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**ASYMPTOTIC BEHAVIOR  
OF VISCOSITY SOLUTIONS  
OF FIRST-ORDER  
HAMILTON JACOBI EQUATIONS**

**Guy BARLES**

**Mai 1985**

# ASYMPTOTIC BEHAVIOR OF VISCOSITY SOLUTIONS OF FIRST-ORDER HAMILTON JACOBI EQUATIONS

*Guy BARLES (\*)*

**ABSTRACT:** We study the asymptotic behavior when  $t \rightarrow +\infty$  of viscosity solutions to the Cauchy problems for first-order Hamilton Jacobi Equations. Our results concern the existence and the nature of the limit as also some estimates on the rate of convergence.

**Key-words:** First-order Hamilton Jacobi Equations, viscosity solutions, Cauchy problem, asymptotic behavior, rate of convergence.

# COMPORTEMENT ASYMPTOTIQUE DE SOLUTIONS DE VISCOSITE D'EQUATIONS DE HAMILTON JACOBI DU PREMIER ORDRE

*Guy BARLES (\*)*

**RESUME:** Nous étudions le comportement asymptotique lorsque  $t \rightarrow +\infty$  des solutions de viscosité des problèmes de Cauchy pour les équations de Hamilton Jacobi du premier ordre. Nos résultats portent sur l'existence et la nature de la limite ainsi que sur des estimations de la vitesse de convergence.

**Mots-clés:** Equations de Hamilton Jacobi du premier ordre, solutions de viscosité, problème de Cauchy, comportement asymptotique, vitesse de convergence.

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ASYMPTOTIC BEHAVIOR OF VISCOSITY SOLUTIONS OF

FIRST ORDER HAMILTON JACOBI EQUATIONS :

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Introduction.

We consider here the following problems

$$(1) \quad \begin{cases} \frac{\partial u}{\partial t} + H(x,t,u,Du) = 0 & \text{in } \Omega \times (0,\infty) \\ u(x,0) = u_0(x) \text{ in } \Omega, \quad u(x,t) = \varphi(x,t) & \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

where  $\Omega$  is any open subset of  $\mathbb{R}^N$ ,  $H(x,t,r,p)$  is a given continuous function which is called the Hamiltonian,  $u_0$  and  $\varphi$  are given functions (initial and boundary conditions). When we take  $\Omega = \mathbb{R}^N$ , there is no boundary conditions.

In this work, we are interested in asymptotic behavior when  $t \rightarrow +\infty$  of viscosity solutions of (1). Our results extend or complement those obtained by S.N. Kruzkov [7,8] or P.L. Lions [9]. Before detailing these results, let us mention that a complete presentation of the notion of viscosity solution may be found in M.G. Crandall and P.L. Lions [4], M.G. Crandall, L.C. Evans and P.L. Lions [3] or P.L. Lions [9], and that existence results for (1) may be found in P.L. Lions [9,10], M.G. Crandall and P.L. Lions [5], P.E. Souganidis [11] and G. Barles [1,2].

In section I, we treat the case when  $\Omega = \mathbb{R}^N$ ,  $u_0$  is uniformly continuous and  $H$  depends only on  $p$ . This section contains two parts. In the first part, we assume that  $H(p) \geq H(0) = 0$  in  $\mathbb{R}^N$ ; we prove that if there exists  $v$  a uniformly continuous viscosity solution of

$$H(Dv) = 0 \quad \text{in } \mathbb{R}^N$$

such that  $u_0 - v$  is bounded from below, then  $u(x,t)$  converges when  $t \uparrow +\infty$  to  $u_\infty$  which is the maximum viscosity solution of :

$$(2) \quad \begin{cases} H(D\omega) = 0 \\ \omega \leq u_0 \end{cases} \quad \text{in } \mathbb{R}^N$$

This result extends the one proved by P.L. Lions in [9] where  $u_0$  was supposed to be bounded (and so one can take  $v = 0$ ). In the second part, we assume that the gradient of  $u_0$  is bounded. We describe the behavior of  $u(x,t)$  when  $t \uparrow +\infty$  under assumptions which concern the behavior of  $Du_0(x)$  when  $|x| \rightarrow +\infty$  and the differentiability of  $H$ .

In section II, we complement the result obtained in the first part of section I by giving some estimates on the rate of convergence of  $u(x,t)$  to  $u_\infty(x)$ . These results are obtained essentially under assumptions which concern the shape of the zeros of  $H$ , the behavior of  $H$  and  $u_0$ . In this part, an essential tool is the Oleinik-Lax formula (cf. P.L. Lions [9]).

In section III, we consider the case when  $\Omega \neq \mathbb{R}^N$ . We give two kinds of results : first when  $\Omega$  is convex,  $H$  depends only on  $p$ , is continuous positive and  $H(0) = 0$  and if (1) has a viscosity solution  $u$ , we give results of the same sort as in section II. Let us just mention that

we also need some assumptions on the boundary condition  $\varphi$  and particularly an assumption on the rate of convergence of  $\varphi$  when  $t \uparrow +\infty$ . Then we treat the case of general Hamiltonians and general domains : we prove that, if there exists a viscosity solution  $u$  of (1) and under assumptions concerning the convergence of  $H$  when  $t \uparrow +\infty$  and on its dependence on  $r$ ,  $u_\infty(x) = \lim_{t \uparrow \infty} u(x,t)$  exists and we give some estimates on the rate of convergence.

The methods used in this paper are the classical methods used in the study of viscosity solutions of Hamilton Jacobi equations : here it is essentially the building of convenient viscosity sub- and super-solutions, the Oleinik-Lax formula and the use of comparizon results for viscosity solutions. We refer the reader to the bibliography for details on such results and methods.

The plan is as follows :

I . Asymptotics in  $\mathbb{R}^N$  : general results.

I.1 Results for nonnegative Hamiltonians.

I.2 Some results for lipschitz initial conditions.

II . Asymptotics in  $\mathbb{R}^N$  : some estimates on the rate of convergence of  $u$  to  $u_\infty$  .

III . Asymptotics in bounded or unbounded domains :

III.1 The case of convex domains.

III.2 One result on general domains.

I . Asymptotics in  $\mathbb{R}^N$  : general results.

We denote by  $UC(O)$  (resp.  $BUC(O)$ ) the space of uniformly continuous functions in  $O$  (resp. bounded uniformly continuous). Let  $u \in UC(\mathbb{R}^N \times (0, \infty))$  the unique viscosity solution of :

$$(3) \quad \begin{cases} \frac{\partial u}{\partial t} + H(Du) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) \in UC(\mathbb{R}^N) \end{cases}$$

where  $H$  is continuous.

Our aim is to obtain general results on the behavior of  $u(x, t)$  when  $t \uparrow +\infty$ .

I.1. Results for nonnegative Hamiltonians :

Our result is the following :

Theorem I.1 : Let  $H \in C(\mathbb{R}^N)$  and  $u_0 \in UC(\mathbb{R}^N)$  such that  $H(p) \geq H(0) = 0$ . Assume that there exists  $v \in UC(\mathbb{R}^N)$  viscosity solution of

$$H(Dv) = 0 \quad \text{in } \mathbb{R}^N$$

such that  $u_0 - v$  is bounded from below, then  $u(x, t)$  converges in  $C(\mathbb{R}^N)^*$  when  $t \uparrow +\infty$  to  $u_\infty(x)$  which is the maximum viscosity solution of

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(\*) i.e. uniformly on compact subsets of  $\mathbb{R}^N$ .

$$(2) \quad \begin{cases} H(D\omega) = 0 \\ \omega \leq u_0 \end{cases} \quad \text{in } \mathbb{R}^N .$$

Remark I.1 : The positivity of  $H$  is quite necessary as one can see by looking at the following example :

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x, 0) = \sin(x) \end{cases}$$

then  $u(x, t) = \sin(x+t)$  .

In the same way the existence of  $v$  is necessary. Indeed take  $H(p) = |p|$  and  $u_0$  not bounded from below ; then  $u$  is given by the Oleinik-Lax formula :

$$u(x, t) = \inf_{|y-x| \leq t} (u_0(y))$$

and so  $u(x, t) \rightarrow -\infty$  when  $t \uparrow +\infty$  . Now it suffices to remark that every viscosity solution of

$$|Dv| = 0 \quad \text{in } \mathbb{R}^N$$

is a constant and so  $u_0 - v$  is not bounded from below.

Remark I.2 : This result generalizes those obtained by P.L. Lions in [9] where  $u_0$  is assumed to be bounded and so one can take  $v = 0$  . In theorem I.1, the existence of  $v$  ensured the existence of the maximal viscosity solution of (2) which is automatic if  $u_0$  is bounded from below.



