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► **To cite this version:**

Jean-Pierre Merlet. Parallel manipulators. Part 2: Theory. Singular configurations and Grassmann geometry. [Research Report] RR-0791, INRIA. 1988, pp.69. inria-00075760

**HAL Id: inria-00075760**

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Submitted on 24 May 2006

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# **PARALLEL MANIPULATORS**

## **Part 2 : Theory**

Singular configurations and Grassmann Geometry

**Jean-Pierre MERLET**

Rapport de Recherche INRIA N° 791, Février 1988

**Résumé:** Les manipulateurs parallèles sont des robots où les différents segments ne sont pas placés successivement à partir de la base vers la "main", mais au contraire tous connectés directement à la fois à la base et à la main. Ils présentent un grand intérêt pour la réalisation d'opérations mécaniques ainsi qu'une alternative aux articulations classiques utilisées dans les manipulateurs actuels. Nous avons exposé dans une première partie les propriétés cinématiques et cinétiques des manipulateurs parallèles. Nous étudions ici la possibilité éventuelle d'avoir plusieurs positions de la main pour des longueurs fixées des segments. Dans ce cas le manipulateur devient non rigide: on est dans une configuration singulière. Le manipulateur n'est alors plus commandable. On sait que trouver ces positions revient à la recherche des racines du déterminant de la jacobienne. Mais cette méthode n'est pas utilisable ici vu la complexité de cette matrice. Nous proposons une autre approche basée sur la géométrie des lignes de Grassmann (parfois appelée aussi géométrie des lignes de Plücker). Si l'on considère que l'ensemble des lignes de  $P^3$  présente une structure de variété de rang au plus 6 on montre qu'une structure telle qu'un robot parallèle sera non rigide si la variété engendrée par les lignes associées aux segments est dégénérée. L'intérêt de cette approche est que l'on sait caractériser **géométriquement** toutes les cas de dégénérescence. On cherchera donc à trouver les configurations du manipulateur qui satisfont à ces conditions.

Dans cette étude partielle on montre des conditions de singularité inédites. Les cas restant à étudier sont exposés.

**Summary:** Parallel manipulators have a specific mechanical architecture where all the links are connected both at the basis and at the gripper of the robot. This kind of manipulator has a better positioning ability than the classical robot. We have addressed in the first part of this paper the direct kinematics problems. We have shown that it may be suspected that in some case for a given set of links lengths more than one solution may be given to this problem: we have thus a singular configuration for the manipulator. To determine these singular configurations the classical method is to find the roots of the determinant of the jacobian matrix. However in our case the jacobian matrix is too complicated to find these roots. We propose here a new method based on Grassmann line-geometry. If we consider the set of lines of  $P^3$ , it constitutes a linear variety of rank 6. It can be easily shown that a singular configuration is obtained when the variety spanned by the lines associated to the robot links has a rank less than 6. An important feature of this geometry is that each degeneracy case can be described by simple **geometric** features.

Thus the difficult problem is partitionned in a small number of geometric problems.

Although all the cases has not been treated yet we propose new singular configurations found with this method.

# 1 Parallel manipulator

## 1.1 Introduction

We deal here with the study of fully parallel manipulator like the model presented in Figure 1.

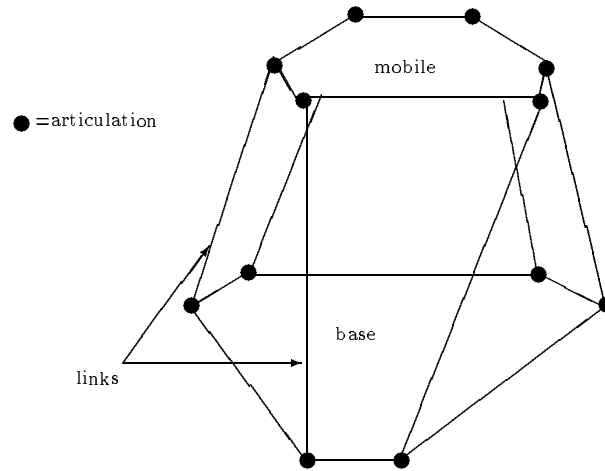


Figure 1: fully parallel manipulator: the links have a variable length

Basically it consists in two plates connected by 6 articulated links. In the following chapters the smaller plate will be called the *mobile* and the greater (which is in general fixed) will be called the *base*. In each link there is at least one actuator. In the case of one actuator by link we have a so-called *fully parallel* manipulator.

Some manipulator of this type have been designed or studied since a long time. The first one, to the author's knowledge, was designed for testing tyres (see Mc Gough in Stewart paper [16]). But the main use of this mechanical architecture consists in the flight simulator (see for example Stewart [16], Watson [17], Baret [15]). The first design as a manipulator system has been done by Mac Callion in 1979 for an assembly workstation [5] but Minsky [9] has presented in the early 70's some design related to various mechanical architectures. Some other researcher have also addressed this problem: Reboulet [11], Inoue [4], Tanaka (as a platform for movies camera!) [12], Fichter [1], Yang [13], Mohamed [8], Zamanov [14]. Even a commercial manipulator was sold by Marconi under the name "Gadfly" for the assembly of integrated circuits. This kind of manipulator have a great positionning ability and are very convenient for force-feedback command (see Merlet [6]). A prototype of parallel

manipulator is currently under development at INRIA. The architecture of this manipulator is based on the prototype developed at the CERT-DERA laboratory in Toulouse with a design which has been modified by the author. In the first part of this paper we have presented various features and problem involved by the parallel architecture. We will deal here with the special problem of singular configurations.<sup>1</sup>

## 1.2 Notation

We introduce the absolute frame R with origin C and a relative frame  $R_b$  fixed to the mobile with origin O (see Figure 2). The rotation matrix relating a

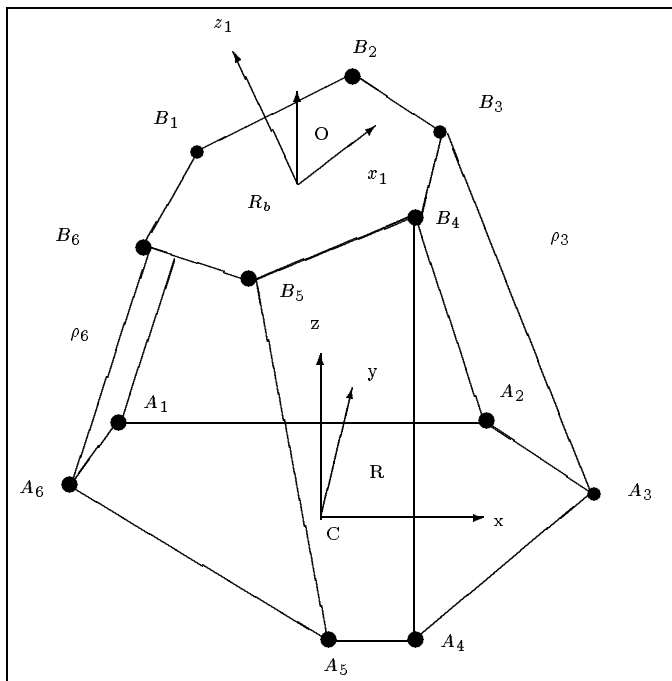


Figure 2: notation

vector in  $R_b$  to the same vector in R will be denoted by  $M$  with:

$$M = \begin{pmatrix} v1 & v2 & v3 \\ v4 & v5 & v6 \\ v7 & v8 & v9 \end{pmatrix}$$

<sup>1</sup>A lot of calculation (in particular the results presented in the Appendices) has been obtained with the aid of MACSYMA [26], a large symbolic manipulation program developed at MIT or with REDUCE.



