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**ON MULTISTEP APPROXIMATION
OF SEMIGROUPS
IN BANACH SPACES**

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ON MULTISTEP APPROXIMATION OF SEMIGROUPS IN BANACH SPACES
APPROXIMATION DE SEMI-GROUPES DANS DES ESPACES DE BANACH PAR DES
METHODES MULTIPAS

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Abstract : We consider a multistep rational approximation of a bounded, strongly continuous semigroup on a Banach space. We study the convergence when the time step converges to zero under a weak form of A-stability assumption.

Résumé : Etant donné un semi-groupe fortement continu dans un espace de Banach et son approximation par une méthode rationnelle multipas, nous étudions la convergence sous une hypothèse de A-stabilité affaiblie.

ON MULTISTEP APPROXIMATION OF SEMIGROUPS
IN BANACH SPACES

by Michel Crouzeix

Abstract : we consider a multistep rational approximation of a bounded, strongly continuous semigroup on a Banach space. We study the convergence when the time step converges to zero under a weak form of A -stability assumption.

1. Introduction.

Given a uniformly bounded, strongly continuous semigroup e^{tA} on a Banach space X :

$$\|e^{tA}\| \leq C_0, \quad (1)$$

and u_0 in X , we approximate the value $u(t) = e^{tA} u_0$ at the time $t_n = n \Delta t$ by the solution u_n of the q -step method

$$u_{n+1} = \sum_{j=0}^{q-1} r_j(\Delta t A) u_{n-j}, \quad n \geq q-1, \quad (2)$$

starting from the procedure

$$u_j = d_j(\Delta t A) u_0, \quad j=1, \dots, q-1. \quad (3)$$

where $r_j(z), d_j(z)$ are rational functions uniformly bounded for $\operatorname{Re} z \leq 0$ and $\Delta t > 0$ denotes the time step.

We first consider the scalar case where $X = \mathbb{C}$, $u_0 = 1$ and $A = \lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \leq 0$. In order to obtain a global error $u(t_n) - u_n = O(\Delta t^p)$, it is natural to assume a local error in (2) and in (3) of order $p+1$, that is to say, setting $z = \lambda \Delta t$,

$$e^{qz} - \sum_{j=0}^{q-1} r_j(z) e^{(q-j-1)z} = O(z^{p+1}) \quad (4)$$

and

$$e^{jz} - d_j(z) = O(z^{p+1}), \quad j=1, \dots, q-1. \quad (5)$$

We introduce the polynomial

$$P(x; z) = x^q - \sum_{j=0}^{q-1} r_j(z) x^{q-j-1}, \quad (6)$$

associated with the linear recursion formula (2); then Condition (4) may be rewritten as

$$P(e^z; z) = O(z^{p+1}). \quad (7)$$

We shall say that Scheme (2) is A-stable with defect $k \geq 0$ if :
 for all z with $\operatorname{Re} z \leq 0$, the roots of $P(\cdot; z)$ lie in the unit disk and the multiplicities of the unimodular roots are less than $k+1$.

The integer k is chosen as small as possible. We remark that, from (7), 1 is a root of $P(\cdot; 0)$. The case $k = 0$ corresponds to the classical notion of A-stability.

Remark : With each monic polynomial $P(\cdot; z)$, whose coefficients are rational functions of z , we can associate a method of Form (2); so, we can construct high order schemes in a simple way :

- if the method associated with the polynomial $P(\cdot; z)$ is of order p and A-stable with defect k , the method associated with $P(\cdot; z)^m$ is of order $mp+m-1$ and A-stable with defect $km+m-1$. Similarly, if another method, associated with $Q(\cdot; z)$, is of order q and A-stable with defect r , the method associated with $P(\cdot; z)Q(\cdot; z)$ is of order $p+q+1$ and A-stable with defect $k+r+1$.

The idea to consider this kind of construction was born during some discussions with G.A. Baker, see also [1].

In this note we prove the following theorem :

Theorem 1. *If Scheme (2) is A-stable with defect k , if Conditions (5) and (7) are satisfied and if u_0 belongs to $D(A^{p+1})$, then the following inequality holds*

$$\|u(t_n) - u_n\|_X \leq C_1 C_0 t_n^{k+1} \Delta t^{p-k} \|A^{p+1} u_0\|_X,$$

where the constant C_1 depends only on the rational functions r_j and d_j .

