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abstract : This paper is an attempt to define effective behaviour algebras and effective (i.e. provable) behaviour equivalences. It offers thereby a defense and illustration of the thesis asserting the identity between effective infinitary objects and $\sum_1^1$ sets (and transformations). In this framework, an effective equivalence of indiscernibility is defined, and its logical and topological properties are investigated.

ALGEBRES EFFECTIVES DE COMPORTEMENTS

Résumé : Cet article tente de définir des algèbres effectives de comportements, et des équivalences effectives (c.a.d. prouvables) sur ces algèbres. Il offre par là même une défense et une illustration de la thèse selon laquelle les objets infinitaires effectifs sont essentiellement les ensembles $\sum_1^1$ de $\mathbb{N} \to \mathbb{N}$ (et leurs transformations). Dans ce cadre, nous introduisons une équivalence effective d'indiscernabilité, et étudions ses propriétés logiques et topologiques.

I Introduction :

Recent work on behaviour algebras such as CCS,TCSP,ACP...

has renewed the study of effective transitions systems (t.s

for short) and their equivalences [1,2,3,4]. We feel that these

equivalences are either too liberal or too strict with respect

to infinite behaviours of t.s . Indeed, equivalences really

accounting for infinite behaviours fail in the following sense:

they have no complete proof systems in first order logic, even

with recursive infinitary rules[5], such as complete induction .

Our goal is to fill in this gap .

By an effective family of t.s [7], we mean a quintuple

(S,ψ_S,A,ψ_A,f) where ψ_S : $S \longrightarrow \mathbb{N}$, resp. ψ_A : $A \longrightarrow \mathbb{N}$ are

bijective codings for states, resp. actions , and f is a

recursive function enumerating triples $< \psi_S(s),\psi_A(a),\psi_S(s')>$,

standing for transitions

$$s \xrightarrow{a} s'$$. A transition system of

the family is then given by an initial state s_0.

By an effective algebra of t.s, we mean a family of t.s

whose set of states S is an algebra of terms (the operators of

the algebra are then induced by the corresponding operators on

terms) . In such algebras, one may introduce computations of a

t.s as sequences of transitions

$$(s_i \xrightarrow{\ a_i\ } s_{i+1}) \ , \ i \leqslant \gamma$$

between terms, where $\gamma \in \mathbb{N} \cup \{\omega\}$. Clearly, for any term of the algebra, identified with the corresponding transition system, the set of computations of that term may be represented as a Σ_1^1 set of $(\mathbb{N} \rightarrow \mathbb{N})$, i.e a set of the form [6]

$$\{f/(\exists g)(\forall i)R(f,g,i)\} \text{ where } R \text{ is recursive relation } .$$

Further, the operators of the algebra induce operators on Σ_1^1 sets, recursive on their Σ_1^1 codes . The same holds for the alternative behaviour algebra, obtained by erasing states from computations and still holds if some actions are considered as invisible and thus are erased from traces (e.g τ in CCS) .

Conversely, we know effective algebras of t.s, namely MEIJE resp. CCS, in which any Σ_1^1 set of $(\mathbb{N} \rightarrow \mathbb{N})$ represents the behaviour (complete traces resp. visible traces) of a term, given from the code of the set by a recursive procedure (for MEIJE, this results mainly from the two theorems by de Simone queted in [7]) .

This comes as a confirmation of the following thesis :

* Effective infinitary objects are (at most) Σ_1^1
* Effective operations on effectived infinitary objects are Σ_1^1 transformations, i.e they are union additive extensions of relations whose graph is Σ_1^1 .

On behalf of this thesis, we exclude from effective behaviour

algebras the factor algebras (of t.s) obtained by greatest bisimulations [8]: not only classes of these equivalences are Π_2^1 sets of terms, but the quotient of an effective transition system by its greatest bisimulation is not always an effective t.s .

In the sequel, we study only linear behaviour of t.s : such behaviours are essentially the Σ_1^1 sets (of $(\mathbb{N} \rightarrow \mathbb{N})$) .

