proved easily that $u' > 0$ and $v' < 0$ on $\mathbb{R}$. □

Notice that the couple (u,v) defined by Proposition 1.6 is solution of (0.1) if and only if $u(-\infty)=0$ and $v(-\infty)=1$. In the next two remarks we derive very simple conditions for a couple (u,v) satisfying (1.13) to be a solution of (0.1).

Remark 1.10 : Let (u,v) satisfy (1.13). There are two prossibilities : either there exists $\underline{x} < 0$ with $u(\underline{x})=0$, or $\forall x < 0$ one has $u(x) > 0$. In the latter case it is easily shown that (u,v) is a solution of (0.1) ; in this situation, we set $\underline{x} = -\infty$. Thus, in any case, there exists an unique $\underline{x} \in [-\infty, 0[$ such that $u(\underline{x}) = 0$; then one has $\forall x \in]\underline{x}, +\infty[$ $u'(x) > 0$ and $u \in \mathscr{C}^0([\underline{x}, +\infty], [0,1])$. Moreover $u'(\underline{x})=0 \Leftrightarrow \underline{x} = -\infty \Leftrightarrow$ (u,v) is solution of (0.1).

Remark 1.11 : To (u,v) satisfying (1.13) we shall henceforth associate the unique function $h \in \mathscr{C}([0,1]) \cap \mathscr{C}^1(]0,1[)$ defined by :

$$(1.16) \quad \begin{cases} \forall s \in]0,1[\quad h(s)=u'(u^{-1}(s)) \text{ and } \dfrac{d}{ds} h(s) = c - \dfrac{v(u^{-1}(s))f(s)}{h(s)}, \\ h(1)=0, \qquad h(0)=u'(\underline{x}), \end{cases}$$

where $\underline{x}$ is defined in Remark 1.10. Then $h(0) = 0 \Leftrightarrow (u,v)$ is a solution of (0.1).

We are now ready to complete the proof of Theorem 1.1 (ii). Let $z_0 \in]0,1[$ be fixed. For any $c > 0$ let us consider the couple (u,v) defined by Proposition 1.6 and the function h associated to (u,v) by (1.16). We have :

$$\forall s \in]0,1[\quad h'(s) \geq c - \frac{\Lambda' f(s)(1-s)}{h(s)} \geq c - \frac{\Lambda' \sigma s}{h(s)}$$

We claim that, if $c \geq 2\sqrt{\sigma\Lambda'}$, one has $h(0)=0$ (this will complete the proof according to Remark 1.11). Indeed let $c \geq 2\sqrt{\sigma\Lambda'}$. The linear function $H(s) = \dfrac{c + \sqrt{c^2 - 4\Lambda'\sigma}}{2} s$ is such that $\forall s \in]0,1[$ $H'(s) = c - \dfrac{\Lambda'\sigma s}{H(s)}$. Thus, since $h(1) < H(1)$, it is easily shown that $h \leq H$ on $]0,1]$. Hence $h(0) \leq H(0) = 0$. □

c) Proof of Theorem 1.1 (iii)

One has the following more general result :

Lemma 1.12 : We assume that the condition (0.4) holds. We suppose furthermore that f is of class $\mathscr{C}^1$ in a neighbourhood of 1. Then $\forall z_0 \in]0,1[$, $\forall c > 0$, there exists one and only one couple (u,v) satisfying (1.13).

Remark 1.13 : Proposition 1.6 dealt with the existence of such a couple (u,v). We shall here check the uniqueness under an additional hypothesis of regularity on f in a neighbourhood of 1.

Proof : To prove Lemma 1.12, we replace the system of two differential equations of the second order considered up to now by a system of three differential equations of the first order. Then we perform a local analysis about a singular point, which will enable us to prove uniqueness.

More precisely, by assumption, there exist $\delta \in]0,1[$ and $\tilde{f}:[0,1+\delta[\to \mathbb{R}$ such that $\tilde{f} \in \mathscr{C}^1(]1-\delta,1+\delta[)$ and $\tilde{f} = f$ on $[0,1]$. Let us introduce the vector field F defined on $]0,1+\delta[\times \mathbb{R} \times \mathbb{R}$ by :

$$(1.17) \qquad F((x_1,x_2,x_3))=(c(x_1-1)+x_3,\ \frac{c}{\lambda}x_2-\frac{1}{\lambda}x_3,-\tilde{f}(x_1)x_2)$$

Let (u,v) be a couple satisfying (1.13). Let us denote $w \equiv -\lambda v' + cv$. One has $\forall x \in \mathbb{R}$ $w'(x) < 0$, and $w(+\infty)= 0$. Thus $X = (u,v,w)$ is such that :

$$(1.18) \qquad \begin{cases} \dfrac{dX}{dx} = F(X) \text{ on } \mathbb{R}, \\ X(+\infty) = (1,0,0), \\ \exists x_o,\ \forall x \geq x_o, \quad X(x) \in \{(x_1,x_2,x_3) \mid x_1 < 1,\ x_2 > 0,\ x_3 > 0\}. \end{cases}$$

Next consider the singular point (1,0,0); the derivative DF ((1,0,0)) of F at (1,0,0) has two positive eigenvalues and one negative eigenvalue. Thus the Stable Manifold Theorem (see [3]) yields the uniqueness (up to a translation of the origin) of a vector field $X \in \mathscr{C}^1(\mathbb{R}, \mathbb{R}^3)$ for which (1.18) holds.
This implies that, if (u_i,v_i), $i = 1,2$, satisfy (1.13), then there exists $\delta \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ $u_1(x) = u_2(x+\delta)$ and $v_1(x)= v_2(x+\delta)$. It is now straightforward to conclude the proof, since, $\forall i = 1,2$, $u_i' > 0$ and $u_i(0)=z_o$. □

The proof of Theorem 1.1 is now complete.

1.2 The role of the ignition temperature.

In this section, we show that we can obtain a solution of (0.1) by passing to the limit as $\upsilon \to 0$ in the solutions of the model with ignition temperature. We then

study the asymptotic behaviour of this solution, as the activation energy goes to $+\infty$.

1.2.1 Construction of a solution of (0.1).

Let us fix some $z_o \in]0,1[$ and, for any $\theta \in]0,1[$, let us consider the problem :

$$(1.19)\begin{cases} \text{To find } u,\ v \in \mathscr{C}^1(\mathbb{R},[0,1]) \text{ and } c \geq 0 \text{ such that :} \\ -u''+c\,u' = f(u)\chi_{[\theta,1]}(u)v, \\ -\lambda v''+c\,v' = -f(u)\chi_{[\theta,1]}(u)v, \\ u(-\infty)=0,\quad u(0)=z_o,\quad u(+\infty)=1, \\ v(-\infty)=1,\quad v(+\infty)=0, \end{cases}$$

where $\chi_{[\theta,1]}$ denotes the characteristic function of $[\theta,1]$.