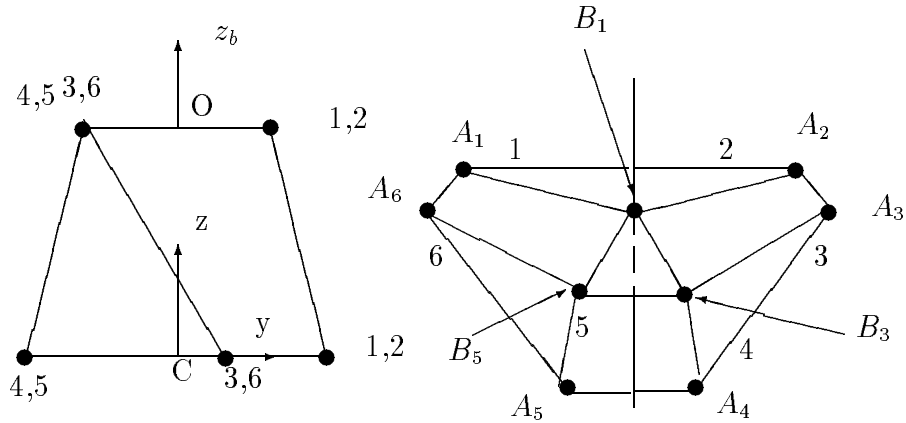


Figure 4: triangular simplified symmetric manipulator TSSM

is presented on Figure 4. In this case the articulation points on the mobile are located only in three different positions. The last interesting design is presented in Figure 5 : basically it is a further simplification of the TSSM where the couples of articulations on the base have the same center of rotation: we will call this manipulator the minimal simplified symmetric manipulator (MSSM). It must be noted that a practical realisation of a MSSM is very difficult.

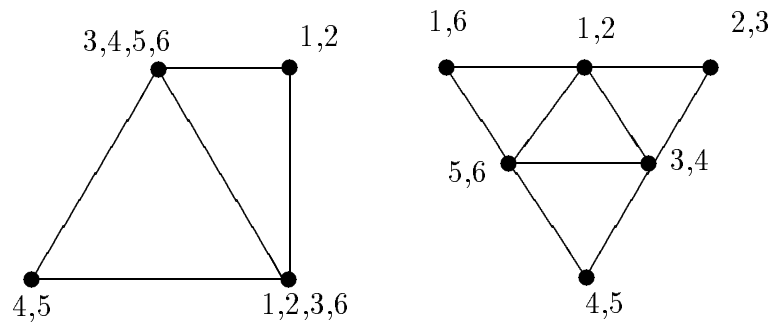


Figure 5: the minimal simplified symmetric manipulator (MSSM)

### 1.3 Singular configurations and the jacobian matrix

The fundamental relations relating the links lengths to the position and orientation of the mobile is:

$$\rho^2 = (x_o - x_a + x.v1 + y.v2 + z.v3)^2 + (y_o - y_a + x.v4 + y.v5 + z.v6)^2 + (z_o - z_a + x.v7 + y.v8 + z.v9)^2 \quad (1)$$

Thus we have to solve a system of 6 non-linear equations of type (1) to get the position of the mobile  $x_o, y_o, z_o$  and its orientations from the links lengths. The solution is unique if the rank of the jacobian matrix  $J$  of this system is equal to 6 with:

$$J = \left( \left( \frac{\partial \rho}{\partial \mathbf{q}} \right) \right) \quad (2)$$

with  $\mathbf{q}$  the position and orientation parameters vector. Note that this matrix is in fact the inverse jacobian (in a robotics sense) of the manipulator. The symbolic computation of the determinant of  $J$  is rather tedious ( see [7] for the formulation of this determinant). Mac Callion [5] used a numerical deflation method to find all the roots of the determinant. Mac Callion have found up to nine roots to this determinant all outside the range of the links lengths and Hunt has shown that there can be up to 16 roots [3]. Thus we need another method to find the singular configurations of the parallel manipulator.

## 2 Plücker coordinates of lines, rigidity and geometry

It is well known that a line can be described by its Plücker coordinates. Let us introduce briefly these coordinates. We consider two points on a line, say  $M_1$  and  $M_2$ , and a reference frame  $R_0$  which origin is  $O$  (see Figure 6). Let us consider now the two three dimensional vectors  $\mathbf{S}$  and  $\mathbf{M}$  defined by :

$$\mathbf{S} = \mathbf{M}_1 \mathbf{M}_2$$

$$\mathbf{M} = \mathbf{O} \mathbf{M}_1 \wedge \mathbf{O} \mathbf{M}_2 = \mathbf{O} \mathbf{M}_2 \wedge \mathbf{S} = \mathbf{O} \mathbf{M}_1 \wedge \mathbf{S}$$

If we assemble these vectors to form a 6-dimensional vector we get the vector  $\mathbf{U}$  of the Plücker-coordinates of this line.

$$\mathbf{U} = [S_x, S_y, S_z, M_x, M_y, M_z]$$

It is useful to introduce the normalized vector  $\mathbf{U}'$  defined by :

$$\mathbf{U}' = \frac{\mathbf{U}}{\|\mathbf{S}\|} = [S'_x, S'_y, S'_z, M'_x, M'_y, M'_z]$$



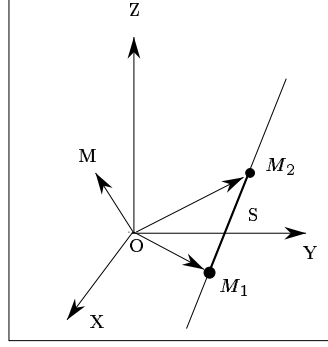


Figure 6: Plücker coordinates

It may be seen that the first three components of this vector are the components of the unit vector  $\mathbf{n}_i$  of the line. The last three components are given by :

$$\mathbf{OM} \wedge \mathbf{n}_i$$

M being any point of the line. We will show now that the matrix P constituted by

$$P = ((U'_1, U'_2, \dots, U'_6))$$

,where  $U'_i$  is the coordinate vector of line  $i$ , is identical to the transpose of the matrix  $J$ . We introduce the articular force vector  $\mathbf{f}$  i.e. the axial forces which are exerted by the links on the mobile plate. Let us consider now the external force vector  $\mathbf{F}$  and torque vector  $\mathbf{M}$  about the origin C acting on the mobile and let  $\mathcal{T} = [\mathbf{F}, \mathbf{M}]$  be the generalized force vector. It is well known that :

$$\mathcal{T} = J^T \mathbf{f}$$

The system being in equilibrium we may write :

$$\sum_{i=1}^{i=6} f_i \mathbf{n}_i = \sum_{i=1}^{i=6} f_i \mathbf{S}'_i = \mathbf{F}$$

$$\sum_{i=1}^{i=6} \mathbf{CB}_i \wedge f_i \mathbf{n}_i = \mathbf{M}$$

where  $\wedge$  denotes the cross product. If we choose C as origin for the reference point of the Plücker frame we are able to write from the two previous equations:

$$P\mathbf{f} = \mathcal{T} \quad (3)$$

and therefore :

$$J^T = P$$

Equation 3 is a linear system of equations in the unknown  $f_i$ . If the system is rigid this means that whatever are the generalized forces it exists one solution to this system ( and in our case the solution will be unique). It will be true if the matrix P is of full rank which is equivalent to say that the Plücker vectors are linearly independent.

