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**ON THE EQUIVALENCE
OF A-STABILITY
AND G-STABILITY**

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On the equivalence of A-stability and G-stability

Equivalence entre A-stabilité et G-stabilité

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Abstract : We prove an algebraic result which gives a new characterization of the A-stability property for linear multistep methods. This enables us to obtain a new simple proof of the equivalence between A-stability and G-stability for one-leg methods. Using the algebraic result, we also give an alternative proof of the second Dahlquist barrier.

Résumé : On montre un résultat algébrique qui fournit une nouvelle caractérisation de la A-stabilité pour les méthodes linéaires à pas multiples. Nous en déduisons une démonstration simple de l'équivalence entre A- et G-stabilité ainsi que la seconde barrière de Dahlquist.

" On the equivalence of A-stability and G-stability."

by Claudio Baiocchi and Michel Crouzeix.

Abstract : We prove an algebraic result which gives a new characterization of the A-stability property for linear multistep methods. This enables us to obtain a new simple proof of the equivalence between A-stability and G-stability for one-leg methods. Using the algebraic result, we also give an alternative proof of the second Dahlquist barrier.

1- The algebraic result.

Given two polynomials ρ and σ , of degree $r \geq 1$, with real coefficients, we consider the properties

$$(A) \quad " |z| \geq 1 \Rightarrow \operatorname{Re} \rho(z) \sigma(\bar{z}) \geq 0 "$$

and

there exist $(r+1)$ real polynomials p_1, \dots, p_r and q with

$$(B) \quad \deg(p_k) \leq r-1 \text{ and } \deg(q) \leq r, \text{ such that, for arbitrary } z, w \in \mathbb{C},$$

$$\rho(z) \sigma(w) + \rho(w) \sigma(z) + (1 - zw) \sum_{k=1}^r p_k(z) p_k(w) = q(z) q(w).$$

Theorem 1 . *The properties (A) and (B) are equivalent ; furthermore, if the polynomials ρ and σ are relatively prime, then the polynomials p_k are linearly independent .*

Proof . Implication (B) \Rightarrow (A) is obtained by taking $w = \bar{z}$. For the converse, we can assume that the polynomials ρ and σ are relatively prime, otherwise we may divide them by their common factor.

As a first step, we show that there exists a polynomial q , with real coefficients and $\deg(q) \leq r$, such that

$$E(z) \equiv \rho(z) \sigma(1/z) + \rho(1/z) \sigma(z) = q(z) q(1/z) . \quad (1)$$

Indeed, if we exclude the trivial case $E \equiv 0$, the roots of E are different from zero and their number (counted with their multiplicities) cannot exceed $2r$; since $E(z) = E(1/z)$, if z_j is a root of E with multiplicity n_j , $1/z_j$ is also a root of E with multiplicity n_j ; when $|z| = 1$, property (A) yields $E(z) = 2 \operatorname{Re} \rho(z) \sigma(\bar{z}) \geq 0$, therefore the multiplicities of the unimodular roots of E are even. Then E can be written as follows :

$$E(z) = a_0 \prod_{j: |z_j| \geq 1} (z - z_j)^{m_j} (1/z - \bar{z}_j)^{m_j},$$

where $m_j = n_j$ if $|z_j| > 1$ and $m_j = n_j/2$ if $|z_j| = 1$. We have $a_0 > 0$ since $E(z) \geq 0$ for $|z| = 1$. Then we can choose $q(z) = \sqrt{a_0} \prod (z - z_j)^{m_j}$; the coefficients of the polynomial q are real, since each complex root is associated with its conjugate.

Now the relation (1) is satisfied; this means that the polynomial $q(z)q(w) - \rho(z)\sigma(w) - \rho(w)\sigma(z)$ is equal to zero when $zw = 1$. It can then be written as follows

$$q(z)q(w) - \rho(z)\sigma(w) - \rho(w)\sigma(z) = (1 - zw) \sum_{k,\ell=1}^r g_{k\ell} z^{\ell-1} w^{k-1}. \quad (2)$$

By lemma 2 below, there exists a real matrix $A = (a_{ij})$, such that $G = A^T A$; the theorem follows by taking

$$p_k(z) = \sum_{i=1}^r a_{ki} z^{i-1}.$$

Since the matrix A is regular, the polynomials p_k are linearly independent.