Remark I.3 : The same result holds if we assume that  $H(p) \leq H(0) = 0$  in  $\mathbb{R}^N$  and  $v - u_0$  bounded from below. But in this case  $u$  converges to  $u_\infty$  which is the minimum viscosity solution of :

$$\begin{cases} H(D\omega) = 0 \\ \omega \geq u_0 \end{cases} \quad \text{in } \mathbb{R}^N$$

Now let us mention an important extension of theorem I.1.

Corollary I.1 : Let  $H \in C(\mathbb{R}^N)$  and  $u_0 \in UC(\mathbb{R}^N)$  satisfying  $\|Du_0\|_{L^\infty(\mathbb{R}^N)} < +\infty$ . Assume that there exists  $v \in UC(\mathbb{R}^N)$  viscosity solution of :

$$H(Dv) = \min_{|p| \leq \|Du_0\|_\infty} H(p)$$

so that  $u_0 - v$  is bounded from below, then  $u(x, t) + \min_{|p| \leq \|Du_0\|_\infty} H(p) \cdot t$  converges in  $C(\mathbb{R}^N)$  when  $t \uparrow +\infty$  to the maximum viscosity solution of :

$$\begin{cases} H(D\omega) = \min_{|p| \leq \|Du_0\|_\infty} H(p) \\ \omega \leq u_0 \end{cases}$$

Remark I.4 : In the setting of remark I.3, the same result as corollary I.1 holds if we assume that there exists  $v \in UC(\mathbb{R}^N)$  viscosity solution of

$$H(Dv) = \max_{|p| \leq \|Du_0\|_\infty} H(p) \quad \text{in } \mathbb{R}^N$$

such that  $v - u_0$  is bounded from below. Then  $u(x,t) + \frac{\text{Max}_{|p| \leq \|Du_0\|_\infty} H(p) \cdot t}{\|Du_0\|_\infty}$  converges in  $C(\mathbb{R}^N)$  to the minimum viscosity solution of :

$$\begin{cases} H(D\omega) = \text{Max}_{|p| \leq \|Du_0\|_\infty} H(p) \\ \omega \geq u_0 \end{cases}$$

Many extensions of the same kind can be made, particularly if  $H$  is bounded.

Proof of theorem I.1 : The proof is almost exactly the same as those given by P.L. Lions in [9]. But we give the proof for the sake of completeness. The proof consists in showing that  $u(x,t)$  is compact in  $C(\mathbb{R}^N)$ . First we prove that  $u-v$  is bounded from below. Consider  $\omega(x,t) = v(x) - \|(u_0 - v)^-\|_{L^\infty(\mathbb{R}^N)}$  ;  $\omega$  is viscosity solution of :

$$\begin{cases} \frac{\partial \omega}{\partial t} + H(D\omega) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ \omega(x,0) = v(x) - \|(u_0 - v)^-\|_{L^\infty(\mathbb{R}^N)} \end{cases}$$

and  $\omega(x,0) \leq u(x,0)$ . So, by comparison results for viscosity solution (see [3], [4], [5], [6], [9]), we have

$$v(x) - \|(u_0 - v)^-\|_{L^\infty(\mathbb{R}^N)} \leq u(x,t)$$

Moreover, since  $H \geq 0$ , we have

$$u(x,t) \leq u(x,s) \leq u_0(x) \quad \forall x \in \mathbb{R}^N, \forall 0 \leq s \leq t$$

So, we get :  $-\|u_0 - v\|_\infty \leq u(x,t) - v(x) \leq u_0(x) - v(x)$ . Therefore,  $u - v$  is bounded on compact subsets of  $\mathbb{R}^N$  uniformly with respect to  $t$ .

Now if  $\rho_z$  denotes the modulus of continuity of  $z$  we are going to prove that :

$$(4) \quad \rho_u(h) \leq \rho_{u_0}(h) \quad , \quad \forall h > 0$$

For that, it suffices to remark that for all  $h > 0$  and  $e \in \mathbb{R}^N$ ,  $|e| = 1$   $u(x+h \cdot e, t)$  is viscosity solution of

$$\begin{cases} \frac{\partial z}{\partial t} + H(Dz) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ z(x, 0) = u_0(x+h \cdot e) \end{cases}$$

Then, by comparison results for viscosity solution (see [3], [4], [5], [6] and [9]) we have :

$$\|u(x+he, t) - u(x, t)\|_{L^\infty(\mathbb{R}^N \times (0, \infty))} \leq \|u_0(x+he) - u_0(x)\|_{L^\infty(\mathbb{R}^N)}$$

which gives (4). Then  $u(\cdot, t)$  is compact in  $C(\mathbb{R}^N)$  and so it exists a sequence  $t_n \uparrow +\infty$  such that  $u(\cdot, t_n)$  converges in  $C(\mathbb{R}^N)$  (i.e. uniformly on compact subsets) to a continuous function  $u_\infty$ . But since  $u$  is decreasing with respect to  $t$ ,  $u(\cdot, t)$  converges in  $C(\mathbb{R}^N)$  to  $u_\infty$  when  $t \uparrow \infty$ .

In order to prove that  $u_\infty$  is a viscosity solution of

$$H(Du_\infty) = 0 \quad \text{in } \mathbb{R}^N$$

let us give a different argument from the one in [9].

Let us consider  $\omega_T(x,t)$  viscosity solution of :

$$\begin{cases} \frac{\partial \omega_T}{\partial t} + H(D\omega_T) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ \omega_T(x,0) = u(x,T) & \text{in } \mathbb{R}^N \end{cases}$$

By uniqueness result, we have  $\omega_T(x,t) = u(x,t+T)$ . Now let  $T \uparrow +\infty$ ,  $\omega_T$  converges uniformly on compact subset of  $\mathbb{R}^N \times (0, \infty)$  to  $u_\infty(x)$  which is viscosity solution of

$$\begin{cases} \frac{\partial z}{\partial t} + H(Dz) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ z(x,0) = u_\infty(x) \end{cases}$$

by stability results for viscosity solutions. And so we have the result. It remains to prove that  $u_\infty$  is the maximum viscosity solution of (2). First we have to remark that there exists at least one solution for (2) which is  $v - \|(u_0 - v)^-\|_{L^\infty(\mathbb{R}^N)}$ . Now let  $\omega$  a viscosity solution of (2),  $\omega$  is viscosity solution of

$$\begin{cases} \frac{\partial \omega}{\partial t} + H(D\omega) = 0 \\ \omega(x,0) = \omega(x) \end{cases}$$

Now by comparison result for viscosity solutions (see [3], [4], [5], [6], [9]) we have :  $u(x,t) \geq \omega(x)$  and so  $u_\infty(x) \geq \omega(x)$ . This ends the proof of Theorem I.1.

We do not give The proof of Corollary I.1 which is a straightforward adaptation of the ideas of the proof of Theorem I.1 in the setting of the following remark :

Remark I.5 : This remark is due to M.G. Crandall and P.L. Lions [4].  
 Let  $u \in UC(\mathbb{R}^N \times (0, \infty))$  satisfy  $\|Du\|_{L^\infty(\mathbb{R}^N \times (0, \infty))}$  and let  
 $\varphi \in C^1(\mathbb{R}^N \times (0, \infty))$  then if  $(x_0, t_0)$  is a maximum (or a minimum) point  
 of  $u - \varphi$ , we have :

$$|D\varphi(x_0, t_0)| \leq \|Du\|_{L^\infty(\mathbb{R}^N \times (0, \infty))}$$

I.2. Some results for lipschitz initial conditions :

In this part, we are interested in problems like

$$(5) \quad \begin{cases} \frac{\partial u}{\partial t} + H(Du) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

where  $H$  is continuous and  $u_0$  is lipschitz and  $Du_0(x) \rightarrow p$  when  $|x| \rightarrow +\infty$ . The main result is the following :

Theorem I.2 : Let  $H \in C(\mathbb{R}^N)$ , differentiable at some  $p \in \mathbb{R}^N$  with  $H'(p) \neq 0$ . Let  $u_0$  satisfy :

$$(6) \quad \|Du_0\|_{L^\infty(\mathbb{R}^N)} < +\infty$$

$$(7) \quad |Du_0(x) - p| = O(|x|^{-\alpha}), \quad \text{when } |x| \rightarrow +\infty, \text{ where } \alpha > 1$$

and let  $u$  the viscosity solution of (5). Then, we have

$$(8) \quad u(x, t) - p \cdot x + H(p) \cdot t \rightarrow u_\infty \in \mathbb{R} \text{ uniformly on compact subsets of } \mathbb{R}^N \text{ when } t \uparrow +\infty, \text{ where } u_\infty = \lim_{|x| \rightarrow +\infty} (u_0(x) - p \cdot x).$$

Remark I.6 : A typical example is  $u_0(x) = p \cdot x$  and we know that  $u(x, t)$  is given by

$$u(x, t) = p \cdot x - t \cdot H(p).$$

After the proof we shall give other comments on the assumptions.