Let us assume now that these vectors belong to a vector space  $V_6$  and we consider the one-dimensional subspaces of  $V_6$  as points of a projective  $P_5$ . Then every line  $g$  in  $P_3$  corresponds to exactly one point  $\mathbf{G}$  in  $P_5$ .

It is well known that point  $\mathbf{G}$  belongs to a quadric  $Q_p$  (see [21], [25], [18]). Indeed we have for every line of  $P_3$  :

$$S_x M_x + S_y M_y + S_z M_z = 0$$

This equation defines the quadric  $Q_p$  which is called the *Grassmannian* or the *Plücker quadric*. At this point we have defined a one-to-one relation between the set of lines in the real  $P_3$  and the quadric  $Q_p$  in  $P_5$ . The rank of this mapping is 6 (there is at most 6 independent Plücker vectors).

Let us consider now the various sub-spaces of  $P_5$  (or more precisely their intersection with  $Q_p$ ). We get various varieties which rank ranges from 0 to 6. As a matter of example a point in  $P_5$  ( rank=1) corresponds to a line in  $P_3$ . As for  $Q_p$  (which represents the set of line of  $P_3$ ) it is defined through 6 linearly independent Plücker vectors and is therefore of rank 6.

Let us come back to the rigidity of a parallel manipulator. We have seen that this manipulator is rigid (and therefore not in a singular configuration) if and only if the 6 lines are linearly independent. Therefore any subset spanned by  $n$  lines must have a rank equal to  $n$ . At this point the problem is far to be solved because we are not able to find the generalized coordinates of the mobile for which there is a linear dependency between the  $n$  Plücker vectors. But we will see that these dependencies can be described by geometric considerations.

### 3 Grassmann Geometry

The varieties of lines has been studied by H. Grassmann (1809-1877). The purpose of this study was to find geometric characterization of each varieties. We will introduce now the various results which can be found in [22] or with more mathematical justifications in [25].

Let us begin with the varieties of rank 0 through 3 (Figure 7 ) We have first the empty set of rank 0. Then the *point* (rank=1) which is a line in the 3D

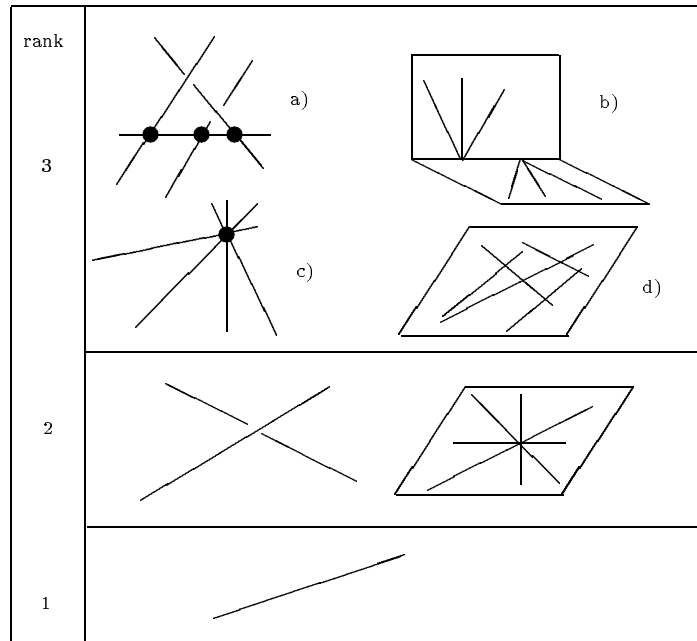


Figure 7: Grassmann varieties of rank 1,2,3

space. The *lines*(rank=2) are either a pair of skew lines in  $R^3$  or a flat pencil of lines: those lying in a plane and passing through some point on that plane.

The *planes* (rank=3) are of four types:

- all lines in a plane (3d)
- all lines through a point (3c)
- the union of two flat pencils having a line in common but lying in distinct planes and with distinct centers (3b)
- a regulus (3a)

Let us define the regulus. Let three skew lines in space and consider the

set of lines which pass through the three lines : this set of line build a surface which is an hyperboloïd of one sheet (a quadric surface, Figure 8) and is called a **regulus**. Each line belonging to the regulus is called a *generator* of the regulus.

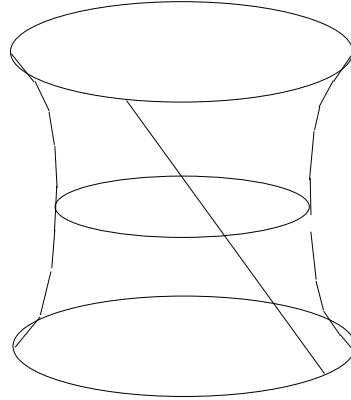


Figure 8: hyperboloid of one sheet

It is shown in [23],[25] that this surface is *doubly ruled*. This mean that it exists two reguli (a regulus and its "complementary" regulus) which generate the same surface or that each point on the surface is on more than one line. The only double ruled surface are the plane, the hyperboloïd of one sheet and the hyperbolic paraboloid.

Let us come back to the regulus: we have seen that there are two families of straight lines on the hyperboloïd and each family covers the surface completely. A line on this surface is dependent of the lines of either the regulus or the complementary regulus. An interesting property is that a line of one family intersects all the lines of the other family and that any two lines of the same family are mutually skew (see [24] for the hairy details).

Another interesting property of the generators is that their equations are easily written if we use homogeneous coordinates  $(x_0, x_1, y_0, y_1)$  based on a pair of skew lines X,Y which contain the reference points  $X_0, X_1, Y_0, Y_1$ . As a matter of example let us consider two points  $A_1, B_1$  belonging to X and  $A_2, B_2$  belonging to Y. The homogeneous coordinates of a point M are  $x_0, x_1, y_0, y_1$  such that:

$$\mathbf{OM} = x_0\mathbf{OA}_1 + x_1\mathbf{OB}_1 + y_0\mathbf{OA}_2 + y_1\mathbf{OB}_2 \quad (4)$$

We shall write  $x$  (resp.  $y$ ) for the column vector of the  $x_i$  (resp.  $y_i$ ). We note that any line which is skew to Y has a matrix equation of the form:

$$y = Ax \quad (5)$$

where  $A$  is a uniquely defined  $2 \times 2$  matrix of constants. Let us consider a regulus  $R$  with generators  $X, Y$ , and a third line  $L$ . The matrix equation of  $L$  with respect to the reference frame  $X, Y$  is

$$y = A_L x \quad (6)$$

An important result is that any generator of  $R$  has a matrix equation of the form:

$$y = \lambda A_L x \quad (7)$$

where  $\lambda$  is a real constant

Let us describe now the varieties of higher rank of the Grassmann geometry (Figure 9). Varieties of dimension 4 are called *congruences* and are of four types:

- a linear spread generated by 4 skew lines i.e. no lines meet the regulus generated by the three other lines in a proper point (*elliptic congruence*, 4a)
- all the lines concurrent with two skew lines (*hyperbolic congruence*, 4b)
- a one-parameter family of flat pencil, having one line in common and forming a variety (*parabolic congruence*, 4c)
- all the lines in a plane or passing through one point in that plane (*degenerate congruence*, 4d)

Varieties of dimension 5 are called *complexes* and are of two types:

- *non singular*: generated by 5 independent skew lines (5a)
- *singular* (or *special*): all the lines meeting one given line (5b)

The complexes may be defined to be the set of lines which are dependent of a skew pentagon. An interesting property of the special complex is that four of the vertices of the skew pentagon are coplanar. As for a general complex its geometric characterization is that through any point of the space there is one and only one flat pencil of line such that all the lines which belong to the pencil belong also to the complex. In other words all the lines of a complex which are coplanar intersect one point.

