



Tabulations of the closure tensors for the M.P.P. model of turbulence

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TABULATIONS OF THE CLOSURE TENSORS FOR THE M.P.P. MODEL OF TURBULENCE

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**TABULATIONS OF THE CLOSURE TENSORS
FOR THE
M.P.P. MODEL OF TURBULENCE**

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TABULATIONS OF THE CLOSURE TENSORS FOR THE M.P.P. MODEL OF TURBULENCE

Abstract.

We study the approximations of the closure tensors, vectors and functions appearing in the MPP turbulence model. The model gives a complete system of equations for the mean velocities $u(x, t)$, the turbulent kinetic energy $q(x, t)$, the turbulent helicity $h(x, t)$ and the inverse of the lagrangian coordinates $a(x, t)$. The ratio h/q plays an important role in this model.

For 1-D mean flows, we deduce the structures of all closure terms as function of two parameters, one of them is the ratio h/q . Frame invariance for the averaged system leads us to reduce the number of parameters to compute the Reynolds stress tensor; we use this property to relate the computations of 2-D and 3-D mean flows with the 1-D case.

TABULATIONS DU TENSEUR DE FERMETURE DANS LE MODELE DE TURBULENCE M.P.P.

Résumé.

On étudie les approximations des tenseurs, vecteurs et fonctions de fermeture du modèle MPP de turbulence. Le modèle donne un système d'équations complet reliant les vitesses moyennes $u(x, t)$, l'énergie cinétique turbulente $q(x, t)$, l'hélicité turbulente $h(x, t)$ et les coordonnées lagrangiennes $a(x, t)$. Le rapport $\beta = h/q$ joue un rôle important dans le modèle.

Dans le cas d'un écoulement moyen monodimensionnel, on peut déduire les structures de tous les termes de fermeture en fonction de deux paramètres, dont un est le rapport h/q . Si le système moyen est invariant par changement de repère, on peut réduire le nombre des paramètres pour le calcul du tenseur de Reynolds; en utilisant cette propriété, on relie les calculs d'écoulements moyens 2-D et 3-D avec le cas monodimensionnel.

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Notations

- Ω : domain in \mathbf{R}^3
 Y : cube $[-\pi, \pi]^3$
 Z : interval $[0, +\infty[$
 $x = (x_1, x_2, x_3)$: space variable ($x \in \Omega$)
 t : time variable ($t > 0$)
 $y = (y_1, y_2, y_3)$: space variable for the microstructures ($y \in Y$)
 τ : time variable for the microstructures ($\tau > 0$)
 $u(x, t)$: velocities of the fluid at the point x and at instant t , ($u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))^T$)
 $p(x, t)$: pressure
 ϵ : ratio of the small to the large scale
 $\epsilon^{1/2} w^o(x/\epsilon, x)$: initial fluctuating field of velocities
 $a(x, t)$: inverse of the Lagrangian coordinates
 $u_{,t}$: $\partial u / \partial t$
 $u_{i,j}$: $\partial u_i / \partial x_j$, $1 \leq i, j \leq 3$
 $u_{i,J}$: $\partial u_i / \partial y_j$, $1 \leq i, j \leq 3$
 ∇a : matrix which elements are $(\nabla a)_{ij} = a_{j,i}$, $1 \leq i, j \leq 3$
 C : $\nabla a^T \nabla a$
 $u \otimes v$: matrix which elements are $u_i v_j$, $1 \leq i, j \leq 3$
 $tr(B)$: trace of the matrix B , B_{ii}
 $\nabla \cdot B$: vector which components are $B_{ji,j}$, $i = 1, 2, 3$
 $w(y, \tau; x, t)$: turbulent velocities (y - periodic in Y)
 $\pi(y, \tau; x, t)$: turbulent pressure (y - periodic in Y)
 $u^{(k)}(y, \tau; x, t)$: k -order turbulent velocities (y - periodic in Y)
 $p^{(k)}(y, \tau; x, t)$: k -order turbulent pressure (y - periodic in Y)
 $\tilde{\nabla}_y w = (\nabla a \nabla_y) w$: matrix which elements are $[(\nabla a \nabla_y) w]_{i,j} = a_{k,i} w_{j,K}$
 $\tilde{\nabla}_y \times w = (\nabla a \nabla_y) \times w$: vector which components are $[(\nabla a \nabla_y) \times w]_i = \epsilon_{ikn} a_{j,k} w_{n,J}$ where ϵ_{ikn} is the third order alternating tensor, i.e.

$$\epsilon_{ikn} = \begin{cases} 1 & \text{if } ikn \text{ is in 12312,} \\ -1 & \text{if } ikn \text{ is in 13213,} \\ 0 & \text{if two subscripts are repeated.} \end{cases}$$

$A : B$: inner product of matrices A and B , i.e. $tr(B^T A) = A_{ij} B_{ij}$

$\langle w \rangle$: mean in Y , $\langle w \rangle = 1/(2\pi)^3 \int_Y w dy$

$\langle\langle w \rangle\rangle$: mean in $Y \times Z$, $\langle\langle w \rangle\rangle = \lim_{\theta \rightarrow \infty} 1/\theta \int_0^\theta \langle w \rangle d\tau$

$\tilde{w} = \nabla a^T w$: canonic microstructure flow

$g(x, t)$: mean turbulent kinetic energy $1/2 \langle |w|^2 \rangle = 1/2 \langle \tilde{w} C^{-1} \tilde{w} \rangle$

$h(x, t)$: mean turbulent helicity $\langle w \cdot \tilde{\nabla}_y \times w \rangle = \langle (\nabla a^{-T} \tilde{w}) \cdot (\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}) \rangle$

$\tilde{w}'(y, \tau; x, t) = q^{-1/2} \tilde{w}(y, q^{-1/2} \tau; x, t)$: normalized canonic microstructure flow

$\pi'(y, \tau; x, t) = q^{-1} \pi(y, q^{-1/2} \tau; x, t)$: normalized turbulent pressure

In this definitions, we have used the summation convention over repeated indices.

O. INTRODUCTION.

Turbulent flow, i.e. with rapidly varying structures in space and time, is a very complex mathematical problem. The numerical simulation of this kind of fluids cannot be obtain directly due to restricted capacity of actual computers.

Consider the Navier-Stokes equations which govern the velocities $\mathbf{u}(x, t)$ and the pressure $p(x, t)$ for incompressible viscous Newtonian fluids :

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \nu \Delta \mathbf{u} = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times]0, T[\quad (0.1)$$

$$\mathbf{u}(x, 0) = \mathbf{u}^o(x) \quad \text{in } \Omega \quad (0.2)$$

$$\mathbf{u} = \mathbf{u}_\Gamma \quad \text{in } \Gamma \times]0, T[\quad (0.3)$$

Here Ω is the domain occupied by the fluid, Γ its boundary, $]0, T[$ the time interval, \mathbf{u}^o stands for the initial velocities of the fluid and \mathbf{u}_Γ the velocities on the walls. We suppose that the characteristic length of Ω , \mathbf{u}^o and/or \mathbf{u}_Γ are unity (the variables have been rescaled). Then ν represents the inverse of the Reynolds number ($Re = \nu^{-1}$). If ν tends to zero, the flow becomes more and more oscillating in x and t . An argument based on dimensional analysis shows that the smallest oscillations in space of \mathbf{u} are on a scale of order $\nu^{3/4}$ (cf. Monin-Yaglom [1]). Thus a numerical simulation of (0.1)-(0.3) allowing us to represent the smallest structures must have a grid of size $h = O(\nu^{3/4})$; turbulence models are otherwise needed.

0.1. Goals of the report.

In this paper, we study a mathematical model of turbulence developped by McLaughlin, Papanicolaou, and Pironneau ([2], [3]). The model uses the asymptotic expansion of homogenization theory (see [4]). The main hypothesis of the model consists in assuming that the large scale developped by the mean field and the small scale related to the fluctuating field are far apart. Turbulence is simulated by an oscillating initial condition in the Navier-Stokes equation

$$\mathbf{u}^\varepsilon(x, 0) = \mathbf{u}^o(x) + \mathbf{w}^o\left(\frac{x}{\varepsilon}, x\right) \quad (0.4)$$

where

- ε is the ratio between the large and the small scale,
- \mathbf{u}^o is the initial mean field,
- \mathbf{w}^o is the initial fluctuating field.

In [3], equations are given for the mean flow $\mathbf{u}(x, t)$, the mean turbulent kinetic energy $q(x, t)$ and the mean turbulent helicity $h(x, t)$. These three averages verify a non linear system of convection-diffusion equations involving the lagrangian coordinates relative to the mean flow. The coefficients of the equations can be computed, in principle, by solving an Euler equation for the microstructure.

We examine in this paper the computation of the Reynolds stress tensor from the microstructure equation, its approximation in terms of two parameters, and we show that, in some cases, we can eliminate the equation for the helicity.

Our goal is to compute mean values of moments of $\tilde{\mathbf{w}}$, solution of:

$$\left. \begin{array}{l} \tilde{w}_{,\tau} + (\tilde{w} \cdot \nabla_y) \tilde{w} + C \nabla_y \pi = 0, \quad \nabla_y \cdot \tilde{w} = 0 \quad \text{in } Y \times Z \\ \tilde{w} \text{ is } y\text{-periodic in } Y, \quad \tilde{w}(y, 0) = w^0(y; x) \quad \text{in } Y \\ \langle \tilde{w} \rangle = 0 \\ \frac{1}{2} \langle \tilde{w} C^{-1} \tilde{w} \rangle = q \\ \langle (\nabla a^{-T} \tilde{w}) \cdot (\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}) \rangle = h \end{array} \right\} \quad (0.5)$$

where

- $Y =]-\pi, \pi[^3, Z =]0, +\infty[$

- $\langle f \rangle = \frac{1}{(2\pi)^3} \int_Y f dy$

- $a(x, t)$ is solution of problem

$$a_{,t} + (u \cdot \nabla) a = 0, \quad a(x, 0) = x,$$

- $C = \nabla a^T \nabla a$,

- $u(x, t)$ is the mean flow,

- $q(x, t)$ is the mean turbulent energy,

- $h(x, t)$ is the mean turbulent helicity.

The functions $q(x, t)$ and $h(x, t)$ are knowns in (0.5), so we can write the general solution of (0.5) as

$$\tilde{w}(y, \tau; x, \nabla a, q, h). \quad (0.6)$$

It is too expensive to tabulate $y - \tau$ mean moments of \tilde{w} in terms of $\nabla a, q, h$, so we shall show that less parameters are independent.

We can show that \tilde{w} can be computed as a function of the ratio $\beta = h/q$ only (chapter 2). The numerical simulations of (0.5) have been made in the stationary case ($\tilde{w}_{,\tau} = 0$) and for matrices ∇a of the form

$$\nabla a = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (0.7)$$

which correspond to fluids with one-dimensional mean flow, i.e.

$$u(x, t) = (u_1(x_2, t), 0, 0)^T. \quad (0.8)$$

In this way, we can rewrite (0.6) as $\tilde{w}(y; \alpha, \beta)$, and the numerical computations are made depending on parameters α and β .

In chapter 3, we study the stress tensors, vectors and functions

$$R = \langle w \otimes w \rangle$$

$$S = 2 \langle w \otimes r \rangle$$

$$V_q = \langle (\pi + \frac{1}{2}|w|^2)w \rangle \quad V_h = 2 \langle (\pi + \frac{1}{2}|w|^2)r \rangle$$

$$\psi_q = \langle (\nabla a \nabla_y)w : (\nabla a \nabla_y)w \rangle \quad \psi_h = 2 \langle (\nabla a \nabla_y)w : (\nabla a \nabla_y)r \rangle$$

$$w = \nabla a^{-T} \tilde{w}, \quad r = (\nabla a \nabla_y) \times w$$

as functions of α and β . All these functions appear in the nonlinear system verified by $u(x, t)$, $q(x, t)$ and $h(x, t)$ (in fact these are closure terms in our model). We give a power expansion in α and β for R , S , V_q , V_h , ψ_q and ψ_h , deduced from some properties of (\tilde{w}, π) , (symmetries, evenness-oddness in α, β).

In chapter 4, we compute numerically the coefficients of the expansions obtained in chapter 3, and so we have a numerical approximation for the closure terms. Finally we show that these computations can be used in more general cases than (0.8) (two-dimensional and three-dimensional mean flows) if we assume a frame invariance hypothesis for the averaged system (chapter 3). In the conclusion, we summarize the results and give reasonable formulae and graphs for the closure terms of the turbulence model of [3].

CHAPTER 1 : Description of the model.

1.1. Navier-Stokes equations with rapidly varying initial data.

The model studies a mathematical problem for a Reynolds closure: the small structures are introduced in the initial data of the Navier-Stokes equations.

As in [5] we consider the following problem:

$$u_{,t}^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon + \nabla p^\varepsilon - \mu \varepsilon^2 \Delta u^\varepsilon = 0, \quad \nabla \cdot u^\varepsilon = 0 \quad \text{in } \Omega \times]0, T[\quad (1.1)$$

$$u^\varepsilon(x, 0) = u^o(x) + \varepsilon^{1/3} w^o\left(\frac{x}{\varepsilon}, x\right) \quad (1.2)$$

where ε is a small dimensionless parameter and $\mu = 0(1)$, here u^o represents the mean initial velocity and $\varepsilon^{1/3} w^o$ the initial fluctuating field.

The parameter ε is the ratio of the small to the large scale so (1.2) tells us that initially we have a flow with two scales (in [9], equation for the temperature $\theta^\varepsilon(x, t)$ is also considered). The analysis of u^ε will be made up to times of order $\varepsilon^{-2/3}$ beyond which the separation of scales cannot be maintained. We assume that $w^o(y, x)$ is a periodic function in $y \in Y =]-\pi, \pi[^3$ and has mean zero, i.e.

$$\langle w^0 \rangle = \frac{1}{|Y|} \int_Y w^0(y, x) dy = 0, \quad \forall x \in \Omega$$

We also assume that $\text{Supp}_x(w(y, x)) \subset \Omega$, so that no boundary layer interactions will be concerned here.

In an attempt to study the limit case $\varepsilon \rightarrow 0$ of u^ε and p^ε , an asymptotic expansion was suggested in [3]:

$$u^\varepsilon(x, t) = u(x, t) + \varepsilon^{1/3} w\left(\frac{a(x, t)}{\varepsilon}, \frac{t}{\varepsilon^{2/3}}; x, t\right) + \varepsilon^{2/3} u^{(1)}\left(\frac{a(x, t)}{\varepsilon}, \frac{t}{\varepsilon^{2/3}}; x, t\right) + \varepsilon u^{(2)}\left(\frac{a(x, t)}{\varepsilon}, \frac{t}{\varepsilon^{2/3}}; x, t\right) + \dots \quad (1.3)$$

$$p^\varepsilon(x, t) = p(x, t) + \varepsilon^{2/3} \pi\left(\frac{a(x, t)}{\varepsilon}, \frac{t}{\varepsilon^{2/3}}; x, t\right) + \varepsilon p^{(1)}\left(\frac{a(x, t)}{\varepsilon}, \frac{t}{\varepsilon^{2/3}}; x, t\right) + \varepsilon^{4/3} p^{(2)}\left(\frac{a(x, t)}{\varepsilon}, \frac{t}{\varepsilon^{2/3}}; x, t\right) + \dots \quad (1.4)$$

This ansatz is compatible with Kolgoromov's scales (see [2]).

In (1.3), (1.4) $a(x, t)$ represents the inverse of the Lagrangian coordinates relative to the mean flow $u(x, t)$, i.e.

$$a_{,t} + (u \cdot \nabla) a = 0, \quad a(x, 0) = x \quad (1.5)$$

and functions $w(y, \tau; x, t), u^{(1)}(y, \tau; x, t), u^{(2)}(y, \tau; x, t), \dots, \pi(y, \tau; x, t), p^{(1)}(y, \tau; x, t), \dots$ are assumed to be periodic in the y variable in $Y =]-\pi, \pi[$ with zero mean in $Y \times Z$, that is

$$\langle\langle w(\cdot, \cdot; x, t) \rangle\rangle = \lim_{\theta \rightarrow +\infty} \frac{1}{(2\pi)^3 \theta} \int_0^\theta \int_Y w(y, \tau; x, t) dy d\tau = 0 \quad (1.6)$$

Using (1.3), (1.4) into (1.1) yields

$$\begin{aligned} & \varepsilon^{-2/3}[w_{i,K}(a_{k,t} + u_j a_{k,j})] + \\ & + \varepsilon^{-1/3}[w_{i,\tau} + u_{i,K}^{(1)}(a_{k,t} + u_j a_{k,j}) + w_j w_{i,K} a_{k,j} + \pi_{,K} a_{k,i}] + \\ & + \varepsilon^0[u_{i,t} + u_{i,\tau}^{(1)} + u_{i,K}^{(2)}(a_{k,t} + u_j a_{k,j}) + u_j u_{i,j} + a_{k,j}(w_j u_{i,K}^{(1)} + u_j^{(1)} w_{i,K}) + p_{,i} + p_{,K}^{(1)} a_{k,i}] + \\ & + \varepsilon^{1/3}[w_{i,t} + u_{i,\tau}^{(2)} + u_{i,K}^{(3)}(a_{k,t} + u_j a_{k,j}) + u_j w_{i,j} + w_j u_{i,j}] + \\ & + a_{k,j}(u_{i,K}^{(2)} w_j + w_{i,K} u_j^{(2)}) + u_j^{(1)} u_{i,K}^{(1)} a_{k,j} + p_{,K}^{(2)} a_{k,j} - \mu w_{i,KL} a_{l,j} a_{k,j}] + \\ & + \varepsilon^{2/3}[u_{i,t}^{(1)} + u_{i,\tau}^{(3)} + u_{i,K}^{(4)}(a_{k,t} + u_j a_{k,j}) + u_j u_{i,j}^{(1)} + u_{i,j} u_j^{(1)} + a_{k,j}(u_{i,K}^{(3)} w_j + w_{i,K} u_j^{(3)}) + w_j w_{i,j} + \\ & + a_{k,j}(u_{i,K}^{(2)} u_j^{(1)} + u_{i,K}^{(1)} u_j^{(2)}) + p_{,K}^{(3)} a_{k,i} + \pi_{,i} - \mu u_{i,KL}^{(1)} a_{k,j} a_{l,j}] + \dots = 0 \end{aligned} \quad (1.7)$$

$$\varepsilon^{-2/3} a_{k,j} w_{j,K} + \varepsilon^{-1/3} a_{k,j} u_{j,K}^{(1)} + u_{i,i} + a_{k,j} u_{j,K}^{(2)} + \varepsilon^{1/3}(w_{i,i} + a_{k,j} u_{j,K}^{(3)}) + \dots = 0 \quad (1.8)$$

Here derivatives with respect to y are denoted by capital indices. We also use the summation convention over repeated indices even when one is a lower case and the other a capital index.

Expressions (1.7), (1.8) can be rewritten as follows :

$$\begin{aligned} & \varepsilon^{-1/3} T(\tilde{w}, \pi; C) + f^{(1)} + L(\tilde{u}^{(1)}, \tilde{w}, p^{(1)}, C) + \\ & + \varepsilon^{1/3}[f^{(2)} + L(\tilde{u}^{(2)}, \tilde{w}, p^{(2)}, C)] + \varepsilon^{2/3}[f^{(3)} + L(\tilde{u}^{(3)}, \tilde{w}, p^{(3)}, C)] + \dots = 0 \end{aligned} \quad (1.9)$$

$$\begin{aligned} & \varepsilon^{-2/3} \nabla_y \cdot \tilde{w} + \varepsilon^{-1/3} \nabla_y \cdot \tilde{u}^{(1)} + \nabla \cdot u + \nabla_y \cdot \tilde{u}^{(2)} + \\ & + \varepsilon^{1/3}(\nabla \cdot w + \nabla_y \cdot \tilde{u}^{(3)}) + \varepsilon^{2/3}(\nabla \cdot u^{(1)} + \nabla_y \cdot \tilde{u}^{(4)}) + \dots = 0 \end{aligned} \quad (1.10)$$

where

$$C = \nabla a^T \nabla a, \quad \tilde{w} = \nabla a^T w, \quad \tilde{u}^{(1)} = \nabla a^T u^{(1)}, \quad \tilde{u}^{(2)} = \nabla a^T u^{(2)}, \dots \quad (1.11)$$

$$f^{(1)} = \nabla a^T [u_{,t} + (u \cdot \nabla) u + \nabla p] \quad (1.12)$$

$$f^{(2)} = \nabla a^T [w_t + (u \cdot \nabla) w + (w \cdot \nabla) u + (\tilde{u}^{(1)} \cdot \nabla_y) u^{(1)} - \mu(\nabla a \nabla_y).((\nabla a \nabla_y) w)] \quad (1.13)$$

$$\begin{aligned} f^{(3)} = & \nabla a^T [u_{,t}^{(1)} + (u \cdot \nabla) u^{(1)} + (u^{(1)} \cdot \nabla) u + (\tilde{u}^{(1)} \cdot \nabla_y) u^{(2)} + \\ & + (\tilde{u}^{(2)} \cdot \nabla_y) u^{(1)} + w \nabla w + \nabla \pi - \mu(\nabla a \nabla_y).((\nabla a \nabla_y) u^{(1)})] \end{aligned} \quad (1.14)$$

and

$$T(\tilde{w}, \pi, C) = \tilde{w}_{,t} + (\tilde{w} \cdot \nabla_y) \tilde{w} + C \nabla_y \pi \quad (1.15)$$

$$L(\tilde{u}, \tilde{w}, p, C) = \tilde{u}_{,t} + (\tilde{u} \cdot \nabla_y) \tilde{w} + (\tilde{w} \cdot \nabla_y) \tilde{u} + C \nabla_y p \quad (1.16)$$

The differential operator T is a generalized Euler operator (note the matrix C in front of $\nabla_y \pi$) and L is the differential operator obtained by linearization of T . A cascade of equations is found with the operators T and L .

$$\tilde{w}_{,t} + (\tilde{w} \cdot \nabla_y) \tilde{w} + C \nabla_y \pi = 0, \quad \nabla_y \cdot \tilde{w} = 0 \quad (1.17)$$

$$\tilde{u}_{,t}^{(i)} + (\tilde{w} \cdot \nabla_y) \tilde{u}^{(i)} + (\tilde{u}^{(i)} \cdot \nabla_y) \tilde{w} + C \nabla_y p^{(i)} = -f^{(i)}, \quad \nabla_y \cdot \tilde{u}^{(i)} = -g^{(i)} \quad (i \geq 1) \quad (1.18)$$

System (1.17)-(1.18) cannot be verified simultaneously unless $\{f^{(i)}, g^{(i)}\}$ verifies certain compatibility conditions.