It appears from the last remark that the existence of complete proof systems for behavioural equivalences of t.s reduces to the corresponding problem for equivalences between Σ_1^1 sets .

Traditionally, proof systems used in Computer Science are defined within first-order logic with recursive infinitary rules (these rules are generally used for complete induction on $\mathbb{N}$ - algorithmic logic- or on more structured domains - Scott's complete induction). We claim that this logical framework is powerful enough to formalize infinitary properties which are not finitely approximable and thus not inductive in the usual sense . Although no complete axiomatization for equality between Σ_1^1 sets can be obtained in this logical fragment [5], the problem is trivial in any fragment of second-order logic . We introduce in the next section a coarser equivalence between Σ_1^1 sets, which is even a congruence for Σ_1^1 transformations, and show that it has a complete proof system

in RL (first order logic with recursive infinitary rules) .

Furthermore, we believe that this equivalence is the finest

equivalence between Σ_1^1 sets provable in RL .

II Indiscernibility as a provable congruence on Σ_1^1 sets :

1) Σ_1^1 sets and transformations :

In the sequel, f,g... range over $\mathcal{N} = (\mathbb{N} \to \mathbb{N})$; $A,B,...$ over $P(\mathcal{N})$.

Similarly, $i,j,...$ range over $\mathbb{N}$; $a,b,...$ over $P(\mathbb{N})$.

A subset A of $\mathcal{N}$ (resp. a subset a of $\mathbb{N}$) is Σ_1^1 iff there

exists a recursive relation R such that

$$A = \{f / (\exists g)(\forall i) R(f,g,i)\}$$

(resp. $a = \{k / (\exists g)(\forall i) R(k,g,i)\}$) .

A Σ_1^1 transformation Φ on $P(\mathcal{N})$ is the union additive extension

of a function $\varphi: \mathcal{N} \longrightarrow P(\mathcal{N})$ such that

$$\{(g,f) / g \in \varphi f\} \text{ is } \Sigma_1^1 .$$

It is easy to see that Σ_1^1 transformations preserve Σ_1^1 sets .

2) Σ_1^1 indiscernibility :

For A,B Σ_1^1 subsets of $\mathcal{N}$, we say that A and B are Σ_1^1

indiscernible ($A \sim_\Sigma B$) iff

for any Σ_1^1 subset C of $\mathcal{N}$, $A \cap C = \emptyset \Longleftrightarrow B \cap C = \emptyset$.

Clearly, $\sim_\Sigma$ is an equivalence, and moreover a congruence with

respect to Σ_1^1 transformations . This follows from two immediate

properties :

(i) The class Σ_1^1 is closed under intersection

(ii) For any Σ_1^1 transformation Φ,

$$\Phi^{-1}(A) = \{f / (\exists g \in A) g \in \varphi(f)\} \text{ is a } \Sigma_1^1 \text{ transformation }.$$

3) Recursive logic :

We consider Ω, a denumerable and possibly multisorted

algebra of terms with variables, and $L(\Omega)$, a first order

language constructed in the usual way from Ω and some finite

family of relational symbols .

A proof system Σ with recursive infinitary rules is a triple

$\langle AX, R, IR \rangle$ where :

* AX is a finite consistent set of formulas from $L(\Omega)$ (the

axioms) .

* R is a set of finitary inference rules, written $A \Leftarrow A_1, \ldots, A_n$

with $A, A_1, \ldots, A_n$ belonging to $L(\Omega)$.

* IR is a set of recursive infinitary rules, written

$A \Leftarrow A_1, \ldots, A_i, \ldots$ each of which is associated with an effective

procedure P such that $P(i) = A_i$ for all i in $\mathbb{N}$.

Proofs in Σ may be defined inductively as finite-path trees

whose terminal roots are the concluding stepd of proof .