The proof of this theorem uses the mathematical framework described in the paper of Brenner and Thomée [4] and some technical lemmas; the case of analytical semigroups is considered in Section 3 and the hilbertian case in Section 4. For related works, see [2], [3], [6], [7].

2. Proof of Theorem 1.

We introduce the Frobenius matrix $R(z)$ associated with $P(\cdot; z)$ and the corresponding linear operator $R(\Delta t A)$ in $\mathcal{L}(X^q, X^q)$ defined by

$$R(z) = \begin{pmatrix} r_0 & r_1 & \dots & r_{q-1} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \quad R(\Delta t A) = \begin{pmatrix} R_0 & R_1 & \dots & R_{q-1} \\ I & 0 & \dots & 0 \\ 0 & I & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & I & 0 \end{pmatrix}$$

where r_j stands for $r_j(z)$ and R_j for $r_j(\Delta t A)$; we also consider the vectors in X^q of approximate and exact solutions :

where λ_{s_j} stands for $\lambda_{s_j}(z)$. Part a) follows from inequalities

$$\|J_S(z)^n\| \leq \rho(J_S(z))^n + n \rho(J_S(z))^{n-1} + \dots + \binom{n}{k_s} \rho(J_S(z))^{n-k_s}$$

and

$$\|R_S(z)^n\| \leq C \|J_S(z)^n\|.$$

b) From (10), we deduce

$$\left\| \frac{d}{dz} (R_S(z))^n \right\| \leq C \max_s (\|R_S(z)^n\| + \left\| \frac{d}{dz} (R_S(z))^n \right\|);$$

the only difficulty is to obtain an estimate of $\left\| \frac{d}{dz} (R_S(z))^n \right\|$ when

$\rho(R_S(z_0)) = \rho(R(z_0))$. But, in this case the matrix $P_S(z)$ may be chosen differentiable and therefore

$$\left\| \frac{d}{dz} (R_S(z))^n \right\| \leq C (\|J_S(z)^n\| + \left\| \frac{d}{dz} (J_S(z))^n \right\|);$$

for the matrix $J_S(z)^n$ we have the following upper bound

$$\left\| \frac{d}{dz} (J_S(z))^n \right\| \leq C [n \rho(J_S(z))^{n-1} + \dots + \binom{n}{k_s} (n - k_s - 1) \rho(J_S(z))^{n-k_s-1}].$$

Lemma 3. *If Scheme (2) is A-stable with defect k , the unimodular eigenvalues of $R(iy)$, when y is real, are twice differentiable near y .*

Proof: First, we remark that the eigenvalues of $R(z)$ are the roots of the polynomial $P(\cdot; z)$. Without loss of generality, we give the proof only for the case $y = 0$ and for the eigenvalues which converge to 1 as $z \rightarrow 0$. These eigenvalues can be written as Puiseux series

$$\lambda_j(z) = 1 + a_j z^{r_j} + b_j z^{s_j} + o(z^{s_j}),$$

where $0 < r_j < s_j$ are rational and $a_j \neq 0$. Since $|\lambda_j(z)| \leq 1$ when $\operatorname{Re} z \leq 0$, we cannot have $r_j > 1$; neither can $r_j = p_j / q_j$ be < 1 , since each determination of z^{1/q_j} has to be considered. Therefore we have $r_j = 1$ and $a_j > 0$. Considering the case $z = iy$, we must have $0 \leq 1 - \operatorname{Re} \lambda_j(iy) \approx -\operatorname{Re}(b_j (iy)^{s_j})$, which implies $s_j \geq 2$.

Using the compactness of $\bar{\mathbf{R}}$ and the previous lemma, we obtain the following corollary.

Corollary 4. *If the scheme is A-stable with defect k , there exists a constant C such that*

$$\begin{aligned} \|R(iy)^n\| &\leq C n^k, \\ \forall n \geq 1, \forall y \in \mathbf{R}, \\ \left\| \frac{d}{dy} R(iy)^n \right\| &\leq C n^{k-1}. \end{aligned}$$