Proof of theorem I.2 : First we prove the existence of  $u_\infty$ . Let us define  $v$  by :

$$v(x) = u_0(x) - p \cdot x$$

By (6) and (7), it is easy to prove that there exists  $C \in \mathbb{R}$  such that :

$$|Dv(x)| \leq C \cdot (1 + |x|^\alpha)^{-1}$$

and thus  $v$  is bounded on  $\mathbb{R}^N$ . We are going to estimate  $|v(x) - v(y)|$  when  $|x|$  and  $|y| \rightarrow +\infty$  in order to prove that  $v$  converges as  $|x| \rightarrow +\infty$ .

Step 1 :  $y = \lambda \cdot x$  with  $\lambda > 1$ . We have :

$$\begin{aligned} |v(\lambda x) - v(x)| &= \left| \int_0^1 Dv((1+t(\lambda-1))x) \cdot (\lambda-1) \cdot x \, dt \right| \\ &\leq \int_0^1 \frac{C(\lambda-1)|x| \, dt}{1 + |(t(\lambda-1)+1)x|^\alpha} \end{aligned}$$

We make the change of variable :  $s = (t(\lambda-1)+1)|x|$  :

$$|v(\lambda x) - v(x)| \leq \int_{|x|}^{|\lambda x|} \frac{C \, ds}{1 + |s|^\alpha}$$

Remark I.7 : Since the function  $s \rightarrow \frac{1}{1+|s|^\alpha}$  is integrable the quantity

$$\int_{|x|}^{|\lambda x|} \frac{C \, ds}{1 + |s|^\alpha} \rightarrow 0 \text{ when } |x| \rightarrow +\infty.$$

Step 2 :  $|y| = |x|$ . There exists a rotation  $R(\theta)$  such that :

$$y = R(\theta) \cdot x$$

$$\begin{aligned} |v(y) - v(x)| &= \left| \int_0^1 Dv(R(s\theta)x) \cdot \frac{d}{ds} (R(s\theta)x) \, ds \right| \\ &\leq \int_0^1 \frac{C}{1 + |R(s\theta)x|^\alpha} \left| \frac{d}{ds} (R(s\theta)x) \right| \, ds \end{aligned}$$

But  $\left| \frac{d}{ds} (R(s\theta)x) \right| = \theta$ , hence we have :

$$|v(y) - v(x)| \leq \frac{C \cdot 2\pi}{1 + |x|^\alpha}$$

Now we can conclude : let  $y$  and  $x \in \mathbb{R}^N$  and assume  $|y| \geq |x|$ , then :

$$|v(y) - v(x)| \leq \left| v(y) - v\left(|x| \frac{y}{|y|}\right) \right| + \left| v\left(\frac{|x|}{|y|} y\right) - v(x) \right|$$

By steps 1 and 2, we can estimate the right-hand side by

$$|v(y) - v(x)| \leq \int_{|x|}^{|y|} C(1+s^\alpha)^{-1} \, ds + 2\pi C(1+|x|^\alpha)^{-1}$$

This inequality (and remark I.7) proves that  $v$  has a limit when  $|x| \rightarrow +\infty$  denoted by  $u_\infty$ .

Now we prove (8). We denote by  $z$  the function :

$$z(x,t) = u(x,t) - p \cdot x + H(p) \cdot t - u_\infty$$

We now observe the following : using step 1 and letting  $\lambda \rightarrow +\infty$  we obtain

$$(9) \quad |z(x,0)| = |v(x) - u_\infty| \leq \int_{|x|}^{+\infty} C \cdot (1+s^\alpha)^{-1} \, ds$$



and it is easy to prove that  $z$  is a viscosity solution of :

$$\begin{cases} \frac{\partial z}{\partial t} + H(Dz+p) - H(p) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ z(x,0) = u_0(x) - p \cdot x - u_\infty \end{cases}$$

We are going to prove that :

$$\lim_{t \uparrow +\infty} z(x,t) \geq 0$$

by constructing a convenient viscosity subsolution.

Let us denote by  $\psi(x,t)$  the quantity  $(t+1)^\beta - 1 - \varepsilon \cdot \frac{H'(p)}{|H'(p)|} \cdot x$ .

We have the following result :

Lemma I.1 : The function  $\omega(x,t) = - \int_{\psi(x,t)}^{+\infty} C(1+|s|^\alpha)^{-1} ds$  is a viscosity subsolution of

$$\frac{\partial v}{\partial t} + H(Dv+p) - H(p) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty)$$

for some positive constants  $\varepsilon$  and  $\beta$ , ( $\varepsilon < 1$ ).

Moreover,  $\omega(x,0) \leq z(x,0)$ .

Let us complete the proof of Theorem I.2 by using the result of Lemma I.1 : by lemma I.1 and using a comparison result for viscosity solution (see [3], [4], [5], [6], [9]) we have

$$\omega(x,t) \leq z(x,t)$$

Therefore :

$$\lim_{t \uparrow +\infty} z(x, t) \geq \lim_{t \uparrow +\infty} \omega(x, t) = 0$$

and it is easy to see that these limits are uniform on compact subsets of  $\mathbb{R}^N$ . In a similar way we prove that :

$$\overline{\lim}_{t \uparrow +\infty} z(x, t) \leq 0$$

by using  $\tilde{\omega}$  defined by :

$$\tilde{\omega}(x, t) = \int_{\tilde{\psi}(x, t)}^{+\infty} C(1+|s|^\alpha)^{-1} ds$$

where  $\tilde{\psi}(x, t) = (t+1)^\beta - 1 + \varepsilon \cdot \frac{H'(p)}{|H'(p)|} \cdot x$ .

This ends the proof of theorem I.2.

Now we prove Lemma I.1.

First we prove  $\omega(x, 0) \leq z(x, 0)$ . Since  $-\varepsilon \frac{H'(p)}{|H'(p)|} \cdot x \leq \varepsilon + |x| \leq |x|$ , we have

$$\int_{-\varepsilon \frac{H'(p)}{|H'(p)|} \cdot x}^{+\infty} C(1+|s|^\alpha)^{-1} ds \geq \int_{|x|}^{+\infty} C(1+|s|^\alpha)^{-1} ds$$

and then, by (9) :  $\omega(x, 0) \leq z(x, 0)$ .

Now we prove that  $\omega$  is a viscosity solution of

$$\frac{\partial v}{\partial t} + H(Dv+p) - H(p) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty)$$

Since  $\omega \in C^\infty(\mathbb{R}^N \times (0, \infty))$ , it suffices to prove :

$$\frac{\partial \omega}{\partial t} + H(D\omega + p) - H(p) \leq 0$$

we have :

$$D\omega(x, t) = D\psi(x, t) \cdot C \cdot (1 + |\psi(x, t)|^\alpha)^{-1}$$

Since  $D\psi(x, t) = -\varepsilon \frac{H'(p)}{|H'(p)|}$ , we have

$$\|D\omega\|_\infty \leq C \cdot \varepsilon$$

Now we compute  $\frac{\partial \omega}{\partial t} + H(D\omega + p) - H(p)$ . Using the fact that  $H$  is differentiable at  $p$ , we have uniformly in  $(x, t)$  :

$$\frac{\partial \omega}{\partial t} + H(D\omega + p) - H(p) = \frac{C \cdot \beta(t+1)^{\beta-1}}{1 + |\psi(x, t)|^\alpha} - \frac{C \cdot \varepsilon \cdot |H'(p)|}{1 + |\psi(x, t)|^\alpha} + o\left(\frac{\varepsilon}{1 + |\psi|^\alpha}\right)$$

Hence we have :

$$\frac{\partial \omega}{\partial t} + H(D\omega + p) - H(p) = \frac{C}{1 + |\psi|^\alpha} \left( \beta(t+1)^{\beta-1} - \varepsilon |H'(p)| + o(\varepsilon) \right)$$

Let us choose  $\varepsilon > 0$  such that  $\varepsilon^{-1} o(\varepsilon) \leq \frac{|H'(p)|}{2}$  :

$$\frac{\partial \omega}{\partial t} + H(D\omega + p) - H(p) \leq \frac{C}{1 + |\psi|^\alpha} \left( \beta(t+1)^{\beta-1} - \frac{\varepsilon |H'(p)|}{2} \right)$$

We choose  $\beta = \inf \left( 1, \frac{\varepsilon |H'(p)|}{2} \right)$  and the proof is complete.