In [3] three sets are found

$$<< f^{(i)} + \tilde{w} g^{(i)} >> = 0, \quad < g^{(i)} > = 0 \quad (1.19)$$

$$<< \tilde{w} C^{-1} f^{(i)} + (\frac{1}{2} |w|^2 + \pi) g^{(i)} >> = 0 \quad (1.20)$$

$$<< (\nabla a \nabla_y) \times w \cdot f^{(i)} >> = 0 \quad (1.21)$$

The number of equations like (1.19)-(1.21) necessary to make (1.18) solvable is strongly connected with the number of invariants of (1.17) (see [6]). We know three invariants for this system :

- the mean velocity

$$< \tilde{w} > = 0 \quad (1.22)$$

- the mean kinetic energy

$$q = \frac{1}{2} \langle \tilde{w} C^{-1} \tilde{w} \rangle = \frac{1}{2} \langle |w|^2 \rangle \quad (1.23)$$

- the mean helicity

$$h = \langle (\nabla a^{-T} \tilde{w}) \cdot (\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}) \rangle = \langle w \cdot (\nabla a \nabla_y) \times w \rangle \quad (1.24)$$

REMARK.- The symbol $\langle . \rangle$ stands for the mean in Y , i.e. $\langle w \rangle = \frac{1}{|Y|} \int_Y w(y, x) dy$. We know that the moments in (1.22)-(1.24) do not depend on τ ; in this case we have $\langle . \rangle = \langle\langle . \rangle\rangle$, where $\langle\langle . \rangle\rangle$ is the average symbol in $Y \times Z$ defined in (1.6).

1.2. The system of the effective equations.

The expressions of $f^{(i)}$, given by (1.12), (1.13), ..., and $g^{(i)}$, defined by annuling every term in (1.10) are implemented in (1.15)-(1.21); in this way, we can obtain a system of nonlinear advection diffusion equations involving $u(x, t)$, $q(x, t)$ and $h(x, t)$ given by

$$u_{,t} + (u \cdot \nabla) u + \nabla p + \varepsilon^{2/3} \nabla \cdot R = 0, \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times]0, T[\quad (1.25)$$

$$q_{,t} + u \nabla q + R : \nabla u + \mu \psi_q + \varepsilon^{1/3} \nabla \cdot V_q = 0 \quad (1.26)$$

$$h_{,t} + u \nabla h + S : \nabla u + \mu \psi_h + \varepsilon^{1/3} \nabla \cdot V_h = 0 \quad (1.27)$$

+ initial conditions + boundary conditions

where tensors R , S , V_q , V_h and functions ψ_q , ψ_h are defined by

$$\left. \begin{aligned} R &= \langle\langle w \otimes w \rangle\rangle & S &= \langle\langle w \otimes r + r \otimes w \rangle\rangle = 2 \langle\langle r \otimes w \rangle\rangle \\ V_q &= \langle\langle (\frac{1}{2}|w|^2 + \pi)w \rangle\rangle & V_h &= 2 \langle\langle (\frac{1}{2}|w|^2 + \pi)\tilde{\nabla}_y \times w + \frac{1}{2}w \times w_{,\tau} \rangle\rangle \\ \psi_q &= \langle\langle \tilde{\nabla}_y w : \tilde{\nabla}_y w \rangle\rangle & \psi_h &= 2 \langle\langle \tilde{\nabla}_y r : \tilde{\nabla}_y w \rangle\rangle \end{aligned} \right\} \quad (1.28)$$

$$\tilde{\nabla}_y = \nabla a \nabla_y, \quad r = \tilde{\nabla}_y \times w, \quad w = \nabla a^{-T} \tilde{w}, \quad \tilde{r} = \nabla a^T r$$

\tilde{w} verifying the canonic microstructure problem

$$\left. \begin{array}{l}
\tilde{w}_{,t} + (\tilde{w} \cdot \nabla_y) \tilde{w} + C \nabla_y \pi = 0, \quad \nabla_y \cdot \tilde{w} = 0 \quad \text{in } Y \times Z \\
\tilde{w} \quad y - \text{periodic}, \quad \frac{\partial \tilde{w}}{\partial n} \Big|_{\Gamma_i^+} + \frac{\partial \tilde{w}}{\partial n} \Big|_{\Gamma_i^-} = 0; \quad \pi \quad y - \text{periodic} \\
\tilde{w}(y, 0) = w^0(y, x) \\
< \tilde{w} > = 0, \quad < \pi > = 0, \\
\frac{1}{2} < \tilde{w} C^{-1} \tilde{w} > = q, \\
< \tilde{w} C^{-1} \tilde{r} > = h
\end{array} \right\} \quad (1.29)$$

and $a(x, t)$ solution of

$$a_{,t} + (u \cdot \nabla) a = 0, \quad a(x, 0) = x \quad (1.30)$$

Relations (1.28) show that the model gives an expression for the Reynolds stress tensor and other coefficients in the averaged equations. These coefficients depend on the turbulent fluctuation w and on the lagrangian coordinates via the microstructure problem (1.29) with three invariants. So we can notice the importance of (1.29).

Unfortunately, it is a very complex problem. However we can study some special cases : time independent solutions \tilde{w} , one-dimensional mean flows, etc. For this case, we are able to give an expression for the closure terms (1.28) by studing directly (1.29).

A preliminary analysis done in [3] has revealed that $\nabla \cdot <> w \otimes w >$ is not a dissipative term. The dissipation terms are found in the next term in the asymptotic expansion. In practice, it is important to include these terms (see the $k - \varepsilon$ model [10], [11]), and equation (1.25) becomes :

$$u_{,t} + (u \cdot \nabla) u + \nabla p + \varepsilon^{2/3} \nabla \cdot <> w \otimes w > + \varepsilon^{4/3} \nabla \cdot <> w \otimes u^{(2)} + u^{(2)} \otimes w > = 0, \quad \nabla \cdot u = 0$$

where $u^{(2)} = \nabla a^{-T} \tilde{u}^{(2)}$, and $\tilde{u}^{(2)}$ verifying the linear canonical microstructure problem relative to operator L given in (1.16)

$$L(\tilde{u}^{(2)}, \tilde{w}, p^{(2)}, C) = f^{(2)}$$

with

$$f^{(2)} = -\nabla a^T [w_{,t} + (u \cdot \nabla) w + (w \cdot \nabla) u - \mu(\nabla a \nabla_y) \cdot ((\nabla a \nabla_y) w)]$$

In the same way, equations for the kinetic energy $q(x, t)$ and helicity $h(x, t)$ are affected by $u^{(2)}$ as new closure tensors, linear in $u^{(2)}$, are involved in their expressions (see [6]).

CHAPTER 2 : The canonical microstructure problem

We analyze in this chapter certain properties for the canonical fluctuation \tilde{w} derived from system (1.29). We shall use the following convention :

If a pair of functions (\tilde{w}, π) is y -periodic in $Y =]-\pi, \pi[^3$ and continuously differentiable, then \tilde{w}, π satisfy

$$\tilde{w}_i(y, \tau)|_{\Gamma_j^+} = \tilde{w}_i(y, \tau)|_{\Gamma_j^-}, \quad 1 \leq i, j \leq 3, \quad \forall \tau \in Z \quad (2.1)$$

$$\frac{\partial \tilde{w}_i}{\partial n}|_{\Gamma_j^+} + \frac{\partial \tilde{w}_i}{\partial n}|_{\Gamma_j^-} = 0, \quad 1 \leq i, j \leq 3, \quad \forall \tau \in Z \quad (2.2)$$

$$\pi(y, \tau)|_{\Gamma_j^+} = \pi(y, \tau)|_{\Gamma_j^-}, \quad 1 \leq j \leq 3, \quad \forall \tau \in Z \quad (2.3)$$

where Γ_j^+ and Γ_j^- are opposite faces in Y and $n|_{\Gamma_j^+}, n|_{\Gamma_j^-}$ their outward normal unitary vectors respectively (so $n|_{\Gamma_j^+} = -n|_{\Gamma_j^-}$).

For a function $\tilde{v} \in L^1(Y \times [0, \theta])$ for all $\theta > 0$, we note

$$<< \tilde{v} >> = \lim_{\theta \rightarrow \infty} \frac{1}{(2\pi)^3 \theta} \int_0^\theta \int_Y \tilde{v}(y, \tau) dy d\tau \quad (2.4)$$

and if :

$$\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)^T \quad (2.5)$$

then we put

$$<< \tilde{w} >> = (<< \tilde{w}_1 >>, << \tilde{w}_2 >>, << \tilde{w}_3 >>)^T \quad (2.6)$$

2.1. Statement of the problem.

Consider

$$\left. \begin{array}{l} \tilde{w}_{,\tau} + (\tilde{w} \cdot \nabla_y) \tilde{w} + C \nabla_y \pi = 0, \quad \nabla_y \cdot \tilde{w} = 0 \quad \text{in } Y \times Z \\ (\tilde{w}, \pi) \quad y - \text{periodic in } Y, \quad \tilde{w}(y, 0) = w^0(y, x) \\ \langle \tilde{w} \rangle = 0, \quad \langle \pi \rangle = 0 \\ \frac{1}{2} \langle \tilde{w} C^{-1} \tilde{w} \rangle = q \\ \langle (\nabla a^{-T} \tilde{w}) \cdot (\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}) \rangle = h \end{array} \right\} \quad (2.7)$$

we wish to find

$$\left. \begin{array}{l} R_{ij} = \langle \langle w_i w_j \rangle \rangle \quad S_{ij} = 2 \langle \langle r_i w_j \rangle \rangle \\ V_q^i = \langle \langle (\frac{1}{2}|w|^2 + \pi) w_i \rangle \rangle \quad V_h^i = 2 \langle \langle (\frac{1}{2}|w|^2 + \pi) r_i + \frac{1}{2}(w \times w_{,\tau})_i \rangle \rangle \\ \psi_q = \langle \langle \tilde{\nabla}_y w : \tilde{\nabla}_y w \rangle \rangle \quad \psi_h = 2 \langle \langle \tilde{\nabla}_y r : \tilde{\nabla}_y w \rangle \rangle \end{array} \right\} \quad (2.8)$$

with

$$w = \nabla a^{-T} \tilde{w}, \quad \tilde{\nabla}_y = \nabla a \nabla_y, \quad r = \tilde{\nabla}_y \times w$$

REMARK.- Since C , q and h depend only on the variables $\{x, t\}$ and the differential equation in (2.7) is relative to the variables $\{y, \tau\}$, we can consider these three functions as parameters of (2.7).

2.2. The dependence on C .

At first sight, problem (2.7) depends on matrix ∇a : it appears in the equation (2.7a), in the definition of the kinetic turbulent energy q (2.7d), and in the expression of the turbulent helicity h (2.7e). But expressions (2.7a), (2.7d) depend on $C = \nabla a^T \nabla a$ rather than ∇a , so it would be nice if this is still true for (2.7e).

In fact, we can show, after some algebra, that the vector \tilde{r} given by

$$\tilde{r} = \nabla a^T r = \nabla a^T (\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}) \quad (2.9)$$

is equal to

$$\tilde{r} = \nabla_y \times (C^{-1} \tilde{w}) \quad (2.10)$$

and so

$$\langle (\nabla a^{-T} \tilde{w}) \cdot (\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}) \rangle = \langle (C^{-1} \tilde{w}) \cdot \nabla_y \times (C^{-1} \tilde{w}) \rangle \quad (2.11)$$

In this way, we can rewrite (2.7) as

$$\left. \begin{aligned} \tilde{w}_{,\tau} + (\tilde{w} \cdot \nabla_y) \tilde{w} + C \nabla_y \pi = 0, & \quad \nabla_y \cdot \tilde{w} = 0 \quad \text{in } Y \times Z \\ (\tilde{w}, \pi) \quad y - \text{periodic in } Y, \quad \tilde{w}(y, 0) = w^0(y, x) \\ & \quad \langle \tilde{w} \rangle = 0, \quad \langle \pi \rangle = 0 \\ & \quad \frac{1}{2} \langle \tilde{w} C^{-1} \tilde{w} \rangle = q \\ & \quad \langle (C^{-1} \tilde{w}) \cdot \nabla_y \times (C^{-1} \tilde{w}) \rangle = h \end{aligned} \right\} \quad (2.12)$$

which depends on C . This property will be used in the study of the invariance of certain tensors when a hypothesis of local isotropic turbulence is assumed (see paragraph 3.5).

REMARK.- It can also be showed that

$$\langle (\nabla a^{-T} \tilde{w}) \cdot (\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}) \rangle = \langle \tilde{w} \cdot (C \nabla_y) \times \tilde{w} \rangle \quad (2.13)$$

which yields another simple expression for the helicity. Notice that in (2.11),(2.13) we are using implicitly that $\det(\nabla a) = 1$.

2.3. Non-uniqueness of solutions.

Let \tilde{w} be given by

$$\tilde{w} = \operatorname{Re}(\hat{w} e^{ik \cdot y}) \quad (2.14)$$

where

$$\hat{w} = c + id, \quad c, d \in \mathbf{R}^3, k \in \mathbf{Z}_*^3$$

Condition $\nabla_y \cdot \tilde{w} = 0$ implies

$$k \cdot c = k \cdot d = 0 \quad (2.15)$$

We have also

$$(\tilde{w} \cdot \nabla_y) \tilde{w} = \operatorname{Re}(i(\tilde{w} \cdot k) \hat{w} e^{ik \cdot y}) = 0 \quad (2.16)$$

$$\nabla_y \cdot \tilde{w} = 0 \quad (2.17)$$

In this case, the invariants of \tilde{w} are given by

$$\langle \tilde{w} \rangle = 0 \quad (2.18)$$

$$\frac{1}{2} \langle \tilde{w} C^{-1} \tilde{w} \rangle = \frac{1}{4} ((c, C^{-1} c) + (d, C^{-1} d)) = q \quad (2.19)$$

$$\langle (\nabla a^{-T} \tilde{w}) \cdot (\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}) \rangle = \langle (C^{-1} \tilde{w}) \cdot \nabla_y \times (C^{-1} \tilde{w}) \rangle = (C^{-1} d) \cdot k \times (C^{-1} c) = h \quad (2.20)$$

putting

$$c = \gamma \bar{c}, \quad d = \delta \bar{d}$$

where \bar{c} and \bar{d} are such that

$$(\bar{c}, C^{-1} \bar{c}) = (\bar{d}, C^{-1} \bar{d}) = 1 \quad (2.21)$$

then (2.19)-(2.20) becomes

$$\left. \begin{aligned} \gamma^2 + \delta^2 &= 4q \\ \gamma \delta &= h/H(k) \end{aligned} \right\} \quad (2.22)$$

where

$$H(k) = (C^{-1} \bar{d}) \cdot k \times (C^{-1} \bar{c}) \quad (2.23)$$

REMARK.- If c and d are independents, and condition (2.15) holds, then we have

$$H(k) = (C^{-1} \bar{d}) \cdot k \times (C^{-1} \bar{c}) \neq 0.$$

The solutions of (2.22) are given by

$$\gamma = \pm \sqrt{2q \pm \sqrt{(2q)^2 - \left(\frac{h}{H}\right)^2}}, \quad \delta = \pm \sqrt{2q \mp \sqrt{(2q)^2 - \left(\frac{h}{H}\right)^2}} \quad (2.24)$$

hence $\gamma, \delta \in \mathbf{R}$ if and only if

$$2q \geq \left| \frac{h}{H(k)} \right| \quad (2.25)$$

Since $|H(k)| \rightarrow +\infty$ as $|k| \rightarrow +\infty$, we have that (2.25) will be verified if $|k|$ is big enough ; so for every $k \in \mathbf{Z}_*^3$ verifying (2.20) we have 4 solutions of the form

$$\tilde{w} = q^{1/2} \gamma \left(\frac{h}{q}, C, k \right) \frac{c}{(c, C^{-1} c)^{1/2}} \cos(k \cdot y) - q^{1/2} \delta \left(\frac{h}{q}, C, k \right) \frac{d}{(d, C^{-1} d)^{1/2}} \sin(k \cdot y) \quad (2.26)$$

with

$$\gamma(s, C, k) = \pm \sqrt{2 \pm \sqrt{4 - \left(\frac{s}{H(C, k)}\right)^2}}, \quad \delta(s, C, k) = \pm \sqrt{2 \mp \sqrt{4 - \left(\frac{s}{H(C, k)}\right)^2}}$$

$$k \cdot c = k \cdot d = 0$$

$$H(C, k) = \frac{(C^{-1} d) \cdot k \times (C^{-1} c)}{(c, C^{-1} c)^{1/2} \cdot (d, C^{-1} d)^{1/2}}$$

Notice that solutions given in (2.26) are not isolated: condition (2.21) does not determine c and d . This was also the case of the Beltrami fields studied in [7] where these were planar. However, for c and d fixed, \tilde{w} depend continuously as function of matrix C .

Consequently, problem (2.7) has infinitely many solutions. Nevertheless, we know that the canonical fluctuation may verify certain symmetries connected to C . So we shall look for solution(s) verifying these symmetries since this can limit the set of solutions \tilde{w} of (2.7).

2.4. Symmetries of the canonical fluctuations.

Solutions of system (2.7) can admit certain symmetries. These symmetries depend on the matrix C . Indeed, let $\tilde{w}(y, \tau) = \tilde{w}(y, \tau; C, q, h)$ be a solution of (2.7). We introduce the new fluctuations \tilde{W} and pressure P defined as

$$\tilde{W}(y, \tau) = Q^T \tilde{w}(Qy, \tau), \quad P(y, \tau) = \pi(Qy, \tau) \quad (2.27)$$

where Q is an orthogonal matrix ($Q^T Q = I$) leaving the cube Y invariant (i.e. $QY = Y$).

Let also assume that the initial fluctuation verifies $\tilde{W}(y, 0) = Q^T \tilde{w}(Qy, 0) = w^0(y, z)$

Therefore (\tilde{W}, P) verifies the following system

$$\left. \begin{array}{l} \tilde{W}_{,\tau} + (\tilde{W} \cdot \nabla_y) \tilde{W} + C^* \nabla P = 0, \quad \nabla_y \cdot \tilde{W} = 0 \quad \text{in } Y \times Z \\ (\tilde{W}, P) \quad y - \text{periodic in } Y, \quad \tilde{W}(y, 0) = w^0(y, z) \\ < \tilde{W} > = 0, \quad < P > = 0 \\ \frac{1}{2} < \tilde{W} (C^*)^{-1} \tilde{W} > = q \\ \det(Q) < \tilde{W} \cdot (C^* \nabla_y) \times \tilde{W} > = h \end{array} \right\} \quad (2.28)$$

where $C^* = Q^T C Q$

If Q is such that

$$\det(Q) = 1, \quad C = Q^T C Q = C^* \quad (2.29)$$

then, problems (2.7) and (2.27) are the same. This leads us to the study of solutions of (2.7) which verify these symmetries. In this case, we would have

$$\tilde{w}(y) = Q^T \tilde{w}(Qy) \quad (2.30)$$

for all orthogonal matrices Q verifying (2.29) and leaving invariant the cube Y .

Unfortunately, all those conditions for Q are very restrictive and they can be satisfied only for certain matrices C .

In the easiest case, we have $C = I$ (at least this is true initially at $t = 0$, $\nabla a(x, 0) = I$) and condition (2.29) is verified for 24 matrices ! the group generated by matrices Q_1 , Q_2 and Q_3 :

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad (2.31)$$

The case

$$C = \begin{pmatrix} 1 + \alpha^2 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

will be studied later (chapter 3) and condition (2.29) is true only for

$$Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.5. The dependence on h/q .

We can relate $\tilde{w}(y, \tau; C, q_1, h_1)$ to $\tilde{w}(y, \tau; C, q_2, h_2)$ if q_1, h_1, q_2, h_2 are such that

$$\frac{h_1}{q_1} = \frac{h_2}{q_2} \quad (2.32)$$

Indeed putting

$$w^*(y, \tau) = \lambda \tilde{w}(y, \lambda \tau; C, q_1, h_1), \quad \pi^*(y, \tau) = \lambda^2 \pi(y, \lambda \tau; C, q_1, h_1) \quad (2.33)$$

then (w^*, π^*) verify

$$\left. \begin{aligned} w_{,\tau}^* + (w^* \nabla_y) w^* + C \nabla_y \pi^* &= 0, \quad \nabla_y \cdot w^* = 0 \quad \text{in } Y \times Z \\ (w^*, \pi^*) \quad y - \text{periodic} \quad \text{in } Y \\ &\quad \left. \begin{aligned} < w^* > &= 0 \\ \frac{1}{2} < w^* C^{-1} w^* &= \lambda^2 q_1 \\ < (\nabla a^{-T} w^*) \cdot (\nabla a \nabla_y) \times (\nabla a^{-T} w^*) &= \lambda^2 h_1 \end{aligned} \right\} \end{aligned} \right\} \quad (2.34)$$

If condition (2.32) holds then we can choose

$$\lambda = \sqrt{\frac{q_2}{q_1}} = \sqrt{\frac{h_2}{h_1}} \quad (2.35)$$

This property leads us to compute $w^*(y, \tau; C, h/q)$ only.