Now we resume the study of the convergence estimates. Since $H_1(z) = (R(z) - e^z I)(e^{(q-1)z}, \dots, e^z, 1)^T = (P(e^z; z), 0, \dots, 0)^T = O(z^{p+1})$, we obtain from (9) and (7)

$$H_n(z) = \sum_{j=0}^{n-1} e^j z R(z)^{n-j-1} H_1(z) = O(z^{p+1}).$$

Similarly, using (5), we have $G_n(z) = O(z^{p+1})$. Therefore the functions

$$\tilde{H}_n(z) = H_n(z)/z^{p+1}, \quad \tilde{G}_n(z) = G_n(z)/z^{p+1} \quad (11)$$

are analytic and uniformly bounded in the halfplane $\operatorname{Re} z \leq 0$. Using the background described in Brenner-Thomée, there exist two functions \mathfrak{H}_n and \mathfrak{G}_n in $L^1(\mathbb{R})^q$ such that, for $\operatorname{Re} z \leq 0$,

$$\tilde{H}_n(z) = \int_0^{\infty} e^{tz} \mathfrak{H}_n(t) dt, \quad \tilde{G}_n(z) = \int_0^{\infty} e^{tz} \mathfrak{G}_n(t) dt, \quad (12)$$

and

$$\mathfrak{H}_n(t) = \mathfrak{G}_n(t) = 0 \text{ for } t < 0. \quad (13)$$

We define the two operators in $L(X^q, X^q)$

$$\tilde{H}_n(\Delta t A) = \int_0^{\infty} e^{t \Delta t A} \mathfrak{H}_n(t) dt, \quad \tilde{G}_n(\Delta t A) = \int_0^{\infty} e^{t \Delta t A} \mathfrak{G}_n(t) dt, \quad (14)$$

and we have from (8) and (11), if u_0 is in $D(A^{p+1})$,

$$U_n - V_n = \Delta t^{p+1} \tilde{H}_n(\Delta t A) A^{p+1} u_0 + \Delta t^{p+1} \tilde{G}_n(\Delta t A) A^{p+1} u_0.$$

Now, Theorem 1 is a simple consequence of the following lemma:

Lemma 5. *Under the hypotheses of Theorem 1, there exists a constant C such that*

$$\|\tilde{H}_n(\Delta t A)\| \leq C C_0 n^{k+1} \text{ and } \|\tilde{G}_n(\Delta t A)\| \leq C C_0 n^{k+1/2}.$$

Proof. We have, from (14) and (1),

$$\|\tilde{H}_n(\Delta t A)\| \leq C_0 \|\mathfrak{H}_n\|_{L^1(\mathbb{R})^q} \text{ and } \|\tilde{G}_n(\Delta t A)\| \leq C_0 \|\mathfrak{G}_n\|_{L^1(\mathbb{R})^q},$$

therefore it is sufficient to prove that

$$\|\mathfrak{H}_n\|_{L^1(\mathbb{R})^q} \leq C n^{k+1} \text{ and } \|\mathfrak{G}_n\|_{L^1(\mathbb{R})^q} \leq C n^{k+1/2}. \quad (15)$$

Let us introduce the functions : $\hat{\mathfrak{H}}_n(y) = H_n(iy)$, $\hat{\mathfrak{G}}_n(y) = G_n(iy)$;
 from (12) and (13), $\hat{\mathfrak{H}}_n$ and $\hat{\mathfrak{G}}_n$ are the Fourier transforms of \mathfrak{H}_n and \mathfrak{G}_n .
 A Carlson's inequality gives

$$\|\hat{\mathfrak{G}}_n\|_{L^1(\mathbb{R})^q} \leq C \|\hat{\mathfrak{G}}_n\|_{L^2(\mathbb{R})^q}^{1/2} \|\hat{\mathfrak{G}}_n'\|_{L^2(\mathbb{R})^q}^{1/2} ;$$

from Corollary 4, we have

$$\|\hat{\mathfrak{G}}_n(y)\| \leq C n^k \min\left(\frac{1}{|y|^{p+1}}, 1\right)$$

and

$$\|\hat{\mathfrak{G}}_n'(y)\| \leq C n^{k+1} \min\left(\frac{1}{|y|^{p+1}}, 1\right)$$

therefore

$$\|\hat{\mathfrak{G}}_n\|_{L^2(\mathbb{R})^q} \leq C n^k, \quad \|\hat{\mathfrak{G}}_n'\|_{L^2(\mathbb{R})^q} \leq C n^{k+1},$$

which shows the second inequality in (15).