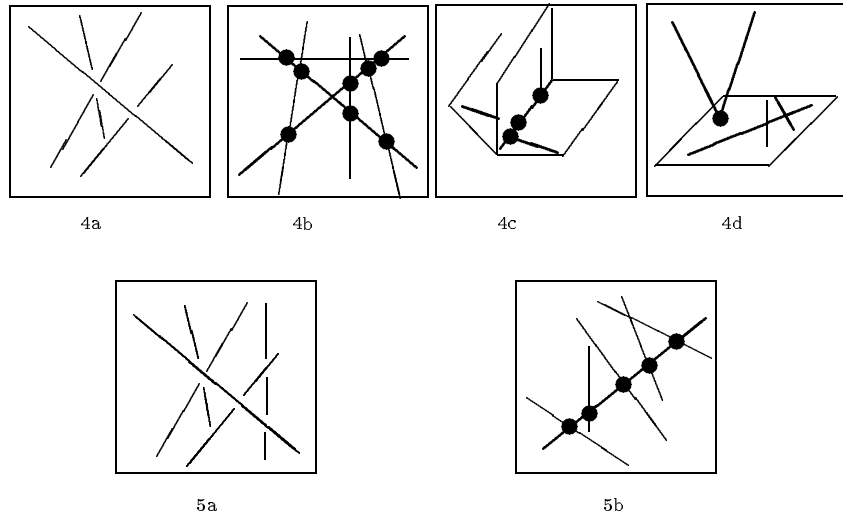


Figure 9: Grassmann varieties of rank 4,5

## 4 An example: the 2D parallel manipulator

Let us consider a basic example: a 2D parallel manipulator (Figure 10 ) In this

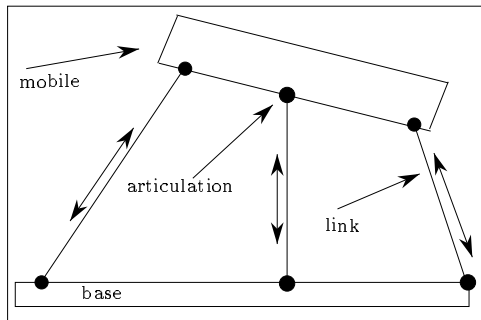


Figure 10: 2D parallel manipulator

case we have three segments and these three lines must constitute a variety of Grassmann of rank 3. Thus we will consider any subset of 1,2,3 segments and determine the condition for which any such subset has a rank 1,2,3.

The case of 1 and 2 segments are rather trivial: for one line we have only to verify that this line exists and for two lines that the lines are distinct. This is clearly the case if we except the configuration where the base and the mobile

are collinear.

We will consider now the whole system of three bars. By reference to Figure 7 we can see that the only possibility for a system of three bars to be a 2-rank Grassmann variety is obtained when the three lines cross the same point (Figure 11). In particular if the mobile and the base are homotetic we

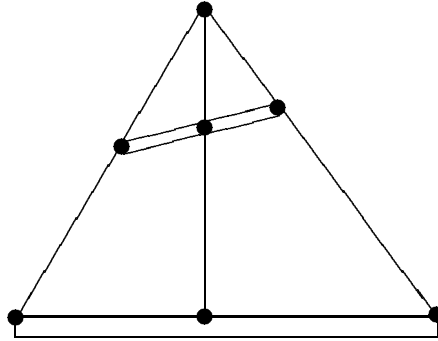


Figure 11: singular configuration for the 2D parallel manipulator

get a rather disturbing singular configuration when the base and the mobile are parallel, whatever is their relative position.

Another design is straightforward to avoid the above singular configuration (Figure 12).

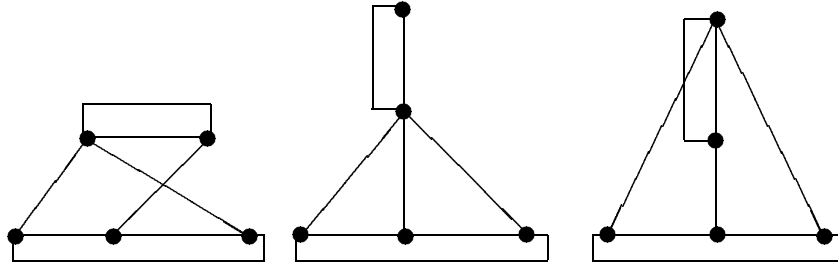


Figure 12: a 2D parallel manipulator and its two singular configurations

We can see here that practically, whatever is the position and orientation of the mobile the three segments cannot cross the same point. We see on the figure the two possible singular configurations which cannot be reached by a real manipulator. Furthermore this design is also interesting because we

can solve the difficult direct kinematics problem i.e. find the position and orientation of the mobile for a given set of links lengths.

## 5 Study of the MSSM

### 5.1 Introduction

For the general case we will make two assumptions :

- a segment cannot lie in the plane of the base . A consequence of this assumption is that the mobile and the base cannot be coplanar.
- the links lengths cannot be equal to zero (i.e. an articulation point on the mobile and the base cannot be in the same location).

We will begin the study of the rigidity of the parallel manipulator by the simplest case: the MSSM (Figure 13).

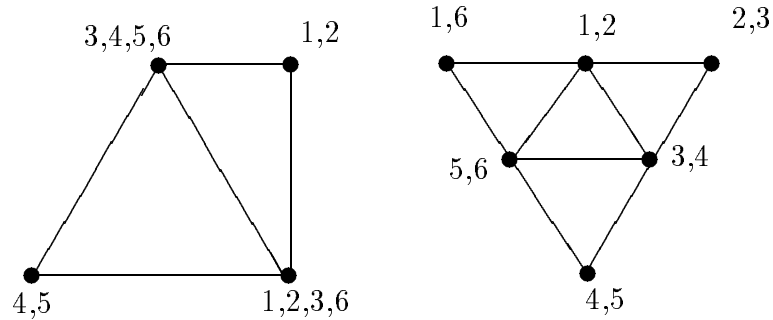


Figure 13: the minimal simplified symmetric manipulator (MSSM)

### 5.2 Case by case study

#### 5.2.1 Subsets of 2,3 bars

**Subset of 2 bars** In the case of 2 bars we have only to count if there is 2 distinct bars in our mechanism. Thus this verification is trivial.

**Subset of 3 bars** In the case of 3 bars the rank of the system is 2 if the lines belong to a flat pencil of lines i.e. are in a plane and pass through some point on that line.



Let us notice first that for any set of 3 lines which may be coplanar we have two lines which have a common articulation point on the mobile (say  $B_i$ ), and the third line has a common articulation point on the base (say  $A_j$ ) with one of the preceding lines. For example consider the set 1,2,3: lines 1,2 have a common point on the mobile ( $B_1$ ) and lines 2,3 a common point on the base ( $A_2$ ).

All the lines being distincts it is thus impossible that three lines have a common intersection point except if the base and the mobile are coplanar or if the articulation point on the base and the mobile have the same location (but this configuration cannot be reached by a real manipulator). In fact the coplanarity of the base and the mobile is a general case of singularity which is excluded in practical applications.

Thus 3 lines of a MSSM cannot constitute a flat pencil of line.