Lemma 2. *The symmetric matrix $G = (g_{k\ell})$ is positive definite.*

Proof. For a multiple root z of $\rho + \lambda\sigma$, we have $\rho(z) + \lambda\sigma(z) = \rho'(z) + \lambda\sigma'(z) = 0$, whence $\rho(z)\sigma'(z) - \rho'(z)\sigma(z) = 0$; so that the multiple roots have a finite number of possible values. Since the polynomials ρ and σ are supposed relatively prime, the number of values of λ such that the polynomial $\rho(z) + \lambda\sigma(z)$ has multiple roots is finite. It follows that there exist a real number λ and r distinct complex numbers z_i satisfying,

$$\lambda > 0, \quad \rho(z_i) + \lambda\sigma(z_i) = 0 \quad \text{and} \quad \sigma(z_i) \neq 0, \quad \text{for } i = 1, \dots, r. \quad (3)$$

Furthermore, property (A) yields

$$|z_i| < 1, \quad \text{for } i = 1, \dots, r. \quad (4)$$

Now we consider the Vandermonde, $r \times r$, matrix

$$V = (v_{ij}) \quad \text{where} \quad v_{ij} = z_j^{i-1};$$

from (1) we deduce $V^* G V = B = (b_{ij})$, where

$$\begin{aligned} b_{ij} &= [q(\bar{z}_i)q(z_j) - \rho(\bar{z}_i)\sigma(z_j) - \rho(z_i)\sigma(\bar{z}_i)] / (1 - \bar{z}_i z_j) \\ &= [q(\bar{z}_i)q(z_j) + 2\lambda\sigma(\bar{z}_i)\sigma(z_j)] / (1 - \bar{z}_i z_j). \end{aligned}$$

The bound (4) allows the expansion

$$b_{ij} = \sum_{m=0}^{\infty} [\bar{z}_i^m q(\bar{z}_i) q(z_j) z_j^{m+2} + 2 \lambda \bar{z}_i^m \sigma(\bar{z}_i) \sigma(z_j) z_j^m] .$$

Since $\sigma(z_i) \neq 0$, for $i = 1, \dots, r$, we deduce that the matrix B , and consequently the matrix G , is positive definite .

2. Application 1. The equivalence of A- and G-stability.

In order to approximate the solution of differential equations, we consider the linear multistep method associated with the pair of polynomials (ρ, σ) . For a sequence $y_0, y_1, \dots, y_n, \dots$ of complex numbers, we denote by E the unitary shift operator : $E y_n = y_{n+1}$. According to G. Dahlquist, the (ρ, σ) method is A-stable if, for all complex numbers λ such that $\text{Re } \lambda < 0$ and all sequences y_n satisfying $\rho(E) y_n = \lambda \sigma(E) y_n$, we have $\lim_{n \rightarrow \infty} y_n = 0$; or equivalently

$$\forall \lambda \in \mathbf{C}, \text{Re } \lambda < 0 \text{ and } \rho(z) - \lambda \sigma(z) = 0 \Rightarrow |z| < 1 .$$

Therefore, A-stability implies property (A), (and is equivalent to it, when the polynomials ρ and σ are relatively prime).

Consider the column vector $Y_n = (y_n, y_{n+1}, \dots, y_{n+r-1})^T$ in \mathbf{C}^r ; according to G. Dahlquist, the (ρ, σ) method is G-stable if there exists a symmetric, positive definite matrix G such that

$$\text{Re } \rho(E) y_n \cdot \overline{\sigma(E) y_n} \leq 0 \Rightarrow Y_{n+1}^* G Y_{n+1} \leq Y_n^* G Y_n . \quad (5)$$

We have seen that the property (B) implies (2), therefore

$$q(E) y_n \cdot \overline{q(E) y_n} - 2 \text{Re } \rho(E) y_n \cdot \overline{\sigma(E) y_n} = Y_n^* G Y_n - Y_{n+1}^* G Y_{n+1} ,$$

which gives (5). Conversely, using the sequence $y_n = z^n$, we obtain the property (A) from (5). We get in another way, the following result of G. Dahlquist .

Theorem 3. *If the polynomials ρ and σ are relatively prime, A-stability and G-stability are equivalent.*

We remark also that the proof of theorem 1 provides a way to construct the matrix G .

3. Application 2. The second barrier of Dahlquist.

Theorem 4. *The order of accuracy of a linear multistep A-stable method is limited to two. For the method of order two, the best possible error constant is $1/12$.*

Proof. The order of a (ρ, σ) method is 2 if

$$\rho(e^z) / \sigma(e^z) = z - c z^3 + O(z^4), \quad (\text{as } z \rightarrow 0); \quad (6)$$

it is greater than 2 when the error constant c is zero. Furthermore A-stability and (6) imply property (B) and $\sigma(1) \neq 0$. Setting

$$\pi_k(z) = \rho_k(e^z) / \sigma(e^z), \quad \varphi(z) = \rho(e^z) / \sigma(e^z),$$

we obtain

$$z + w - c(z^3 + w^3) + (1 - e^{z+w}) \sum_{k=1}^r \pi_k(z) \pi_k(w) = \varphi(z) \varphi(w) + O(z^4 + w^4).$$

If we identify the coefficients in the Taylor expansion, we deduce that

$$\varphi(0) = \varphi'(0) = 0,$$

$$\sum_{k=1}^r \pi_k(0)^2 = 1, \quad \sum_{k=1}^r \pi_k(0) \pi_k'(0) = -1/2, \quad \sum_{k=1}^r \pi_k'(0)^2 = c + 1/6.$$

From the Cauchy-Schwarz inequality $1/4 \leq c + 1/6$, whence $c \geq 1/12$.

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