A formula φ is provable in Σ if :

* φ is an instance of a provable formula ψ ,

* φ belongs to AX ,

* $\varphi = \sigma(A)$ and there are substitutions $\sigma_1, \ldots, \sigma_n$; $\tau_1, \ldots, \tau_n$

such that for all i , $\sigma = \sigma_i \tau_i$ and $\tau_i(A_i)$ is provable in Σ and

$A \Leftarrow A_1 , \ldots , A_n$ belongs to R ,

    * $\varphi = \sigma(A)$ and there are substitutions σ_i , τ_i ($i \in \mathbb{N}$) such that for all i , $\sigma = \sigma_i \tau_i$ and $\tau_i(A_i)$ is provable in Σ and $A \Leftarrow A_1 , \ldots , A_i , \ldots$ belongs to IR .

4) <u>A theorem</u> :

<u>Theorem</u> : There exists a complete proof system for $\sim_\Sigma$ in RL .

5) <u>Complements</u> [6] :

    * <u>Kleene normal form theorem for Σ_1^1 sets</u> . For $k \geqslant 0$ and $l \geqslant 0$, there exist recursive predicates

$$T'_{k+1,1}(z, f_1, \ldots, f_k, g, x_1, \ldots, x_1, w)$$

where z, $x_1, \ldots, x_1, w$ are integer variables, such that for any Σ_1^1 set R, $R \subseteq \mathcal{N}^k \times \mathbb{N}^l$ there is a z such that

  $R = \{\langle f_1, \ldots, f_k, x_1, \ldots, x_1 \rangle / (\exists g)(\forall w) \text{ not } T'_{k+1}(z, f, g, x, w)\}$

z is then called a Σ_1^1 index for R .

    * <u>Π_1^1 , Δ_1^1</u> .

A subset A of $\mathcal{N}$ (resp. a of $\mathbb{N}$) is a Π_1^1 set iff $\mathcal{N} - A$ is a Σ_1^1 set (resp. $\mathbb{N} - a$ is Σ_1^1 set) .

A subset A of $\mathcal{N}$ (resp. a of $\mathbb{N}$) is a Δ_1^1 set iff it is both a Σ_1^1 and a Π_1^1 set .

    * <u>Π_1^1-completeness</u> .

A set A is one-one reducible to a set B ($A \leqslant_1 B$) if there is a one-one recursive function f such that

    $(\forall x) [x \in A \iff fx \in B]$.

A set A is Π_1^1 complete if and only if $B \leqslant_1 A$ for every Π_1^1 set B.

The set T of finite path trees is Π_1^1 complete (and thus the same holds for proof trees in RL) .

6) Hints for the proof of the theorem :

<u>Notations</u> : $\Sigma_f(x)$ is the Σ_1^1 subset of $\mathcal{N}$ of Σ_1^1 index x .

$\Sigma_N(x)$ is the Σ_1^1 subset of $\mathbb{N}$ of Σ_1^1 index x .

<u>Lemma 1</u> : Problems $\Sigma_f(x) \sim_\Sigma \Sigma_f(y)$ and $\Sigma_N(x') = \Sigma_N(y')$ are

effectively inter-reducible .

<u>Notation</u> : For any theory Θ, RL(Θ) stands for the following

assertion :

" There is a complete proof system for Θ in RL "

<u>Lemma 2</u> : RL($\Sigma_N(x) = \Sigma_N(y)$) is equivalent to

RL($(\forall f)(\exists j) T'_{1,1}(z,f,j)$) & RL($(\exists f)(\forall j)$not $T'_{1,1}(z,f,j)$)

(The recursive infinitary rule

$$\Sigma_N(x) = \Sigma_N(y) \Leftarrow (i\in\Sigma_N(x)\Longleftrightarrow i\in\Sigma_N(y))_{i \in \mathbb{N}}$$

is clearly valid and complete) .

<u>Lemma 3</u> : RL($(\exists f)(\forall j)$ not $T'_{1,1}(z,f,j)$)

(Introduce the infinitary rule

$(\exists f)(\forall j)$not $T'_{1,1}(z,f,j) \Leftarrow ($ not $T'_{0,i+1}(h(x,i),y_0,\ldots,y_i,i))_{i\in\mathbb{N}}$

where h(x,i) is the recursive function such that

$T'_{1,1}(x,f,i) \Longleftrightarrow T'_{0,i+1}(h(x,i),y_0,\ldots,y_i,i)$ if f(j) = y_j .