From the study of the model with ignition temperature (see [1]), we know that, $\forall\theta\in]0,1[$, (1.19) has got a solution $(u_\theta, v_\theta, c_\theta)$; moreover :

$$\begin{cases} 0 < u_\theta,\ v_\theta < 1,\quad 0 < u'_\theta \leq c_\theta,\quad -\dfrac{c_\theta}{\lambda} \leq v'_\theta < 0 \text{ on } \mathbb{R}, \\ c_\theta^2 \geq 2\Lambda\displaystyle\int_\theta^1 f(s)(1-s)ds \quad (\text{where } \Lambda \text{ is given by (1.2)}), \\ u_\theta \text{ and } v_\theta \text{ satisfy the relations (1.7), (1.8) and (1.9).} \end{cases}$$

Our main result is then the following :

Theorem 1.14 : *Under condition* (0.4), $\forall z_o\in]0,1[$, *there exists a sequence* $(\theta_n)_n$, *with* $\lim \theta_n=0$, *such that the solution* $(u_{\theta_n}, v_{\theta_n}, c_{\theta_n})$ *of* (1.19) *converges to a solution* (u_o,v_o,c_o) *of* (0.1) *in* $\mathscr{C}^2(\mathbb{R})\times\mathscr{C}^2(\mathbb{R})\times\mathbb{R}$.

Proof : According to the results that we recalled above , it suffices to prove that c_θ is bounded independently of $\theta\in]0,1[$, in order to obtain estimates of $(u_\theta,v_\theta,c_\theta)$ independent of $\theta \leq \theta_o$. This will be proved in the next lemma :

Lemma 1.15 : $\forall\theta \in]0,1[\quad c_\theta < 2\sqrt{\Lambda' \sup_{x\in]0,1[} \frac{f(x)(1-x)}{x}}$.

Proof : $\forall\theta\in]0,1[$, let us associate to $(u_\theta,v_\theta,c_\theta)$ the unique function $h_\theta\in\mathscr{C}^0([0,1])\cap\mathscr{C}^1([0,1[-\{\theta\})$ defined by :

$$(1.20)\begin{cases} \forall s\in]0,1[& h_\theta(s) = u'_\theta\ (u_\theta^{-1}(s)), \qquad h_\theta(1)=h_\theta(0)=0, \\ \forall s\in[\theta,1] & h'_\theta(s) = c_\theta - \dfrac{f(s)v_\theta(u_\theta^{-1}(s))}{h_\theta(s)} \ . \end{cases}$$

Since $h_\theta(\theta) = c_\theta\theta$ and $h_\theta(1)=0$, there exists $\theta_1\in]0,1[$ such that $\forall s\in[\theta,\theta_1[\ h_\theta(s)>\frac{c_\theta}{2}s$ and $h_\theta(\theta_1) = \frac{c_\theta}{2}\theta_1$. We remark that $h'(\theta_1)\le\frac{c_\theta}{2}$. On the other hand we know, from the equation satisfied by h_θ, that : $h'_\theta(\theta_1) \ge c_\theta - \frac{\Lambda' f(\theta_1)(1-\theta_1)}{h_\theta(\theta_1)}$. Combining these two inequalities, we conclude the proof. □

Lemma 1.15 implies the existence of a sequence (θ_n) with $\lim\theta_n=0$ such that $(u_{\theta_n},v_{\theta_n},c_{\theta_n})$ converges to (u_o,v_o,c_o) in $\mathscr{C}^1_{loc}(\mathbb{R})\times\mathscr{C}^1_{loc}(\mathbb{R})\times\mathbb{R}$.

It remains to show that (u_o,v_o,c_o) is a solution of (0.1).

$\forall$ n, let us define x_n by $u_{\theta_n}(x_n) = \theta_n$. We claim that $\lim x_n = -\infty$. Indeed, should there exist a subsequence (x_{n_k}) with $\lim x_{n_k} = \tilde{x} > -\infty$, then we would have necessarily :

$$\begin{cases} -u''_o + c_o u'_o = f(u_o)v_o \text{ on } \mathbb{R}, \\ u_o(0)=z_o, \qquad u_o = 0 \text{ sur }]-\infty,\tilde{x}] \end{cases}$$

which is impossible, by the uniqueness in the initial value problem for O.D.E..

Now, since $\lim x_n=-\infty$, we see that :

$$\begin{cases} -u''_o + c_o u'_o = f(u_o)v_o \\ -\lambda v''_o + cv'_o = -f(u_o)v_o \end{cases} \text{ on } \mathbb{R}$$

On the other hand, from the qualitative properties satisfied by $(u_\theta, v_\theta, c_\theta)$, we deduce :

$$\begin{cases} -u_o' + c_o u_o - \lambda v_o' + c_o v_o = c_o \text{ on } \mathbb{R}, \\ 0 \le u_o \le 1, \quad u_o' \ge 0, \quad 0 \le v_o \le 1, \quad v_o' \le 0 \text{ on } \mathbb{R}, \\ u_o(0) = z_o . \end{cases}$$

This implies, as above, (see Proposition 1.6), that (u_o, v_o, c_o) is a solution of (0.1). Finally, it is straightforward to prove the convergence in $\mathscr{C}^2(\mathbb{R}) \times \mathscr{C}^2(\mathbb{R}) \times \mathbb{R}$ □

Theorem 1.14 asserts that by truncating f by 0 on $[0,\theta]$ and then by letting $\theta \to 0$ in the model with ignition temperature, one can construct a solution of (0.1) ; in other words, there exists a solution (u_o, v_o, c_o) of (0.1) having the following property (denoted by (P)) :

(P) There exists a sequence $(\theta_n)_n$ with $\lim \theta_n = 0$ such that the solution $(u_{\theta_n}, v_{\theta_n}, c_{\theta_n})$ of (1.19) converges to (u_o, v_o, c_o) in $\mathscr{C}^2(\mathbb{R}) \times \mathscr{C}^2(\mathbb{R}) \times \mathbb{R}$.

Actually, one does not know whether there exists or not a unique solution of (0.1) with the property (P). We will see in Section 2.2.2 below, that, at least in the case $\lambda < 1$, there is one and only one solution of (0.1) satisfying (P).

Anyway, in the remainder of this section, we will show a particular property of the solutions of (0.1) satisfying (P), which will prove to be of crucial interest in the asymptotic analysis (see Section 1.2.2 below).

<u>*Proposition 1.16*</u> : *Let condition* (0.4) *hold. We assume furthermore that* f *is of class* $\mathscr{C}^1$ *in a neighbourhood of* 1 *and that* f *is differentiable at* 0 *with* $f'(0) = 0$. *Let* (u_o, v_o, c_o) *be a solution of* (0.1) *satisfying the property* (P). *Then the function* h_o *associated to* (u_o, v_o, c_o) *by* (1.16) *is differentiable at* 0 *and* $h_o'(0) = c_o$.

<u>Remark 1.17</u> : This proposition generalizes a previously known result for the scalar case ($\lambda=1$) (see, for example, Johnson [5], Marion[7], Uchiyama[9]).

Proof of Proposition 1.16: Let (u_o,v_o,c_o) satisfy (P) ; let us denote $c_n=c_{\theta_n}$. By considering the equation satisfied by h_o, it is easily seen that $h_o'(0)$ exists and is equal either to c_o or to 0.

Thus it remains to show that $h_o'(0)=c_o$. We first remark that (P) implies that $\forall s\in]0,1[\quad h_o(s) = \lim\ h_{\theta_n}(s)$ (where h_θ is defined by (1.20)).