2.6. A new expression for the system of the effective equations.

According to section 2.5, we can express the system (1.25)-(1.30) in the following manner

$$\left. \begin{array}{l} u_{,t} + (u \cdot \nabla) u + \nabla p + \varepsilon^{2/3} \nabla \cdot (q R') = 0, \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times]0, T[\\ q_{,t} + u \nabla q + q R' : \nabla u + \mu q \psi'_q + \varepsilon^{1/3} \nabla \cdot (q^{3/2} V'_q) = 0 \quad \text{in } \Omega \times]0, T[\\ h_{,t} + u \nabla h + q S' : \nabla u + \mu q \psi'_h + \varepsilon^{1/3} \nabla \cdot (q^{3/2} V'_h) = 0 \quad \text{in } \Omega \times]0, T[\\ a_{,t} + (u \cdot \nabla) a = 0 \quad \text{in } \Omega \times]0, T[\end{array} \right\} \quad (2.36)$$

$$\left. \begin{array}{l} u(x, 0) = u^o(x) \quad \text{in } \Omega \\ q(x, 0) = \frac{1}{2} \langle |w^o|^2 \rangle \quad \text{in } \Omega \\ h(x, 0) = \langle w^o \cdot \nabla \times w^o \rangle \quad \text{in } \Omega \\ a(x, 0) = x \quad \text{in } \Omega \end{array} \right\} \quad (2.37)$$

where

$$\left. \begin{array}{l} R' = \langle \langle w' \otimes w' \rangle \rangle \quad S' = 2 \langle \langle \tilde{\nabla}_y \times w' \otimes w' \rangle \rangle \\ V'_q = \langle \langle (\frac{1}{2}|w'|^2 + \pi') w' \rangle \rangle \quad V'_h = 2 \langle \langle (\frac{1}{2}|w'|^2 + \pi') \tilde{\nabla}_y \times w' \rangle \rangle \\ \psi'_q = \langle \langle \tilde{\nabla}_y w' : \tilde{\nabla}_y w' \rangle \rangle \quad \psi'_h = 2 \langle \langle \tilde{\nabla}_y (\tilde{\nabla}_y \times w') : \tilde{\nabla}_y w' \rangle \rangle \end{array} \right\} \quad (2.38)$$

$$w' = \nabla a^{-T} \tilde{w}'$$

\tilde{w}' being solution of the problem

$$\left. \begin{array}{l} \tilde{w}'_{,\tau} + (\tilde{w}' \cdot \nabla_y) \tilde{w}' + C \nabla_y \pi' = 0, \quad \nabla_y \cdot \tilde{w}' = 0 \quad \text{in } Y \times Z \\ (\tilde{w}', \pi') \quad y - \text{periodic}, \tilde{w}'(y, 0) = w^0(y; x) \quad \text{in } Y \\ \langle \tilde{w}' \rangle = 0 \\ \frac{1}{2} \langle \tilde{w}' C^{-1} \tilde{w}' \rangle = 1 \\ \langle \tilde{w}' \cdot (C \nabla_y) \times \tilde{w}' \rangle = h/q \end{array} \right\} \quad (2.39)$$

CHAPTER 3 : Fluid flows with one-dimensional mean

In this chapter we study the structures of tensors (2.33) in the case of fluid flows with one dimensional mean, i.e.

$$\mathbf{u}(x_1, x_2, x_3; t) = (u_1(x_2, t), 0, 0)^T \quad (3.1)$$

To find the structures of these tensors we will look for the symmetries of the canonical fluctuations that can be deduced from (3.1).

At first sight, it would seem that (3.1) is a very special case that could not play an important role in our study. However, we shall see that the study of two-dimensional mean flows can be reduced to the one-dimensional case if we assume frame invariance for the averaged system. The study of fluid flows of the form (3.1) can also be used to compute the 3D case.

Along this chapter we assume that the fluctuating field \tilde{w}' is stationnary ($\tilde{w}'_{,\tau} = 0$, that is, the initial fluctuation w'^0 verifies the stationnary Euler equation.)

3.1. The lagrangian coordinates.

If we assume that the mean field $\mathbf{u}(x, t)$ is of the form (3.1), then we have an explicit expression for $a(x, t)$ (the inverse of the lagrangian coordinates) :

$$\left. \begin{array}{l} a_1(x, t) = x_1 - \int_0^t u_1(x_2, s) ds \\ a_2(x, t) = x_2 \\ a_3(x, t) = x_3 \end{array} \right\} \quad (3.2)$$

hence

$$\nabla a = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \nabla a^T \nabla a = \begin{pmatrix} 1 + \alpha^2 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.3)$$

where

$$\alpha = \alpha(x, t) = - \int_0^t u_{1,2}(x_2, s) ds \quad (3.4)$$

3.2. Evenness-Oddness properties of the canonical fluctuations.

Let $\tilde{w}'(y_1, y_2, y_3; \alpha, \beta)$ be the general solution of (2.34) with α defined by (3.4) and $\beta = h/q$.

Let

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad Q_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad Q_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad Q_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

REMARK. All these matrices leave Y invariant.

Implementing $Q = Q_1, \dots, Q_6$ in (2.19)-(2.27) we deduce that

If $Q = Q_1$

$$\left. \begin{array}{l} \tilde{w}'_1(y_1, y_2, y_3; \alpha, \beta) = \tilde{w}'_1(y_1, y_2, -y_3; \alpha, -\beta) \\ \tilde{w}'_2(y_1, y_2, y_3; \alpha, \beta) = \tilde{w}'_2(y_1, y_2, -y_3; \alpha, -\beta) \\ \tilde{w}'_3(y_1, y_2, y_3; \alpha, \beta) = -\tilde{w}'_3(y_1, y_2, -y_3; \alpha, -\beta) \\ \pi'(y_1, y_2, y_3; \alpha, \beta) = \pi'(y_1, y_2, -y_3; \alpha, -\beta) \end{array} \right\} \quad (3.5)$$

If $Q = Q_2$

$$\left. \begin{array}{l} \tilde{w}'_1(y_1, y_2, y_3; \alpha, \beta) = \tilde{w}'_1(y_1, -y_2, y_3; -\alpha, -\beta) \\ \tilde{w}'_2(y_1, y_2, y_3; \alpha, \beta) = -\tilde{w}'_2(y_1, -y_2, y_3; -\alpha, -\beta) \\ \tilde{w}'_3(y_1, y_2, y_3; \alpha, \beta) = \tilde{w}'_3(y_1, -y_2, y_3; -\alpha, -\beta) \\ \pi'(y_1, y_2, y_3; \alpha, \beta) = \pi'(y_1, -y_2, y_3; -\alpha, -\beta) \end{array} \right\} \quad (3.6)$$

If $Q = Q_3$

$$\left. \begin{array}{l} \tilde{w}'_1(y_1, y_2, y_3; \alpha, \beta) = -\tilde{w}'_1(-y_1, y_2, y_3; -\alpha, -\beta) \\ \tilde{w}'_2(y_1, y_2, y_3; \alpha, \beta) = \tilde{w}'_2(-y_1, y_2, y_3; -\alpha, -\beta) \\ \tilde{w}'_3(y_1, y_2, y_3; \alpha, \beta) = \tilde{w}'_3(-y_1, y_2, y_3; -\alpha, -\beta) \\ \pi'(y_1, y_2, y_3; \alpha, \beta) = \pi'(-y_1, y_2, y_3; -\alpha, -\beta) \end{array} \right\} \quad (3.7)$$

If $Q = Q_4$

$$\left. \begin{array}{l} \tilde{w}'_1(y_1, y_2, y_3; \alpha, \beta) = -\tilde{w}'_1(-y_1, -y_2, y_3; \alpha, \beta) \\ \tilde{w}'_2(y_1, y_2, y_3; \alpha, \beta) = -\tilde{w}'_2(-y_1, -y_2, y_3; \alpha, \beta) \\ \tilde{w}'_3(y_1, y_2, y_3; \alpha, \beta) = \tilde{w}'_3(-y_1, -y_2, y_3; \alpha, \beta) \\ \pi'(y_1, y_2, y_3; \alpha, \beta) = \pi'(-y_1, -y_2, y_3; \alpha, \beta) \end{array} \right\} \quad (3.8)$$

If $Q = Q_5$

$$\left. \begin{array}{l} \tilde{w}'_1(y_1, y_2, y_3; \alpha, \beta) = -\tilde{w}'_1(-y_1, y_2, -y_3; -\alpha, \beta) \\ w'_2(y_1, y_2, y_3; \alpha, \beta) = \tilde{w}'_2(-y_1, y_2, -y_3; -\alpha, \beta) \\ \tilde{w}'_3(y_1, y_2, y_3; \alpha, \beta) = -\tilde{w}'_3(-y_1, y_2, -y_3; -\alpha, \beta) \\ \pi'(y_1, y_2, y_3; \alpha, \beta) = \pi'(-y_1, y_2, -y_3; -\alpha, \beta) \end{array} \right\} \quad (3.9)$$

If $Q = Q_6$

$$\left. \begin{array}{l} \tilde{w}'_1(y_1, y_2, y_3; \alpha, \beta) = \tilde{w}'_1(y_1, -y_2, -y_3; -\alpha, \beta) \\ \tilde{w}'_2(y_1, y_2, y_3; \alpha, \beta) = -\tilde{w}'_2(y_1, -y_2, -y_3; -\alpha, \beta) \\ \tilde{w}'_3(y_1, y_2, y_3; \alpha, \beta) = -w'_3(y_1, -y_2, -y_3; -\alpha, \beta) \\ \pi'(y_1, y_2, y_3; \alpha, \beta) = \pi'(y_1, -y_2, -y_3; -\alpha, \beta) \end{array} \right\} \quad (3.10)$$

It is evident that (3.5), (3.6), (3.7) \Rightarrow (3.8), (3.9), (3.10).

3.3. Evenness-Oddness of the closure tensors.

Proposition 3.3.1.- *The Reynolds stress tensor $R = \langle w \otimes w \rangle$ can be written as*

$$R = q \nabla a^{-T} \langle \tilde{w}' \otimes \tilde{w}' \rangle \nabla a^{-1} \quad (3.11)$$

where

$$\langle \tilde{w}' \otimes \tilde{w}' \rangle = \begin{pmatrix} \rho_{11} & \rho_{12} & 0 \\ \rho_{12} & \rho_{22} & 0 \\ 0 & 0 & \rho_{33} \end{pmatrix} \quad (3.12)$$

and functions $\rho_{11}, \rho_{22}, \rho_{33}, \rho_{12}$ satisfy

$$\rho_{ii}(\alpha, \beta) = \rho_{ii}(-\alpha, \beta), \quad i = 1, 2, 3 \quad (3.13)$$

$$\rho_{ii}(\alpha, \beta) = \rho_{ii}(\alpha, -\beta), \quad i = 1, 2, 3 \quad (3.14)$$

$$\rho_{12}(\alpha, \beta) = -\rho_{12}(-\alpha, \beta) \quad (3.15)$$

$$\rho_{12}(\alpha, \beta) = \rho_{12}(\alpha, -\beta) \quad (3.16)$$

$$\rho_{11} + \rho_{22} + \rho_{33} = 2 \quad (3.17)$$

Proof.- By definition

$$\rho_{ij} = \frac{1}{|Y|} \int_Y \tilde{w}'_i(y; \alpha, \beta) \tilde{w}'_j(y; \alpha, \beta) dy \quad (3.18)$$

Making the changes $y^* = Q_1y$, $y^{**} = Q_5y$ in (3.18) and using (3.5) and (3.9) respectively, we deduce (3.13)-(3.16).

Moreover, from (3.9), we have

$$\rho_{13}(\alpha, \beta) = \rho_{13}(-\alpha, \beta)$$

$$\rho_{23}(\alpha, \beta) = -\rho_{23}(-\alpha, \beta)$$

on the other hand, from (3.10) we have

$$\rho_{13}(\alpha, \beta) = -\rho_{13}(-\alpha, \beta)$$

$$\rho_{23}(\alpha, \beta) = \rho_{23}(-\alpha, \beta)$$

Thus

$$\rho_{13} = \rho_{23} = 0$$

Equality (3.17) follows from (2.34d). Finally $w = q^{1/2} \nabla a^{-T} \tilde{w}'$ yield (3.11).

Proposition 3.3.2.- The vector function $V_q = < (\pi + \frac{1}{2}|w|^2)w >$ is of the form

$$V_q = q^{3/2} \begin{pmatrix} 0 \\ 0 \\ d_3(\alpha, \beta) \end{pmatrix} \quad (3.19)$$

with

$$d_3(\alpha, \beta) = -d_3(-\alpha, \beta) \quad (3.20)$$

$$d_3(\alpha, \beta) = -d_3(\alpha, -\beta) \quad (3.21)$$

Proof.- Putting

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}(\alpha, \beta) = < (\pi' + \frac{1}{2}\tilde{w}'C^{-1}\tilde{w}')\tilde{w}' > (\alpha, \beta) \quad (3.22)$$

by definition of V_q in (2.8), we can write

$$V_q = q^{3/2} \nabla a^{-T} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}(\alpha, \beta). \quad (3.23)$$

From (3.22), we have

$$\left. \begin{aligned} d_1(\alpha, \beta) &= < (\pi' + \frac{1}{2}|\tilde{w}'|^2)\tilde{w}'_1 + \alpha^2\tilde{w}'_2\tilde{w}'_1 - 2\alpha\tilde{w}'_1\tilde{w}'_2\tilde{w}'_1 > \\ d_2(\alpha, \beta) &= < (\pi' + \frac{1}{2}|\tilde{w}'|^2)\tilde{w}'_2 + \alpha^2\tilde{w}'_1\tilde{w}'_2 - 2\alpha\tilde{w}'_1\tilde{w}'_2\tilde{w}'_2 > \\ d_3(\alpha, \beta) &= < (\pi' + \frac{1}{2}|\tilde{w}'|^2)\tilde{w}'_3 + \alpha^2\tilde{w}'_2\tilde{w}'_3 - 2\alpha\tilde{w}'_1\tilde{w}'_2\tilde{w}'_3 > \end{aligned} \right\} \quad (3.24)$$

Making $y^* = Q_5y$, $y^{**} = Q_6y$ in (3.24) and using (3.9) and (3.10) respectively, we deduce

$$d_1 = d_2 = 0$$

$$d_3(\alpha, \beta) = -d_3(-\alpha, \beta)$$

the change $y^{***} = Q_1y$ in (3.24) and (3.5) yield

$$d_3(\alpha, \beta) = -d_3(\alpha, -\beta).$$

Proposition 3.3.3.- *The stress tensor $S = 2 < w \otimes r >$ is of the form*

$$S = 2q \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \quad (3.25)$$

and functions $\sigma_{ij}(\alpha, \beta)$ verify

$$\sigma_{ii}(\alpha, \beta) = \sigma_{ii}(-\alpha, \beta) \quad i = 1, 2, 3 \quad (3.26)$$

$$\sigma_{12}(\alpha, \beta) = -\sigma_{12}(-\alpha, \beta) \quad (3.27)$$

$$\sigma_{ij}(\alpha, \beta) = -\sigma_{ij}(\alpha, -\beta), \quad i, j = 1, 2, 3 \quad (3.28)$$

$$\sigma_{11} + \sigma_{22} + \sigma_{33} = \frac{h}{q} \quad (3.29)$$

Proof.- The components of $\nabla a^{-T}\tilde{w}'$ and $(\nabla a\nabla_y) \times (\nabla a^{-T}\tilde{w}')$ are given respectively by

$$\begin{pmatrix} \tilde{w}'_1 - \alpha\tilde{w}'_2 \\ \tilde{w}'_2 \\ \tilde{w}'_3 \end{pmatrix}, \quad \begin{pmatrix} \alpha\tilde{w}'_{3,1} + \tilde{w}'_{3,2} - \tilde{w}'_{2,3} \\ -(\tilde{w}'_{3,1} - \tilde{w}'_{1,3} + \alpha\tilde{w}'_{2,3}) \\ \tilde{w}'_{2,1} - \alpha\tilde{w}'_{1,1} + \alpha^2\tilde{w}'_{2,1} - \tilde{w}'_{1,2} + \alpha\tilde{w}'_{2,2} \end{pmatrix} \quad (3.30)$$

Putting

$$S = 2q \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} (= 2 < \nabla a^{-T} \tilde{w}' \otimes (\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}') >) \quad (3.31)$$

and making the same changes of variables for all integral expressions of σ_{ij} as in Proposition 1, we obtain (3.26)-(3.28) and

$$\sigma_{13} = \sigma_{23} = 0$$

Condition (3.29) yields from (2.34).

REMARK.- Tensor S is not a symmetric tensor in principle. This is true if $\nabla_y \cdot \tilde{w}' = 0$, which is the case.

Proposition 3.3.4.- *The vector function $V_h = 2 < (\pi + \frac{1}{2}|w|^2)r >$ is of the form*

$$V_h = q^{3/2} \begin{pmatrix} 0 \\ 0 \\ e_3(\alpha, \beta) \end{pmatrix} \quad (3.32)$$

with

$$e_3(\alpha, \beta) = -e_3(-\alpha, \beta) \quad (3.33)$$

$$e_3(\alpha, \beta) = e_3(\alpha, -\beta). \quad (3.34)$$

Proof.- Putting

$$V_h = q^{3/2} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (\alpha, \beta) = 2q^{3/2} < (\pi' + \frac{1}{2}\tilde{w}' C^{-1} \tilde{w}').(\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}') > \quad (3.35)$$

thus

$$e_1(\alpha, \beta) = 2 < (\pi' + \frac{1}{2}|\tilde{w}'|^2 + \alpha^2 \tilde{w}_2'^2 - 2\alpha \tilde{w}_1' \tilde{w}_2').(\alpha \tilde{w}_{3,1}' + \tilde{w}_{3,2}' - \tilde{w}_{2,3}') >$$

$$e_2(\alpha, \beta) = 2 < (\pi' + \frac{1}{2}|\tilde{w}'|^2 + \alpha^2 \tilde{w}_2'^2 - 2\alpha \tilde{w}_1' \tilde{w}_2').(-\tilde{w}_{3,1}' - \tilde{w}_{1,3}' + \alpha \tilde{w}_{2,3}') >$$

$$e_3(\alpha, \beta) = 2 < (\pi' + \frac{1}{2}|\tilde{w}'|^2 + \alpha^2 \tilde{w}'_2^2 - 2\alpha \tilde{w}'_1 \tilde{w}'_2).(\tilde{w}'_{2,1} - \alpha \tilde{w}'_{1,1} + \alpha^2 \tilde{w}'_{2,1} - \tilde{w}'_{1,2} + \alpha \tilde{w}'_{2,2}) >$$

Now equalities (3.33), (3.34) can be demonstrated as in proposition 3.3.2.

Proposition 3.3.5.- Functions $\psi_q = < (\nabla a \nabla_y) w : (\nabla a \nabla_y) w >$ and $\psi_h = 2 < (\nabla a \nabla_y) w : (\nabla a \nabla_y) r >$ are of the form

$$\psi_q = qf(\alpha, \beta) \quad (3.36)$$

$$\psi_h = qg(\alpha, \beta) \quad (3.37)$$

with

$$f(\alpha, \beta) = f(-\alpha, \beta) \quad (3.38)$$

$$f(\alpha, \beta) = f(\alpha, -\beta) \quad (3.39)$$

$$g(\alpha, \beta) = g(-\alpha, \beta) \quad (3.40)$$

$$g(\alpha, \beta) = -g(\alpha, -\beta) \quad (3.41)$$

Proof.- Putting

$$\begin{aligned} f(\alpha, \beta) = & < (\nabla a \nabla_y)(\nabla a^{-T} \tilde{w}') : (\nabla a \nabla_y)(\nabla a^{-T} \tilde{w}') > = \\ = & < (\tilde{w}'_{1,1} - \alpha \tilde{w}'_{2,1})^2 + \tilde{w}'_{2,1}^2 + \tilde{w}'_{3,1}^2 + \\ & + (\alpha \tilde{w}'_{1,1} - \alpha^2 \tilde{w}'_{2,1} + \tilde{w}'_{1,2} - \alpha \tilde{w}'_{2,2})^2 + (\alpha \tilde{w}'_{2,1} + \tilde{w}'_{2,2})^2 + (\alpha \tilde{w}'_{3,1} + \tilde{w}'_{3,2})^2 + \\ & + (\tilde{w}'_{1,3} - \alpha \tilde{w}'_{2,3})^2 + \tilde{w}'_{2,3}^2 + \tilde{w}'_{3,3}^2 > \end{aligned} \quad (3.42)$$

$$\begin{aligned} g(\alpha, \beta) = & < (\nabla a \nabla_y)[(\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}')] : (\nabla a \nabla_y)(\nabla a^{-T} \tilde{w}') > = \\ & < (\alpha \tilde{w}'_{3,11} + \tilde{w}'_{3,21} - \tilde{w}'_{2,31}).(\tilde{w}'_{1,1} - \alpha \tilde{w}'_{2,1}) + \\ & + (\alpha \tilde{w}'_{3,12} + \tilde{w}'_{3,22} - \tilde{w}'_{2,32}).(\tilde{w}'_{1,2} - \alpha \tilde{w}'_{2,2}) + \\ & + [\alpha(\alpha \tilde{w}'_{3,11} + \tilde{w}'_{3,21} - \tilde{w}'_{2,31}) + \alpha \tilde{w}'_{3,12} + \tilde{w}'_{3,22} - \tilde{w}'_{2,32}].[\alpha(\tilde{w}'_{1,1} - \alpha \tilde{w}'_{2,1}) + (\tilde{w}'_{1,2} - \alpha \tilde{w}'_{2,2})] + \\ & + [\alpha(-\tilde{w}'_{3,11} - \tilde{w}'_{1,31} + \alpha \tilde{w}'_{2,31}) - (\tilde{w}'_{3,12} - \tilde{w}'_{1,32} + \alpha \tilde{w}'_{2,32})].[\alpha \tilde{w}'_{2,1} + \tilde{w}'_{2,2}] + \\ & + [\tilde{w}'_{2,13} - \alpha \tilde{w}'_{1,13} + \alpha^2 \tilde{w}'_{2,13} - \tilde{w}'_{1,23} + \alpha \tilde{w}'_{2,23}].\tilde{w}'_{3,3} > \end{aligned} \quad (3.43)$$

and equalities (3.38)-(3.41) can be deduced applying the changes of variables in (3.42),(3.43) already used.