<u>Lemma 4</u> : RL ($(\forall f)(\exists j) T'_{1,1}(x,f,j)$)

(Let FPT(φ_z) mean : the recursive function φ_z of index z is

the characteristic function of a finite path tree . Then

(i) there is a recursive function p such that

$$(\forall f)(\exists j) T'_{1,1}(x,f,j) \Longleftrightarrow FPT(\varphi_{p(x)})$$

(ii) FPT(φ_z) is provable with the help of the recursive

infinitary rule :

FPT(φ_z) $\Leftarrow$ ($\varphi_z(i) = 0$ or FPT$(\varphi_{h(z,i)})$)$_{i \in N}$

where $\varphi_{h(z,y0)}(k) = \varphi_z(l)$ for $k = \langle y_1, \ldots, y_n \rangle$ and $l = \langle y_0, \ldots, y_n \rangle$

III Effective behaviour algebras :

* Summary :

On behalf of our thesis on effective infinitary objects and

transformations (cf. Section I) an effective behaviour

algebra must satisfy :

    (i) the behaviour of a term is a Σ_1^1 subset of N

    (ii) the operators in duced by contexts are Σ_1^1

transformations .

So any effective algebra factors through $\sim_z$ into another

effective behaviour algebra whose equality is provable - we

conjecture that $\sim_z$ is the finest equivalence with this nice

property .

We delineate in the remaining of this section two generic

behaviour algebras which are effective according to our thesis

(and cover all models cuurently developped for linear

behaviour of concurrent systems) :

* the first model is generic for initial linear models (sets

of infinitary transitions sequences)

* the second model is generic for the usual abstraction

therefrom (e.g sets of infinitary traces) .

(a) A generic framework for operational models :

Programming languages under concern are usual term algebras

with additional combinators for recursion . Among the latter,

one may think of the the rec combinator used in most process

algebras - e.g rec(x).t(x) - or more generally of the let-rec

combinator of ML .

A possible way to define effective transition systems whose

states are terms of the programming language, is to follow the

well known method of structured operational semantics [9]due to

G.Plotkin . As far as all the side conditions of the inference

rules are partial recursive (which is certainly reasonable) ,

this operational definition results in one global transition

system per programming languge, but it is clear that one may

also consider the result as an effective algebra of transition

systems (defined in the introduction) . Since transition

systems are faithfully represented by corresponding sets of

(infinitary) transition sequences, th above algebra of

transition systems induces an effective poweralgebra of

transition sequences .

Let the infinitary sequences of labelled transitions between

terms be represented as usual in $\mathcal{N}$; then, the elements of the

power algebra become Σ_1^1 subsets of $\mathcal{N}$ and the operators of the

poweralgebra are recursive on their Σ_1^1 indexes . But even

better, we know from recent work that these operators are union

additive extensions of operators $\varphi : \mathscr{N}^n \longrightarrow P(\mathscr{N})$ whose 'graph'

($\{\langle x,y \rangle / y \in \varphi x\}$) is Σ_1^1 - so they are Σ_1^1 transformations .

Now, as regards combinators for recursion, we have shown

that they lead naturally to greatest fixpoints with respect to

inclusion if the recursive definition satisfies the Greibach

condition . But $\lambda U . \mathrm{gfp}_X \, F(U,X)$ is a Σ_1^1 transformation if $F(U,X)$