Next, let us associate to c_n the unique couple (u_n,v_n) satisfying (1.13) (see Lemma 1.12) and let us introduce also the function h_n associated to (u_n,v_n,c_n) by (1.16). Let x_n be such that $u_{\theta_n}(x_n)=\theta_n$. Then, according to the Stable Manifold Theorem, we must have $u_n=u_{\theta_n}$ on $[x_n,+\infty]$. Thus $\tilde{h}_n= h_{\theta_n}$ on $[\theta_n,1]$.

These two remarks imply, on one hand, that $\tilde{h}_n(\theta_n) = c_n\ \theta_n$, and, on the other hand, that $\forall s\in\ ]0,1[\ h_o(s) = \lim \tilde{h}_n(s)$. We can now easily show that $h_o'(0)=c_o$. Indeed, since $\tilde{h}_n(\theta_n)=c_n\ \theta_n$ and $\tilde{h}_n' < c_n$ on $]0,1[$, one has $\tilde{h}_n(0) > 0$. As $\tilde{h}_n(1)=0$, there exists y_n with the property that $\tilde{h}_n(y_n) = \frac{c_n}{2}\ y_n$ and $\forall s\in[0,y_n[\quad \tilde{h}_n(s)> \frac{c_n}{2}\ s$. We claim that there are $\alpha > 0$ and $n_o \in\mathbb{N}$ such that, $\forall n \geq n_o$, one has $y_n \geq \alpha$. Indeed, arguing by contradiction, there exists a subsequence of $(y_n)_n$, again denoted $(y_n)_n$, with $\lim y_n=0$. Since, $\forall n$, $\tilde{h}_n'(y_n) \geq c_n - \dfrac{f(y_n)(1-y_n)}{\tilde{h}_n(y_n)}$, this yields $\lim \tilde{h}_n'(y_n) \geq c_o$, which is impossible, since, $\forall n$, $\tilde{h}_n'(y_n) \leq \frac{c_n}{2}$. Now, $\forall s\in]0,\alpha[$, $h_o(s)=\lim \tilde{h}_n(s) \geq \frac{c_o}{2}\ s$, and thus $h_o'(0)=c_o$. The proof of Proposition 1.16 is thereby complete. □

1.2.2 Asymptotic analysis

In this section, we assume that f depends on a parameter $\varepsilon > 0$ and we let $f = f_\varepsilon$. The corresponding solution of (0.1) obtained in Theorem 1.14 (for some fixed $\lambda>0$) will be denoted by $(u_{o,\varepsilon},v_{o,\varepsilon},c_{o,\varepsilon})$ and we shall investigate the asymptotic behaviour of this solution as $\varepsilon\to 0$. The following conditions will be assumed on the family of functions f_ε :

(1.21) For each $\varepsilon > 0$, f_ε satisfies condition (0.4),

(1.22) For each $\varepsilon > 0$, f_ε is of class $\mathscr{C}^1$ in a neighbourhood of 1 and f_ε is differentiable at 0 with $f_\varepsilon'(0)=0$,

$$(1.23)\quad \begin{aligned} &\exists\theta_\varepsilon,\ 0 \le \theta_\varepsilon \le 1 \text{ such that } \lim_{\varepsilon\searrow 0} \uparrow \theta_\varepsilon = 1 \text{ and} \\ &\lim_{\varepsilon\to 0} \left\{ \sup_{x\in]0,\theta_\varepsilon]} \frac{f_\varepsilon(x)(1-x)}{x} \right\} = 0, \end{aligned}$$

$$(1.24)\quad \lim_{\varepsilon\to 0} \int_0^1 f_\varepsilon(s)(1-s)ds = m > 0, \quad m < +\infty.$$

Remark 1.18 : The typical function f_ε, provided by the Arrhenius law, is $f_\varepsilon(x) = \frac{1}{\varepsilon^2} \exp\left(\frac{1}{\varepsilon}\ \frac{x-1}{x}\right)$. Recall that ε represents the inverse of the activation energy.

Our main result is the following :

Theorem 1.19 : *Under conditions* (1.21), (1.22), (1.23), (1.24) *and with the above notations* :

(i) $\underline{\lim}_{\varepsilon\to 0}\ c_{o,\varepsilon} \ge \sqrt{2\ \Lambda\ m}$ *and* $\overline{\lim}_{\varepsilon\to 0}\ c_{o,\varepsilon} \le \sqrt{2\Lambda' m}$.

(ii) *Let* $(\varepsilon_n)_n$ *be a sequence with the property that* $\lim\ \varepsilon_n = 0$ *and* c_{o,ε_n} *converges to some* c_o. *Then* $(u_{o,\varepsilon_n}, v_{o,\varepsilon_n})$ *converges to* (u_o, v_o) *in* $\mathscr{C}^0(\mathbb{R}) \times \mathscr{C}^0(\mathbb{R})$ *where* (u_o, v_o) *is uniquely determined by* :

$$\begin{cases} -u_o'' + c_o\, u_o' = c_o\ \delta_{x = x_o}, \\ -\lambda v_o'' + c_o\, v_o' = -c_o\ \delta_{x = x_o}, \\ u_o(-\infty) = 0, \quad u_o(+\infty) = 1, \\ v_o(-\infty) = 1, \quad v_o(+\infty) = 0, \end{cases}$$

and x_o *is uniquely determined by the condition* $u_o(0) = z_o$.

Remark 1.20 : A considerable part of the literature on combustion is devoted to formal asymptotic limits for infinite activation energy (see, for example, Buckmaster and Ludford [2] or Williams [10]). Theorem 1.19 proves that the solution

$(u_{o,\varepsilon}, v_{o,\varepsilon}, c_{o,\varepsilon})$ has got the expected asymptotic behaviour. This result is analogous to the one obtained by Berestycki, Nicolaenko and Scheurer [1] for the model with ignition temperature.
The asymptotic analysis of the other solutions of (0.1) is an open question. A partial answer will be given, in Section 2.3 below, in the case $\lambda < 1$.

Remark 1.21 : Based on the formal asymptotic calculations, we conjecture that $c_o = \sqrt{\frac{2m}{\lambda}}$.

Remark 1.22 : The assumptions of Theorem 1.19 could probably be weakened.

Proof of Theorem 1.19 : First, recall that, from Lemma 1.5, we know that

$$\forall \varepsilon > 0 \qquad c_{o,\varepsilon} > \sqrt{2\Lambda \int_0^1 f_\varepsilon(s)(1-s)\, ds}, \text{ which implies :}$$

$$\underline{\lim}_{\varepsilon \to 0} \; c_{o,\varepsilon} \geq \sqrt{2\Lambda m}.$$

$\forall \varepsilon > 0$, let us denote by $h_{o,\varepsilon}$ the function associated to $(u_{o,\varepsilon}, v_{o,\varepsilon}, c_{o,\varepsilon})$ by (1.16). In the next lemma, we investigate the asymptotic limit of $h_{o,\varepsilon}$, as $\varepsilon \to 0$, and we prove the second part of (i).