In order to prove the first inequality, we introduce the notations

$$S(z) = \begin{pmatrix} s_0 & s_1 & \dots & s_{q-1} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad D(z) = \begin{pmatrix} e^{(q-1)z} & & & \\ & \ddots & & \\ & & 0 & \\ & & & e^z \\ & & & & 1 \end{pmatrix}$$

where s_j stands for $s_j(z) = r_j(z) e^{-(j-1)z}$. We have

$R(z) = e^z D(z) S(z) D(z)^{-1}$, therefore

$$\hat{\mathfrak{H}}_n(y) = \frac{1}{(iy)^{p+1}} e^{niy} D(iy) (S(iy)^{n-1}) E = e^{niy} D(iy) \hat{T}_n(y) \quad (16)$$

where $E = (1, 1, \dots, 1)^T$ and

$$\hat{T}_n(y) = \frac{1}{(iy)^{p+1}} (S(iy)^n - I) E. \quad (17)$$

We remark that \hat{T}_n is the Fourier transform of T_n which, from (16), is related to \mathfrak{H}_n by the relation, (on the j^{th} component),

$$\mathfrak{H}_n(t)_j = T_n(t - (n+q-j))_j$$

therefore, using a Carlson's inequality, we obtain

$$\|\mathfrak{H}_n\|_{L^1(\mathbb{R})^q} \leq C \|T_n\|_{L^1(\mathbb{R})^q} \leq C \|\hat{T}_n\|_{L^2(\mathbb{R})^q}^{1/2} \|\hat{T}_n'\|_{L^2(\mathbb{R})^q}^{1/2} .$$

The following lemma completes the proof of the first inequality in (15).

LEMMA 6 . If the multiplicity of the root 1 of $P(\cdot; 0)$ is $k+1$, then we have

$$\|\hat{T}_n\|_{L^2(\mathbb{R})} \leq C n^{k+1-1/(2r+2)} \quad \text{and} \quad \|\hat{T}_n'\|_{L^2(\mathbb{R})} \leq C n^{k+1+1/(2r+2)},$$

where r is a rational (which will be defined in the proof).

If the multiplicity is less than or equal to k , then

$$\|\hat{T}_n\|_{L^2(\mathbb{R})} \leq C n^k \quad \text{and} \quad \|\hat{T}_n'\|_{L^2(\mathbb{R})} \leq C n^{k+1}.$$

Proof . We consider only the case where the multiplicity of the root 1 is $k+1$, the other case is easier. Relation (17) and Corollary 4 imply

$$\|\hat{T}_n(y)\| \leq C n^k / |y|^{p+1} \quad \text{and} \quad \|\hat{T}_n'(y)\| \leq C(n^k / |y|^{p+2} + n^{k+1} / |y|^{p+1}). \quad (18)$$

In the definition of $S(z)$, we remark that $\mu(z)$ is an eigenvalue of $S(z)$ if and only if $\lambda(z) = e^z \mu(z)$ is an eigenvalue of $R(z)$, and that the algebraic multiplicity is preserved. From Lemma 2, we can find, in a neighborhood \mathcal{V} of zero on the imaginary axis, an invertible and twice differentiable matrix $H(z)$ such that

$$\forall z \in \mathcal{V}, \quad S(z) = H(z)^{-1} \begin{pmatrix} A(z) & 0 \\ 0 & B(z) \end{pmatrix} H(z), \quad (19)$$

where the matrix $B(z) - I$ is invertible and the $(k+1) \times (k+1)$ matrix $A(z)$ has the form

$$A(z) = \begin{pmatrix} \mu_0(z) & & & \\ 1 & \mu_1(z) & & 0 \\ & 1 & \ddots & \\ 0 & & \ddots & 1 & \mu_k(z) \end{pmatrix}, \quad \text{with} \quad \mu_j(0) = 1.$$

Now we introduce the twice differentiable vectors $A_n(z)$, $B_n(z)$, $a(z)$, $b(z)$, $\beta(z)$ by

$$\begin{pmatrix} A_n(z) \\ B_n(z) \end{pmatrix} = H(z) \hat{T}_n(z), \quad \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = H(z) E, \quad \beta(z) = \frac{1}{z^{p+1}} b(z); \quad (20)$$

then

$$A_n(z) = \frac{1}{z^{p+1}} (A(z)^n - I) a(z), \quad B_n(z) = (B(z)^n - I) \beta(z). \quad (21)$$

Since $\beta(z) = (B(z) - I)^{-1} B_1(z)$ is twice differentiable, we obtain

$$\|B_n(iy)\| \leq C n^k, \quad \|B_n'(iy)\| \leq C n^{k+1}, \quad \text{for } iy \in \mathcal{V}. \quad (22)$$