Remark I.8 : Looking at the proof, we can see that  $H'(p)$  exists and is different from zero is an essential assumption. And it is not surprising :

consider the case when  $H$  is convex, continuous and  $H'(p) = 0$  then we have :

$$H(p+q) - H(p) \geq H'(p) \cdot q = 0$$

If we denote by  $G(q) = H(p+q) - H(p)$ ,  $G$  is convex continuous positive and  $G(0) = 0$ . In this case, the behavior of  $z(x,t)$  is described in part A and may be quite different from the behavior of  $z$  described in this part.

Remark I.9 : Let us observe that if we add a "perturbation  $\varphi$  ( $\in D(\mathbb{R}^N)$  for example) to  $u_0$ , this does not change the asymptotic behavior of  $u(x,t)$ . It is a striking difference with the context of part A.

Remark I.10 : The assumption  $|Du_0(y) - p| = O(|y|^{-\alpha})$  with  $\alpha > 1$  when  $|y| \rightarrow +\infty$  is necessary in order to have such a behavior. More precisely, if  $H \in W_{loc}^{1,\infty}(\mathbb{R}^N)$  and if  $u_0$  satisfy :

$$(10) \quad \|Du_0\|_{L^\infty(\mathbb{R}^N)} < +\infty$$

$$(11) \quad |Du_0(x) - p \cdot x| = O(|x|^{-\alpha}) \text{ when } |x| \rightarrow +\infty, \text{ with } \alpha \leq 1.$$

Let  $u$  the viscosity solution of (5), then we have :

$$|u(x,t) - p \cdot x + t H(p)| \leq C t^{1-\alpha} + C \quad \text{if } \alpha < 1$$

or

$$|u(x,t) - p \cdot x + t H(p)| \leq C \cdot \text{Log } t + C \quad \text{if } \alpha = 1.$$

This result is optimal (for the exponent  $1 - \alpha$ ) as one can see by looking at some example like :

$$\begin{cases} u_t - u_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x, 0) = \int_0^x \frac{ds}{1+|s|^\alpha} \end{cases}$$

where  $u(x, t) = \int_0^{x+t} \frac{ds}{1+|s|^\alpha}$ .

Let us briefly describe the ideas of the proof of this result. The proof consists in estimating the function  $z$  given by :

$$z(x, t) = u(x, t) - p \cdot x + t H(p)$$

As in the proof of theorem I.2,  $z$  is a viscosity solution of

$$\begin{cases} \frac{\partial z}{\partial t} + H(Dz+p) - H(p) = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ z(x, 0) = u_0(x) - p \cdot x \end{cases}$$

Using (10) and (11), we show easily that

$$|Dz(x, 0)| \leq \frac{C}{1+|x|^\alpha}$$

and so  $|z(x, 0)| \leq C_1 |x|^{1-\alpha} + C_2$ .

Now since  $H \in W_{loc}^{1, \infty}(\mathbb{R}^N)$  and  $\|Dz\|_\infty < +\infty$ ,  $z$  is respectively viscosity sub- and supersolution of

$$(12) \quad \frac{\partial z}{\partial t} - E \cdot |Dz| = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty)$$

$$(13) \quad \frac{\partial z}{\partial t} + E \cdot |Dz| = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty)$$

Now let us consider  $v$  viscosity solution of (12) with  $v(x,0) = z(x,0)$  and  $v$  viscosity solution of (13) with  $v_1(x,0) = z(x,0)$ . Using comparison results for viscosity solutions, we have

$$(14) \quad v_1 \leq z \leq v_2$$

To conclude it suffices to use the estimate above on  $z(x,0)$  and the Oleinik-Lax formula which gives  $v_1$  and  $v_2$  (cf. remark II.4). Then :

$$\inf_{|y-x| \leq Et} (v_1(x,0)) \leq z(x,t) \leq - \inf_{|y-x| \leq Et} (v_2(x,0))$$

Therefore :

$$- C_1(|x|+Et)^{1-\alpha} - C_2 \leq z(x,t) \leq C_1(|x|+Et)^{1-\alpha} + C_2$$

This ends the proof.

II . Asymptotics in  $\mathbb{R}^N$  : some estimates on the rate of convergence of

$u$  to  $u_\infty$  .

Our purpose is to give some estimates on the rate of convergence of  $u$  to  $u_\infty$  when  $u_0$  is bounded from below. Let us first remark that as in P.L. Lions [ 9 ], when  $H$  is convex or if  $\{p \mid H(p) = 0\}$  is convex,  $u_\infty$  is given by :

$$(15) \quad u_\infty(x) = \inf_{y \in \mathbb{R}^N} \left( u_0(y) + \sup_{H(p) = 0} (x-y|p) \right)$$

(See P.L. Lions [ 9 ] for the proof of this claim).

So in all this section, we will suppose that  $C = \{p \mid H(p) = 0\}$  is convex and we assume that  $\inf_{\mathbb{R}^N} u_0 = 0$  (this is no loss of generality because when we change  $u_0$  in  $u_0 - \inf_{\mathbb{R}^N} u_0$ ,  $u$  is changed in  $u - \inf_{\mathbb{R}^N} u_0$ , by the uniqueness result). We will also use the following assumptions and notations :

$$(16) \quad H(q) \geq A(d(q,C))^v \quad \text{where } A > 0, v > 0 \text{ and}$$

$$d(q,C) = \inf_{p \in C} (|p-q|)$$

$$(17) \quad \Lambda = \{x \in \mathbb{R}^N \mid u_0(x) = 0\}$$

$$(18) \quad |u_0(x)| \leq M \cdot |x|^{-\beta} \quad \text{for } |x| \geq 1 .$$

Remark II.1 : When  $C = \{0\}$ , the only viscosity solutions of :

$$H(Dv) = 0 \quad \text{in } \mathbb{R}^N$$

are the constants and  $u_0$  has to be bounded from below if  $u_0$  exists. So in this case we do not lose any generality.

The results depend strongly on the shape of  $C$  and we list the various situations in three different results.

Proposition II.1 : We assume  $C = \{0\}$ . Let  $H$  satisfy (16) with  $v \geq 1$ . Then,  $u_\infty = 0$  and :

(i) If  $\Lambda \neq \emptyset$  and if  $v > 1$

$$0 \leq u(x,t) \leq (At)^{-1/v-1} \psi(v) (d(x,\Lambda))^{v/v-1}$$

where  $\psi(v) = \left(\frac{v-1}{v}\right) v^{-1/v-1}$ ; while if  $\Lambda \neq \emptyset$  and  $v = 1$

$$u(x,t) = 0 \quad \text{when} \quad d(x,\Lambda) \leq At.$$

(ii) If  $u_0$  satisfies (18) :

$$0 \leq u(x,t) \leq B \cdot t^{-\gamma}$$

where  $\gamma = -\beta[1+(\beta+1)(v-1)]^{-1}$  and  $B$  is a constant which depends on  $A, v, x, \beta$ .

Proposition II.2 : We assume that  $C$  is a neighborhood of  $0$ . Let  $H$  satisfy (16) with  $v \geq 1$  and assume that

$$C \supset \{x / |x| \leq \varepsilon\}, \quad \varepsilon > 0. \quad \text{Then :}$$



- (i) If  $\nu > 1$  :  $|u(x,t) - u_\infty(x)| \leq (At)^{-1/\nu-1} \psi(\nu) \left( \frac{u_0(x)}{\varepsilon} \right)^{\nu/\nu-1}$
- (ii) If  $\nu = 1$  :  $u(x,t) = u_\infty(x)$  for  $t \geq \frac{u_0(x)}{A \cdot \varepsilon}$

Before giving the third result, let us give some notations and definitions :

$$D = \left\{ e \in \mathbb{R}^N / \sup_{H(p)=0} (e|p) = 0 \right\}$$

Notice that if  $C$  is not a neighborhood of  $0$ ,  $D$  is not equal to  $\{0\}$ . Then we introduce  $E$  defined by :

$$E = \Lambda + D = \{x / x = y + e, y \in \Lambda, e \in D\}$$

Remark II.2 : If  $C = \{0\}$ , we have  $D = \mathbb{R}^N$  and  $E = \mathbb{R}^N$ .

Proposition II.3 : We assume that  $C$  is not a neighborhood of  $0$ .

Let  $H$  satisfy (16) with  $\nu \geq 1$  and let  $u_0$  satisfy (18), then we have  $u_\infty = 0$  and :

(i) If  $\Lambda \neq \emptyset$  and if  $x \in E$  :

$$0 \leq u(x,t) \leq (At)^{-1/\nu-1} \psi(\nu) (g(x))^{\nu/\nu-1} \quad \text{if } \nu > 1$$

$$u(x,t) = 0 \quad \text{when } g(x) \leq At \quad \text{if } \nu = 1$$

where  $g(x) = \text{Inf} \{ |e| / e \in D, x-e \in \Lambda \}$

(ii) If  $\Lambda = \emptyset$  or if  $x \notin E$  :

$$0 \leq u(x,t) \leq B t^{-\gamma}$$

where  $\gamma = -\beta [1+(\beta+1)(\nu-1)]^{-1}$  and  $B$  is a constant which depends on  $A, \nu, x, \beta$ .