Lemma 1.23 : $\forall s \in [0,1[\quad \lim_{\varepsilon \to 0} (h_{o,\varepsilon}(s) - c_{o,\varepsilon} s) = 0,$

$$\overline{\lim}_{\varepsilon \to 0} \; c_{o,\varepsilon} \leq \sqrt{2\Lambda' m} \; .$$

Proof : $\forall \varepsilon > 0$, let us set $\sigma(\varepsilon) = \sup_{x \in]0,\theta_\varepsilon]} \frac{f_\varepsilon(x)(1-x)}{x}$. $h_{o,\varepsilon}$ satisfies :

$$\forall s \in]0,\theta_\varepsilon] \quad h'_{o,\varepsilon}(s) \geq c_{o,\varepsilon} - \frac{\Lambda' f_\varepsilon(s)(1-s)}{h_{o,\varepsilon}(s)} \geq c_{o,\varepsilon} - \frac{\Lambda' \sigma(\varepsilon) s}{h_{o,\varepsilon}(s)}$$

Since, from Proposition 1.16, one has $h'_{o,\varepsilon}(0) = c_{o,\varepsilon}$, it is easily seen as above (see the last part of the proof of Theorem 1.1 (ii)) that :

$$\forall s \in [0,\theta_\varepsilon] \qquad h_{o,\varepsilon}(s) \geq \frac{1}{2}\left(c_{o,\varepsilon} + \sqrt{c_{o,\varepsilon}^2 - 4\Lambda'\sigma(\varepsilon)}\right) s$$

Therefore, we have :

$$\forall s\in[0,\theta_\varepsilon],\ 0 \geq h_{o,\varepsilon}(s)-c_{o,\varepsilon}\, s \geq \frac{1}{2}\left(-c_{o,\varepsilon}+\sqrt{c_{o,\varepsilon}^2-4\Lambda'\sigma(\varepsilon)}\right)s$$

Thus, since we have $\underline{\lim}_{\varepsilon\to 0}\ c_{o,\varepsilon}>0$ and $\lim_{\varepsilon\to 0}\sigma(\varepsilon)=0$, we obtain on one hand that $\lim_{\varepsilon\to 0}$ $(h_{o,\varepsilon}(\theta_\varepsilon)-c_{o,\varepsilon}\,\theta_\varepsilon)=0$ and on the other hand that $\forall s\in[0,1[$ $\lim_{\varepsilon\to 0}(h_{o,\varepsilon}(s)-c_{o,\varepsilon}s)=0$.

It remains to show that $\overline{\lim}_{\varepsilon\to 0}\ c_{o,\varepsilon}\leq\sqrt{2\Lambda' m}$. $\forall\varepsilon>0$ let x_ε be defined by $u_{o,\varepsilon}(x_\varepsilon)=\theta_\varepsilon$. Since $u'_{o,\varepsilon}(x_\varepsilon)=h_{o,\varepsilon}(\theta_\varepsilon)$, we deduce from above that $\overline{\lim}\ c_{o,\varepsilon}=\overline{\lim}\ u'_{o,\varepsilon}(x_\varepsilon)$. Now, writing (1.12) with $a=x_\varepsilon$ and $b=+\infty$, we derive : $\frac{1}{2}u'_{o,\varepsilon}(x_\varepsilon)^2\leq\Lambda'\int_{\theta_\varepsilon}^1 f_\varepsilon(s)(1-s)ds$

Therefore one has : $\overline{\lim}_{\varepsilon\to 0}\ u'_{o,\varepsilon}(x_\varepsilon)\leq\sqrt{2\Lambda' m}$. The proof of the lemma is thereby complete. □

We are now ready to complete the proof of Theorem 1.19. Let $(\varepsilon_n)_n$ be a sequence with the property that c_{o,ε_n} converges to some c_o. We shall denote by $u_n=u_{o,\varepsilon_n}$, $v_n=v_{o,\varepsilon_n}$, $f_n=f_{\varepsilon_n}$ and $x_n=x_{\varepsilon_n}$ (we recall that $u_{o,\varepsilon_n}(x_{\varepsilon_n})=\theta_{\varepsilon_n}$). We claim that there exist $n_o\in\mathbb{N}$ and $y\in\mathbb{R}$ such that $\forall n\geq n_o$ one has $x_n\leq y$. Indeed by setting $a=0$ and $b=x$ in (1.12), we obtain : $\forall x\in[0,x_n]$ $\frac{1}{2}u_n'^2(x)\geq$ $\frac{1}{2}u_n'^2(0)-\Lambda'\int_{z_o}^{\theta_{\varepsilon_n}}f_n(s)(1-s)ds$. According to Lemma 1.23, $\lim u_n'(0)=c_o\,z_o$. Since

$\lim\int_{z_o}^{\theta_{\varepsilon_n}}f_n(s)(1-s)ds=0$, we see that there exist $\eta>0$ and $n_o\in\mathbb{N}$ such that

$\forall n\geq n_o$, $\forall x\in[0,x_n]$, one has $u'_n(x)\geq\eta$. This proves our claim, since $\forall n\geq n_o$

$1-z_o\geq u_n(x_n)-u_n(0)\geq\eta\, x_n$.

We can now see that $u_n\to 1$ in $\mathscr{C}^0([y,+\infty[)$. Moreover, from (1.9), we obtain $v_n\to 0$ in $\mathscr{C}^0([y,+\infty[)$.

Next, for any $k\in\mathbb{N}^*$, let us consider the interval $[-k,y]$. Then, from Lemma 1.4, we know that u_n and v_n are bounded in $H^1([-k,y])$ independent of $n\geq n_1$. Hence there exists a subsequence denoted by $(n_\ell)_\ell$, such that u_{n_ℓ}, v_{n_ℓ} converge weakly in

$H^1([-k,y])$ and strongly in $\mathscr{C}^0([-k,y])$ to u_o,v_o and, moreover, $x_{n_\ell} \to x_o$. We remark that $u_o=1$ and $v_o=0$ on $[x_o,y]$ (hence $x_o > 0$ since $u_o(0)=z_o$). Besides, from (1.23), we know that u_{n_ℓ} and v_{n_ℓ} are bounded in $H^2([-k,x_o-\delta])$ independent of $\ell \geq \ell_1$. Thus we can derive $-u_o'' + c_o u_o' = 0$ and $-\lambda v_o'' + c_o v_o' = 0$ on $[-k,x_o[$. Lastly we know from Lemma 1.23 that $u_o'(0) = \lim u_n'(0) = c_o z_o$ and we remark that :

$$-u_o'(0)+c_o u_o(0) -\lambda v_o'(0) + c_o v_o(0) = \lim(-u_{n_\ell}'(0) + c_{n_\ell} u_{n_\ell}(0)-\lambda v_{n_\ell}'(0)+ c_{n_\ell} v_{n_\ell}(0))=c_o$$

Hence, finally, (u_o,v_o) satisfies :

$$\begin{cases} -u_o' + c_o u_o = 0 \text{ and } -\lambda v_o' + c_o v_o = c_o \text{ on } [-k,x_o[\ , \\ u_o = 1 \text{ and } v_o = 0 \text{ on } [x_o,y]. \end{cases}$$

Since $u_o \in \mathscr{C}^0([-n,y])$, we see that $x_o = -\frac{1}{c_o}$ Log z_o and so x_o and (u_o,v_o) are independent of the subsequence $(n_\ell)_\ell$ considered.