### 5.2.2 Subsets of 4 bars

**Degeneracy of type 3d** In this case four lines are coplanar.

The first possible case is obtained when the base and the mobile are coplanar. But other cases can be found. Let us suppose that lines 2,3,4 are coplanar. We may rotate then the mobile around the line B2B4 until the point B5 lie in the plane defined by 2,3,4 (Figure 14). In this case line 2,3,4,5 are coplanar. The appendix 1 in section 7, page 27, gives the conditions which must be

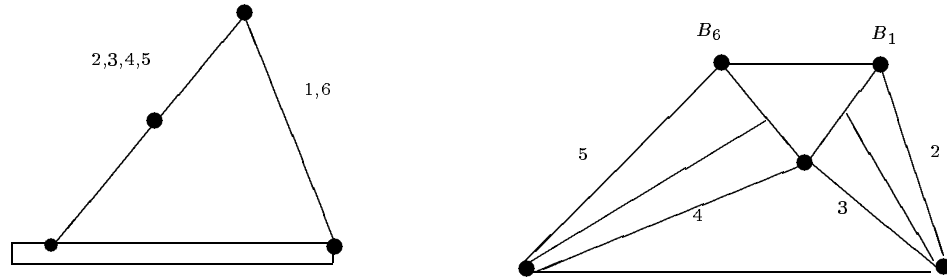


Figure 14: singular configuration of type 3d

satisfied by the parameters if the system is in such singular configurations. We get the following equation:

$$(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0 \quad (8)$$

which express the coplanarity of 2-3-4. The coplanarity of 5 with 2-3-4 is more difficult to express.

This configuration is known as Hunt singular configuration. We will see later that this configuration yields also to a special complex for the 6 lines.

An open problem is to determine if, for a given design of a MSSM (i.e. the position of the articulation points being given), and for a given range of the links lengths such a singular configuration lie in the working area of the MSSM.

Let us make a useful remark for the following part : in this case we may notice that the 6 lines cross one line ( ligne  $B_4B_5$  in the above example).

**Degeneracy of type 3c** In this case we will have four lines passing through the same point.

Let us remark first that among a set of four lines we have two lines (say T,U) with a common intersection point  $A_i$  on the base and two (say V,W) on the mobile (point  $B_j$ ). The lines being distinct they cannot have any other intersection point. We have assumed that every  $A_i$  are different from the  $B_j$  and thus any set of four lines of a MSSM cannot have a common intersection point.

**Degeneracy of type 3b** In this case 4 lines constitute two flat pencils having a line in common but lying in distinct planes (Figure 15).

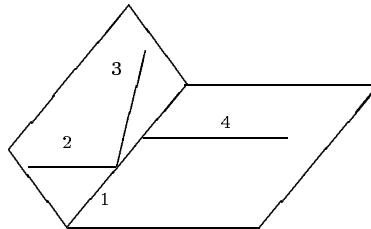


Figure 15: 3-dimensional Grassmann variety of type 3b

In this case we must have three coplanar lines which constitute a flat pencil. But we have seen in the part devoted to the subset of three lines that it does not exist three lines such that they constitute a flat pencil. Thus this configuration is not to be considered.

**Degeneracy of type 3a** The problem is to find 4 lines which are on the same regulus.

The hyperboloid of one sheet has two regulus  $\mathfrak{R}_1$  and  $\mathfrak{R}_2$  and we denote by (1) the family of lines which are spanned by  $\mathfrak{R}_1$  and (2) the family of lines

spanned by  $\mathfrak{R}_2$ . Remember that each lines of (1) has an intersection point with every line of (2) and none with the other lines of (1).

Let us suppose that lines 1 belongs to (1). Whatever is the configuration of the MSSM line 2 intersects the line 1 and thus must belong to (2). Lines 3 intersects line 2 and therefore belongs to (1). Applying the same reasoning for the other lines it is clear that lines 1,3,5 belong to  $\mathfrak{R}_1$  and line 2,4,6 belong to  $\mathfrak{R}_2$  whatever is the configuration of the MSSM. Thus it is impossible that 4 lines of the MSSM belongs to the same regulus.

### 5.2.3 Subsets of 5 bars

**configuration 4d** All the five lines are in a plane or pass through one point of this plane.

Let us notice first that we cannot have five coplanar lines : indeed we have at most 4 collinear articulation points on the base and thus at most 4 coplanar lines.

Consider now the case of 4 coplanar lines. We have investigated this case for the singular configuration of type 3d. It may be seen on Figure 14 that in this case the fifth line cross the plane at its articulation point on the mobile. Thus the singular configuration in this case is identical to the singular configuration of type 3d.

Let us suppose now that three lines are coplanar.

We notice that the three articulation points on the base must be collinear. We can now distinguish two cases. The first one is obtained when two of these lines have a common articulation on the mobile (2,3,4 for example) and the second one none of the three lines have a common articulation point ( 2,3,5 for example). In this later case 4 will be coplanar to 2,3,5 and therefore 4 lines are coplanar. The above result can be applied.

In the first case line 5 crosses the plane at point  $A_5$  and line 1 at point  $B_2$ . Thus lines 1,5 can intersect the plane at the same point if  $B_2=A_5$ . For the lines 6,5 we get by the same manner  $B_6 = A_5$  or 5 in the plane 2,3,4 (but this is configuration 3d). As for the lines 1,6 they cannot have a common point with the plane 2,3,4. We assume that these cases cannot be realized by real manipulators.

The last case to be considered is when 3 lines cross at a same point a plane spanned by two others lines. Without loss of generality consider the plane spanned by 1,2. Lines 3,6 cross the plane in two different points  $A_1$  and  $A_2$  and thus cannot belong both to the set of lines to be considered. Its remains thus two sets (1,2,3,4,5) or (1,2,4,5,6). In this case the intersection point with the plane is either  $A_3$  or  $A_6$ . But among the sets of three non coplanar lines there is two line which have a common point ( $B_3$  or  $B_5$ ). Thus these lines

cannot cross in the same time an other point in plane 1,2.

**configuration 4c** In this case three lines must constitute a flat pencil of lines and therefore this configuration is not to be considered.

**configuration 4b** Five lines must pass through two skew lines (Figure 16).

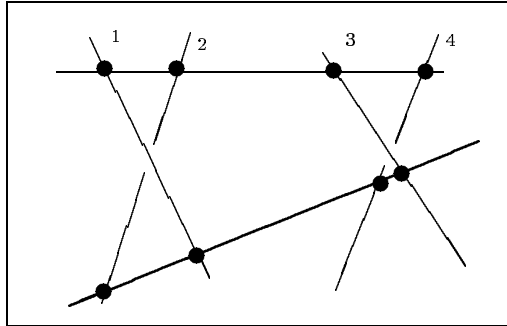


Figure 16:

We will consider without loss of generality lines 1,2,3,4,5. Let us notice first that there is no two skew lines which can intersect three coplanar lines.

Let us consider lines 1,2,3,4 : if 1,2,3 are coplanar we cannot find skew lines intersecting 1,2,3. If 1,2,3 are not coplanar and if a line intersect lines 1-2-3 then this line lie in the plane 2-3 and pass through the articulation point of 1,2 on the mobile or lie in the plane 1-2 and intersect 3 at its common point with 2 on the base (Figure 17) or is the line crossing both  $B_1$  and  $B_3$ .

But a line which has a common point with lines 4,5 either lie in the plane 4,5 or crosses the articulation point common to 4,5 on the base. In the former case a line crossing line 1,2,3 and 4,5 must be coplanar to 4,5 and 1,2 or to 4,5 and 2,3: this is clearly impossible except if line 2,3,4 are coplanar. Indeed a line crossing  $B_2$  and  $A_4$  will intersect 5 segments (Figure 18) but in this case we have shown that is impossible to find 2 skew lines crossing 2,3,4.

In the latter case let us notice that there is no line in the planes 1,2 or 2,3 crossing the articulation point common to 4,5. Thus no line in the above plane can cross at the same time 1,2,3,4,5.