is a Σ_1^1 transformation of U and X . To sum up, all contexts in

the programming language induce Σ_1^1 transformations in the

corresponding poweralgebra, which is thus an effective
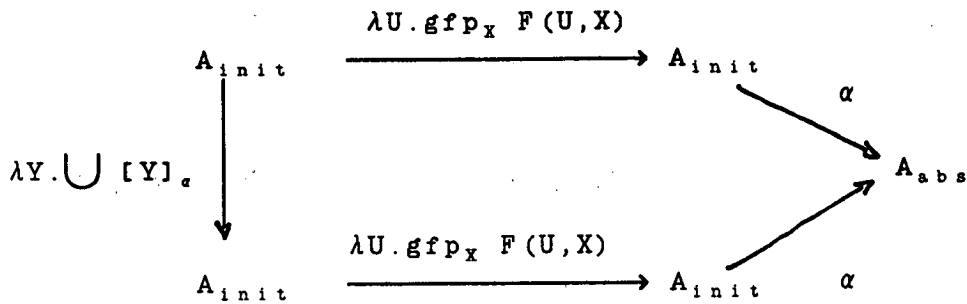
behaviour algebra .

The same holds, independently of structural operational

semantics, as long as the operational specification of the

operators on terms (resp. of the combinators for recursion)

gives rise to Σ_1^1 transformations (resp. to greatest fixpoints

with respect to inclusion) .

(b) A generic framework for abstraction :

We consider here behaviour algebras derived from the aboved

initial algebras through morphisms of poweralgebras, which

forget irrelevant details from computations . More precisely,

we assume abstraction morphisms α of the form $\alpha = \beta_o \gamma$, where γ

is any Σ_1^1 transformation (e.g erasing of states, or extraction

if infinitely repeated states...) and β is a closure operator

with respect to some recursive order on $\gamma(\mathscr{N})$, chosen among the

following family which reflects the three versions of powerdomains : downwards closure, convex closure, upwards closure .

Beside being morphisms of algebras, we ask our abstraction morphisms to satisfy the requirement expressed by the commutation of the diagram

$$
\begin{array}{ccc}
A_{init} & \xrightarrow{\ \lambda U.gfp_X\ F(U,X)\ } & A_{init} \\
& & \qquad\searrow^{\alpha} \\
\lambda Y.\bigcup [Y]_\alpha \downarrow & & \qquad\qquad A_{abs} \\
& & \qquad\nearrow \\
A_{init} & \xrightarrow{\ \lambda U.gfp_X\ F(U,X)\ } & A_{init} \quad \alpha
\end{array}
$$

where . A_{init} is the behaviour algebra defined in II a)

  . A_{abs} is $\alpha(A_{init})$ and

  . $\lambda Y.\bigcup [Y]_\alpha$ is the operator $\lambda Y.\alpha^{-1}(\alpha(Y))$

The intuition behind the diagram is the following : given a recursively defined program $C[u]$ with subprogram u, you may as often as you wish, during an execution of $C[u]$, substitute without damage successive versions u_n of u such that $u_n \sim_\alpha u$, where $\sim_\alpha$ is the obvious equivalence (dynamic adaptation of concurrent programs) . Any morphism α satisfying the above yields an effective behaviour algebra A_{abs}, because the derived interpretations for recursive contexts are then Σ_1^1 transformations .

Examples : 1) γ may be the erasure of states, and β may be

closure under prefix-ordering .

2) γ may also erase some actions considered as

invisible (e.g τ in CCS or ACP_τ)

3) See [10] for a more complex abstraction yielding a

model of CCS in which

$$u = v \text{ iff } u \text{ and } v \text{ have identical sets}$$

of infinitary interactions with any program t set

in parallel.

IV Topological and metric properties of indiscernibility :

a) The closure operator induced by $\sim_\Sigma$:

For A a Σ_1^1 subset of $\mathcal{N}$, let $A^\frown$ be the the union of all Σ_1^1

subsets B of $\mathcal{N}$ such that $A\sim_\Sigma B$.

The following characterizations of $A^\frown$ are an easy consequence

of a result by Louveau [11] :

$A^\frown = \cap$ {C Π_1^1 subset of $\mathcal{N}$ / $A \underline{\subset} C$} = $\cap$ {C Δ_1^1 subset of $\mathcal{N}$ / $A \underline{\subset} C$}

and $A^\frown$ is the largest Σ_1^1 set in the equivalence class of A .