Remark II.3 : The above results extend or complement those obtained by S.N. Kruzkov [7,8] or P.L. Lions [9] and are optimal as one can see by looking at some examples. For instance, consider the following problem

$$\begin{cases} \frac{\partial u}{\partial t} + \left| \frac{\partial u}{\partial x} \right|^\nu = 0 & \text{in } \mathbb{R} \times (0, \infty), \quad \nu > 1 \\ u(x,0) = |x| \end{cases}$$

The viscosity solution is given by the Oleinik-Lax formula :

$$u(x,t) = \inf_{y \in \mathbb{R}} (\psi(y,t))$$

where  $\psi(y,t) = |y| + (\nu-1) \nu^{-\nu/\nu-1} t^{-1/\nu-1} |x-y|^{\nu/\nu-1}$ .

Now let us take  $x > 0$ ; then the infimum is achieved for  $y \in (0,x)$  because  $\psi(-y,t) < \psi(y,t)$  if  $y < 0$  and  $\psi(2x-y,t) < \psi(y,t)$  if  $y > x$ . So we have to compute the infimum on  $(0,x)$  of :

$$y + (\nu-1) \nu^{-\nu/\nu-1} t^{-1/\nu-1} (x-y)^{\nu/\nu-1}$$

let us remark that, as  $u(x,t) \rightarrow 0$  when  $t \uparrow +\infty$ , for  $t$  larger enough we may only consider  $y$  such that  $y \leq \frac{x}{2}$  because  $\psi(y,t) \geq y$ . Then, either the infimum is achieved in  $0$  or we have :

$$\frac{\partial \psi}{\partial y}(y, t) = 0$$

So :

$$1 - (\nu t)^{-1/\nu-1} (x-y)^{1/\nu-1} = 0$$

This implies :  $x-y = \nu t$  which is impossible for  $t$  larger enough. Then the Infimum is achieved in 0 and

$$u(x, t) = (\nu-1) \nu^{-\nu/\nu-1} t^{-1/\nu-1} |x|^{\nu/\nu-1}$$

This example shows that the first result of proposition II.1 is optimal. The same kind of computation by taking  $H(p) = (d(p, B(0, 1)))^\nu$  or  $|(p|A)|^\nu$  with  $u_0(x) = |x|$  or  $(1+|x|)^{-\beta}$ .

Remark II.4 : We have the same result if  $H$  is nonpositive and if  $H(-p) = H(p)$  for all  $p \in \mathbb{R}^N$ . It suffices to remark that, if  $S_H$  is the semi-group associated to  $H$ , we have :

$$S_{-H}(t) u_0 = - S_H(t) (-u_0)$$

Remark II.5 : We shall often use in the following the dual function of  $F(q) = |p|^\nu$  (with  $\nu \geq 1$ ), let us just recall that  $F^*$  is given by :

$$F^*(p) = \sup_{q \in \mathbb{R}^N} ((p(q) - F(q))) = \psi(\nu) |p|^{\nu/\nu-1} \quad (\nu > 1)$$

where  $\psi(\nu) = (\nu-1) \cdot \nu^{-\nu/\nu-1}$ , and if  $\nu = 1$ , we have

$$F^*(p) = \begin{cases} +\infty & \text{if } |p| > 1 \\ 0 & \text{if } |p| \leq 1 \end{cases}$$

Proof of Proposition II.1 : Since  $u_0(x) \geq 0$  and since 0 is viscosity solution of  $u_t + H(Du) = 0$  in  $\mathbb{R}^N \times (0, \infty)$ , we have by comparison results for viscosity solution (see [3], [4], [5], [6], [9])

$$0 \leq u(x, t)$$

Moreover by (16),  $u$  is viscosity subsolution of

$$(19) \quad \begin{cases} \frac{\partial u}{\partial t} + A|Du|^v = 0 & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

then, if we denote by  $\tilde{u}$  the viscosity solution of (19), we have by the comparison result for viscosity solutions :

$$u(x, t) \leq \tilde{u}(x, t)$$

Now it suffices to over estimate  $\tilde{u}(x, t)$ . It is easy because  $\tilde{u}$  is given by the Oleinik-Lax formula

if  $v \geq 1$  :

$$(20) \quad \tilde{u}(x, t) = \inf_{y \in \mathbb{R}^N} \left\{ u_0(y) + \psi(v) (At)^{-1/v-1} |x-y|^{v/v-1} \right\}$$

if  $v = 1$  :

$$(21) \quad \tilde{u}(x, t) = \inf_{|y-x| \leq At} (u_0(y))$$

If  $\Lambda \neq \emptyset$ , let  $x_0 \in \Lambda$  such that  $|x-x_0| = d(x, \Lambda)$ . We take  $y = x_0$  in (8)

$$\tilde{u}(x, t) \leq u_0(x_0) + \psi(v) (At)^{-1/v-1} |x_0-x|^{v/v-1}$$

"
  
0

which gives the first part of (i). And if  $At \geq |x-x_0|$ , we can take  $y = x_0$  in (21) which gives the second part of (i).

If  $\Lambda = \emptyset$  and if  $u_0$  satisfies (18),  $\tilde{u}$  satisfies :

$$(22) \quad \tilde{u}(x,t) \leq \inf_{y \in \mathbb{R}^N} (M|y|^{-\beta} + \psi(v) (At)^{-1/v-1} |x-y|^{v/v-1})$$

One has to compute an optimal "y" to take in (22). We give :

$$|y| \sim t^\delta$$

where  $\delta = (1 + (\beta+1)(v-1))^{-1}$ . Then from (22) we have :

$$\tilde{u}(x,t) = O(t^{-\beta\delta})$$

This ends the proof of proposition I.1.

Proof of proposition II.2 : The beginning of the proof is the same as in the proof of proposition II.1. We have

$$0 \leq u(x,t) \leq \tilde{u}(x,t)$$

where  $\tilde{u}$  is the viscosity solution of

$$\begin{cases} \frac{\partial u}{\partial t} + A(d(Du, C))^v = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Notice also that, since  $u(x,t) \uparrow u_\infty(x)$ , we have

$$(23) \quad u_\infty(x) \leq u(x,t) \leq \tilde{u}(x,t)$$

Since the zeros of  $H$  and  $d(\cdot, C)$  are the same,  $\tilde{u}$  converges to  $u_\infty$  when  $t \uparrow +\infty$  by (15) and by (23), we have

$$0 \leq u(x, t) - u_\infty(x) \leq \tilde{u}(x, t) - u_\infty(x)$$

We are going to estimate  $\tilde{u}(x, t) - u_\infty(x)$ . Since  $(d(\cdot, C))^v$  is convex  $\tilde{u}$  is given by the Oleinik-Lax formula

$$\tilde{u}(x, t) = \inf_{y \in \mathbb{R}^N} \left[ u_0(y) + \sup_{q \in \mathbb{R}^N} ((x-y|q) - At \cdot (d(q, C))^v) \right]$$

which is also :

$$\tilde{u}(x, t) = \inf_{y \in \mathbb{R}^N} \sup_{q \in \mathbb{R}^N} \sup_{q_0 \in C} (u_0(y) + (x-y|q) - At|q-q_0|^v)$$

because  $d(q, C) = \inf_{q_0 \in C} |q - q_0|$ .