Thus there is only one line (one edge of the mobile) which can cross a set of 5 lines and therefore it is impossible to find 2 skew lines crossing a set of 5 lines.

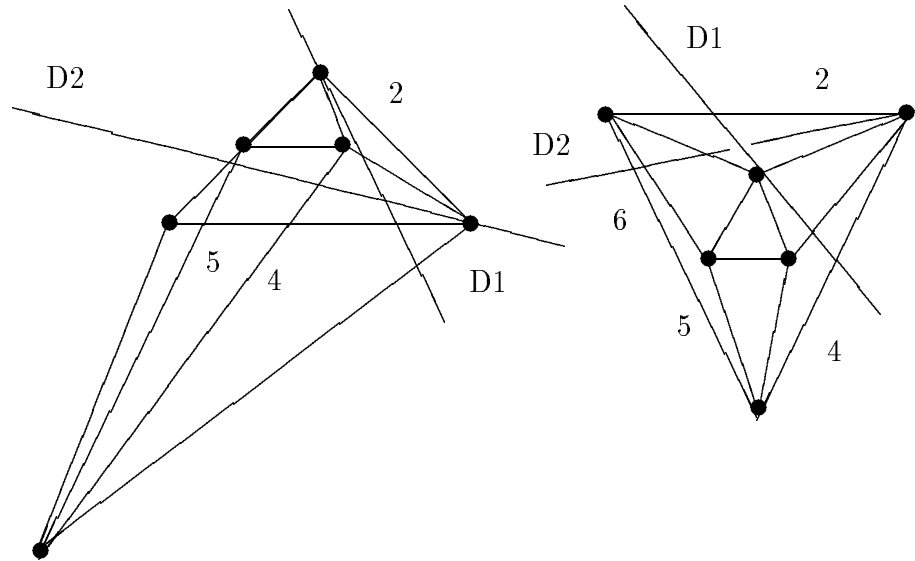


Figure 17: 2 skew lines intersecting line 1,2,3

**configuration 4a** In this case one line is dependent from four lines. Among this four lines none of them intersect the regulus spanned by the three others in a proper point.

Let us consider lines 1,2,3,4,5. From this set we consider 3 skew lines which spanned a regulus. There is only one set (1,3,5) where the lines may be skew. But the lines 2 or 4 intersect one line of the above set and therefore we cannot find 4 lines which meet the required condition.

#### 5.2.4 Subsets of 6 bars

**Configuration 5b** 6 lines cross the same line D in space.

We may justify one more time Hunt's singular configuration described for 3d type(Figure 20). In this case we see that all the segments pass through the line  $B_4B_5$ .

We have seen for the 4b case that one line can have a common intersection point with five links if and only if :

- this line is an edge of the mobile and 4 segments are coplanar (these segments are successive).
- 3 segments are coplanar and the intersection line crosses the articulation points on the base which are not common to the segments (see Figure 18).

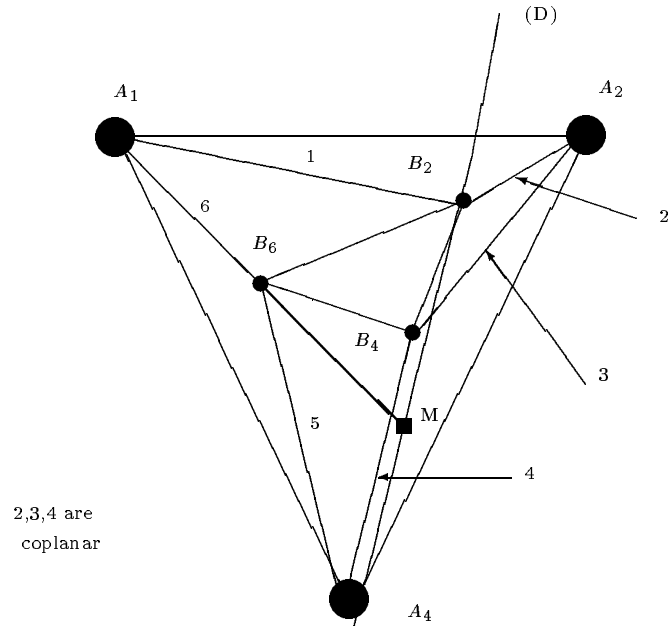


Figure 18: line (D) cross 1,2,3,4,5 if 2,3,4 are coplanar

Let us remark we may have found directly that at least 3 segments must be coplanar. Indeed let us remember that for a special complex 4 vertices of the skew pentagon must be coplanar. The geometry of the MSSM yields then directly that at least there must be 3 coplanar segments.

Let us consider the first case. If we rotate the mobile around the edge  $B_4B_5$  until line 1,2 are coplanar with 6,3 the line based on this edge intersects the 6 links. We find again Hunt's singular configuration. Now we will consider edge  $B_2B_3$  as being the intersection line of 5 segments (1,2,3,4,5). Let us notice that in that case the line based on edge  $B_2B_5$  intersects 1,2,4,5,6 and may intersect 3 if the edge is not parallel to 3. In summary this case yields no more singular configuration than case 4d.

We will consider now the case of three coplanar segments (say 2,3,4) and consider the line (D) intersecting  $B_2$  and  $A_4$ . (D) intersects segments 1,2,3,4,5. Let us rotate now the mobile around its edge  $B_2B_4$ . Let M be the intersection point with the plane spanned by 2,3,4. It may be possible that M belongs to (D) and therefore (D) intersects the 6 segments (Figure 21). The appendix 2 in section 8, page 29, shows that to obtain this configuration we must have

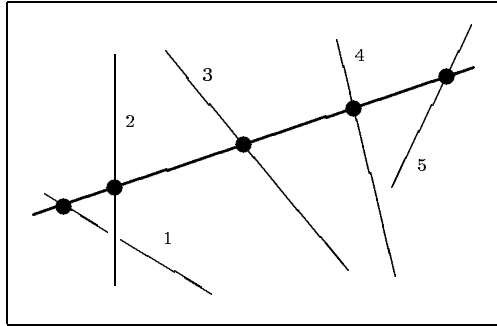


Figure 19: 5-dimensional Grassmann variety of type 5b

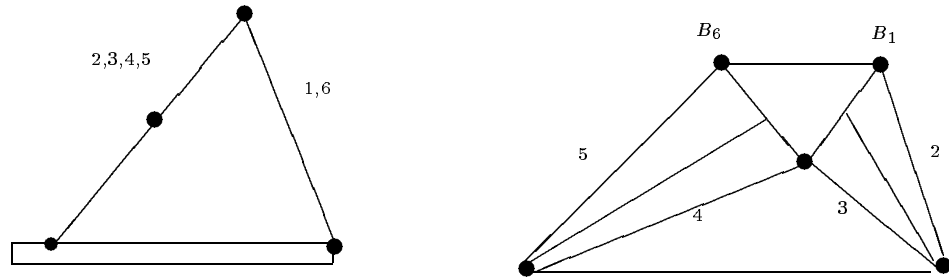


Figure 20: Hunt's singular configuration

:

$$(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0 \quad (9)$$

$$Az_0^2 + Bz_0 + C = 0 \quad (10)$$

**configuration 5a** In this case 6 lines spanned a general complex. FICHTER [1] has shown by using an intuitive method that this will be the case if we rotate the mobile around the vertical axis with an angle  $\frac{\pi}{2}$  or  $-\frac{\pi}{2}$  whatever is the position of its center.

First we will show that if we consider only rotation around a vertical axis the above singular configurations are the only cases where we have a general complex.