Besides, the closure operator $(.)^\frown$ satisfies de Morgan's laws .

b) Generalizing $\sim_\Sigma$ into $\sim_\Delta$:

For A, B Σ_1^1 subsets of $\mathcal{N}$, let $A\sim_\Delta B$ iff for any Δ_1^1 subset C

of $\mathcal{N}$, $A\cap C = \emptyset \Longleftrightarrow B\cap C = \emptyset$.

The above characterization shows that $A\sim_\Sigma B \Longleftrightarrow A\sim_\Delta B$.

We consider from now on generalized definitions of relations

$\sim_\Sigma$ and $\sim_\Delta$ for arbitrary sets A and B . In this generalized

framework where the last equivalence is no longer valid, it appears that $\sim_\Delta$ has nicer topological properties than $\sim_\Sigma$.

For A in P($\mathcal{N}$), let $\overline{A}$ be the intersection of all Δ_1^1 subsets C of $\mathcal{N}$ such that A $\subseteq$ C . Then $(\overline{})$ is a closure operator inducing a topology T_Δ on $\mathcal{N}$,   A$\sim_\Delta$B $\Longleftrightarrow$ $\overline{A}$ = $\overline{B}$, and $\overline{A}$ is the largest Borel set in the equivalence class of A .

c) The topology T_Δ :

T_Δ has a clopen basis, namely the family of Δ_1^1 subsets of $\mathcal{N}$ (but all clopen sets are not Δ_1^1) .

($\mathcal{N}$,T_Δ) is metrizable, but it is neither compact nor locally compact .

The family of isolated points is exactly the family of Δ_1^1 singleton sets.

The latter family is not dense, whereas the Π_1^1 singleton sets form a dense family .

d) Continuous functions :

Although we have no full characterization of continuous functions for T_Δ , we know two important classes of continuous functions, namely :

   (i) funtions whose graph is Σ_1^1 (thus including Σ_1^1
        transformations) ;

   (ii) functions whose graph is the union of a Σ_1^1-indexed family of Δ_1^1 sets (where indexes are Δ_1^1 codes) .

In general, any continuous function is also a closed

mapping, since $A \sim_\Delta B$ implies $fA \sim_\Delta fB$ for continuous f . As a

consequence, $\sim_I$ is a congruence with respect to set-extensions

of T_Δ-continuous functions .

e) A metric for T_Δ :

Let x,y be in $\mathcal{N}$. An ultrametric distance $d(x,y)$ compatible

with T_Δ is defined by

$$d(x,y) = 2^{-n} \text{ where n is the smallest } \Sigma_1^1 \text{ index of a } \Delta_1^1$$

$$\text{set C such that } x \in C \iff y \notin C .$$

$$= 0 \text{ if there is no such separating set (} x = y \text{ in}$$

$$\text{this case) .}$$

But no distance compatible with T_Δ makes $\mathcal{N}$ into a complete

metric space, so it makes sense to find completions of $(\mathcal{N},d)$.

V Directions for further research :

A first subject of investigation is the logic RL, both from

model theoretic and proof theoretic points of view . In

particular, the problem of whether the indiscernibility of

behaviours is liable to RL proofs in a language built around

terms of behaviour algebras, is worth consideration. Another

concern about indiscernibility is to show that it is really the

finest equivalence (on Σ_1^1) provable in RL . We conjecture

that it is also the finest testable equivalence in the sense of

effective binary tests (extending De Nicola and Hennesy's

tests) . Since the separating sets on which relies

indiscernibility are Δ_1^1 sets, one may imagine a strong

connection between tests and Δ_1^1 behaviours . A natural question

is then the operational meaning of Δ_1^1 sets . A related topic is

the search for general (not necessarily profinite)

specification techniques for Σ_1^1 transformations, and more

generally for T_Δ-continuous functions . This suipposes of

course that we obtain beforehand a full characterization of

continuous functions . A last topic, with possible implications

in formal language theory, is the (metric) completion of $(\mathcal{N}, T_\Delta)$.

We have some reasons to believe that bi-infinite words come

here into play .

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