We assume that the order of the eigenvalues $\mu_j(z)$ of $A(z)$ has been chosen in order to ensure

$$0 \leq r = r_0 \leq r_1 \leq \dots \leq r_k,$$

for the first exponent of the Puiseux expansion

$$\mu_j(z) = 1 + a_j z^{r_j+1} + \dots, \quad r_j \in \mathbb{Q}, \quad a_j \neq 0.$$

Noticing that

$$A_n(z) = \sum_{j=0}^{n-1} A(z)^j A_1(z) \quad \text{and} \quad \|A'(z)\| \leq C |z|^\Gamma,$$

we get

$$\|A_n(z)\| \leq C n^{k+1}, \quad \|A_n'(z)\| \leq C n^{k+1} (1 + n |z|^\Gamma). \quad (23)$$

With the notations

$$a(z) = (a_0(z), \dots, a_k(z))^T, \quad e_0 = (1, 0, \dots, 0)^T$$

$$c(z) = (A(z) - I)(a(z) - a_0(z) e_0) = (0, c_1(z), \dots, c_k(z))^T, \quad (24)$$

($c(z) = 0$ when $k = 0$),

we can also write

$$A_n(z) = \frac{1}{z^{p+1}} \left[a_0(z) (A(z)^n - I) e_0 + \sum_{j=0}^{n-1} A(z)^j c(z) \right]. \quad (25)$$

Since the first row and the first column of $A(z)$ are not used in $A(z)^j c(z)$, this term is bounded by $C j^{k-1} \|c(z)\|$. Then

$$\|A_n(z)\| \leq C n^k (|a_0(z)| + \|c(z)\|) / |z|^{p+1}. \quad (26)$$

Similarly, by derivation of (25), we obtain

$$\|A_n'(z)\| \leq (p+1) \|A_n(z)\| / |z| + C n^k (|a_0'(z)| + \|c'(z)\|) / |z|^{p+1} + C n^{k+1} |z|^\Gamma (|a_0(z)| + \|c(z)\|) / |z|^{p+1}. \quad (27)$$

From the relations

$$(A(z) - I) a(z) = z^{p+1} A_1(z),$$

$$(\mu_0(z) - 1) a_0(z) = z^{p+1} (A_1(z))_0,$$

we get

$$a_0(z) = O(z^{p-r}), \quad a_0'(z) = O(z^{p-r-1}),$$

$$c(z) = z^{p+1} A_1(z) - a_0(z) (A(z) - I) e_0 = O(z^{p-r}),$$

and

$$c'(z) = O(z^{p-r-1}).$$

Together with (26), (27) and (23), it yields

$$\|A_n(z)\| \leq C n^k / |z|^{r+1}, \quad \|A_n'(z)\| \leq C n^{k+1} / |z| \text{ for } z \in \mathcal{V}.$$

Considering (18), (22) and (23), we have proved that, for all $y \in \mathbb{R}$,

$$\|\hat{T}_n(y)\| \leq C n^k \min(n, 1/|y|^{r+1}),$$

$$\|\hat{T}_n'(y)\| \leq C n^{k+1} \min(1+n|y|^r, 1/|y|).$$

The lemma follows from a simple calculation.

3. The case of holomorphic semigroups.

In this section, we make the stronger assumption that A generates a holomorphic semigroup on X ; more precisely we also assume that the spectrum of A is included in the sector S_θ and

$$\forall z \in \mathbb{C} \setminus S_\theta, \quad \|(zI - A)^{-1}\| \leq C / |z|, \quad (28)$$

where $\theta \in]0, \pi/2[$, C is a constant and

$$S_\theta = \{z \in \mathbb{C}; \pi - \theta \leq |\text{Arg } z| \leq \pi \text{ or } z = 0\}.$$

We can make weaker assumptions on the scheme; we assume that the rational functions r_j and d_j are uniformly bounded in S_θ and satisfy

$$P(e^z; z) = O(z^{p+1}) \quad (7)$$

and

$$e^z - d_j(z) = O(z^p), \quad j=1, \dots, q-1; \quad (29)$$

the last requirement is weaker than (5). We suppose also that the method is $A(\theta)$ -stable with defect k , that is to say
for all z belonging to S_θ , the roots of $P(\cdot; z)$ lie in the unit disk and the multiplicities of the unimodular roots are less than $k+1$.