To demonstrate this results we consider the pencils of lines spanned by 1-6, 2-3 and 4-5. Then we consider in each of these pencils the line  $D_i$  which is

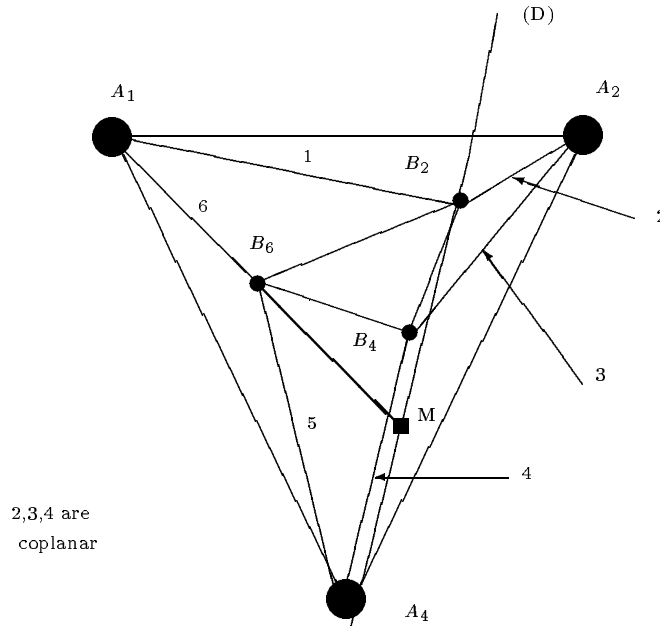


Figure 21: A new singular configuration

coplanar to the base (see Figure 22).

**Condition C1**

*Lines 1-2-3-4-5-6 spanned a general complex if and only if the lines  $D_i$  have a common point.*

*Proof*

From the classical property of a general complex we know that all the lines of a complex which are coplanar have a common point.

Let us consider now the complex spanned by lines 1-2-3-4-6 and let M the common intersection point of  $D_1, D_2$ . Then lines  $A_4M$  belongs to the complex together with the lines of the flat pencil which center is  $A_4$  and lines are 4 and  $A_4M$ . By construction line 5 belongs to this pencil and therefore 5 belong to the complex.

If we apply condition C1 in the case where we have only a rotation around a vertical axis we may show (see the appendix 3 in section 9, page 31) that



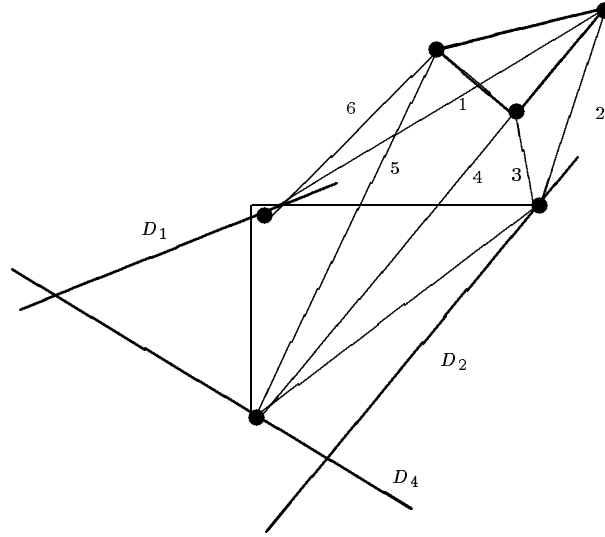


Figure 22:

C1 is equivalent to:

$$2\cos(\psi)z_0(-x_3ya_1 + x_3ya_4 + xa_1y_1 - xa_1y_3) = 0$$

whatever is  $x_0$  and  $y_0$ . The above condition is true only for  $\psi = \frac{\pi}{2}$  or  $-\frac{\pi}{2}$ . Indeed the last term of the above equation will be zero if the edge of the base and the mobile were parallel.

Using the same method we will show however that this not the only case where we get such a singular configuration. In the general case condition C1 yields to a third degree polynomial in the variable  $z_0$  (see the appendix 4 in section 10, page 32). Thus for a given orientation of the mobile and given value of  $x_0$  and  $y_0$  we have at least one singular configuration where the MSSM is a general complex.

Let us consider an example. We choose  $x_0 = 0$ ,  $y_0 = 0$ ,  $\psi = 40^\circ$ ,  $\theta = 40^\circ$ ,  $\phi = 40^\circ$ . In this case the polynomial has three real roots. Figures 23, 24, 25 show the three singular configurations. The first singular configurations can be easily interpreted. We see that lines 4 and 5 are identical. Thus the variety spanned by the lines of the MSSM is in fact spanned by only 5 lines.

## 6 Results : the singular configurations of a MSSM

The table below summarizes the results.

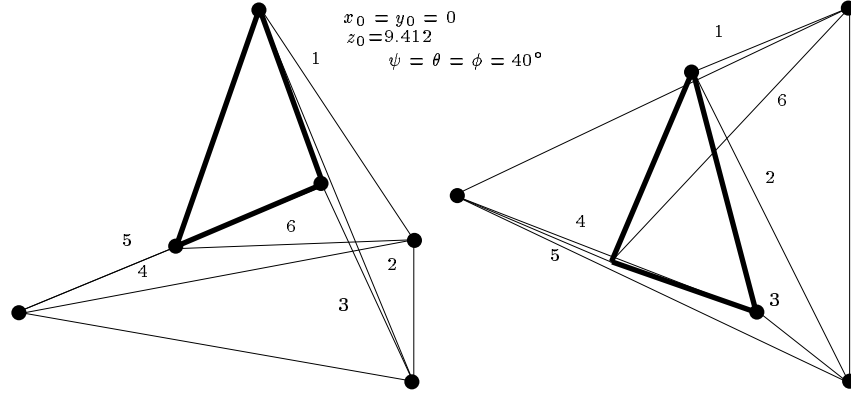


Figure 23: Perspective and top view of the first 5a singular configuration

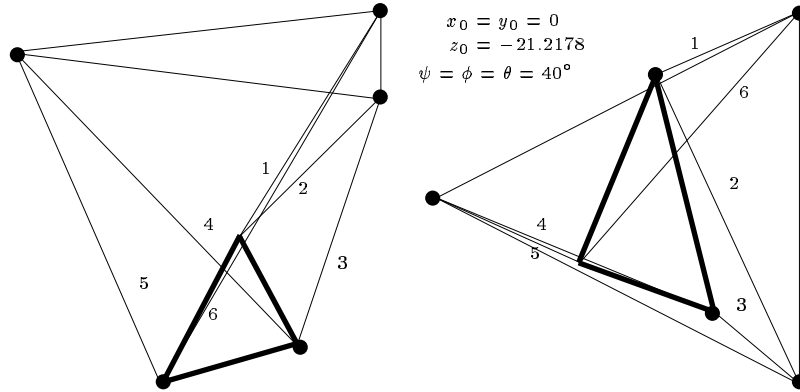


Figure 24: Perspective and top view of the second 5a singular configuration

4 lines coplanar ( <i>Hunt configuration</i> )	$A(y_0, z, \psi, \theta, \phi)x_0 + B(y_0, z_0, \psi, \theta, \phi) = 0$ $(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0$
special complex	$U_1x_0^2 + V_1x_0 + W_1 = 0$ $(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0$
general complex	$\psi = \pm \frac{\pi}{2}, \theta = 0, \phi = 0$
general complex (second case)	$a(x_0, y_0, \psi, \theta, \phi)z_0^3 + b(x_0, y_0, \psi, \theta, \phi)z_0^2$ $+d(x_0, y_0, \psi, \theta, \phi)z_0 + e(x_0, y_0, \psi, \theta, \phi) = 0$