REMARK.- About the dependence on C : in the general case, we can express all tensors as follows

$$R = q \nabla a^{-T} \tilde{R}' \nabla a^{-1}, \quad S = q \nabla a^{-T} \tilde{S}' \nabla a^{-1} \quad (3.44)$$

$$V_q = q^{3/2} \nabla a^{-T} \tilde{V}'_q, \quad V_h = q^{3/2} \nabla a^{-T} \tilde{V}'_h \quad (3.45)$$

$$\psi_q = q\psi'_q(C, \beta), \quad \psi_h = q\psi'_h(C, \beta) \quad (3.46)$$

where

$$\tilde{R}' = \langle \tilde{w}' \otimes \tilde{w}' \rangle (C, \beta), \quad \tilde{S}' = 2 \langle \tilde{w}' \otimes \tilde{r}' \rangle (C, \beta) \quad (3.47)$$

$$\tilde{V}_q' = \langle (\pi' + \frac{1}{2}|\tilde{w}'|^2)\tilde{w}' \rangle (C, \beta), \quad \tilde{V}_h' = 2 \langle (\pi' + \frac{1}{2}|\tilde{w}'|^2)\tilde{r}' \rangle (C, \beta) \quad (3.48)$$

$$\psi_q'(C, \beta) = C_{ij}^{-1} C_{kl} \langle \tilde{w}'_{i,K} \tilde{w}'_{j,L} \rangle (C, \beta), \quad \psi_h'(C, \beta) = 2C_{ij}^{-1} C_{kl} \langle \tilde{w}'_{i,K} \tilde{r}'_{j,L} \rangle (C, \beta) \quad (3.49)$$

$$\tilde{r}' = \nabla a^T (\nabla a \nabla) \times (\nabla a^{-T} \tilde{w}') \quad (3.50)$$

REMARK.- Two dimensional mean flows of the form

$$u(x, t) = (u_1(x_1, x_2; t), u_2(x_1, x_2; t), 0)$$

correspond a matrix C of the form

$$C = \nabla a^T \nabla a = \begin{pmatrix} b & c & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad bd = 1 + c^2, \quad b, d > 0$$

in this case, we can write $\tilde{w} = \tilde{w}(y, \tau; b, c, \beta)$, and it can be shown that the parameter c plays an identical role as α in 1-D mean flows, in the sense that symmetries (3.5)-(3.10) are still right as well as propositions 1,2,3 and 4.

3.4. A power expansion for the closure terms.

Symmetries found in section 2.3 for the canonical fluctuation when $C = I$ ($I : 3 \times 3$ identity matrix) yield

$$\langle w_1 w_1 \rangle = \langle w_2 w_2 \rangle = \langle w_3 w_3 \rangle \quad (3.51)$$

$$\langle w_i w_j \rangle = 0 \quad \text{if} \quad i \neq j \quad (3.52)$$

Since $tr(R) = \langle |w|^2 \rangle = 2q$, we have the value of R taken at $C = I$ (corresponding to $\alpha = 0$)

$$R(I) = \frac{2}{3}qI \quad (3.53)$$

In the same way, we have

$$\langle r_1 w_1 \rangle = \langle r_2 w_2 \rangle = \langle r_3 w_3 \rangle \quad (3.54)$$

$$\langle r_i w_j \rangle = 0 \quad \text{if} \quad i \neq j \quad (3.55)$$

and on the other hand, $\frac{1}{2}tr(S) = h$; so we know the value of S at $C = I$

$$S(I) = \frac{2}{3}hI \quad (3.56)$$

we also have

$$V_q(I) = V_h(I) = 0 \quad (3.57)$$

Identities (3.53), (3.56), (3.57) together with propositions (3.3.1)–(3.3.5) allow us to find the expansions in α and β for tensors R, S vectors V_q, V_h and functions ψ_q, ψ_h given as follows.

Corollary 3.4.1.- Assume that R, S, V_q, V_h, ψ_q and ψ_h smooth enough in $(\alpha, \beta) \in E(0, 0)$; then we have the following expansions :

$$\langle w_1 w_1 \rangle \cong \frac{2}{3}q + b_{11}^{(1)} \alpha^2 q + b_{11}^{(2)} \frac{h^2}{q} \quad (3.58)$$

$$\langle w_2 w_2 \rangle \cong \frac{2}{3}q + b_{22}^{(1)} \alpha^2 q + b_{22}^{(2)} \frac{h^2}{q} \quad (3.59)$$

$$\langle w_3 w_3 \rangle \cong \frac{2}{3}q + b_{33}^{(1)} \alpha^2 q + b_{33}^{(2)} \frac{h^2}{q} \quad (3.60)$$

$$\langle w_1 w_2 \rangle \cong b_{12}^{(1)} \alpha q + b_{12}^{(2)} \alpha \frac{h^2}{q} \quad (3.61)$$

$$\langle r_1 w_1 \rangle \cong \frac{1}{3}h + c_{11} h \alpha^2 \quad (3.62)$$

$$\langle r_2 w_2 \rangle \cong \frac{1}{3}h + c_{22} h \alpha^2 \quad (3.63)$$

$$\langle r_3 w_3 \rangle \cong \frac{1}{3}h + c_{33} h \alpha^2 \quad (3.64)$$

$$\langle r_1 w_2 \rangle \cong c_{12}^{(1)} \alpha h + c_{12}^{(2)} \alpha^3 h + c_{12}^{(3)} \alpha \frac{h^3}{q^2} \quad (3.65)$$

$$\langle (\frac{1}{2}|w|^2 + \pi)w \rangle_3 \cong q^{3/2} (d^{(1)} \alpha h + d^{(2)} \alpha^3 \frac{h}{q} + d^{(3)} \alpha \frac{h^3}{q^3}) \quad (3.66)$$

$$\langle (\frac{1}{2}|w|^2 + \pi)(\nabla a \nabla_y) \times w \rangle_3 \cong q^{3/2} (e^{(1)} \alpha + e^{(2)} \alpha \frac{h^2}{q^2}) \quad (3.67)$$

$$\langle (\nabla a \nabla_y)w : (\nabla a \nabla_y)w \rangle \cong f^{(0)} q + f^{(1)} \alpha^2 \frac{h^2}{q} \quad (3.68)$$

$$\langle (\nabla a \nabla_y)w : (\nabla a \nabla_y)r \rangle \cong g^{(0)} h + g^{(1)} \alpha^2 h. \quad (3.69)$$

So the problem is to find out the values of all coefficients $b_{ij}^{(k)}, c_{jj}, c_{12}^{(i)}, d^{(i)}, e^{(i)}, f^{(i)}$ and $g^{(i)}$ in (3.58)–(3.69).

To do so, we resolve the system (2.39) numerically for some α and $\beta = h/q$; later we compute R, S, V_q, V_h, ψ_q and ψ_h ; finally, we obtain all coefficients $b_{ij}^{(k)}, \dots, g^{(i)}$ by the interpolation of the computed values.

As in [2], from the special form of $u(x, t)$ in (3.1) we can find fluctuations that depend only on (x_2, t) . Hence

$$\nabla \cdot V_q = \nabla \cdot V_h = 0 \quad (3.70)$$

We fixe now our attention in S and ψ_h as they are proportionnal to the helicity h . This means that a zero initial condition for h ($h(x, 0) = 0$) implies, according to (3.70), that

$$h(x, t) = 0, \quad \forall t > 0 \quad (3.71)$$

so that the expression for helicity is neglected, since it is trivially verified.

3.5. Asymptotic behaviour with respect to α .

In this section we study the behaviour of the turbulent field $w(y; \alpha, \beta)$ when α is big enough and β is fixed. In fact, we are interested in tensors R and S , and in functions ψ_q and ψ_h .

Putting $\gamma = \alpha^{-1}$, problem (2.39) becomes

$$\left. \begin{aligned} (w_1 w_{1,1} + w_2 w_{1,2} + w_3 w_{1,3} + \pi_{,1})\gamma + w_2 w_{1,1} &= 0 \\ (w_1 w_{2,1} + w_2 w_{2,2} + w_3 w_{2,3} + \pi_{,2})\gamma + w_2 w_{2,1} + \pi_{,1} &= 0 \\ (w_1 w_{3,1} + w_2 w_{3,2} + w_3 w_{3,3} + \pi_{,3})\gamma + w_2 w_{3,1} &= 0 \\ (w_{1,1} + w_{2,2} + w_{3,3})\gamma + w_{2,1} &= 0 \\ < w_1^2 + w_2^2 + w_3^2 > = 2 \\ < w_1 w_{3,1} - w_3 w_{1,1} + \gamma[w_1(w_{3,2} - w_{2,3}) + w_2(w_{1,3} - w_{3,1}) + w_3(w_{2,1} - w_{1,2})] > = \beta\gamma \end{aligned} \right\} \quad (3.72)$$

and we wish to know the limit

$$\lim_{\gamma \rightarrow 0} w(y; \gamma, \beta) \quad (3.73)$$

We assume that the dependency of w with respect to the parameter γ is analitical; so we have

$$w_i = w_i^0 + \gamma w_i^1 + \frac{1}{2}\gamma^2 w_i^2 + \frac{1}{6}\gamma^3 w_i^3 + \dots \quad (3.74)$$

$$\pi = \pi^0 + \gamma \pi^1 + \frac{1}{2}\gamma^2 \pi^2 + \frac{1}{6}\gamma^3 \pi^3 + \dots \quad (3.75)$$

where

$$w_i^j = \frac{d^j w_i}{d\gamma^j} \Big|_{\gamma=0}, \quad \pi^j = \frac{d^j \pi}{d\gamma^j} \Big|_{\gamma=0} \quad (3.76)$$

Making $\gamma = 0$ in (3.72) we obtain

$$\left. \begin{array}{l} w_2^0 w_{1,1}^0 = 0 \\ w_2^0 w_{2,1}^0 + \pi_{,1}^0 = 0 \\ w_2^0 w_{3,1}^0 = 0 \\ w_{2,1}^0 = 0 \\ \langle (w_1^0)^2 + (w_2^0)^2 + (w_3^0)^2 \rangle = 2 \\ \langle w_1^0 w_{3,1}^0 - w_3^0 w_{1,1}^0 \rangle = 0 \end{array} \right\} \quad (3.77)$$

Two cases have to be into account

- (i) $w_2^0 \neq 0$
- (ii) $w_2^0 = 0$

- (i) $w_2^0 \neq 0$

From (3.77a)-3.77d) we have

$$\left. \begin{array}{l} w_1^0 = w_1^0(y_2, y_3) \\ w_2^0 = w_2^0(y_2, y_3) \\ w_3^0 = w_3^0(y_2, y_3) \\ \pi^0 = \pi^0(y_2, y_3) \end{array} \right\} \quad (3.78)$$

i.e. (w^0, π^0) do not depend on y_1 . By induction we can deduce that (w^j, π^j) , $j \geq 0$, do not depend on y_1 ; consequently (w, π) do not depend on y_1 too. This allows us to rewrite (3.72) as

$$\left. \begin{array}{l} w_2 w_{1,2} + w_3 w_{1,3} = 0 \\ w_2 w_{2,2} + w_3 w_{2,3} + \pi_{,2} = 0 \\ w_2 w_{3,2} + w_3 w_{3,3} + \pi_{,3} = 0 \\ w_{2,2} + w_{3,3} = 0 \\ \langle w_1^2 + w_2^2 + w_3^2 \rangle = 2 \\ \langle w_1(w_{3,2} - w_{2,3}) + w_2 w_{1,3} - w_3 w_{1,2} \rangle = \beta \end{array} \right\} \quad (3.79)$$

i.e. $(w_1, w_2), \pi$ verifies a periodic incompressible Euler problem in the plane $y_2 - y_3$, and w_1 is transported by (w_2, w_3) in the same plane (note that the parameter γ is gone).

If (w_2, w_3) is ergodic in the square $[-\pi, \pi] \times [-\pi, \pi]$ (streamlines are dense in the square), then $w_1 = 0$ (this is because in that case $w_1 = \text{constant}$, and $\langle w_1 \rangle \geq 0$). But $w_1 = 0$ implies $\beta = 0$, being in general $\beta \neq 0$. Hence, ergodicity and no-zero helicity are not agreed with the condition $w_2^0 \neq 0$.

When helicity vanish, we can have $w = (0, w_2, w_3)^T$, and $(w_2, w_3), \pi$ being solution of a periodic incompressible Euler problem in $y_2 - y_3$, and consequently

$$R = q \langle w \otimes w \rangle = q \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_{22} & 0 \\ 0 & 0 & \rho_{33} \end{pmatrix}, \quad \rho_{22} + \rho_{33} = 2 \quad (3.80)$$

$$S \equiv 0 \quad (3.81)$$

$$\psi_q = \langle \nabla w : \nabla w \rangle \quad (3.82)$$

$$\psi_h \equiv 0 \quad (3.83)$$

(ii) $w_2^0 = 0$

In this case, we can find also solutions w that do not depend on γ . Assume $w_2 = 0, \pi_{,1} = 0$, then (3.72) becomes

$$\left. \begin{array}{l} \pi_{,1} = 0 \\ w_1 w_{1,1} + w_3 w_{1,3} = 0 \\ \pi_{,2} = 0 \\ w_1 w_{3,1} + w_3 w_{3,3} + \pi_{,3} = 0 \\ w_{1,1} + w_{3,3} = 0 \\ \langle w_1^2 + w_3^2 \rangle = 2 \\ \langle w_1 w_{3,2} - w_3 w_{1,2} \rangle = \beta \end{array} \right\} \quad (3.84)$$

This problem have solutions of the form

$$w_1(y_1, y_2, y_3) = \phi(y_2) \quad (3.85)$$

$$w_3(y_1, y_2, y_3) = \psi(y_2) \quad (3.86)$$

$$\pi = 0 \quad (3.87)$$

where ϕ and ψ verify

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (\phi^2 + \psi^2) = 2 \quad (3.88)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi \phi' = \frac{\beta}{2} \quad (3.89)$$

So, for $w = (w_1, 0, w_3)^T$ we have

$$R = q \langle w \otimes w \rangle = q \begin{pmatrix} \rho_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho_{33} \end{pmatrix}, \quad \rho_{11} + \rho_{33} = 2 \quad (3.90)$$

$$S = 2q \langle w \otimes r \rangle = q \begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad (3.91)$$

$$\psi_q = q \langle (\nabla a \nabla) w : (\nabla a \nabla) w \rangle = O(1) \quad (3.92)$$

$$\psi_h = q \langle (\nabla a \nabla) w : (\nabla a \nabla) r \rangle = O(1) \quad (3.93)$$

REMARK.- The general solution of the differential system (3.84) is given by

$$w_1 = \kappa(y_3 - \phi(y_2)y_1, y_2)$$

$$w_3 = \phi(y_2)\kappa(y_3 - \phi(y_2)y_1, y_2)$$

$$\pi \equiv 0$$

However, periodicity conditions and the helicity restriction causes $\kappa(s, t) = \kappa_1(t)$, i.e. w_1, w_3 have the form (3.85)-(3.86)

REMARK.- Notice that functions ϕ and ψ are arbitrary (they only verify (3.69)-(3.70)); hence ρ_{11} and ρ_{33} are only subjected up to the restriction $\rho_{11} + \rho_{33} = 2$.

Consider now the general case $w_2 \neq 0, w_2^0 = 0$:

$$w_1 = w_1^0 + \gamma w_1^1 + \frac{1}{2}\gamma^2 w_1^2 + \dots \quad (3.94)$$

$$w_2 = \gamma w_2^1 + \frac{1}{2}\gamma^2 w_2^2 + \dots \quad (3.95)$$

$$w_3 = w_3^0 + \gamma w_3^1 + \frac{1}{2}\gamma^2 w_3^2 + \dots \quad (3.96)$$

hence

$$\begin{aligned}
& \langle w_1 w_1 \rangle = \langle w_1^0 w_1^0 \rangle + O(\gamma^2) \\
& \langle w_2 w_2 \rangle = O(\gamma^2) \\
& \langle w_3 w_3 \rangle = \langle w_3^0 w_3^0 \rangle + O(\gamma^2) \\
& \langle w_1 w_2 \rangle = O(\gamma) \\
& \langle w_1 r_1 \rangle = \langle w_1^0 w_{3,1}^1 \rangle + \langle w_1^1 w_{3,1}^0 \rangle + \langle w_1^0 w_{3,2}^0 \rangle + O(\gamma) \\
& \langle w_2 r_2 \rangle = O(\gamma) \\
& \langle w_3 r_3 \rangle = - \langle w_3^0 w_{1,1}^1 \rangle - \langle w_3^1 w_{1,1}^0 \rangle - \langle w_3^0 w_{1,2}^0 \rangle + O(\gamma) \\
& \langle w_1 r_2 \rangle = O(\gamma) \\
& \psi_q = O(\gamma^{-2}) \\
& \psi_h = O(\gamma^{-3})
\end{aligned} \tag{3.97}$$

and thus

$$\lim_{\gamma \rightarrow 0} R = q \begin{pmatrix} \langle w_1^0 w_1^0 \rangle & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \langle w_3^0 w_3^0 \rangle \end{pmatrix}, \quad \langle w_1^0 w_1^0 \rangle + \langle w_3^0 w_3^0 \rangle = 2 \tag{3.98}$$

$$\lim_{\gamma \rightarrow 0} S = \begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix} \tag{3.99}$$

Notice that limits $\langle w_1 w_1 \rangle, \langle w_3 w_3 \rangle$ are not unique (functions $w = (w_1, 0, w_3)$ given in (3.85), (3.86) are possible limits $w^0 = (w_1^0, 0, w_3^0)$). In the other hand, limits $\langle w_1 r_1 \rangle, \langle w_3 r_3 \rangle$ are determined and their values are both β . Moreover, in any case we have

$$\langle w_1 w_2 \rangle = O(\gamma) \tag{3.100}$$

$$\langle w_1 r_2 \rangle = O(\gamma) \tag{3.101}$$

3.5.1. Conclusion.

If we assume that system (3.72) has isolated solutions $w(y; \gamma, \beta)$ with analitical dependency on γ , we obtain the following results:

(a) When $\beta \neq 0$, any branch $\gamma \rightarrow w(y; \gamma, \beta)$ converges to a function $w^0(y)$ with second component equal to zero ($w^0(y) = (w_1^0(y), 0, w_3^0(y))^T$).

The limits of the mean values $\langle w_1 w_1 \rangle (\gamma, \beta)$, $\langle w_3 w_3 \rangle (\gamma, \beta)$, depend on the branch, but $\langle w_1 r_1 \rangle (\gamma, \beta)$, $\langle w_3 r_3 \rangle (\gamma, \beta)$ both converge to $\beta/2$. Moreover, we have

$$\langle w_1 w_2 \rangle = O(\gamma)$$

$$\langle w_1 r_2 \rangle = O(\gamma)$$

The limit function $w^0(y)$ may depends only on y_2 ,

$(w^0(y) = (w_1^0(y_2), 0, w_3^0(y_2))^T)$, and then

$$\psi_q = O(1)$$

$$\psi_h = O(1)$$

but if $w_{1,1}^0 \neq 0$ or $w_{3,1}^0 \neq 0$, we have

$$\psi_q = O(\gamma^{-2})$$

$$\psi_h = O(\gamma^{-3})$$

(b) When $\beta = 0$ another case may appear: function $w^0(y)$ can be of the form

$$w^0(y) = (0, w_2^0(y_2, y_3), w_3^0(y_2, y_3))^T$$

and (w_2^0, w_3^0) verifying an incompressible Euler equation in the $y_2 - y_3$ variables. We also have

$$\psi_q = O(1)$$

$$\psi_h = 0$$

3.6. Frame invariance.

We can deduce simple expressions for functions R , S , V_q , V_h , ψ_q and ψ_h in the hypothesis of frame invariance for system (2.36). This will lead us to reduce our computations.