We need also the following hypothesis: there exist $\eta > 0$ and μ , $0 < \mu < \cos \theta$, such that

$$\begin{aligned} &\text{for all root } \lambda_j(z) \text{ of } P(\cdot; z) \text{ such that } \lambda_j(0) \text{ is} \\ &\text{unimodular and of multiplicity } k+1, \text{ we have} \\ &\forall z \in S_\theta \text{ with } |z| \leq \eta, \quad |\lambda_j(z)| \leq e^{-\mu|z|}. \end{aligned} \quad (30)$$

We have the following theorems:

Theorem 7. *If the scheme is $A(\theta)$ -stable with defect k , if conditions (7), (28), (29) and (30) are satisfied and if u_0 belongs to $D(A^p)$, then the following inequality holds*

$$\|u(t_n) - u_n\|_X \leq C t_n^k \Delta t^{p-k} \|A^p u_0\|_X$$

Theorem 8. *If the same assumptions are satisfied but u_0 belongs to X , if furthermore, for all $z \in S_\theta - \{0\}$ and for $z = \infty$, the roots of $P(\cdot; z)$ lie in the open unit disk, then the following inequality holds*

$$\|u(t_n) - u_n\|_X \leq C \Delta t^{p-k} / t_n^{p-k} \|u_0\|_X$$

Proof of Theorem 7. We use the same representation formula (8)

$$U_n - V_n = H_n(\Delta t A) u_0 + G_n(\Delta t A) u_0, \quad (8)$$

as for Theorem 1, but instead of (11), we use the functions

$$\tilde{H}_n(z) = H_n(z)/z^p, \quad \tilde{G}_n(z) = G_n(z)/z^p. \quad (31)$$

Then

$$U_n - V_n = \Delta t^p (\tilde{H}_n(\Delta t A) + \tilde{G}_n(\Delta t A)) A^p u_0$$

and

$$\|U_n - V_n\|_X \leq C \Delta t^p (\|\tilde{H}_n(\Delta t A)\| + \|\tilde{G}_n(\Delta t A)\|) \|A^p u_0\|_X \quad (32)$$

Since $\|\tilde{H}_n(z)\| \leq C_n \min(|z|, |z|^{-p})$, we can use the Dunford-Taylor spectral representation

$$\tilde{H}_n(\Delta t A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} (zI - \Delta t A)^{-1} \tilde{H}_n(z) dz, \quad (33)$$

where Γ_θ denotes the oriented boundary of S_θ .

From Lemma 2 and (30), we deduce that

$$\forall z \in \Gamma_\theta, \quad \|R(z)^n\| \leq C n^k,$$

and

$$\forall z \in \Gamma_\theta \text{ with } |z| \leq \eta, \quad \|R(z)^n\| \leq C n^k e^{-\mu n |z|};$$

therefore

$$\forall z \in \Gamma_\theta \text{ with } z \neq 0, \quad \|\tilde{H}_n(z)\| \leq C n^k |z|^{-p}$$

and, since from (9), $\tilde{H}_n(z) = \sum_{j=0}^{n-1} e^{jz} R(z)^{n-j-1} \tilde{H}_1(z) = O(z^{p+1})$.

$$\forall z \in \Gamma_\theta \text{ with } |z| \leq \eta, \quad \|\tilde{H}_n(z)\| \leq C n^{k+1} e^{-\mu n |z|} |z| \quad (34)$$

Then, using (33) and (28),

$$\begin{aligned} \|\tilde{H}_n(\Delta t A)\| &\leq \frac{1}{\pi} \int_0^\infty \frac{C}{r} \tilde{H}_n(r e^{i\theta}) dr, \\ &\leq C \int_0^\eta n^{k+1} e^{-\mu nr} dr + C n^k \int_\eta^\infty r^{-p-1} dr, \\ &\leq C n^k. \end{aligned}$$