Now we are going to estimate  $u_0(y) + (x-y|q) - At|q-q_0|^v$ . (This is the basic argument of the proof) :

$$u_0(y) + (x-y|q) - At|q-q_0|^v = u_0(y) + (x-y|q) + (x-y|q-q_0) - At|q-q_0|^v$$

Now, if  $v > 1$ , we claim that we have :

$$\begin{aligned} u_0(y) + (x-y|q) - At|q-q_0|^v &\leq \\ &\leq u_0(y) + \sup_{q_0 \in C} (x-y|q_0) + \psi(v) (At)^{-1/v-1} |x-y|^{v/v-1} \end{aligned}$$

And if  $v = 1$ , we have :

if  $|y-x| \leq At$  :

$$u_0(y) + (x-y|q) - At|q-q_0|^v \leq u_0(y) + \sup_{q_0 \in C} (x-y|q_0) .$$

Finally :

$$(24) \quad \tilde{u}(x,t) \leq \inf_{y \in \mathbb{R}^N} \left\{ u_0(y) + \sup_{q_0 \in C} (x-y|q_0) + \psi(v)(At)^{-1/v-1} \cdot |x-y|^{v/v-1} \right\} \quad \text{if } v > 1$$

$$(25) \quad \tilde{u}(x,t) \leq \inf_{|y-x| \leq At} \left\{ u_0(y) + \sup_{q_0 \in C} (x-y|q_0) \right\} \quad \text{if } v = 1$$

To conclude, it suffices to remark that we are only interested by the point  $y$  such that :

$$u_0(y) + \sup_{q_0 \in C} (x-y|q_0) + \psi(v)(At)^{-1/v-1} |x-y|^{v/v-1} \leq u_0(x)$$

Then,  $y$  must satisfy

$$\sup_{q_0 \in C} (x-y|q_0) \leq u_0(x) - u_0(y) \leq u_0(x)$$

But  $B(0,\varepsilon) \subset C$ , so the following inequalities hold

$$\varepsilon|x-y| \leq \sup_{q_0 \in C} (x-y|q_0) \leq u_0(x)$$

Finally :

$$(26) \quad |x-y| \leq \frac{u_0(x)}{\varepsilon}$$

And the same estimate holds if  $v = 1$ .

Using (26) in (24), we deduce :

$$\tilde{u}(x,t) \leq \inf_{y \in \mathbb{R}^N} \left[ u_0(y) + \sup_{q_0 \in C} (x-y|q_0) \right] + \\ + \psi(v) (At)^{-1/v-1} \left( \frac{u_0(x)}{\varepsilon} \right)^{v/v-1}$$

Using (26) in (25), we obtain that  $\tilde{u}(x,t) = u_\infty(x)$  for  $t \geq \frac{u_0(x)}{A\varepsilon}$ .  
This ends the proof of proposition II.2.

We do not give the proof of proposition II.3 which is a straightforward adaptation of the ideas of proofs of propositions II.1 and II.2.

Let us finally mention a result when  $H(p) \geq A \cdot |\Gamma|^v$  with  $v < 1$ .

Proposition I.4 : Let  $H$  satisfying :  $H(p) \geq A|p|^v$  with  $v < 1$ , and let  $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ . Then  $u_\infty = 0$  and we have :

(i) If  $\Lambda \neq \emptyset$ ,  $u(x,t) = 0$  if  $t \geq \frac{d(x,\Lambda)}{A} \|Du_0\|_\infty^{1-v}$

(ii) If  $u_0$  satisfies (18) :  $u(x,t) = 0(t^{-\beta})$ .

Proof of proposition I.4 : The idea of the proof is very simple : if  $\|Du_0\|_\infty \leq 1$ , then  $\|Du\|_\infty < 1$  and we have :

$$|Du|^{1/2} \geq |Du|$$

and we can apply the results of proposition II.1.

Remark II.6 : We must recall here that if  $u(\cdot,t) \in W^{1,\infty}(\mathbb{R}^N)$ , the sub-



and superdifferential of  $u(\cdot, t)$  at every point of  $\mathbb{R}^N$  are bounded by  $\|Du\|_\infty$ .

Now let  $u_0 \in W^{1,\infty}(\mathbb{R}^N)$  and let  $\alpha$  defined by :  $\alpha = \|Du_0\|_\infty^{-1}$ . Since  $H(p) \geq A|p|^\nu$ , by comparison results for viscosity solutions (see [3], [4], [5], [6], [9]) we have :

$$u(x, t) \leq \tilde{u}(x, t)$$

where  $\tilde{u}$  is the viscosity solution in  $\mathbb{R}^N \times (0, \infty)$  of :

$$\begin{cases} \frac{\partial u}{\partial t} + A|Du|^\nu = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

Now we are going to estimate  $\tilde{u}$ . We consider  $u_0(\alpha x)$ , the viscosity solution associated to  $v_0(x) = u_0(\alpha x)$  is  $\tilde{u}(\alpha x, \alpha^\nu t)$  by a uniqueness result (see [3], [4], [5], [6], [9]). Moreover, since we have :

$$\|D\tilde{u}\|_\infty \leq \|Du_0\|_\infty$$

then  $\tilde{v}(x, t) = \tilde{u}(\alpha x, \alpha^\nu t)$  satisfies  $\|D\tilde{v}\|_\infty \leq 1$ . By remark II.6, it is clear that  $\tilde{v}$  is a viscosity subsolution of :

$$(27) \quad \begin{cases} \frac{\partial u}{\partial t} + A|Du| = 0 \\ u(x, 0) = u_0(\alpha x) \end{cases}$$

Let  $v$  be the viscosity solution of (27), we have by comparison results for viscosity solutions (see [3], [4], [5], [6], [9])

$$\tilde{v}(x,t) \leq v(x,t)$$

So, we deduce :

$$u(x,t) \leq \tilde{u}(x,t) = \tilde{v}\left(\frac{x}{\alpha}, \frac{t}{\alpha^\nu}\right) \leq v\left(\frac{x}{\alpha}, \frac{t}{\alpha^\nu}\right)$$

But if  $\Lambda \neq \emptyset$ ,  $u_0(\alpha x) = 0$  for  $x \in \frac{\Lambda}{\alpha}$  then :

$$v(x,t) = 0 \quad \text{when } t \geq d\left(x, \frac{\Lambda}{\alpha}\right) \cdot A^{-1}$$

and

$$u(x,t) = 0 \quad \text{when } At \geq d(x, \Lambda) \|Du_0\|_\infty^{1-\nu}$$

Then if  $\Lambda = \emptyset$  and  $u_0$  satisfies (18), we have

$$u(x,t) = B \cdot E^\beta$$

where  $B$  depends on  $A$ ,  $x$ ,  $\beta$  and  $\|Du_0\|_\infty$ .

### III . Asymptotics in bounded or unbounded domains :

In this section, we are interested in the asymptotic behavior of the viscosity solutions of two kinds of problems. First, in convex domains  $\Omega$  , we consider the same type of problems as in section II :

$$(28) \quad \begin{cases} \frac{\partial u}{\partial t} + H(Du) = 0 & \text{in } \Omega \times (0, \infty) \\ u(x, 0) = u_0(x) \text{ in } \Omega, \quad u(x, t) = \varphi(x, t) \text{ on } \partial\Omega \times (0, \infty) \end{cases}$$

$H$  is a continuous positive function,  $H(0) = 0$  .  $u_0$  and  $\varphi$  are bounded uniformly continuous function.

Then, we give a result, (in a non necessary convex domain) similar to those obtained by P.L. Lions in  $\mathbb{R}^N$  , for the following problem :

$$(29) \quad \begin{cases} \frac{\partial u}{\partial t} + H(x, t, u, Du) = 0 & \text{in } \Omega \times (0, \infty) \\ u(x, 0) = u_0(x) \text{ in } \Omega, \quad u(x, t) = \varphi(x, t) \text{ on } \partial\Omega \times (0, \infty) \end{cases}$$

where  $H$  is a continuous function.  $u_0$  and  $\varphi$  are bounded uniformly continuous functions.

#### III.1. The case of convex domains.

We use the following notations and assumptions :

$$(30) \quad Q = \Omega \times (0, \infty) \quad , \quad \partial^0 Q = \partial\Omega \times (0, \infty) \cup \Omega \times \{0\}$$

$$(31) \quad \varphi(x, 0) = u_0(x)$$

$$(32) \quad \Lambda_s = \{x \in \partial\Omega \text{ (or } \Omega \text{ if } s = 0) \mid \varphi(x, s) = 0\}$$

$$(33) \quad |\varphi(x)| \leq M(1+|\alpha|)^{-\beta}, \quad \beta > 0.$$

$$(34) \quad |\varphi(x, t) - \varphi_\infty(x)| \leq B \cdot t^{-\alpha}, \quad \alpha > 0.$$

We always suppose that  $H$  is continuous positive function such that  $C = \{p \mid H(p) = 0\}$  is convex.  $H(0) = 0$ . We also assume that  $\inf_{\partial^0 Q} (\varphi(y, s)) = 0$  (without loss of generality).