Thus, we have shown that $\forall k \in \mathbb{N}^*$, $u_n \to u_o$ and $v_n \to v_o$ in $\mathscr{C}^0([-k,+\infty[)$ where u_o,v_o are uniquely determined by :

$$\begin{cases} -u_o' + c_o u_o = 0 \text{ and } -\lambda v_o' + c_o v_o = c_o \text{ on }]-\infty,x_o[, \\ u_o = 1 \text{ and } v_o = 0 \text{ on } [x_o+\infty[\end{cases}$$

and x_o is uniquely determined by the condition $u_o(0) = z_o$.
It is straightforward to show the convergence in $\mathscr{C}^0(\mathbb{R})$; this concludes the proof of Theorem 1.19. □

1.3 The problem on a bounded interval.

From the viewpoint of the numerical approximation, it is desirable to study the analogous problem to (0.1) on a bounded interval and the way a solution of the latter converges to a solution of (0.1). For $a > 0$, we let $I_a = [-a,+a]$ and we consider the problem :

$$(1.25)\quad \begin{cases} \text{To find } u,v\in\mathscr{C}^2(I_a,\mathbb{R}) \text{ and } c\geq 0 \text{ satisfying :} \\ -u''+cu' = f(u)v \text{ and } -\lambda v''+cv' = -f(u)v \text{ on } I_a, \\ -u'(-a)+cu(-a)=0, \quad u(0)=z_o, \quad u(a)=1, \\ -\lambda v'(-a)+cv(-a)=c, \quad v(a)=0, \end{cases}$$

for some fixed $z_o\in]0,1[$.

The purpose of this section is to show, under condition (0.4), the following :

<u>*Theorem 1.24*</u> : *For any* $a>0$, (1.25) *has (at least) a solution* (u_a,v_a,c_a). *Moreover, there exists a sequence* $(a_n)_n$ *with* $\lim a_n = +\infty$ *with the property that* $(u_{a_n},v_{a_n},c_{a_n})$ *converges to a solution* (u,v,c) *of* (0.1) *in* $\mathscr{C}^2(\mathbb{R})\times\mathscr{C}^2(\mathbb{R})\times\mathbb{R}$; *where* u_{a_n} (resp v_{a_n}) *is extended by* $u_{a_n}(x)=1$ $\forall x\geq a_n$ *and* $u_{a_n}(x)=u_{a_n}(-a_n)$ $\forall x\leq -a_n$ (resp $v_{a_n}(x)=0$ $\forall x\geq a_n$ *and* $v_{a_n}(x)=v_{a_n}(-a_n)$ $\forall x\leq -a_n$)).

<u>Proof</u> : Berestycki, Nicolaenko and Scheurer [1] considered Problem (1.25) in the case of an ignition temperature and their work, which relies on a topological degree argument, can be partly applied here.

Let $X=\mathscr{C}^1(I_a,\mathbb{R})\times\mathscr{C}^1(I_a,\mathbb{R})\times\mathbb{R}$. For any $\tau\in[0,1]$, let $L_\tau : X\to X$ be defined by $L_\tau((u,v,c))=(U,V,c-u(0)+z_o)$, where (U,V) is the unique solution of the following linear system :

$$\begin{cases} -U''+cU' = \tau f(u)v \text{ and } -\lambda V''+cV' = -\tau f(u)v \text{ on } I_a, \\ -U'(-a)+cU(-a)=0, \quad U(a)=1, \\ -\lambda V'(-a)+cV(-a)=c, \quad V(a)=0. \end{cases}$$

If there exist some $\bar{c}>0$ and $R>0$ with the property that $(0,0,0)\notin(I-L_\tau)(\partial\Omega)$ $\forall\tau\in[0,1]$, where $\Omega=\{(u,v,c)\in X \mid \|u\|_{\mathscr{C}^1(I_a)}, \|v\|_{\mathscr{C}^1(I_a)}<R,\ 0<c<\bar{c}\}$, then (1.25) has a solution (see [1]).

In the next two lemmae, we derive the a priori estimates which enable to construct such an open set Ω , and next to pass to the limit as $a \to +\infty$.

<u>Lemma 1.25</u> : *Let* (u_a, v_a, c_a) *be a solution of* (1.25). *Then* :

$c_a > 0$, $u_a > 0$ *on* I_a, $0 < u_a' \le c_a$ *and* $-\frac{c_a}{\lambda} \le v_a' < 0$ *on* I_a,

u_a *and* v_a *satisfy the relations* (1.7), (1.8) *and* (1.9) *on* I_a.

<u>Proof</u> : See previous lemmae or Berestycki, Nicolaenko and Scheurer [1].

<u>Lemma 1.26</u> : *For any* $a > 0$, *let* (u_a, v_a, c_a) *be a solution of* (1.25).

Then :

(i) $c_a \ge \sqrt{2\Lambda \int_{z_o}^{1} f(s)(1-s)\,ds}$.

(ii) *Let* $a \ge a_o > 0$. *Then* $c_a \le \sup\left(2\sqrt{\sigma\Lambda'}, \frac{2}{a_o} \operatorname{Log} \frac{1}{z_o}\right)$, *where* σ *is defined by* (1.1).

<u>Proof</u> : The proof of (i) is similar to Lemma 1.5 and is omitted. Let us check (ii). Let $a \ge a_o > 0$. Then either $c_a < 2\sqrt{\sigma\Lambda'}$, or $c_a \ge 2\sqrt{\sigma\Lambda'}$.

Suppose the latter case occurs. Let $h_a \in \mathscr{C}^1([u_a(-a),1])$ be defined by

$h_a(s) = u_a'(u_a^{-1}(s))$. Then, $\forall s \in [u_a(-a), 1]$, $h_a'(s) \ge c_a - \frac{\Lambda'\sigma s}{h_a(s)}$.

Now, if we set $r_a = \frac{1}{2}\left(c_a + \sqrt{c_a^2 - 4\sigma\Lambda'}\right)$, the function $v_a(s) = r_a s$ satisfies $\frac{d}{ds} v_a(s) = c_a - \frac{\sigma\Lambda' s}{v_a(s)}$. Thus, since $v_a(u_a(-a)) < h_a(u_a(-a))$, we obtain $v_a < h_a$ on $[u_a(-a),1]$, that is $\forall x \in [-a,+a]$ $u_a'(x) > r_a u_a(x)$. Therefore, since $u_a > 0$ on I_a, one has : $\operatorname{Log} \frac{u_a(a)}{u_a(0)} > r_a a \ge r_a a_o$. This yields $c_a \le \frac{2}{a_o} \operatorname{Log} \frac{1}{z_o}$.

Thus, in any case, we have : $c_a \le \sup\left(2\sqrt{\sigma\Lambda'}, \frac{2}{a_o} \operatorname{Log} \frac{1}{z_o}\right)$. □

It is straightforward to deduce from these two lemmae the existence of a solution (u_a, v_a, c_a) of (1.25), for any $a > 0$. Moreover since we obtained a priori estimates independent of $a \ge a_o$, it is possible to pass to the limit for (u_a, v_a, c_a) as $a \to +\infty$ (at least for a sequence) and the obtained limit is a solution of (0.1). The proof of Theorem 1.24 is thereby complete. □

Remark 1.27 : (1.25) is the analogous problem to (0.1) on a bounded interval but it behaves quite differently. Indeed if (u,v,c) is a solution of (1.25), then c is bounded. In the scalar case (λ=1) there is one and only one solution to the problem (1.25) (see[7]). But, on the contrary, the problem (0.1) has a solution of the form (u,v,c) for any large enough c. This difference of behaviour does not occur when dealing with the model with ignition temperature (see [1]).