Let $O_+^3 = \{Q \in M_{3 \times 3} \mid QQ^T = I, \det(Q) = 1\}$ be the set of all orthogonal matrices whose determinant is equal to 1. For $Q \in O_+^3$, we put

$$X = Qx \tag{3.102}$$

$$U(X, t) = Qu(x, t) \tag{3.103}$$

$$E(X, t) = q(x, t) \tag{3.104}$$

equalities (3.102),(3.103) imply

$$A(X, t) = Qa(x, t) \quad (3.105)$$

and thus

$$u\nabla_x q(x, t) = U\nabla_X E(X, t) \quad (3.106)$$

$$(\nabla_x a)(x, t) = Q^T \nabla_X A(X, t) Q \quad (3.107)$$

REMARK.-We will work with the equation for $q(x, t)$, and we will deduce some properties for R' , V'_q , and ψ'_q ; the same arguments can be used for the equation of $h(x, t)$ to obtain the same properties for S' , V'_h , and ψ'_h .

REMARK.- For clarity, we do not write the parameter h/q in the arguments of R' , S' ,...

The equation (2.36b), in terms of X , U , A , and E becomes

$$\begin{aligned} & E_{,t} + U\nabla_X E + ER'(Q^T \nabla_X A Q) : Q^T \nabla_X U Q + \\ & + \mu E \psi'_q(Q^T \nabla_X A^T \nabla_X A Q) + \varepsilon^{1/3} \nabla_X \cdot (E^{3/2} Q V'_q(Q^T \nabla_X A Q)) = 0 \end{aligned} \quad (3.108)$$

On the other hand, the frame invariance hypothesis implies

$$E_{,t} + U\nabla_X E + ER'(\nabla_X A) : \nabla_X U + \mu E \psi'_q(\nabla_X A^T \nabla_X A) + \varepsilon^{1/3} \nabla_X \cdot (E^{3/2} V'_q(\nabla_X A)) = 0 \quad (3.109)$$

hence

$$\begin{aligned} & ER'(Q^T \nabla_X A Q) : Q^T \nabla_X U Q + \mu E \psi'_q(Q^T \nabla_X A^T \nabla_X A Q) + \varepsilon^{1/3} \nabla_X \cdot (E^{3/2} Q V'_q(Q^T \nabla_X A Q)) = \\ & = ER'(\nabla_X A) : \nabla_X U + \mu E \psi'_q(\nabla_X A^T \nabla_X A) + \varepsilon^{1/3} \nabla_X \cdot (E^{3/2} V'_q(\nabla_X A)), \quad \forall Q \in O_+^3 \end{aligned} \quad (3.110)$$

Notice that (3.110) is also true when $\mu = 0$, so

$$\begin{aligned} & ER'(Q^T \nabla_X A Q) : Q^T \nabla_X U Q + \varepsilon^{1/3} \nabla_X \cdot (E^{3/2} Q V'_q(Q^T \nabla_X A Q)) = \\ & = ER'(\nabla_X A) : \nabla_X U + \varepsilon^{1/3} \nabla_X \cdot (E^{3/2} V'_q(\nabla_X A)), \quad \forall Q \in O_+^3 \end{aligned} \quad (3.111)$$

and

$$\psi'_q(Q^T \nabla_X A^T \nabla_X A Q) = \psi'_q(\nabla_X A^T \nabla_X A), \quad \forall Q \in O_+^3 \quad (3.112)$$

The following lemma will be used to get separate expressions for R' , and V'_q

Lemma 3.6.1.- Let $x_0 \in \Omega$, $t_0 > 0$ and $M_0, M_1 \in M_{3 \times 3}$ such that

$$tr(M_0) = 0, \quad det(M_1) = 1;$$

then there exist two fields $u(x, t)$, $a(x, t)$ verifying

$$\nabla \cdot u(x, t) = 0 \quad \text{in } \Omega \quad (3.113)$$

$$a_{,t}(x, t) + (u(x, t) \cdot \nabla) a(x, t) = 0 \quad \text{in } \Omega \quad (3.114)$$

$$a(x, 0) = x \quad \text{in } \Omega \quad (3.115)$$

and

$$\nabla u(x_0, t_0) = M_0 \quad (3.116)$$

$$\nabla a(x_0, t_0) = M_1 \quad (3.117)$$

Proof.- Consider a smooth matrix function $B(t)$ such that

$$B(0) = \Theta, \quad tr(B(t)) = 0 \quad \forall t \geq 0$$

and define $a(x, t)$ and $u(x, t)$ as

$$a(x, t) = e^{B(t)} x, \quad u(x, t) = (\nabla a)^{-T} a_{,t}$$

so we have (3.113)-(3.115). A suitable choice of function $B(t)$ can give conditions (3.116),(3.117). (Note that $\nabla a(x, t) = e^{B^T(t)}$ and that the expression for $\nabla u(x, t)$ contains the matrix $B'(t)$).

We return now to the equation (3.111); by lemma 3.6.1, we can write (3.111) when $\nabla U(x_0, t_0)$ is substituted by $2\nabla U(x_0, t_0)$, i.e.

$$\begin{aligned} 2ER'(Q^T \nabla_X A Q) : Q^T \nabla_X U Q + \varepsilon^{1/3} \nabla_X \cdot (E^{3/2} Q V'_q(Q^T \nabla_X A Q)) = \\ = 2ER'(\nabla_X A) : \nabla_X U + \varepsilon^{1/3} \nabla_X \cdot (E^{3/2} V'_q(\nabla_X A)), \quad \forall Q \in O_+^3 \end{aligned} \quad (3.118)$$

hence

$$R'(\nabla a) : \nabla u = R'(Q \nabla a Q^T) : Q \nabla u Q^T, \quad \forall Q \in O_+^3 \quad (3.119)$$

$$\nabla \cdot (q^{3/2} Q V'_q(\nabla a)) = \nabla \cdot (q^{3/2} V'_q(Q \nabla a Q^T)), \quad \forall Q \in O_+^3 \quad (3.120)$$

By lemma 3.6.1 these expressions are true for any ∇a and ∇u such that $\det(\nabla a) = 1$, $\text{tr}(\nabla u) = 0$, and in consequence

$$R'(\nabla a) = Q^T R'(Q \nabla a Q^T) Q + \lambda(\nabla a, Q) I, \quad \forall Q \in O_+^3 \quad (3.121)$$

$$V'_q(\nabla a) = Q^T V'_q(Q \nabla a Q^T), \quad \forall Q \in O_+^3 \quad (3.122)$$

Taking traces in () we deduce $\lambda \equiv 0$. Finally

$$R'(\nabla a) = Q^T R'(Q \nabla a Q^T) Q, \quad \forall Q \in O_+^3 \quad (3.123)$$

$$V'_q(\nabla a) = Q^T V'_q(Q \nabla a Q^T), \quad \forall Q \in O_+^3$$

Putting

$$\tilde{R}' = \langle \tilde{w}' \otimes \tilde{w}' \rangle, \quad \tilde{V}'_q = \langle (\pi + \frac{1}{2}|\tilde{w}'|^2)\tilde{w}' \rangle, \quad w = q^{1/2} \nabla a^{-T} \tilde{w}' \quad (3.124)$$

we will have

$$R'(\nabla a) = \nabla a^{-T} \tilde{R}'(\nabla a^T \nabla a) \nabla a^{-1} \quad (3.125)$$

$$V'_q(\nabla a) = \nabla a^{-T} \tilde{V}'_q(\nabla a^T \nabla a) \quad (3.126)$$

Using (3.122), (3.123), (3.125) and (3.126), we obtain the same properties for \tilde{R}' and \tilde{V}'_q

$$\tilde{R}'(Q^T C Q) = Q^T \tilde{R}'(C) Q, \quad \forall Q \in O_+^3 \quad (3.127)$$

$$\tilde{V}'_q(Q^T C Q) = Q^T \tilde{V}'_q(C), \quad \forall Q \in O_+^3 \quad (3.128)$$

From (3.128) we deduce that

$$\tilde{V}'_q \equiv 0$$

(firstly, assume C diagonal and take $Q = Q_4, Q_5, Q_6$ as in section 3.2; secondly, consider $Q \in O_+^3$ that diagonalizes C , and then apply the first part).

We can summarize these results in the

Theorem 3.6.2.- Assume that system (2.96) is invariant under changes of coordinates (rotations). Then $\tilde{R}', \tilde{S}', \tilde{V}'_q, \tilde{V}'_h, \psi'_q$ and ψ'_h given in (3.47)-(3.50) verify

$$\tilde{R}'(Q^T C Q) = Q^T \tilde{R}'(C) Q, \quad \forall Q \in O_+^3 \quad (3.129)$$

$$\tilde{S}'(Q^T C Q) = Q^T \tilde{S}'(C) Q, \quad \forall Q \in O_+^3 \quad (3.130)$$

$$\psi'_q(Q^T C Q) = \psi'_q(C), \quad \forall Q \in O_+^3 \quad (3.131)$$

$$\psi'_h(Q^T C Q) = \psi'_h(C), \quad \forall Q \in O_+^3 \quad (3.132)$$

$$\tilde{V}'_q \equiv 0, \quad \tilde{V}'_h \equiv 0 \quad (3.133)$$

From (3.48),(3.133) we have

Corollary 3.6.3.- *The vector functions V'_q and V'_h are identically zero.*

Conditions (3.131),(3.132) show the

Corollary 3.6.4.- *The scalar functions $\psi'_q(C, h/q)$ and $\psi'_h(C, h/q)$ depend only on h/q and on the invariants of C :*

$$\psi'_q = \psi'_q(i_1(C), i_2(C), h/q) \quad (3.134)$$

$$\psi'_h = \psi'_h(i_1(C), i_2(C), h/q) \quad (3.135)$$

where

$$i_1 = \text{tr}(C), \quad i_2 = \text{tr}(\text{Adj}(C))$$

REMARK.- The third invariant of C does not appear because in any case we have

$$i_3(C) = \det(C) = 1.$$

Tensors \tilde{R}' and \tilde{S}' verify the hypothesis of Rivlin-Ericksen's theorem (conditions (3.129), (3.130)), in consequence we deduce the

Corollary 3.6.5.- *The matrix functions \tilde{R}' and \tilde{S}' are of the form*

$$\tilde{R}'(C, h/q) = \tilde{\rho}_o(i_1, i_2, h/q)I + \tilde{\rho}_1(i_1, i_2, h/q)C + \tilde{\rho}_2(i_1, i_2, h/q)C^2 \quad (3.136)$$

$$\tilde{S}'(C, h/q) = \tilde{\sigma}_o(i_1, i_2, h/q)I + \tilde{\sigma}_1(i_1, i_2, h/q)C + \tilde{\sigma}_2(i_1, i_2, h/q)C^2 \quad (3.137)$$

In fact, we are interested in R' and S' rather than \tilde{R}' , \tilde{S}' ; by Corollary (3.6.5) and (3.47) we have the

Corollary 3.6.6.- *The matrix functions R' and S' are of the form*

$$R'(\bar{C}, h/q) = \rho'_o(i_1, i_2, h/q)I + \rho'_1(i_1, i_2, h/q)\bar{C} + \rho'_2(i_1, i_2, h/q)\bar{C}^2 \quad (3.138)$$

$$S'(\bar{C}, h/q) = \sigma'_o(i_1, i_2, h/q)I + \sigma'_1(i_1, i_2, h/q)\bar{C} + \sigma'_2(i_1, i_2, h/q)\bar{C}^2 \quad (3.139)$$

where

$$\bar{C} = \nabla a \nabla a^T$$

and

$$\left. \begin{array}{ll} \rho'_o = \tilde{\rho}_1 + i_2 \tilde{\rho}_o & \sigma'_o = \tilde{\sigma}_1 + i_2 \tilde{\sigma}_o \\ \rho'_1 = \tilde{\rho}_2 - i_1 \tilde{\rho}_o & \sigma'_1 = \tilde{\sigma}_2 - i_1 \tilde{\sigma}_o \\ \rho'_2 = \tilde{\rho}_o & \sigma'_2 = \tilde{\sigma}_o \end{array} \right\} \quad (3.140)$$

Corollary (3.6.6) shows that the tabulations of R' and S' , in the general case, are reduced to the computations of ρ'_o , ρ'_1 , ρ'_2 , σ'_o , σ'_1 and σ'_2 as functions of two parameters only! (and not five)

REMARK.- In flows of the form (3.1) we obtained

$$C = \nabla a^T \nabla a = \begin{pmatrix} 1 + \alpha^2 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

thus

$$i_1 = 3 + \alpha^2, \quad i_2 = 3 + \alpha^2, \quad i_3 = 1 \quad (3.141)$$

so there is only one independent non-constant invariant.

3.6.1. Computations of the 2-D case.

Two-dimensional mean flows with two-dimensional turbulence have a matrix C of the form

$$C = \begin{pmatrix} b & c & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad bd - c^2 = 1, \quad b > 0, \quad d > 0 \quad (3.142)$$

In this case, we have one independent non-constant invariant only

$$i_1 = b + d + 1 \quad (i_1 = i_2, \quad i_3 = 1). \quad (3.143)$$

This property will allow us to relate the computations of 1-D mean flows with the 2-D simulation.

Putting

$$A = \begin{pmatrix} b & c \\ c & d \end{pmatrix} = M_2^T M_2, \quad \nabla a = \begin{pmatrix} M_2 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.144)$$

we have

$$A^2 = (i_1 - 1)A - I \quad (3.145)$$

using (3.78), \tilde{R}' can be written as

$$\tilde{R}' = \begin{pmatrix} \tilde{R}'_2 & 0 \\ 0 & \tilde{R}'_{33} \end{pmatrix}, \quad \tilde{R}'_2 = \tilde{r}_o I + \tilde{r}_1 A, \quad \tilde{R}'_{33} = \tilde{\rho}_o + \tilde{\rho}_1 + \tilde{\rho}_2 \quad (3.146)$$

where

$$\tilde{r}_o = \tilde{\rho}_o - \tilde{\rho}_2, \quad \tilde{r}_1 = \tilde{\rho}_1 + (i_1 - 1)\tilde{\rho}_2 \quad (3.147)$$

REMARK.- The Reynolds stress tensor R can be expressed by

$$R = q \nabla a^{-T} \begin{pmatrix} \tilde{r}_o I + \tilde{r}_1 M_2^T M_2 & 0 \\ 0 & \tilde{R}'_{33} \end{pmatrix} \nabla a^{-1} = q \begin{pmatrix} \tilde{r}_o M_2^{-T} M_2^{-1} + \tilde{r}_1 I & 0 \\ 0 & \tilde{R}'_{33} \end{pmatrix} \quad (3.148)$$

but $M_2^{-T} M_2^{-1} = (i_1 - 1)I - M_2 M_2^T$ and thus

$$\nabla \cdot R = -\nabla \cdot (q \tilde{r}_o \nabla a \nabla a^T) + \nabla r'_1 \quad (3.149)$$

$$R : \nabla u = -q \tilde{r}_o \nabla a \nabla a^T : \nabla u \quad (3.150)$$

and the computations of $\nabla \cdot R$ and $R : \nabla u$ reduce to the computation of the function \tilde{r}_o (the term $\nabla \tilde{r}'_1$ can be included in ∇p).

The parameters $\tilde{\rho}_i$ can be obtained from \tilde{R}' as they verify the linear system

$$\left. \begin{array}{l} 2\tilde{\rho}_o + (i_1 - 1)\tilde{\rho}_1 + [(i_1 - 1)^2 - 2]\tilde{\rho}_2 = \mu_1 \quad (= \tilde{R}'_{11} + \tilde{R}'_{22}) \\ \tilde{\rho}_o + \tilde{\rho}_1 + \tilde{\rho}_2 = \mu_2 \quad (= \tilde{R}'_{33}) \\ i_1 \tilde{\rho}_o + 3\tilde{\rho}_1 + i_1 \tilde{\rho}_2 = 2 \quad (= \langle \tilde{w}' C^{-1} \tilde{w}' \rangle) \end{array} \right\} \quad (3.151)$$

whose determinant $\Delta = -(3 - i_1)^2(1 + i_1)$ is zero for $C = I$.

For $C \neq I$, the solution of (3.151) is given by

$$\tilde{\rho}_o = \frac{2i_1 - \mu_1 - (2i_1 + 1)\mu_2}{(i_1 - 3)(i_1 + 1)}, \quad \tilde{\rho}_1 = \frac{i_1 \mu_2 - 2}{i_1 - 3}, \quad \tilde{\rho}_2 = \frac{2 + \mu_1 - (i_1 + 2)\mu_2}{(i_1 - 3)(i_1 + 1)} \quad (3.152)$$

and by (3.147), (3.152)

$$\tilde{r}_o = \frac{2i_1 - 2 - 2\mu_1 + (1 - i_1)\mu_2}{(i_1 - 3)(i_1 + 1)} \quad (3.153)$$

The same arguments can be used with \tilde{S}' to obtain an expression for $S : \nabla u$. We find

$$S : \nabla u = -q \tilde{s}_0 \nabla a \nabla a^T : \nabla u \quad (3.154)$$

where

$$\tilde{s}_o = \frac{(i_1 - 1)h/q - 2\varsigma_1 + (1 - i_1)\varsigma_2}{(i_1 - 3)(i_1 + 1)} \quad (3.155)$$

and, in turn, ς_1 and ς_2 are defined as

$$\varsigma_1 = \tilde{S}'_{11} + \tilde{S}'_{22}, \quad \varsigma_2 = \tilde{S}'_{33} \quad (3.156)$$

We can use all computations made in 1-D to compute $\mu_1(i_1)$, $\mu_2(i_1)$, $\varsigma_1(i_1)$ and $\varsigma_2(i_1)$.

Thus, to summarize, $\nabla \cdot R$, $R : \nabla u$ and $S : \nabla u$ are given by (), (), () with

$$\tilde{r}_o = \frac{2\alpha^2 + 4 - 2\mu_1 - (2 + \alpha^2)\mu_2}{\alpha^2(\alpha^2 + 4)}, \quad \tilde{s}_o = \frac{(\alpha^2 + 2)h/q - 2\varsigma_1 - (2 + \alpha^2)\varsigma_2}{\alpha^2(\alpha^2 + 4)}, \quad \alpha^2 = b + d - 2 = \text{tr}(\nabla a^T \nabla a) - 3$$

and μ_1 , μ_2 , ς_1 , and ς_2 functions of α and h/q tabulated.

Figures 5, 6 show μ_1 , μ_2 as function of h/q for some values of α .

Figure 7 shows \tilde{r}_o for the same values of α .

3.6.2. Computations of the 3-D case.

For 3-D mean flows, the frame invariance reduces the parameters of C to 2, so tabulations must be made in terms of the two invariants i_1 , i_2 . Thus the 1-D computations are not sufficient to tabulate the moments in all cases. However, we can build up a family of matrices with one non trivial invariant (see [6]); for example, consider

$$C = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_1, \lambda_2, \lambda_3 > 0, \quad \lambda_1 \lambda_2 \lambda_3 = 1 \quad (3.157)$$

It is easy to see that $i_1(C) = i_2(C)$ if and only if $\lambda_i = 1$ for some $i = 1, 2, 3$; so putting $\lambda_3 = 1$, we find

$$\lambda_1 = \lambda_1(\alpha) = 1 + \frac{\alpha^2}{2} + \frac{\alpha}{2}\sqrt{4 + \alpha^2}, \quad \lambda_2 = \lambda_1(-\alpha), \quad (\alpha = \sqrt{\lambda_1 + \lambda_2 - 2}) \quad (3.158)$$

and computations in the 1-D case can be used too.

Equalities (3.138),(3.139) yield systems respectively for the coefficients $\tilde{\rho}_o$, $\tilde{\rho}_1$, $\tilde{\rho}_2$ and $\tilde{\sigma}_o$, $\tilde{\sigma}_1$, $\tilde{\sigma}_2$:

$$\tilde{R}'_{ij} = \tilde{\rho}_o + \tilde{\rho}_1 \lambda_j + \tilde{\rho}_2 \lambda_j^2, \quad 1 \leq j \leq 3 \quad (3.159)$$

$$\tilde{S}'_{jj} = \tilde{\sigma}_o + \tilde{\sigma}_1 \lambda_j + \tilde{\sigma}_2 \lambda_j^2, \quad 1 \leq j \leq 3 \quad (3.160)$$

whose determinant $\Delta = (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)$ is zero for $\alpha = 0$, i.e. $C = I$.

Certain tabulations of $\tilde{\rho}_o, \tilde{\rho}_1, \tilde{\rho}_2$ have been made in [6] for fluctuations \tilde{w} with zero helicity.

3.6.3. Conclusion.

We can reduce the number of parameters in the computation of functions R, S, ψ_q and ψ_h if the averaged system (2.36) is invariant under changes of coordinate systems. These parameters are functions of $i_1 = \text{tr}(C)$ and $i_2 = \text{tr}(\text{Adj}(C))$. In this case, vector functions V_q and V_h vanish.

For 1-D mean flows of the form (3.1), we have $i_1 = i_2$, i.e. one invariant only. Since in 2-D mean flows with 2-D turbulence, we also have $i_1 = i_2$, we can relate the computations made in 1-D to 2-D simulations. In some cases, we can also use the 1-D computations to simulate 3-D mean flows.

CHAPTER 4 : Computations of the canonical microstructure flow: numerical approximations of the closure terms.