We know by the results obtained by P.L. Lions [ 9 ], that if  $H$  is convex and if  $\varphi$  satisfies :

$$(\star) \quad \varphi(x, t) - \varphi(y, s) \leq (t-s) H^* \left( \frac{x-y}{t-s} \right) \quad \text{for all } (x, t), (y, s) \in \partial^0 Q \\ s < t$$

then (28) has a unique viscosity solution  $u$  given by the Oleinik-Lax formula

$$u(x, t) = \inf_{(y, s) \in \partial^0 Q} \left( \varphi(y, s) + (t-s) H^* \left( \frac{x-y}{t-s} \right) \right)$$

Proposition III.1 : Assume that there exists  $u \in BUC(\Omega \times (0, \infty))$  viscosity solution of (28). Then as  $t \uparrow +\infty$ ,  $u(x, t) \downarrow u_\infty(x)$  uniformly on compact subsets to  $u_\infty$ , which is the maximum viscosity solution of

$$\begin{cases} H(Dv) = 0 & \text{in } \Omega \\ v = \varphi_\infty & \text{on } \partial\Omega \\ v \leq u_0 & \text{in } \Omega \end{cases}$$

Moreover if  $H$  is convex (or if  $C$  is convex)  $u_\infty$  is given by :

$$u_\infty(x) = \inf_{y \in \partial\Omega} \left( \varphi_\infty(y) + \sup_{H(p)=0} (x-y | p) \right)$$

The proof is a straightforward adaptation of the ideas of P.L. Lions [9] and so, we do not give the proof.

**Proposition III.2 :** Assume that  $H(p) \geq A \cdot |p|^\nu$  and that there exists  $u \in BUC(\Omega \times (0, \infty))$  viscosity solution of (28). Assume also that  $\varphi$  satisfies (\*) with  $H = A|p|^\nu$  and (34). Then  $u_\infty \equiv 0$  and :

$$(i) \text{ If } \Lambda_s \neq \emptyset : |u(x,t)| \leq (A(t-s))^{-1/\nu-1} \psi(\nu)(d(x, \Lambda_s))^{\nu/\nu-1} \text{ for } \nu > 1$$

$$u(x,t) = 0 \text{ when } A(t-s) \geq d(x, \Lambda_s) \text{ for } \nu = 1$$

(ii) If,  $\forall s > 0$ ,  $\Lambda_s = \emptyset$  : if there exists  $s$  such that  $\varphi(y,s)$  satisfies (33) then :

$$u(x,t) = O(t^{-\delta}) \quad \text{where } \delta = \beta (1+(\nu-1)(\beta+1))^{-1}$$

for  $\nu \geq 1$ .

(iii) If,  $\forall s > 0$ ,  $\Lambda_s = \emptyset$ , then :

$$u(x,t) = O(t^{-\gamma}) \quad \gamma = \inf(\alpha, (\nu-1)^{-1})$$

Now when  $\Omega$  is bounded, we can give a result true for any convex  $C$  :

**Proposition III.3 :** Assume that  $H(p) \geq A(d(p,C))^\nu$  and that there exists  $u \in BUC(\Omega \times (0, \infty))$  viscosity solution of (28). Assume also that  $\varphi$  satisfies (\*) with  $H = A(d(p,C))^\nu$  and (34). Then we have :

$$(i) \quad \underline{\text{If } v > 1} : \quad u(x,t) - u_\infty(x) = O(t^{-\gamma})$$

with  $\gamma = \text{Inf}(\alpha, 1/v-1)$ .

$$(ii) \quad \underline{\text{If } v = 1} : \quad |u(x,t) - u_\infty(x)| = O(t^{-\alpha}) .$$

Remark III.1 : The above results extend or complement those obtained by S.N. Kruzkov [ 7 ] or [ 8 ] and P.L. Lions [ 9 ] and are optimal as one can see by looking at some examples like  $F(p) = |p|^v$  or  $(d(p, B(0,1)))^v$  or  $|(p|A)|^v$ .

Remark III.2 : As in remark II.4, we can say that we have the same results with  $H$  negative and such that  $H(p) = H(-p)$ .

We just give the proof of proposition III.3. The proof of proposition III.2 is a straightforward adaptation of the proofs of the section II.

Proof of Proposition III.3 : We only give the proof when  $v > 1$ . By the same argument as in the proof of proposition I.2, we have

$$u_\infty(x) \leq u(x,t) \leq \tilde{u}(x,t)$$

where  $\tilde{u}$  is the viscosity solution of :

$$\begin{cases} \frac{\partial u}{\partial t} + A(d(Du, C))^v = 0 & \text{in } Q . \\ u(x,0) = u_0(x) \text{ in } \Omega , \quad u(x,t) = \varphi(x,t) \text{ on } \partial\Omega \times (0, \infty) \end{cases}$$

$\tilde{u}$  exists because  $\varphi$  satisfies  $(\star)$  with  $A(d(p, C))^v$  (see P.L. Lions [ 9 ]), and  $\tilde{u}_\infty = u_\infty$  (with obvious notations).

In the same way as in the proof of proposition II.2, we have :

$$\tilde{u}(x,t) \leq \inf_{(y,s) \in \partial^0 Q} \left( \varphi(y,s) + \sup_{H(p)=0} (x-y/p) + \right. \\ \left. + \psi(v) (A(t-s))^{-1/v-1} |x-y|^{v/v-1} \right)$$

To conclude, it suffices to remark that  $|x-y| \leq \text{diam}(\Omega)$  and take  $s = t/2$  then  $|\varphi(y,s) - \varphi_\infty(y)| \leq B \left(\frac{t}{2}\right)^{-\alpha}$ .

Therefore :

$$\tilde{u}(x,t) \leq \underbrace{\inf_{y \in \partial\Omega} \left( \varphi_\infty(y) + \sup_{H(p)=0} (x-y/p) \right)}_{\text{III}} + B \left(\frac{t}{2}\right)^{-\alpha} + \psi(v) \left(\frac{At}{2}\right)^{-1/v-1} \cdot (\text{diam } \Omega)^{v/v-1} \\ u_\infty(x)$$

This ends the proof of proposition III.3.

III.2. One result on general domains :

We will use the following assumptions :

$$(35) \quad H \text{ is uniformly continuous on } \bar{\Omega} \times (0, T) \times (-R, R) \times \bar{B}_R \\ \text{for all } R, T < +\infty .$$

$$(36) \quad \forall R < \infty, \exists \gamma_R > 0 \text{ such that for } x \in \bar{\Omega}, t \in (0, \infty), p \in \mathbb{R}^N, \\ -R \leq s \leq t \leq R,$$

$$H(x, t, r, p) - H(x, t, s, p) \geq \gamma_R(r-s)$$

$$(37) \quad \varphi(x, t) \rightarrow \varphi_\infty(x) \text{ uniformly on } \partial\Omega$$

$$(38) \quad H(x, t, r, p) \rightarrow H_\infty(x, r, p) \text{ uniformly on compact subsets of } \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$$

$$(38') \quad H(x, t, r, p) \rightarrow H_\infty(x, r, p) \text{ uniformly on } \bar{\Omega} \times (-R, R) \times \mathbb{R}^N \text{ for all} \\ 0 < R < +\infty .$$

$$(38'') \quad H(x, t, r, p) \rightarrow H_\infty(x, r, p) \text{ uniformly on } \bar{\Omega} \times (-R, R) \times \bar{B}_R \text{ for all} \\ 0 < R < +\infty .$$

$$(39) \quad u \in BUC(\Omega \times (0, \infty)) \text{ viscosity solution of :}$$

$$(40) \quad \begin{cases} \frac{\partial u}{\partial t} + H(x, t, u, Du) = 0 & \text{in } \Omega \times (0, \infty) \\ u(x, 0) = u_0(x) \text{ in } \Omega, \quad u(x, t) = \varphi(x, t) \text{ on } \partial\Omega \times (0, \infty) \end{cases}$$

where  $u_0$  and  $\varphi$  are bounded uniformly continuous functions.

$$(41) \quad \limsup_{\varepsilon \uparrow 0} \left\{ |H(x, t, r, p) - H(y, t, r, p)| / |x-y|(1+|p|) \leq \varepsilon, |r| \leq R, \right. \\ \left. t \geq 0 \right\} = 0 \text{ for all } R < +\infty .$$



**Proposition III.4 :** Under assumptions (35), (36), (37), (38), (39) and (41)  $u$  converges uniformly on compact subsets of  $\bar{\Omega}$  to  $u_\infty$ , when  $t \uparrow +\infty$  where  $u_\infty$  is the unique viscosity solution of :

$$(P_\infty) \quad \begin{cases} H_\infty(x, u, Du) = 0 & \text{in } \Omega \\ u = \varphi_\infty & \text{on } \partial\Omega \end{cases}$$