2. CASE OF THE LEWIS NUMBER LARGER THAN 1.

In this section, we assume that λ (which represents the inverse of the Lewis number) satisfies

(2.1) $0 < \lambda < 1$,

and that f satisfies hypothesis (0.4) and

(2.2) f is of class $\mathscr{C}^1$ in a neighbourhood of 1.

Under these assumptions, we complete in Section 2.2, results of Section 1, namely the study of the solutions of (0.1) and the comparison with the model with ignition temperature. This is possible thanks to a monotony lemma that we first prove in Section 2.1. Lastly, in Section 2.3, we investigate the high activation energy limits.

2.1 Monotony lemma.

This lemma is an additional result to those of Section 1.1. Let some $z_0 \in]0,1[$ be fixed. In Section 1.1, we associated to any $c > 0$ a triplet (u_c, v_c, h_c) in the following manner :

- (u_c, v_c) is the unique couple satisfying (1.13) (see lemma 1.12),
- h_c is then defined by (1.16).

In the sequel, a key-role will be played by the following proposition :

Proposition 2.1 (*Monotony lemma*) : *Under conditions* (0.4), (2.1), (2.2) *and* $0 < c_1 < c_2$, *one has* $h_{c_2}(s) < h_{c_1}(s) \quad \forall s \in]0,1[$.

Remark 2.2 : Proposition 2.1 extends a result previously known in the scalar case (see Johnson[5], Marion [7]).

Corollary 2.3 : *Under conditions* (0.4), (2.1), (2.2) *and* $0 < c_1 < c_2$, *one has* $u_{c_2}(x) > u_{c_1}(x) \quad \forall x < 0$ *and* $u_{c_2}(x) < u_{c_1}(x) \quad \forall x > 0$.

Proof of Proposition 2.1 : We remark that, since $\lambda < 1$, one has, from (1.2), $\Lambda = 1$ and $\Lambda' = \frac{1}{\lambda}$. Thus (1.9) yields $\forall c > 0 \quad u_c + v_c - 1 > 0$ on $\mathbb{R}$ (it is straightforward to verify the strict inequality).

For any $c > 0$, let us define $l_c \mathscr{C}([0,1]) \cap \mathscr{C}^1(]0,1[)$ by :

$$\begin{cases} \forall s \in]0,1[& l_c(s) = (u_c + v_c - 1)\ (u_c^{-1}(s)) = s + v_c(u_c^{-1}(s)) - 1, \\ l_c(1)=0, & l_c(0) = v_c\ (u_c^{-1}(0)) - 1. \end{cases}$$

Hence, (h_c, l_c) satisfies the differential system :

$$\begin{cases} & \dfrac{d}{ds} h_c(s) = c - \dfrac{f(s)(1-s)}{h_c(s)} - \dfrac{f(s) l_c(s)}{h_c(s)}, \\ \forall s \in]0,1[, & \\ & \dfrac{d}{ds} l_c(s) = 1 - \dfrac{1}{\lambda} + \dfrac{c}{\lambda} \dfrac{l_c(s)}{h_c(s)}. \end{cases}$$

<u>Lemma 2.4</u> : *For any* $c > 0$, h_c and l_c *are differentiable at 1. Moreover:*

$$(2.3) \qquad h_c'(1) = \frac{1}{2}\left(\frac{c}{\lambda} - \sqrt{\frac{c^2}{\lambda^2} + \frac{4f(1)}{\lambda}} \right),$$

$$(2.4) \qquad l_c'(1) = \left(1 - \frac{1}{\lambda}\right) \left(1 + \frac{c}{f(1)}\ h_c'(1)\right).$$

<u>Proof</u> : Let us denote by $u=u_c$, $v=v_c$, $h=h_c$ and $l=l_c$. If we suppose that h and l are differentiable at 1, then we deduce easily from the differential system for (h,l) that $h'(1)$ and $l'(1)$ are given by (2.4) and (2.5).

Thus the main point is to prove the differentiability of h and l at 1.

To prove the differentiability of h, we consider the dynamical system that we introduced in Lemma 1.12 and we again apply the Stable Manifold Theorem. Indeed, recall that, if we denote by $w = -\lambda v' + cv$, then $X = (u,v,w)$ satisfies (1.18), where the vector field F is defined by (1.17). We saw that the derivative $DF((1,0,0))$ of F at $(1,0,0)$ has two > 0 eigenvalues and one < 0 eigenvalue. Moreover, the eigenspace E^- associated to the < 0 eigenvalue is spanned by the vector :

$$(a_1, a_2, a_3) = \left(\frac{2\lambda}{-\sqrt{c^2 + 4\lambda f(1)} - 2\lambda c + c}, \frac{2}{c + \sqrt{c^2 + 4\lambda f(1)}}, 1 \right)$$

The Stable Manifold Theorem asserts that the tangent space at $(1,0,0)$ of the local

stable manifold is E^-. Hence, one has $\lim_{x\to+\infty} \frac{w(x)}{u(x)-1} = \frac{a_3}{a_1}$; since $u'=c(u-1)+w$,

this leads to $\lim_{x\to+\infty} \frac{u'(x)}{u(x)-1} = \frac{a_3}{a_1} + c$. Thus h is differentiable at 1 and moreover (2.3) is showed.

It remains to prove the differentiability of l at 1. Let us argue by contradiction and let $m \in \left]\underline{\lim}_{s\to 1} \frac{l(s)}{s-1}, \overline{\lim}_{s\to 1} \frac{l(s)}{s-1}\right[$. Then, from the equation satisfied by l, one deduces $m = \left(1-\frac{1}{\lambda}\right)\left\{1-\frac{c}{\lambda h'(1)}\right\}^{-1}$, which is impossible. Hence l is differentiable at 1 and one sees easily that $l'(1)$ is equal to the above expression for m ; combining this expression and (2.3), one obtains (2.4). □

<u>End of the proof of Proposition 2.1</u> : Let $0 < c_1 < c_2$ and, $\forall i = 1,2$, let us denote by $h_i = h_{c_i}$ and $l_i = l_{c_i}$.

From Lemma 2.4, we know that $h_1'(1) < h_2'(1)$ and $l_1'(1) < l_2'(1)$. Thus, since $h_i > 0$ and $l_i > 0$ on $]0,1[$, there exists some $s_o \in]0,1[$ with the property that :

(2.5) $h_1 > h_2$ and $l_1 > l_2$ on $]s_o,1[$.

Let us argue by contradiction and prove that (2.5) yields $h_1 > h_2$ and $l_1 > l_2$ on $]0,1[$. Indeed, if not, there exists some $\alpha \in]0,1[$ such that $h_1 > h_2$ and $l_1 > l_2$ on $]\alpha,1[$, and, $h_1(\alpha) = h_2(\alpha)$, or $l_1(\alpha) = l_2(\alpha)$. In the first case, taking the difference of the equations for h_1 and h_2, one has :

$$h_2'(\alpha) - h_1'(\alpha) = c_2 - c_1 - \frac{f(\alpha)}{h_2(\alpha)} \left(l_2(\alpha) - l_1(\alpha)\right) > 0$$

which is impossible. In the same way, in the latter case, we obtain a contradiction since :

$$l_2'(\alpha) - l_1'(\alpha) = \frac{1}{\lambda} l_2(\alpha) \left(\frac{c_2}{h_2(\alpha)} - \frac{c_1}{h_1(\alpha)}\right) > 0.$$

This completes the proof of Proposition 2.1. □

<u>Remark 2.5</u> : The assumption $\lambda < 1$ is crucial for this last argument. Notice that, if $\lambda > 1$, then $l_i < 0$.