4.1. Numerical solution by least squares.

The stationary microstructure problem (2.39) has been solved numerically by a least squares method, which is set up as follows :

$$0 = \min_{\tilde{w}' \in W - \{0\}} J(\tilde{w}') = E\left(\frac{\tilde{w}'}{\sqrt{\langle \tilde{w}' C^{-1} \tilde{w}' \rangle}}\right) + K \left| \frac{\langle (\nabla a^{-T} \tilde{w}').(\nabla a \nabla_y) \times (\nabla a^{-T} \tilde{w}') \rangle}{\langle \tilde{w}' C^{-1} \tilde{w}' \rangle} - \frac{h}{2q} \right|^2 \quad (4.1)$$

where W is the set of periodic solenoidal fields in $[H^1(Y)]^3$ with zero mean; K is a penalty constant and the functional $E(\tilde{w}')$ is, as in [8]

$$E(\tilde{w}') = \int_Y \nabla \epsilon(\tilde{w}') : \nabla \epsilon(\tilde{w}') dy \quad (4.2)$$

where $\epsilon(\tilde{w}')$ is the solution of

$$-\Delta_y \epsilon = C^{-1} \nabla_y \cdot (\tilde{w}' \otimes \tilde{w}') + \nabla_y \pi', \quad \epsilon \text{ } y - \text{periodic}, \quad \langle \epsilon \rangle = 0 \quad (4.3)$$

and, in turn, π is the solution of

$$-\nabla_y \cdot (C \nabla_y \pi') = \nabla_y \cdot (\tilde{w}' \nabla_y \tilde{w}'), \quad \pi' \text{ } y - \text{periodic}, \quad \langle \pi \rangle = 0 \quad (4.4)$$

The equivalence of problems (2.39) and (4.1) is established in [8]. Here we add a penalty term for the treatment of the helicity.

For the resolution of (4.1) a discretization of W by the finite element method is made (see [6],[8] for details) and the discret problem

$$\min_{\tilde{w}'_h \in W_h - \{0\}} J(\tilde{w}'_h) \quad (4.5)$$

is minimized by a conjugate gradient algorithm. FFT methods are used to compute the solutions of the linear systems appearing in the CG algorithm (as a consequence of the periodic boundary conditions).

4.2. Numerical results.

All these computations have been made for matrices ∇a of the form (3.3) and depending on the parameters α and $\beta = h/q$.

The different tabulations made show us that all parity-imparity properties obtained in section 3.2 are verified numerically. We have taken $K = 1$ the penalty constant and the same initialization $\tilde{w}_h^{(0)}$ for all these experiments (we construct $\tilde{w}_h^{(0)}$ that verifies the condition (2.25)). The function $\tilde{w}_h^{(0)}$ has zero helicity.

To obtain $J(\tilde{w}_h^{(n)}) = 10^{-3}J(\tilde{w}_h^{(0)})$ it needs between 15 to 25 iterations according to the value $\|C - I\|$; we have taken 20 iterations in all cases, and the average CPU time per iteration was 90 secondes of CRAY-1. The convergence of $J(\tilde{w}_h^{(n)})$ towards zero is shown on figs. 1, 2.

Table 1 and 2 show the convergence to the expected helicity for differents values of α and h/q .

$\alpha = 0.01$		$\alpha = 0.1$	
h/q	h/q	h/q	h/q
<i>desired</i>	<i>obtained</i>	<i>desired</i>	<i>obtained</i>
0.01	0.0086	0.01	0.0109
0.1	0.1007	0.1	0.107
0.4	0.4009	0.4	0.3995
0.7	0.701	0.7	0.698
3.0	3.002	3.0	2.98
7.0	6.998	7.0	6.86

Table 1

Table 2

Thus, we get good approximations of the helicity for small values of α .

Tables 3, 4 and 5 show the approximations of the closure tensors for some values of α and β . Functions $\langle w_1 w_2 \rangle (\alpha, h/q)$ and $\sigma_{12}(\alpha, h/q)$ are represented in figures 3 and 4.

$\alpha = 0.01$									
h/q	$\langle w_1 w_1 \rangle$	$\langle w_2 w_2 \rangle$	$\langle w_3 w_3 \rangle$	$\langle w_1 w_2 \rangle$	σ_{11}	σ_{22}	σ_{33}	σ_{12}	ψ_q
.0086	.68984	.69575	.6144	.31202	.00205	.00278	.0038	.03736	9.635
.1007	.67372	.67754	.64873	.19065	.03483	.03449	.0314	-.01064	9.321
.4009	.68498	.68961	.6254	.21924	.13283	.13211	.13598	-.40163	19.182
.701	.6607	.6628	.6764	.0856	.2268	.2246	.2493	-.1169	21.068
3.002	.7101	.7152	.5747	.2232	.9960	.9963	1.01	.0148	23.733
6.998	.66643	.66524	.66673	-.01651	2.3325	2.3316	2.3343	-.0597	59.504

Table 3

$\alpha = 0.1$

h/q	$\langle w_1 w_1 \rangle$	$\langle w_2 w_2 \rangle$	$\langle w_3 w_3 \rangle$	$\langle w_1 w_2 \rangle$	σ_{11}	σ_{22}	σ_{33}	σ_{12}	ψ_q
.0109	.76134	.92914	.3095	.7827	-.0051	-.0018	.0178	.0179	7.043
.107	.75275	.87612	.3711	.6960	.0873	.0579	-.0378	-.1337	7.797
.3995	.05092	.04871	1.900	-.0139	.1174	.0995	.0183	-.0977	5.987
.698	.09398	.08878	1.8172	-.0344	.2034	.1774	.3172	-.1554	8.829
2.98	.54154	.53734	.9211	-.1017	.8828	.8636	1.2342	-.1870	20.48
6.86	.6509	.62342	.7257	-.1979	2.2368	2.129	2.5	-.6438	54.18

Table 4

$\alpha = 1.$

h/q	$\langle w_1 w_1 \rangle$	$\langle w_2 w_2 \rangle$	$\langle w_3 w_3 \rangle$	$\langle w_1 w_2 \rangle$	σ_{11}	σ_{22}	σ_{33}	σ_{12}	ψ_q
0.03	.6741	1.0963	.2296	.72061	.0115	.005	.0134	-.0249	11.203
.15	.4255	.5830	.9915	.2593	.3968	-.34	.096	-.0200	11.977
.6827	.06056	.02417	1.9153	-.032	.28172	-.0595	.3414	-.16131	5.410
2.498	.43293	.29705	1.33	-.28582	.81458	.4577	1.2257	-.51762	9.635
4.456	.8977	.46098	.64132	-.59304	1.5746	.7542	2.1275	-.9357	21.729

Table 5

By interpolating the values of these functions (for $\alpha = 0.01, 0.1$ and $h/q = 0.01, 0.1$) we get the coefficients of the expansions (3.58)-(3.69).

$$\langle w_1 w_1 \rangle = \frac{2}{3}q + 233\alpha^2 q - 16013\alpha^2 \frac{h^2}{q} \quad (4.6)$$

$$\langle w_2 w_2 \rangle = \frac{2}{3}q + 292.2\alpha^2 q - 18090\alpha^2 \frac{h^2}{q} \quad (4.7)$$

$$\langle w_3 w_3 \rangle = \frac{2}{3}q - 525.2\alpha^2 q + 34103\alpha^2 \frac{h^2}{q} \quad (4.8)$$

$$\langle w_1 w_2 \rangle = 31.3\alpha q - 1205.7\alpha \frac{h^2}{q} \quad (4.9)$$

$$\langle w_1 r_1 \rangle = \frac{1}{3}h - 73h\alpha^2 \quad (4.10)$$

$$\langle w_2 r_2 \rangle = \frac{1}{3}h - 50h\alpha^2 \quad (4.11)$$

$$\langle w_3 r_3 \rangle = \frac{1}{3}h + 123h\alpha^2 \quad (4.12)$$

$$\langle w_1 r_2 \rangle = 442\alpha h - 42022.5\alpha^3 h - 44254.4\alpha \frac{h^3}{q^2} \quad (4.13)$$

$$e_3 = 2 \langle (\pi + \frac{1}{2}|w|^2)(\nabla a \nabla_y) \times w \rangle_3 = (\frac{16}{5}\alpha - 194\alpha \frac{h^2}{q^2})q^{3/2} \quad (4.14)$$

$$d_3 = \langle (\pi + \frac{1}{2}|w|^2)w \rangle_3 = (-60\alpha h + 5664\alpha^3 h/q + 6072\alpha \frac{h^3}{q^3})q^{3/2} \quad (4.15)$$

$$\psi_q = \langle (\nabla a \nabla_y)w : (\nabla a \nabla_y)w \rangle = 9.6q + 360\alpha^2 q - 13\frac{h^2}{q} \quad (4.16)$$

$$\psi_h = 2 \langle (\nabla a \nabla_y)w : (\nabla a \nabla_y)(\nabla a \nabla_y) \times w \rangle = 1.6h + 138\alpha^2 h \quad (4.17)$$

GENERAL CONCLUSION.

We can summarize the results for the 2-D case by the following :

Let

$$C = \nabla a^T \nabla a = \begin{pmatrix} b & c & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad bd - c^2 = 1, \quad b > 0, \quad d > 0, \quad \alpha^2 = b + d - 2$$

We have then

- Functions $\mu_1, \mu_2, \varsigma_1$ and ς_2 defined in (3.92),(3.156) are given by

$$\left. \begin{aligned} \mu_1 &= \frac{1}{q}(\langle w_1 w_1 \rangle + 2\alpha \langle w_1 w_2 \rangle + (\alpha^2 + 1) \langle w_2 w_2 \rangle) \\ \mu_2 &= \frac{1}{q} \langle w_3 w_3 \rangle \end{aligned} \right\} \quad (4.19)$$

$$\left. \begin{aligned} \varsigma_1 &= \frac{1}{q}(\langle w_1 r_1 \rangle + 2\alpha \langle w_1 r_2 \rangle + (\alpha^2 + 1) \langle w_2 r_2 \rangle) \\ \varsigma_2 &= \frac{1}{q} \langle w_3 r_3 \rangle \end{aligned} \right\} \quad (4.18)$$

using the expansions (4.6)-(4.13), we have an approximation for functions \tilde{r}_o and \tilde{s}_o given in (3.153),(3.155) as follows

$$\tilde{r}_o(\alpha, \frac{h}{q}) = \frac{1}{\alpha^2 + 4}(-124.5 + 59.2\alpha^2 + 4823\frac{h^2}{q^2} + 2077\alpha^2\frac{h^2}{q^2}) \quad (4.20)$$

$$\tilde{s}_o(\alpha, \frac{h}{q}) = \frac{h/q}{\alpha^2 + 4}(-1767.3 + 168090\alpha^2 + 88508.8\frac{h^2}{q^2}) \quad (4.21)$$

These expressions eliminates the singularity of \tilde{r}_o and \tilde{s}_o in $\alpha = 0$ and show that

$$\lim_{\alpha \rightarrow 0} \tilde{r}_o \cong -31.125 + 1205.7\frac{h^2}{q^2} \quad (4.22)$$

$$\lim_{\alpha \rightarrow 0} \tilde{s}_o \cong -\frac{h}{q}(441.825 - 22127\frac{h^2}{q^2}) \quad (4.23)$$

Thus, the Reynolds stress tensors verify

$$\nabla \cdot R = -\nabla \cdot (q \tilde{r}_o (\sqrt{\text{tr}(\nabla a^T \nabla a)} - 3, h/q) \nabla a \nabla a^T) \quad (4.24)$$

$$R : \nabla u = -q \tilde{r}_o (\sqrt{\text{tr}(\nabla a^T \nabla a)} - 3, h/q) \nabla a \nabla a^T : \nabla u \quad (4.25)$$

$$S : \nabla u = -q\tilde{s}_o(\sqrt{tr(\nabla a^T \nabla a)} - 3, h/q) \nabla a \nabla a^T : \nabla u \quad (4.26)$$

and (4.20), (4.21) are reasonable approximations when α and h/q are small enough.

Figures 5, 6, 7, show a certain regularity of functions μ_1, μ_2 and by \tilde{r}_0 for great values of h/q .

-Vector functions V_q and V_h vanish

$$\nabla \cdot V_q \equiv 0 \quad (4.27)$$

$$\nabla \cdot V_h \equiv 0 \quad (4.28)$$

Tensor S is proportionnal to the ratio h/q . This property together with the equation (4.28) and the initial condition for the helicity $h(x, 0) = 0$ yield

$$h(x, t) = 0 \quad \forall (x, t) \in \Omega \times [0, T] \quad (4.29)$$

$$\psi_q \cong 9.6q + 360\alpha^2 q - 13\frac{h^2}{q} \quad (4.30)$$

$$\psi_h \cong 1.6h + 138\alpha^2 h \quad (4.31)$$

In 3-D, we have the approximations for functions $\tilde{\rho}_1, \tilde{\rho}_2$ in (3.159)

$$\tilde{\rho}_1(\alpha, \frac{h}{q}) \cong 1661.4 + 24.6\alpha + 4573\alpha^2 - 105772\frac{h^2}{q^2} - \frac{8.2\alpha}{\sqrt{4 + \alpha^2}} \quad (4.32)$$

$$\tilde{\rho}_2(\alpha, \frac{h}{q}) \cong -827 - 8.2\alpha - 1200\alpha^2 - 52694\frac{h^2}{q^2} + 4.1\alpha\sqrt{4 + \alpha^2} \quad (4.33)$$

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Caption to figures.

Figures 1,2,3 : Descent of the cost $J(\tilde{w}_h^n)$ for different values of the ratio h/q and α .

In all cases, the maximum cost corresponds to the maximum h/q . The descent is rapid till the 8th iteration; from the 9th iteration the descent is almost linear.

Figures 4,5,6 : Function $\langle w_1 w_2 \rangle(\alpha, h/q)$.

Figures 7,8 : Function $\sigma_{12}(\alpha, h/q)$.

Figures 9,11,12 Function $\mu_1(\alpha, h/q) = \langle \tilde{w}'_1 \tilde{w}'_1 \rangle + \langle \tilde{w}'_2 \tilde{w}'_2 \rangle$

Visualization of the curves $h/q \rightarrow \mu_1(\alpha, h/q)$ for some values of α .

Notice that $\mu_1(0, h/q) = 4/3$ for all h/q and that $\mu_1(.001, h/q)$ stays close to this value. When α increases, the difference $\mu_1(\alpha, h/q) - 4/3$ also increases for $|h/q| < \gamma^*(\alpha)$. Numerical computations seem to show

$$\lim_{h/q \rightarrow \infty} \mu_1(\alpha, h/q) = \frac{4}{3}$$

Figure 10 : Function $\mu_2(\alpha, h/q) = \langle \tilde{w}'_3 \tilde{w}'_3 \rangle$

Visualization of the curves $h/q \rightarrow \mu_2(\alpha, h/q)$ $\alpha = 0, 0.01, 0.1, 1$. We can say the same comments made for $\mu_1(\alpha, h/q)$:

- when $\alpha = 0$, we have a constant function

$$\mu_2(0, h/q) = 2/3$$

- numerically

$$\lim_{h/q \rightarrow \infty} \mu_2(\alpha, h/q) = \frac{2}{3}$$

Comparing the figure 5 to figure 6 we notice that

- . $\mu_1(\alpha, h/q)$ convex $\Leftrightarrow \mu_2(\alpha, h/q)$ concave
- . $\mu_1(\alpha, h/q) \geq 2/3 \Leftrightarrow \mu_2(\alpha, h/q) \leq 1/3$.

Figure 13 : Function $\tilde{r}_0(\alpha, h/q)$

Visualization of the curves

$$h/q \rightarrow \tilde{r}_0(0.01, h/q)$$

$$h/q \rightarrow \tilde{r}_0(0.1, h/q)$$

$$h/q \rightarrow \tilde{r}_0(1., h/q)$$

We find

$$\tilde{r}_0(0.01, 0.0086) = -31.20$$

which is a very good approximation for the value obtained in (4.22).

We also notice a regular behaviour of \tilde{r}_0 when $h/q \gg 1$; the convergence of μ_1 and μ_2 implies the convergence of \tilde{r}_0 , i.e.

$$\lim_{h/q \rightarrow \infty} \tilde{r}_0(\alpha, h/q) = \frac{4}{3(\alpha^2 + 4)}$$

Figure 14 : Function $\psi_q = <(\nabla a \nabla_y)w : (\nabla a \nabla_y)w>$ (dissipation)
Visualization of the curves

$$\begin{aligned} h/q &\rightarrow \psi_q(0.01, h/q) \\ h/q &\rightarrow \psi_q(0.1, h/q) \\ h/q &\rightarrow \psi_q(1., h/q) \end{aligned}$$

Our numerical results show that ψ_q decreases on certain (small) interval $(\delta_1(\alpha), \delta_2(\alpha))$. However, on $h/q > \delta_2(\alpha)$, ψ_q increases and

$$\lim_{h/q \rightarrow \infty} \psi_q = +\infty$$

On the other hand, the tabulations show that ψ_q decreases with α (for h/q great enough).

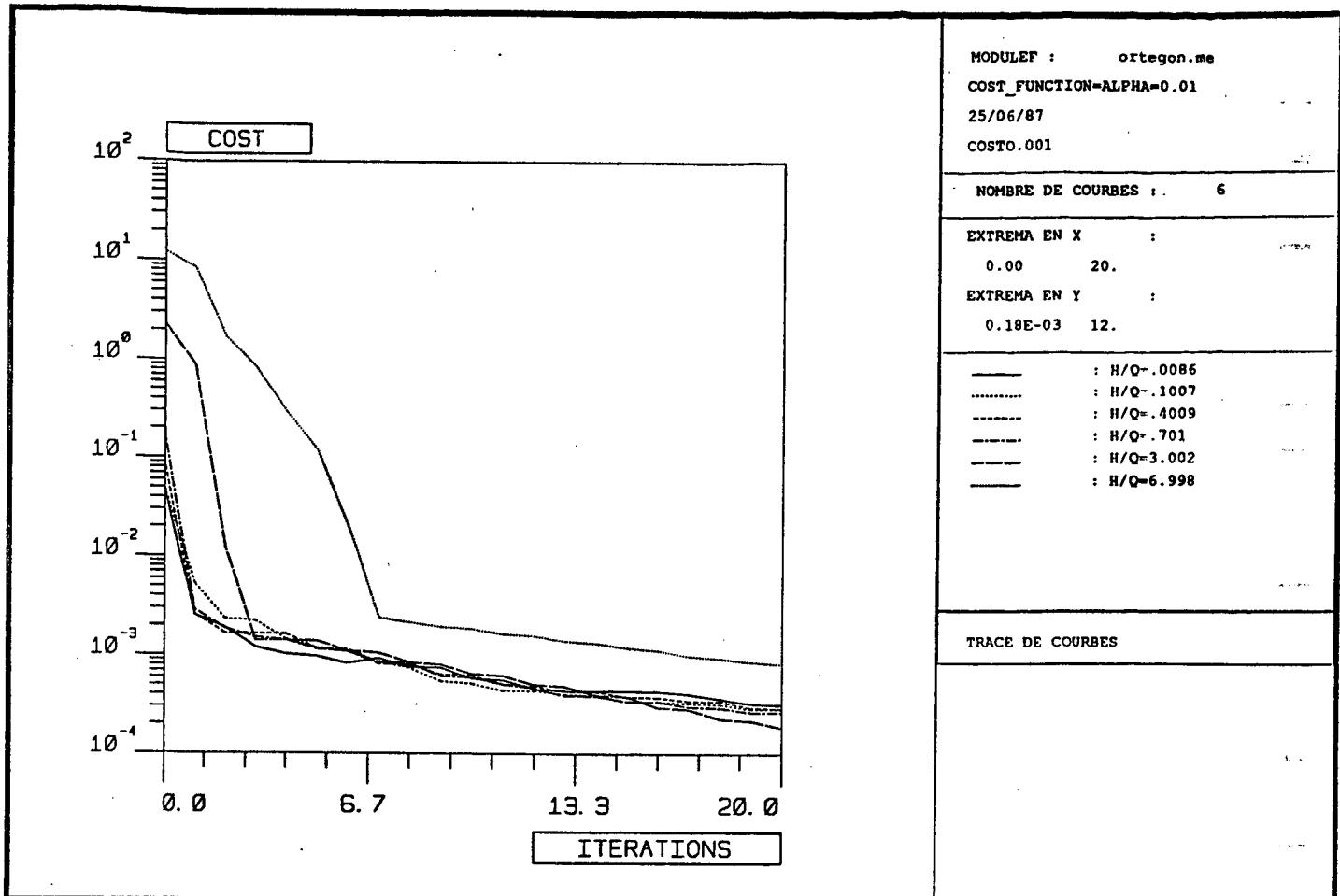
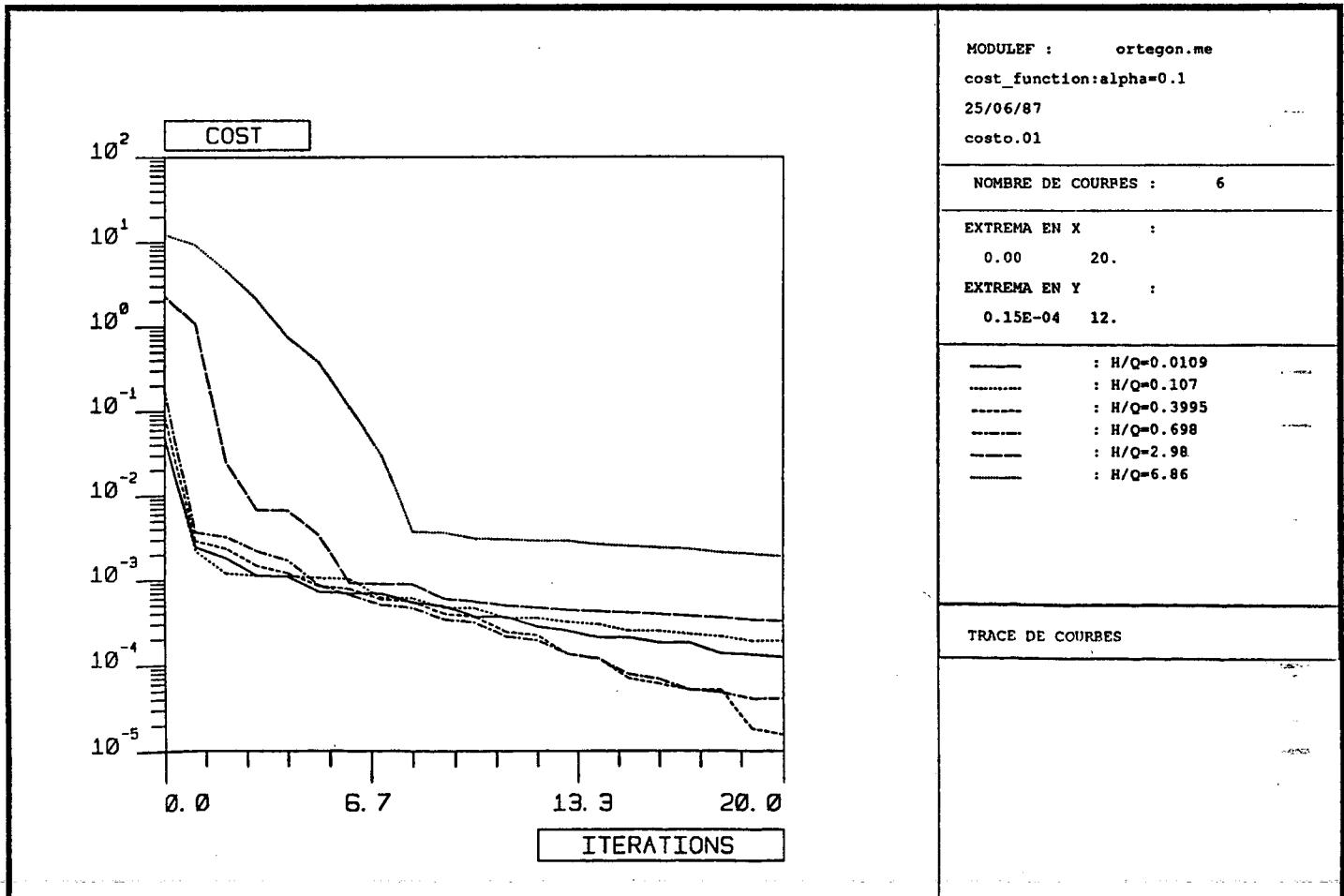