We cannot use directly the Dunford-Taylor spectral representation for $\tilde{G}_n(\Delta t A)$ when $\tilde{G}_n(0) \neq 0$, but we can write

$$\tilde{G}_n(\Delta t A) = e^{n\Delta t A} \tilde{G}_n(0) + \frac{1}{2\pi i} \int_{\Gamma_\theta} (z I - \Delta t A)^{-1} (\tilde{G}_n(z) - e^{nz} \tilde{G}_n(0)) dz;$$

therefore

$$\|\tilde{G}_n(\Delta t A)\| \leq C n^k + \frac{1}{\pi} \int_0^\infty \frac{C}{r} \|\tilde{G}_n(r e^{i\theta}) - \exp(nre^{i\theta}) \tilde{G}_n(0)\| dr,$$

and we obtain easily

$$\forall z \in \Gamma_\theta \text{ with } |z| \geq \eta, \quad \|\tilde{G}_n(z) - e^{nz} \tilde{G}_n(0)\| \leq C n^k |z|^{-p},$$

and

$$\forall z \in \Gamma_\theta \text{ with } |z| \leq \eta, \quad \|\tilde{G}_n(z) - e^{nz} \tilde{G}_n(0)\| \leq C n^{k+1} e^{-\mu n|z|} |z|.$$

Thus, $\|\tilde{G}_n(\Delta t A)\| \leq C n^k$ and Theorem 7 follows from (32).

Proof of Theorem 8. From the representation formula (8), we have

$$\|u(t_n) - u_n\|_X \leq C (\|H_n(\Delta t A)\| + \|G_n(\Delta t A)\|) \|u_0\|_X.$$

Now we write

$$H_n(\Delta t A) = (I - e^{\Delta t A}) H_n(\infty) + \frac{1}{2\pi i} \int_{\Gamma_\theta} (z I - \Delta t A)^{-1} (H_n(z) - (1 - e^z) H_n(\infty)) dz.$$

From Lemma 2, there exists β , $0 < \beta < 1$, such that

$$\forall z \in \Gamma_\theta \text{ with } |z| \geq \eta, \quad \|R(z)^n\| \leq C \beta^n;$$

using also $R(z) - R(\infty) = O(z^{-1})$, (when $z \rightarrow \infty$), we obtain

$$\forall z \in \Gamma_\theta \text{ with } |z| \geq \eta, \quad \|H_n(z) - (1 - e^z) H_n(\infty)\| \leq C n^{k+1} \beta^n |z|^{-1},$$

and from (34)

$\forall z \in \Gamma_\theta$ with $|z| \leq \eta$, $\|H_n(z) - (1 - e^z)H_n(\infty)\| \leq C(n^{k+1}e^{-\mu n|z|}|z|^{p+1} + \beta^n |z|)$.

Therefore

$$\begin{aligned} \|H_n(\Delta t A)\| &\leq C(n^{k+1}\beta^n + \int_0^\eta n^{k+1}e^{-\mu n r} r^p dr), \\ &\leq C(n^{k+1}\beta^n + n^{k-p} \int_0^\infty e^{-\mu t} t^p dt), \\ &\leq C n^{k-p} = C(\Delta t/t_n)^{p-k}. \end{aligned}$$

Similarly we get

$$\forall z \in \Gamma_\theta \text{ with } |z| \geq \eta, \quad \|G_n(z) - (1 - e^z)G_n(\infty)\| \leq C n^{k+1} \beta^n |z|^{-1},$$

and

$\forall z \in \Gamma_\theta$ with $|z| \leq \eta$, $\|G_n(z) - (1 - e^z)G_n(\infty)\| \leq C(n^k e^{-\mu n|z|}|z|^p + \beta^n |z|)$,
which yields

$$\|G_n(\Delta t A)\| \leq C(\Delta t/t_n)^{p-k}$$

and completes the proof of Theorem 8.

4. Remarks.

Theorem 1 is valid in particular when X is a Hilbert space but in this case it may be improved by taking Condition (29) in the place of Condition (5). Furthermore an easier proof can be given; indeed, changing possibly the norm in X , we can assume that e^{tA} is a semigroup of contraction on X , i.e. $\|e^{tA}\| \leq 1$, ($\forall t > 0$); then, from a theorem of von Neumann, we have

$$\|\tilde{H}_n(\Delta t A)\| \leq C \max_{\operatorname{Re} z = 0} \|\tilde{H}_n(z)\| \leq C n^{k+1},$$

similarly, $\|\tilde{G}_n(\Delta t A)\| \leq C n^k$, which completes the proof.

Now, if we assume further that A is a (semidefinite negative) selfadjoint operator, Theorem 7 and Theorem 8 are valid with $\theta = 0$. In this case, the proof becomes very simple since, from the spectral theory,

$$\|H_n(\Delta t A)\| \leq \sup_{x \leq 0} \|H_n(x)\|, \quad \|G_n(\Delta t A)\| \leq \sup_{x \leq 0} \|G_n(x)\|.$$

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