Remark 2.6 : This type of monotony results has been known since the formal works of Kanel'[6], but it is the first time, as far as we know, that a rigorous proof is given.

2.2 Applications

2.2.1 - Determination of the set of solutions of (0.1)

The following result refines that of Theorem 1.1 :

Theorem 2.7 : Under conditions (0.4), (2.1) *and* (2.2) :

(i) *There exists some* $c_o > 0$ *such that* (0.1) *has a solution* (u,v,c) *if and only if* $c \geq c_o$.

(ii) *For any* $c \geq c_o$, *there exists an unique couple* (u,v) *(up to a translation of the origin) such that* (u,v,c) *is solution of* (0.1).

(iii) *Moreover,*

$$\sqrt{2\int_0^1 f(s)(1-s)ds} < c_o \leq 2\sqrt{\frac{1}{\lambda}\sup_{x\in]0,1[}\frac{f(x)(1-x)}{x}}.$$

Proof : Let us set $\mathscr{C} = \{c;(0.1)$ has a solution $(u,v,c)\}$. From Theorem 1.1, we know that there exist $0 < \alpha \leq \beta$ such that $[\beta + \infty[\subset \mathscr{C}$ and $\forall x \in \mathscr{C}$ one has $x \geq \alpha$. We deduce from Corollary 2.3 (see Remark 1.10) that $\mathscr{C}$ is an interval. Hence there exists some $c_o > 0$ such that $\mathscr{C} = (c_o, +\infty[$ and it is easily seen that $c_o \in \mathscr{C}$ by passing to the limit as $c \to c_o$, $c > c_o$, for the solutions of (0.1) ; this concludes the proof of (i). (ii) and (iii) are consequences of Theorem 1.1. □

Remark 2.8 : Theorem 2.7 generalizes a result previously known in the scalar case (see Fife [4], Johnson [5], Marion [7], Uchiyama [9]).

2.2.2 Relationship with the model with ignition temperature.

Before investigating this relationship, let us consider the model with ignition temperature. Under condition (0.2), Berestycki, Nicolaenko and Scheurer proved, for any $\lambda > 0$, the existence of a solution of (0.1). Thanks to the monotony lemma, we shall derive here the uniqueness of such a solution in the case $\lambda < 1$. More precisely, the following theorem holds :

Theorem 2.9 : *We assume that condition* (0.2) *holds with* f *Lipschitz continuous on* $[\theta,1]$ *and of class* $\mathscr{C}^1$ *in a neighbourhood of* 1. *Then, if* $0 < \lambda < 1$, *there exists one and only one solution of* (0.1) *(up to a translation of the origin).*

Proof : To prove the uniqueness, we argue by contradiction. Let (u_1,v_1,c_1) and (u_2,v_2,c_2) be two different solutions of (0.1), with $\forall i=1,2$ $u_i(0)=\theta$. Then according to the Stable Manifold Theorem (see the proof of Lemma 1.12), if $c_1=c_2$, we would have $u_1 = u_2$ and $v_1 = v_2$. Thus we have necessarily $c_1 \neq c_2$.
Suppose $c_1 < c_2$. $\forall i = 1,2$, let us associate h_i to (u_i,v_i,c_i) by (1.20). Then the arguments of the proof of Proposition 2.1 show that $h_2 < h_1$ on $[\theta,1[$. In particular $h_2(\theta) < h_1(\theta)$, which is impossible since $h_i(\theta) = c_i\theta$. □

Remark 2.10 : Theorem 2.7 and Theorem 2.9 emphasize the great difference in the behaviour of the solutions of the two models. Nevertheless the solutions of these two models can be closely connected. It is the purpose of the next theorem.

$\forall\theta\in]0,1[$, let us denote by $(u_\theta,v_\theta,c_\theta)$ the unique solution of (1.19). The following theorem improves Theorem 1.14 :

Theorem 2.11 : *Under conditions* (0.4), (2.1) *and* (2.2) :
(i) The function : $]0,1[\to \mathbb{R}^{+*}$, $\theta \to c_\theta$ *is strictly increasing.*
(ii) $(u_\theta,v_\theta,c_\theta)$ *converges to* (u_o,v_o,c_o) *in* $\mathscr{C}^2(\mathbb{R})\times\mathscr{C}^2(\mathbb{R})\times\mathbb{R}$, *as* $\theta \to 0$, *where* c_o *is defined at Theorem* 2.6 (i) *and* (u_o,v_o,c_o) *is the unique corresponding solution of* (0.1) *with* $u_o(0) = z_o$.

Proof : The monotony lemma enables us to prove this theorem by arguments similar to those of the scalar case ; the proof is omitted (see [5,7] and the a-priori estimates are obtained in Theorem 1.14). □

2.3 Further remarks on the asymptotic analysis.

We assume that f depends on a parameter $\varepsilon > 0$ ($f = f_\varepsilon$) and that the family $(f_\varepsilon)_\varepsilon$ satisfies conditions (1.21), (1.22), (1.23) and (1.24). Let some $z_o \in]0,1[$ be fixed. For any $\varepsilon > 0$, the problem :

$$(2.6)\begin{cases} -u'' + cu' = f_\varepsilon(u)v \\ -\lambda v'' + cv' = -f_\varepsilon(u)v \\ u(-\infty) = 0, \quad u(0) = z_o, \quad u(+\infty) = 1, \\ v(-\infty) = 1, \quad v(+\infty) = 0. \end{cases}$$

has got infinitely many solutions (for $c \geq c_{o,\varepsilon} > 0$). Moreover, $\forall c \geq c_{o,\varepsilon}$ the corresponding solution of (2.6) is unique. In this section, we are concerned with the asymptotic behaviour of these solutions, as $\varepsilon \to 0$.

First, let us deal with the solution corresponding to $c_{o,\varepsilon}$, that we denote by $(u_{o,\varepsilon}, v_{o,\varepsilon}, c_{o,\varepsilon})$:

Proposition 2.12 : *Under conditions* (1.21), (1.22), (1.23) *and* (1.24) :

(i) $\overline{\lim\limits_{\varepsilon\to 0}}\ c_{o,\varepsilon} \leq \sqrt{2\Lambda' m}$, *and* $\underline{\lim\limits_{\varepsilon\to 0}}\ c_{o,\varepsilon} \geq \sqrt{2\Lambda m}$.

(ii) *Let* (ε_n) *be a sequence such that* $\lim \varepsilon_n = 0$ *and* $\lim c_{o,\varepsilon_n} = c_o$. *Then* $(u_{o,\varepsilon_n}, v_{o,\varepsilon_n})$ *converges to* (u_o, v_o) *in* $\mathscr{C}^0(\mathbb{R}) \times \mathscr{C}^0(\mathbb{R})$, *as* $n \to +\infty$, *where* (u_o, v_o) *is uniquely determined by* :

$$\begin{cases} -u_o'' + c_o u_o' = c_o \delta_{x=x_o}, \\ -\lambda v_o'' + c_o v_o' = -c_o \delta_{x=x_o}, \\ u_o(-\infty) = 0, \quad u_o(+\infty) = 1, \\ v_o(-\infty) = 1, \quad v_o(+\infty) = 0, \end{cases}$$

and x_o *is determined by the condition* $u_o(0) = z_o$.