Moreover if  $u \in W^{1,\infty}(\mathbb{R} \times (0, \infty))$  and if (38'') holds or if (38') holds we have :

$$(42) \quad \|u(\cdot, t) - u_\infty(\cdot)\|_{L^\infty(\bar{\Omega})} < \|e^{-\gamma(t-s)}(\varphi(\cdot, s) - \varphi_\infty(x))\|_{L^\infty(\bar{\Omega} \times (T, t))} \vee \left[ 2R_0 e^{-\gamma(t-T)} + \int_T^t e^{-\gamma(t-s)} f(s) ds \right]$$

where  $a \vee b = \sup(a, b)$ ,  $R_0 = \|u\|_{L^\infty(\Omega \times (0, \infty))}$ ,  $\gamma = \gamma_{R_0}$  and

$$f(t) = \|H(x, t+s, r, p) - H(x, t, r, p)\|_{L^\infty(\Omega \times (0, \infty) \times (-R_0, R_0) \times B_{\|Du\|_\infty})}$$

in the first case and

$$f(t) = \|H(x, t+s, r, p) - H(x, t, r, p)\|_{L^\infty(\Omega \times (0, \infty) \times (-R_0, R_0) \times \mathbb{R}^N)}$$

in the second case and

$$T = \inf \{ \theta \mid \forall \theta' \geq \theta, f(\theta') \leq 1 \} .$$

**Remark III.3 :**  $T$  exists because by assumption (38') or (38''),  $f(t) \rightarrow 0$  when  $t \uparrow +\infty$  and the term  $\int_T^t e^{-\gamma(t-s)} \cdot f(s) \cdot ds \rightarrow 0$  when  $t \uparrow +\infty$  because

$$\text{we can estimate it by : } \left( \frac{t-T}{2} \right) \exp\left(-\gamma\left(\frac{t-T}{2}\right)\right) + \sup_{s \geq \frac{t+T}{2}} f(s) .$$

Remark III.4 : This result extends those proved by P.L. Lions [9] in  $\mathbb{R}^N$ .

Remark III.5 : The assumption (36) with  $\gamma_R > 0$  is essential even for the first part of the result as can be seen from the following example :

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x, 0) = \sin(x) & \text{in } \mathbb{R} \end{cases}$$

In this example we have (36) with  $\gamma_R \equiv 0$  and (35), (37), (38) (even (38')!) (39) and (41) are satisfied. But  $u(x, t) = \sin(x+t)$  has not the behavior described in proposition III.4.

Proof of proposition III.4 : Let us take a sequence  $t_n \uparrow +\infty$ . Since  $u \in BUC(\Omega \times (0, \infty))$ ,  $u(\cdot, t_n)$  is compact in  $C(\bar{\Omega})$  by the Ascoli theorem. Hence there exists a subsequence, still denoted by  $t_n$ , such that  $u(\cdot, t_n)$  converges uniformly on compact subsets of  $\bar{\Omega}$  to  $v \in C(\bar{\Omega})$ . We are going to prove that  $u_\infty$ , the unique viscosity solution of  $(P_\infty)$ , exists and that  $v = u_\infty$ . This will imply the result because the uniqueness of  $u_\infty$  and the fact that every sequence  $t_n \uparrow +\infty$  has a subsequence  $t_n$ , such that  $u(\cdot, t_n)$  converges in  $C(\bar{\Omega})$  to  $u_\infty$  imply that  $u(\cdot, t)$  converges in  $C(\bar{\Omega})$  to  $u_\infty$  when  $t \uparrow +\infty$ .

Let us define  $u^n$  by :

$$u^n(x, t) = u(x, t+t_n - T)$$

where  $T > 0$ .  $u^n$  is well-defined for  $n$  larger enough.

Moreover  $u^n(x, T) = u(x, t_n)$  and  $u^n$  is viscosity solution of :

$$\begin{cases} \frac{\partial u^n}{\partial t} + H(x, t+t_n-T, u^n, Du^n) = 0 & \text{in } \Omega \times (0, \infty) \\ u^n(x, 0) = u(x, t_n-T) \text{ in } \Omega, \quad u^n(x, t) = \varphi(x, t+t_n-T) \text{ on } \partial\Omega \end{cases}$$

Since  $u^n$  is compact in  $BUC(\Omega \times (0, \infty))$ , there exists a subsequence, still denoted by  $u^n$ , such that  $u^n$  converges uniformly on compact subsets of  $\bar{\Omega} \times (0, \infty)$  to  $\tilde{u} \in BUC(\Omega \times (0, \infty))$ .

By the stability result for viscosity solution, we obtain that  $\tilde{u}$  is viscosity solution of

$$\frac{\partial \tilde{u}}{\partial t} + H_\infty(x, \tilde{u}, D\tilde{u}) = 0 \quad \text{in } \Omega \times (0, \infty)$$

Moreover  $\tilde{u}(x, t) = \varphi_\infty(x)$  on  $\partial\Omega$  and  $\tilde{u}(x, T) = v(x)$ .

Now the existence of  $u_\infty$  is given by the following Lemma :

**Lemma III.1** :  $\tilde{u}(\cdot, t)$  converges uniformly when  $t \uparrow +\infty$  to  $u_\infty$  which is the unique solution of  $(P_\infty)$ .

First we complete the proof of proposition III.4 using Lemma III.1. We recall that  $z(x, t) = u_\infty(x)$  is, by Lemma III.1, the unique viscosity solution of :

$$\begin{cases} \frac{\partial z}{\partial t} + H_\infty(x, z, Dz) = 0 & \text{in } \Omega \times (0, \infty) \\ z(x, 0) = u_\infty(x) \text{ in } \Omega, \quad z(x, t) = \varphi_\infty(x) \text{ on } \partial\Omega \end{cases}$$

By the comparison result for viscosity solutions (see [3], [4], [5], [6], [9]) we have :

$$\|e^{\gamma t}(\hat{u}-z)\|_{L^\infty(\Omega \times (0,\infty))} \leq \|\hat{u}(x,0)-u_\infty(x)\|_{L^\infty(\Omega)}$$

Therefore :

$$e^{\gamma T} \|v-u_\infty\|_{L^\infty(\Omega)} \leq \|e^{\gamma t}(\hat{u}-z)\|_{L^\infty(\Omega \times (0,\infty))} \leq 2 R_0$$

So :

$$v(x) - u_\infty(x) \leq 2 e^{-\gamma T} R_0, \quad \forall x \in \Omega$$

But  $T$  is arbitrary, so we have  $v = u_\infty$ . And the proof of the first part is complete. We do not give the proof of lemma III.1 which is totally similar to the proof of the second part of proposition III.4 which we give now.

We prove (42) under the assumption (38'). The proof in the case of (38'') and  $u \in W^{1,\infty}(\Omega \times (0,\infty))$  is exactly the same in the setting of Remark I.5.

The proof consists in estimating the quantity :

$$\|u(\cdot, T+t) - u(\cdot, T'+t)\|_{L^\infty(\Omega)}$$

where  $T$  is defined in proposition III.4 and  $T' > T$ .

In order to do this, we remark that  $u(\cdot, T+\cdot)$  is viscosity solution of

$$\begin{cases} \frac{\partial v}{\partial t} + H(x, T+t, v, Dv) = 0 & \text{in } \Omega \times (0, \infty) \\ v(x, 0) = u(x, T) & \text{in } \bar{\Omega}, \quad v(x, t) = \varphi(x, T+t) & \text{on } \partial\Omega \end{cases}$$

And the same result holds for  $u(\cdot, T'+\cdot)$ , changing  $T$  in  $T'$ .

Then  $u(\cdot, T'+\cdot)$  is respectively viscosity sub- and supersolution of :

$$\frac{\partial v}{\partial t} + H(x, T+t, v, Dv) - f(T+t) = 0$$

and

$$\frac{\partial v}{\partial t} + H(x, T+t, v, Dv) + f(T+t) = 0$$

Using comparison results for viscosity solutions (see [3], [4], [5], [6], [9]) we find

$$(43) \quad \|u(\cdot, T=t) - u(\cdot, T'+t)\|_{L^\infty(\Omega)} \leq \|e^{-\gamma(t-s)} (\varphi(\cdot, T+s) - \varphi(\cdot, T'+s))\|_{L^\infty(\Omega \times (0, t))} \\ \left[ 2R_0 e^{-\gamma t} + \int_0^{T+t} e^{-\gamma(t-s)} f(T+s) ds \right]$$

The inequality (43) shows (using remark III.3 and (38')) that  $u(\cdot, t)$  is a Cauchy sequence when  $t \uparrow +\infty$  in  $L^\infty(\Omega)$  and so  $u(\cdot, t)$  converges uniformly in  $\bar{\Omega}$  to a continuous function which we can prove to be  $u_\infty$  by (for example) the method given in the proof of Theorem I.1. Moreover  $u_\infty \in BUC(\Omega)$  and the modulus of continuity of  $u_\infty$  is estimate by the modulus of continuity of  $u$  in space.

Letting  $T' \uparrow +\infty$  in (43) gives the inequality (42), with the change of variable  $\theta = t+T$ .

This ends the proof of proposition III.4. Let us just remark that in the proof of (42) we have given completely the proof of Lemma III.1.

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