Figure 1-2



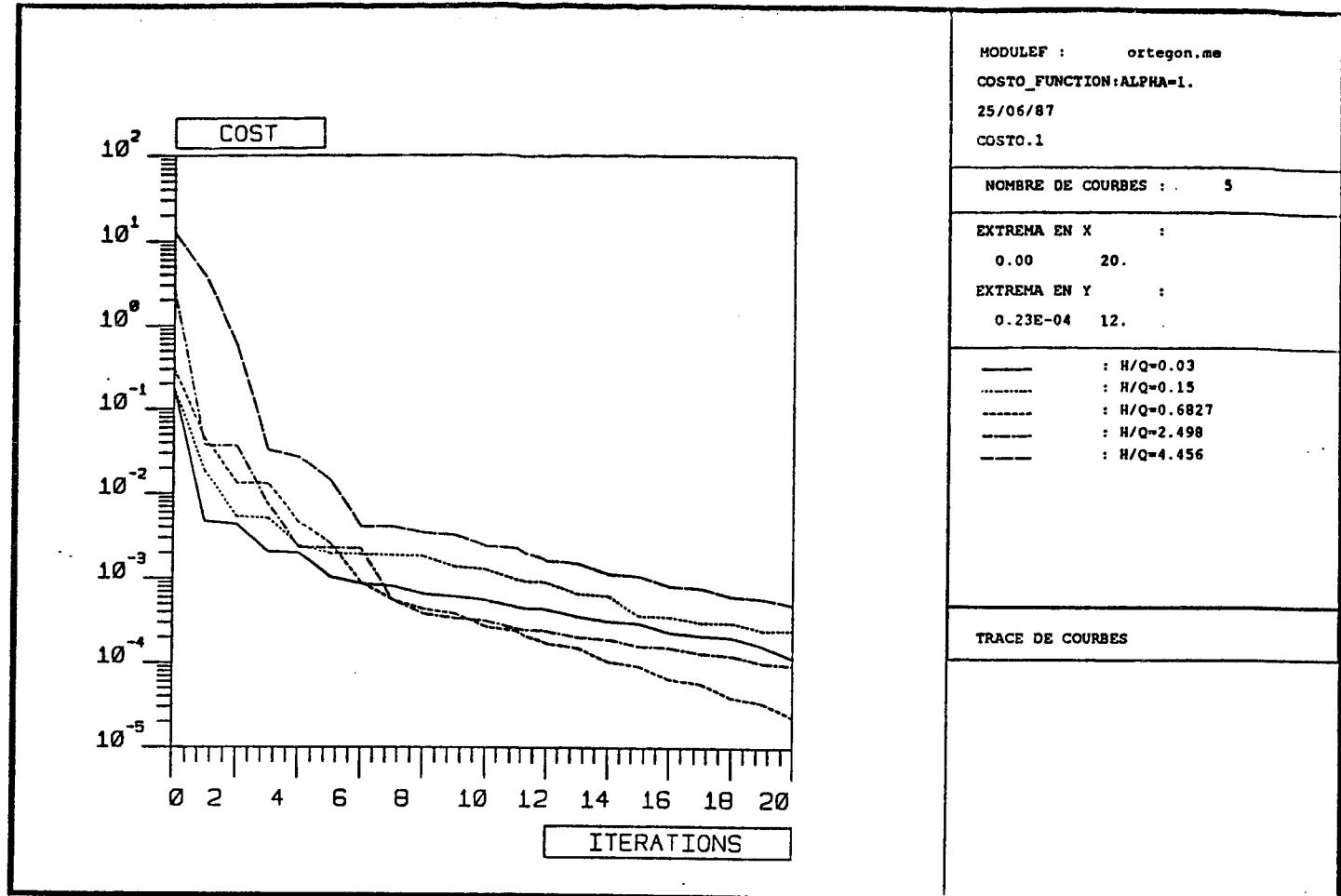
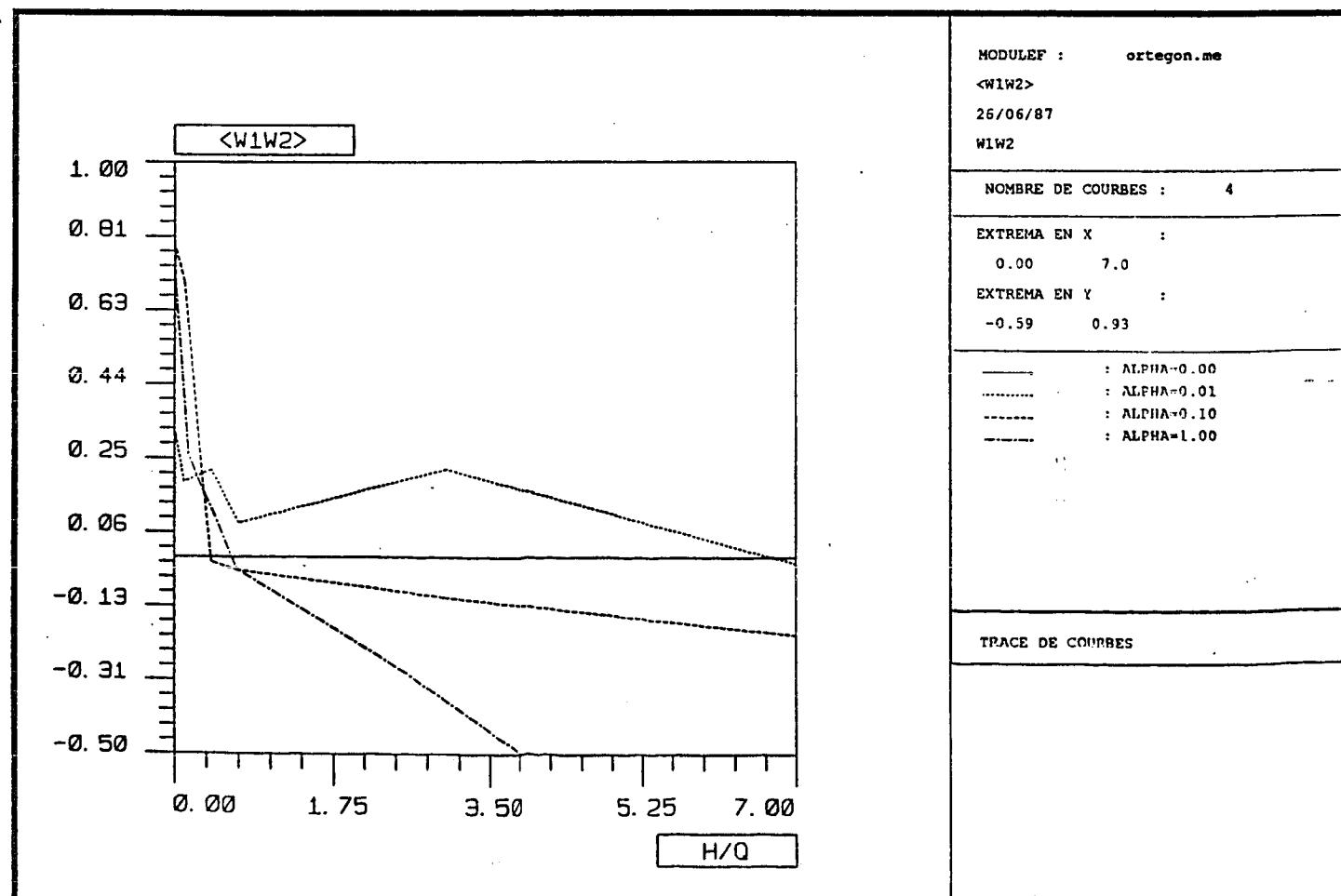