Proof : It is a straighforward consequence of Theorem 2.11 and Theorem 1.19. □

Now, since $\underline{\lim}_{\varepsilon\to 0} c_{o,\varepsilon} \le \sqrt{2\Lambda' m}$, for any $c > \sqrt{2\Lambda' m}$, we can introduce the solution $(u_{c,\varepsilon}, v_{c,\varepsilon}, c)$ of (2.6) (defined for $\varepsilon \le \varepsilon_o$) and we can investigate the asymptotic behaviour of $(u_{c,\varepsilon}, v_{c,\varepsilon})$ as $\varepsilon \to 0$. The following theorem deals with a particular case where the limit of $(u_{c,\varepsilon}, v_{c,\varepsilon})$, as $\varepsilon \to 0$, is entirely determined ; in the general case, we obtain results only for sequences (see Remark 2.15 below).

Theorem 2.13 : Under conditions (1.21), (1.22), (1.23) *and* (1.24), *let some* $c > \sqrt{2\Lambda' m}$ *be fixed and let us denote by* $(u_{c,\varepsilon}, v_{c,\varepsilon}, c)$ *the corresponding solution of* (2.6) *(defined for* $\varepsilon \le \varepsilon_o$*). Then, under the additional condition* $z_o \in \left]0, \frac{c-\sqrt{2m\Lambda'}}{c}\right[$, $(u_{c,\varepsilon}, v_{c,\varepsilon})$ *converges to* (u,v) *in* $\mathscr{C}^0_{loc}(\mathbb{R}) \times \mathscr{C}^0_{loc}(\mathbb{R})$, *as* $\varepsilon \to 0$, *where* u (resp v) *is the function* : $x \to z_o$ (resp : $x \to 1 - z_o$).

Proof : $\forall\, \varepsilon \le \varepsilon_o$, one has $u'_{c,\varepsilon}(z_o) = h_{c,\varepsilon}(z_o)$ where $h_{c,\varepsilon}$ is the function associated to $(u_{c,\varepsilon}, v_{c,\varepsilon}, c)$ by (1.16). Theorem 2.13 is then an easy consequence of the next lemma :

Lemma 2.14 : Under conditions (1.21), (1.22), (1.23) *and* (1.24) , *one has*

(i) $\forall s \in]0,1[\quad \underline{\lim}_{\varepsilon\to 0} h_{c,\varepsilon}(s) \ge \sup\,(0, cs - c + \sqrt{2\Lambda m})$,

(ii) $\forall s \in]0,1[\quad \overline{\lim}_{\varepsilon\to 0} h_{c,\varepsilon}(s) \le \sup\,(0, cs - c + \sqrt{2\Lambda' m})$.

Proof of Lemma 2.14 : Let any $s \in]0,1[$ be fixed. $\forall \varepsilon (\le \varepsilon_o)$, let us denote by $(u_\varepsilon, v_\varepsilon, c)$ the solution of (2.6) with $u_\varepsilon(0) = s$ (u_ε differs from $u_{c,\varepsilon}$ by a translation of the x-axis). Let us write (1.10), (1.11) and (1.12) (for $(u_\varepsilon, v_\varepsilon, c)$ and f_ε) with $a = 0$ and $b = +\infty$; by an argument similar to the one of lemma 1.5, we obtain :

$$\frac{1}{2}(u'_\varepsilon(0))^2 + c(1-s)\, u'_\varepsilon(0) + \frac{c^2}{2}(1-s)^2 - \Lambda \int_s^1 f_\varepsilon(\tau)\,(1-\tau)d\tau \geq 0$$

which yields $u'_\varepsilon(0) \geq -c(1-s) + \sqrt{2\Lambda \int_s^1 f_\varepsilon(\tau)\,(1-\tau)d\tau}$.

It is now straightforward to complete the proof of (i), since $h_{c,\varepsilon}(s) = u'_\varepsilon(0)$.

From Lemma 1.23, we know that the application $h_{o,\varepsilon}$ associated to $(u_{o,\varepsilon}, v_{o,\varepsilon}, c_{o,\varepsilon})$ by (1.16) satisfies :

$\forall s \in [0,1[\quad \lim_{\varepsilon\to 0} (h_{o,\varepsilon}(s) - c_{o,\varepsilon} s) = 0.$

In particular, $\overline{\lim_{\varepsilon\to 0}}\ h_{o,\varepsilon}(s) \leq \overline{\lim_{\varepsilon\to 0}}\ c_{o,\varepsilon} s \leq \sqrt{2\Lambda' m}\ s.$

From the monotony lemma, we deduce $h_{c,\varepsilon} \leq h_{o,\varepsilon}$ on $]0,1[$.

Combining these inequalities, we obtain

(2.7) $\quad \forall s \in]0,1[\quad \overline{\lim_{\varepsilon\to 0}}\ h_{c,\varepsilon}(s) \leq \sqrt{2\Lambda' m}\ s.$

(ii) can then be deduced from (2.7) by an argument similar to the one of the scalar case [7]. □

<u>End of the proof of Theorem 2.13</u> : In the case $z_o \in \left]0, \frac{c-\sqrt{2m\Lambda'}}{c}\right]$, Lemma 2.14 yields $\lim_{\varepsilon\to 0} u'_{c,\varepsilon}(0) = 0$, and the proof is now easily completed. □

<u>Remark 2.15</u> : In the case $z_o \notin \left]0, \frac{c-\sqrt{2m\Lambda'}}{c}\right]$, we know from lemma 2.15 that, for any sequence (ε_n) with $\lim \varepsilon_n = 0$ such that there exists $(u,v) = \lim (u_{c,\varepsilon_n}, v_{c,\varepsilon_n})$, there are two possibilities for (u,v) :

. if $\lim u'_{c,\varepsilon_n}(0) = 0$, (u,v) is the couple defined at Theorem 2.13,

. if $\lim u'_{c,\varepsilon_n}(0) = \delta > 0$, (u,v) is uniquely determined by :

$$\begin{cases} -u' + cu = -\delta + c\, z_o & \text{on }]-\infty, \bar{x}[, \\ -\lambda v' + cv = c(1-z_o) + \delta & \text{on }]-\infty, \bar{x}[, \\ u \equiv 1 \text{ and } v = 0 \text{ on } [\bar{x}\ +\infty[, \end{cases}$$

and $\bar{x}$ is uniquely determined by the condition $u(0) = z_o$.

Moreover, one has :

$$\sup\,(0, c\,z_o - c + \sqrt{2\Lambda m}\,) \leq \delta \leq \sup(0, c\,z_o - c + \sqrt{2\Lambda' m})$$

In any case, one has $u(-\infty) \neq 0$ and $v(-\infty) \neq 1$: the boundary conditions at $-\infty$ are no more satisfied ; this result generalizes the one obtained in the scalar case (we proved in [7] that the asymptotic behaviour of the solutions which do not correspond to the minimal c is not physical : at the limit $\varepsilon = 0$, one does not find any "fresh gas" as one goes towards $-\infty$).

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