Figure 3-4



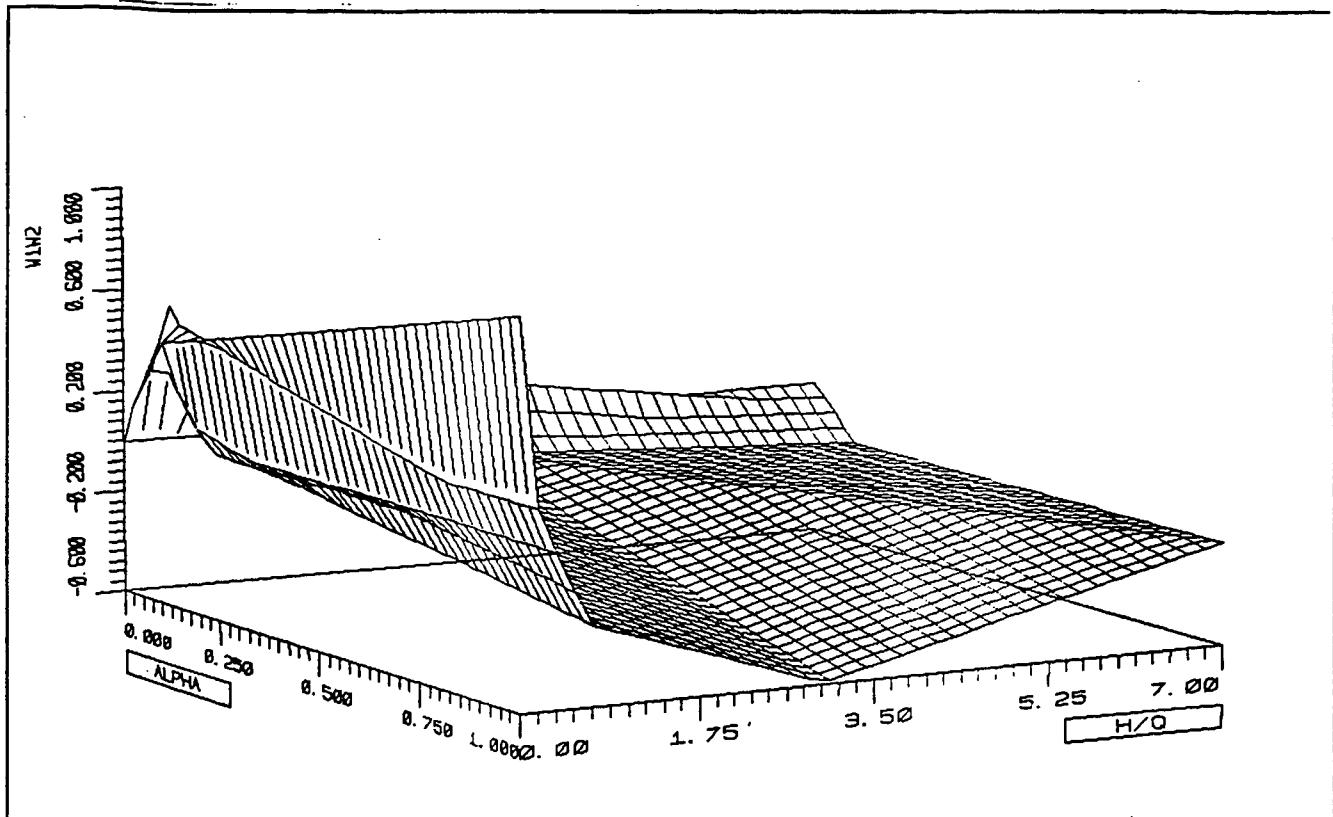


Figure 5

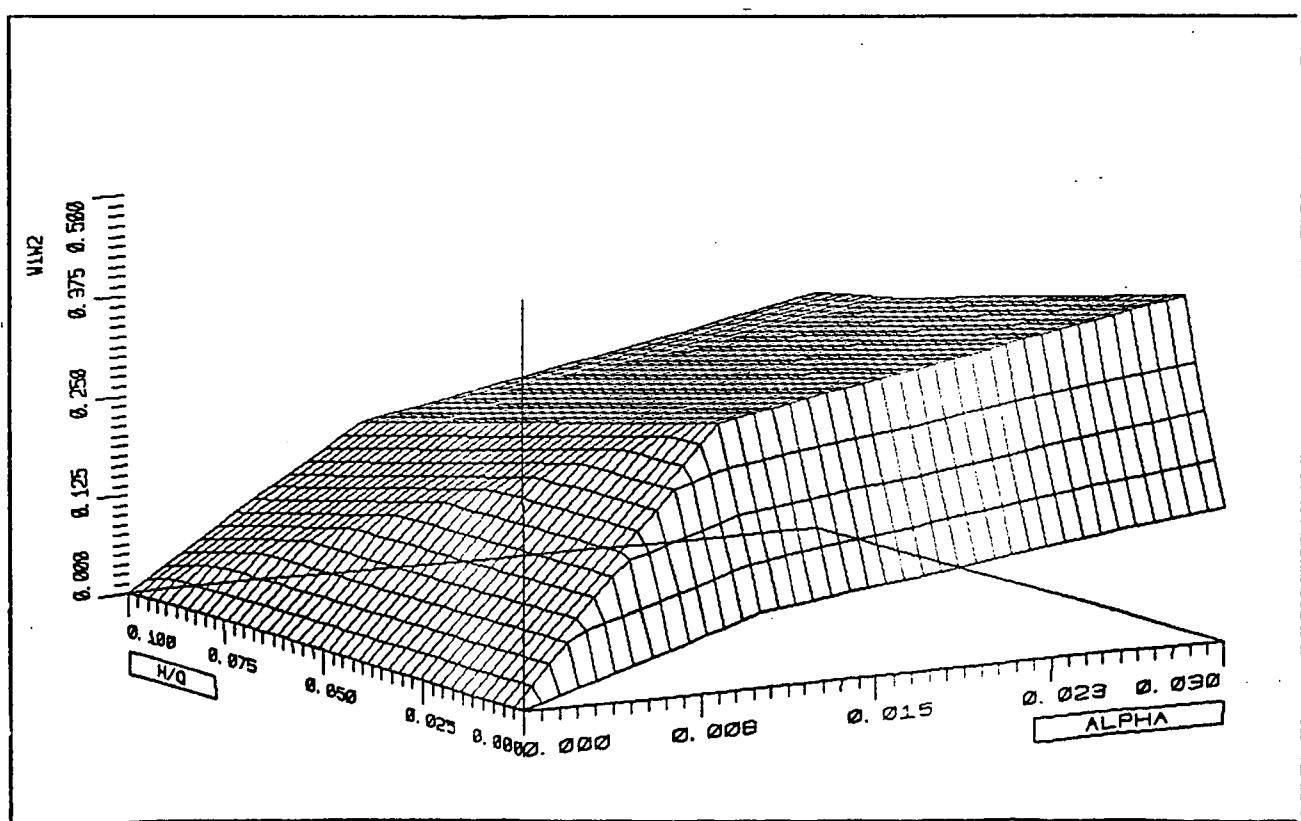


Figure 6

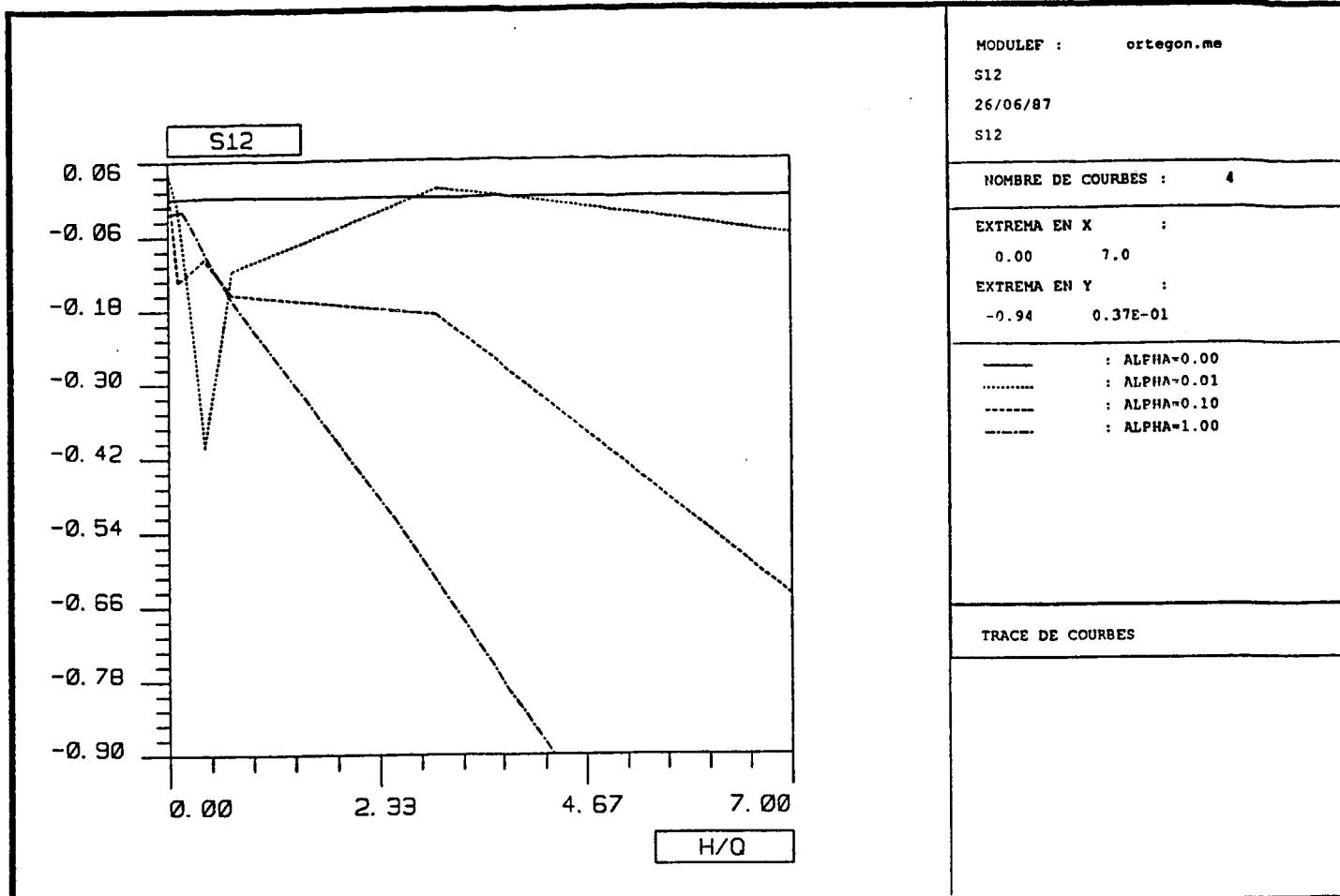


Figure 7

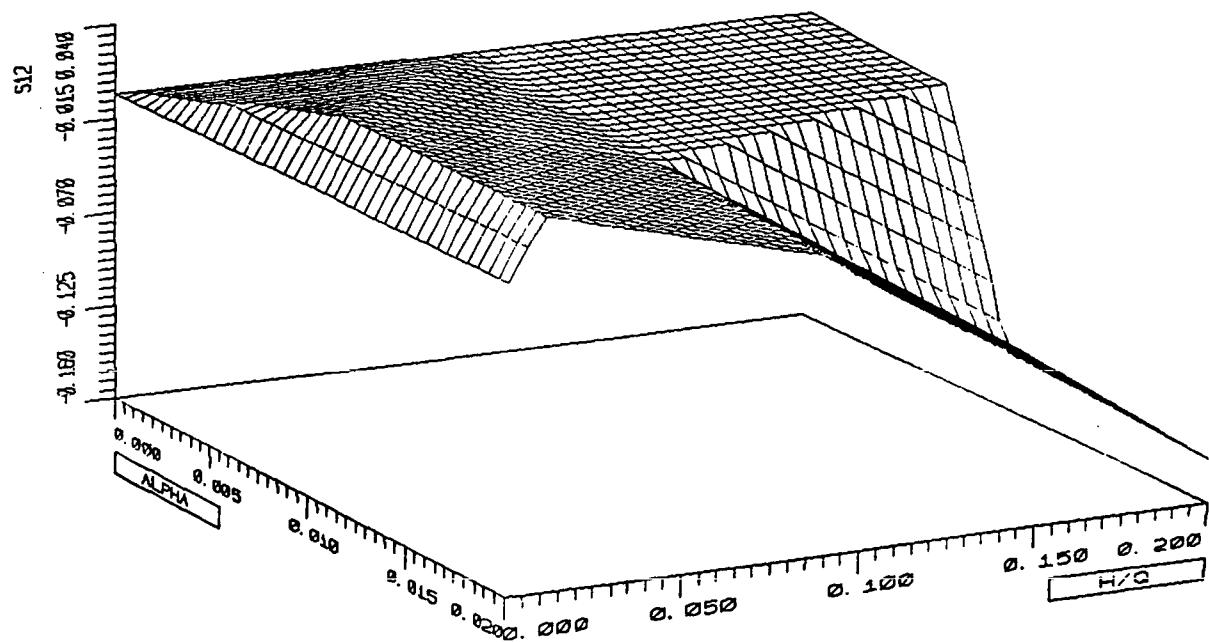


Figure 8

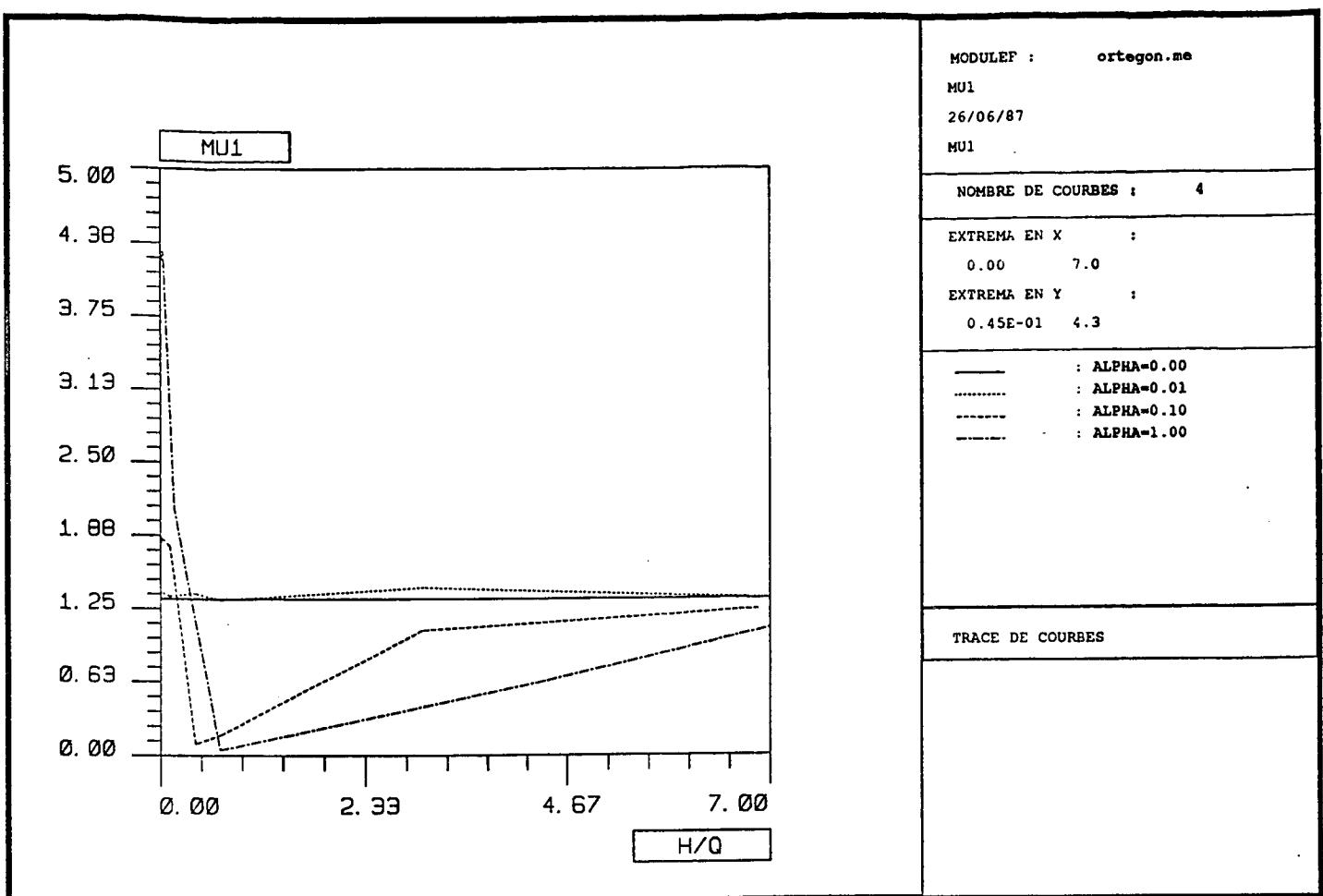
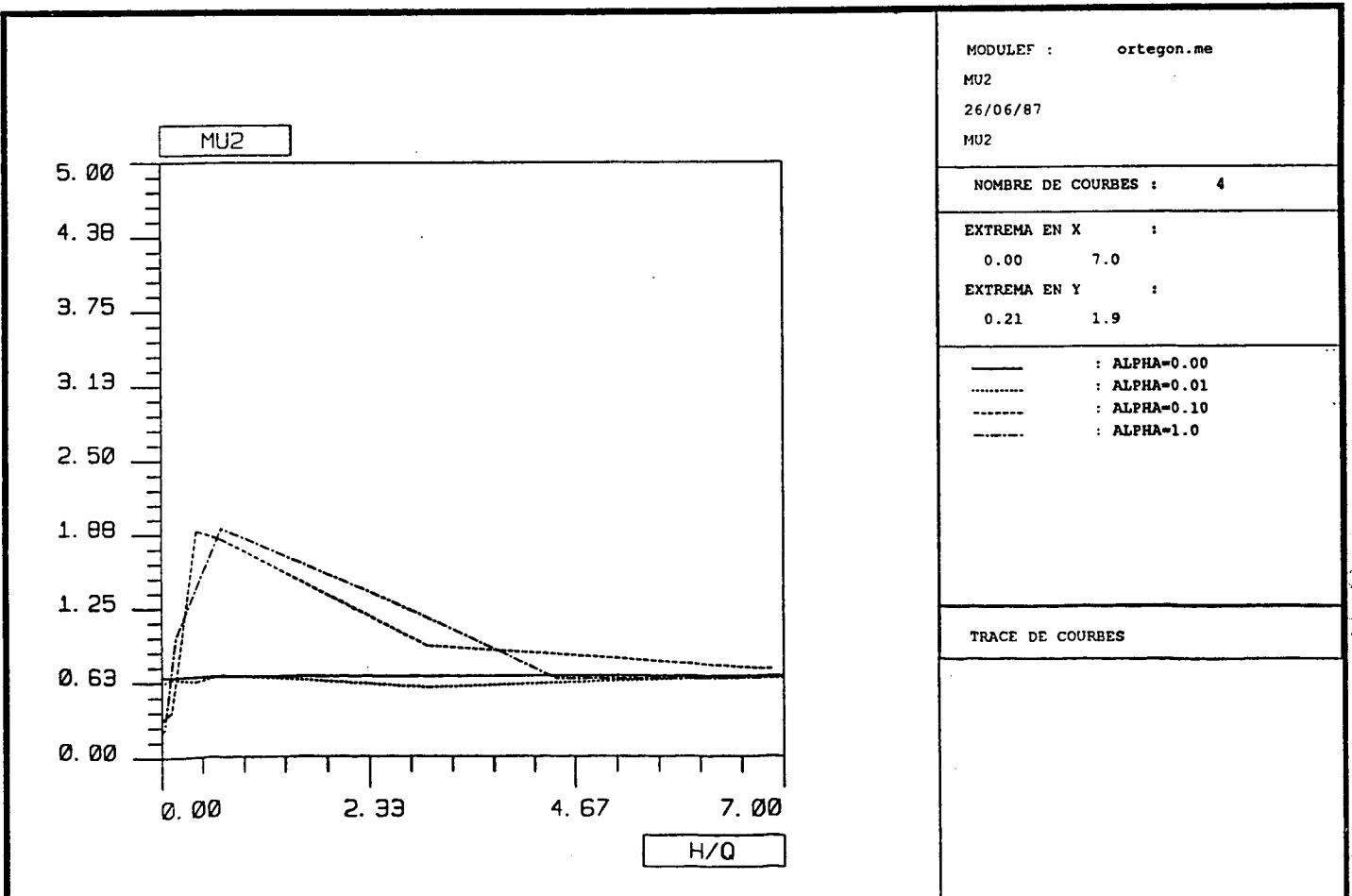


Figure 9-10



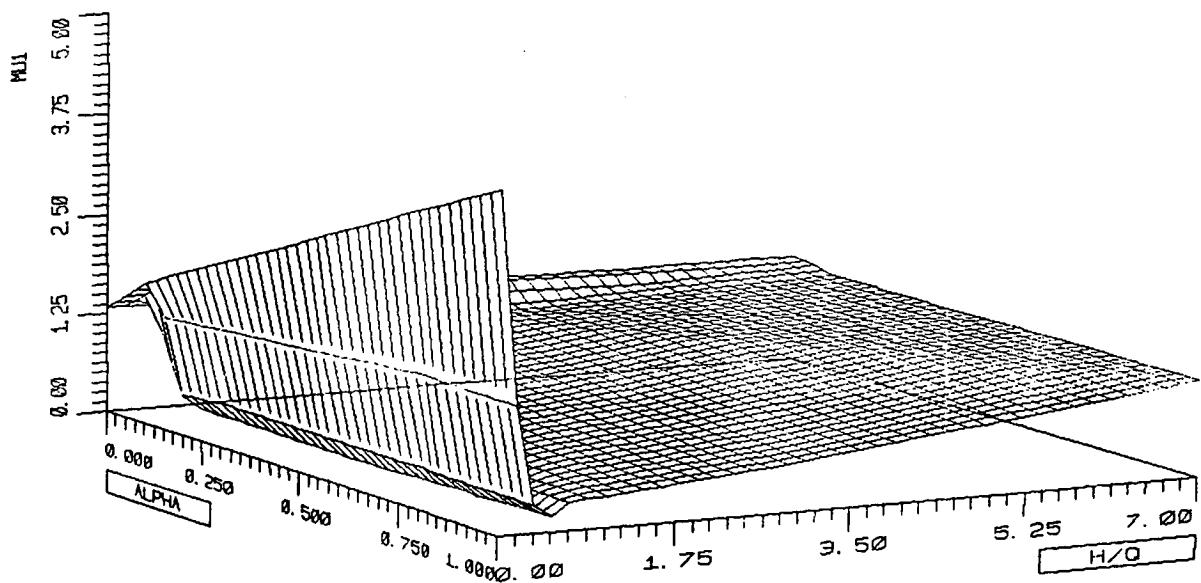


Figure 11

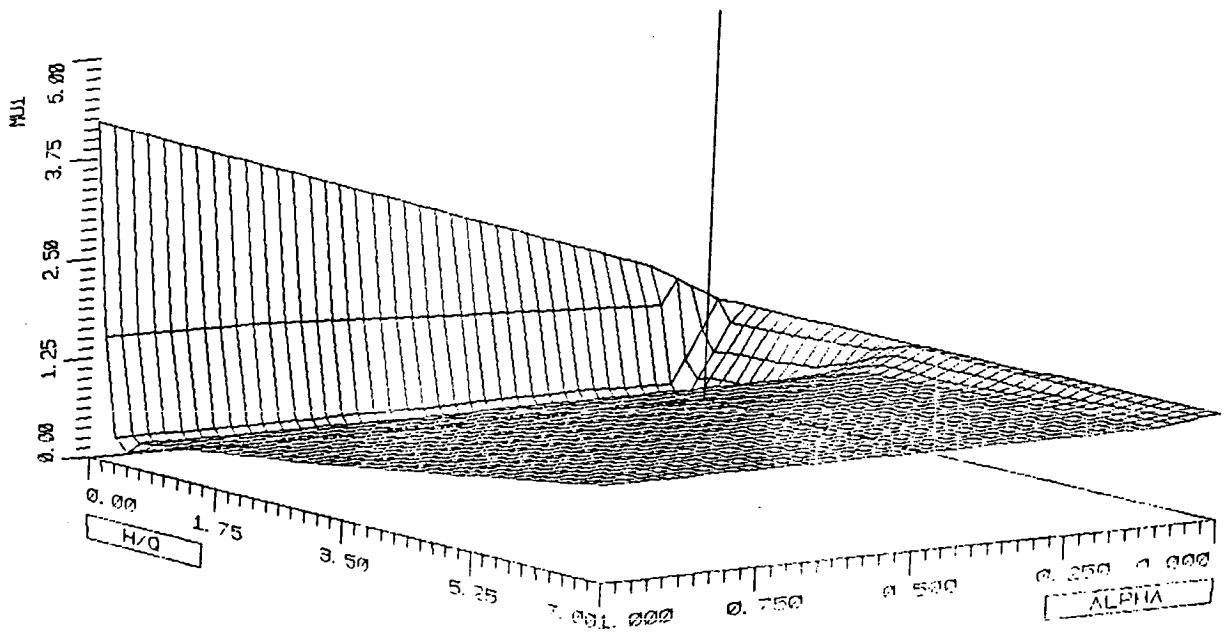


Figure 12

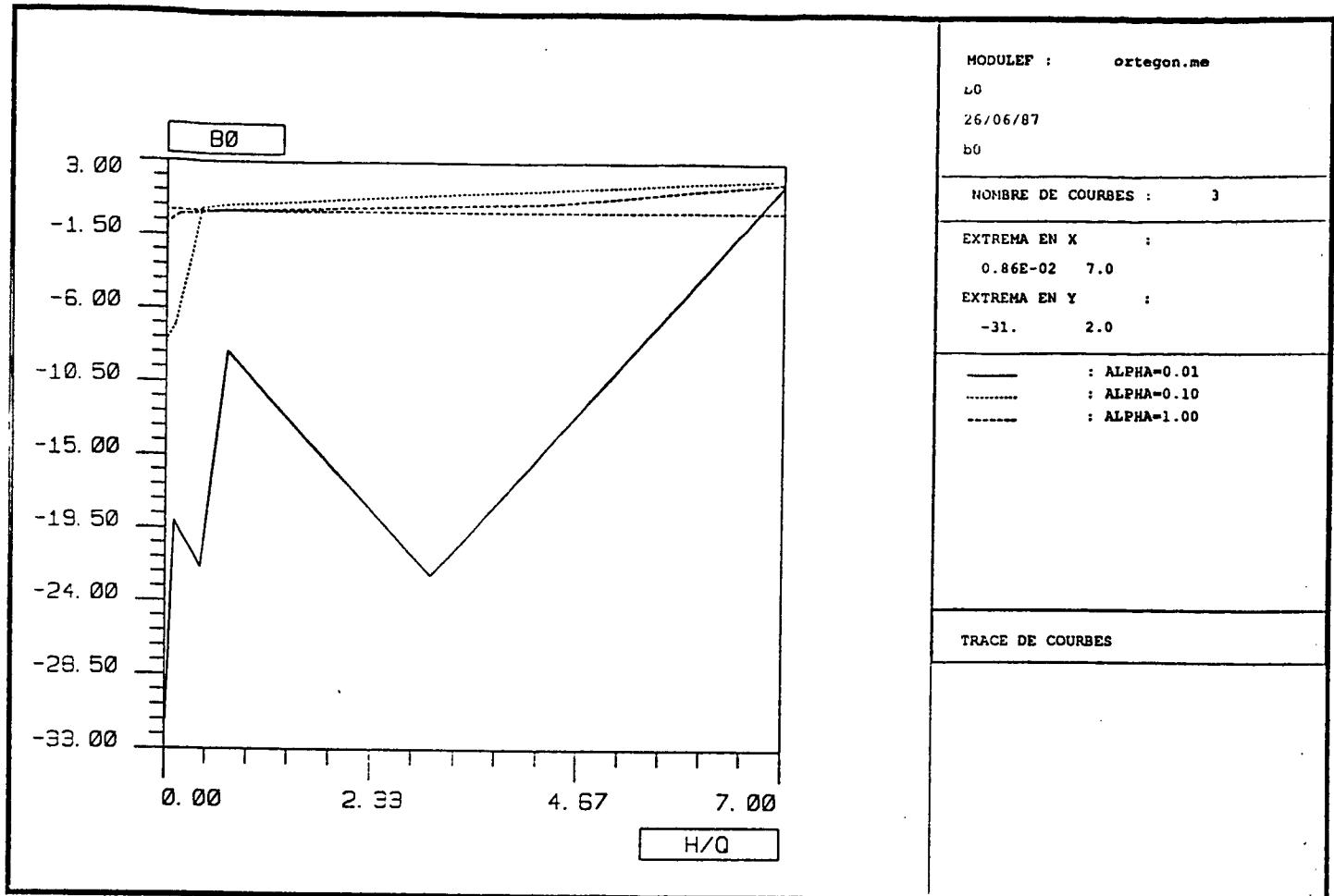
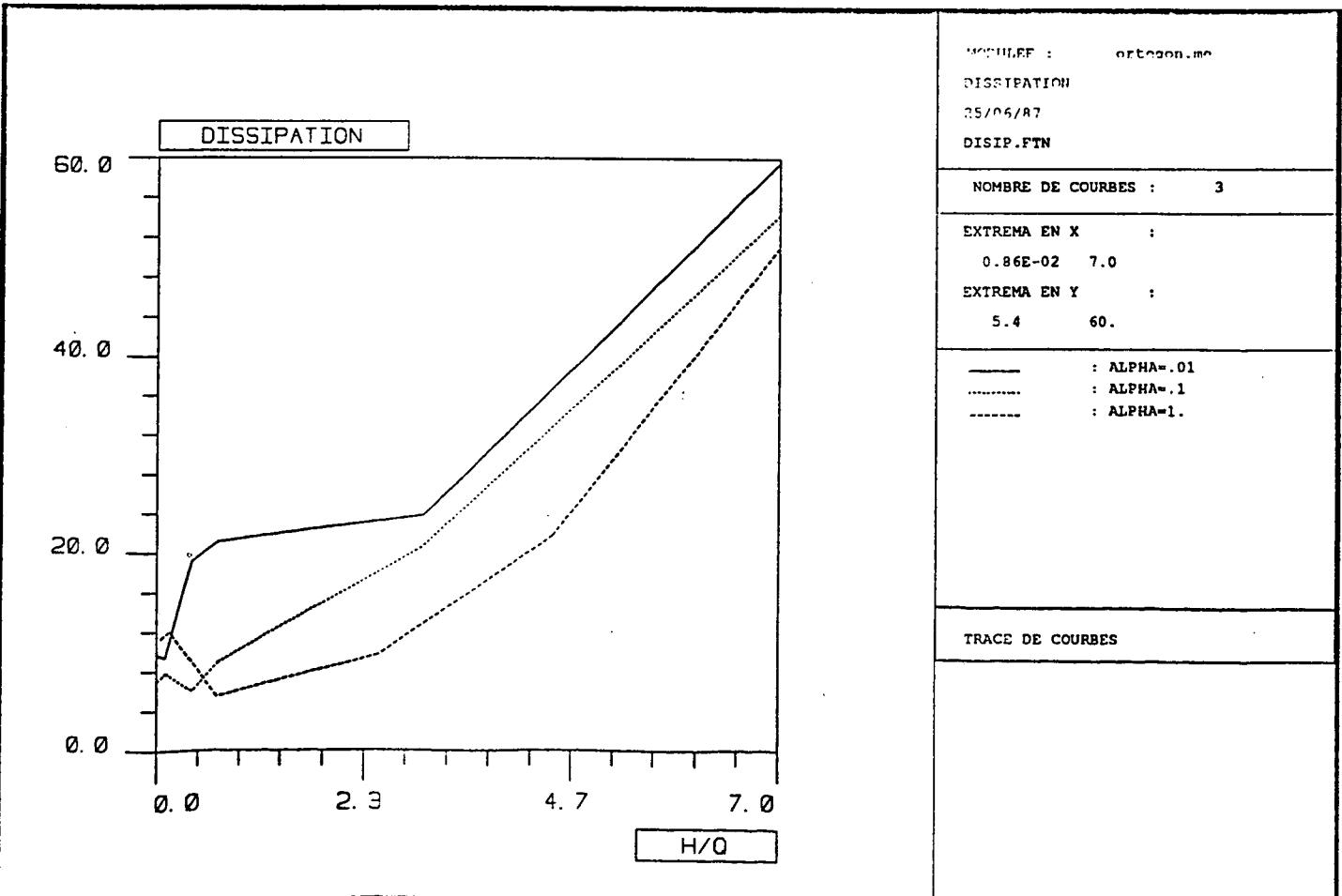


Figure 13-14



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