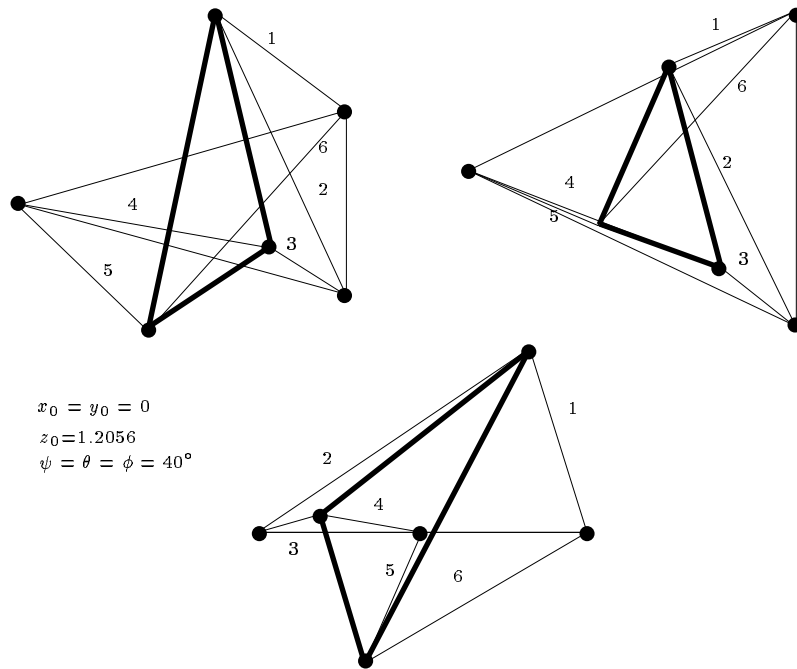


Figure 25: Perspective, top and side view of the third 5a singular configuration

## 7 Appendix

1 )Condition for a MSSM singularity of type 3d

We will consider here that line 2,3,4 are coplanar and that the mobile is rotated around  $B_2B_4$  until point 5 lie in the plane. The others possibilities can be easily deduced from this case by simple rotation.

The condition of 2,3,4 being coplanar is that the projection  $(B_2B_4)^p$  of  $B_2B_4$  on the base plane is parallel to  $A_2A_4$  which yields :

$$(B_2B_4)^p \wedge A_2A_4 = 0 \quad (11)$$

If 5 lie in the plane 2-3-4 then :

$$(A_2B_2 \wedge A_3B_3) \cdot A_5B_5 = 0$$

This two equations define the two constraints on the parameters of the system. We get:

$$(v_2y_3 - v_2y_1 + v_1x_3)(ya_4 - ya_1) + xa_2(v_5y_3 - v_5y_1 + v_4x_3) = 0 \quad (12)$$

```
(c3) /* solve the problem of singularities of type 3d
for the MSSM. Line 2,3,4 are coplanar and point 5
is in the plane of 2,3,4 and thus 2,3,4,5 are coplanar
*/
```

```
/* rotation matrix of the mobile */
```

```
rot: matrix([b1,b2,b3],[b4,b5,b6],[b7,b8,b9])$
```

```
(c4) /* position of the articulation points on the mobile */
```

```
b1r: matrix([0],[y1],[0])$
```

```
(c5) b2r: matrix([0],[y1],[0])$
```

```
(c6) b3r: matrix([x3],[y3],[0])$
```

```
(c7) b4r: matrix([x3],[y3],[0])$
```

```
(c8) b5r: matrix([-x3],[y3],[0])$
```

```
(c9) b6r: matrix([-x3],[y3],[0])$
```

```
(c10) /* position of the articulation points on the base */
```

```
a1: matrix([-xa2],[ya1],[0])$
```

```
(c11) a2: matrix([xa2],[ya1],[0])$
```

```
(c12) a3: matrix([xa2],[ya1],[0])$
```

```
(c13) a4: matrix([0],[ya4],[0])$
```

```
(c14) a5: matrix([0],[ya4],[0])$
```

```
(c15) a6: matrix([-xa2],[ya1],[0])$
```

```
(c16) /* position of the center of the mobile */
```

```
cen: matrix([x0],[y0],[z0])$
```

```
(c17) /* articulation point of the mobile in absolute coordinates */
```

```
bb1: cen+rot.b1r$
```

```
(c18) bb2: cen+rot.b2r$
```

```
(c19) bb3: cen+rot.b3r$
```

```
(c20) bb4: cen+rot.b4r$
```

```
(c21) bb5: cen+rot.b5r$
```

```
(c22) bb6: cen+rot.b6r$
```

```
(c23) a1b1: ev(bb1-a1)$
```

```
(c24) a2b2: ev(bb2-a2)$
```

```
(c25) a3b3: ev(bb3-a3)$
```

```
(c26) a4b4: ev(bb4-a4)$
```

```

(c27) a5b5:ev(bb5-a5)$
(c28) a6b6:ev(bb6-a6)$

(c29) cross(u1,u2,u3):=block(
u3[1]:u1[2]*u2[3]-u1[3]*u2[2],
u3[2]:u1[3]*u2[1]-u1[1]*u2[3],
u3[3]:u1[1]*u2[2]-u1[2]*u2[1]
)$

(c30) dot(u1,u2):=block([p],
p:sum(u1[i]*u2[i],i,1,3),
return(p)
)$
(c31) a2a4:a4-a2$

(c32) /* the projection of B2 on the base plane*/

b2proj:b2$

(c33) b2proj[3,1]:0$

(c34) /* the projection of B4 on the base plane*/

b4proj:b4$

(c35) b4proj[3,1]:0$

(c36) /* the projection of B2B4 on the base plane*/

b2b4proj:b4proj-b2proj$
(c37) /* 2,3,4 coplanar*/

eq1:b2b4proj[1,1]*a2a4[2,1]-b2b4proj[2,1]*a2a4[1,1];

(d37) (v2 y3 - v2 y1 + v1 x3) (ya4 - ya1) + xa2 (v5 y3 - v5 y1 + v4 x3)

(c38) nn1:matrix([0],[0],[0])$

(c39) cross(a2b2,a3b3,nn1)$

(c40) eq2:dot(nn1,a5b5)$

(c41) eq2:eq2[1]$

(c42) ratsimp(eq2,z0);
(d42)((- v2 y3 + v2 y1 - v1 x3) ya4 + (v2 y3 - v2 y1 + v1 x3) ya1
+((2 v1 v5-2 v2 v4) x3-v5 xa2) y3+(v5 xa2+(2 v2 v4 - 2 v1 v5) x3)
y1 - v4 x3 xa2) z0 + ((v8 x0 - v8 xa2) y3
+(v8 xa2 + (v2 v7-v1 v8) x3-v8 x0) y1-v7 x3 xa2 + v7 x0 x3) ya4
+(((2 v1 v8-2 v2 v7) x3-v8 x0) y3+((v2 v7-v1 v8) x3 + v8 x0) y1
- v7 x0 x3) ya1 + ((v8 xa2 + (2 v2 v7 - 2 v1 v8) x3) y0
+ (2 v5 v7 - 2 v4 v8) x3 xa2 + (2 v4 v8 - 2 v5 v7) x0 x3) y3
+(((2 v1 v8-2 v2 v7) x3- v8 xa2) y0 + (v4 v8 - v5 v7) x3 xa2
+ (2 v5 v7 - 2 v4 v8) x0 x3) y1 + v7 x3 xa2 y0

```

## 8 Appendix

2 ) The MSSM as a special complex

In this case 2-3,4 are coplanar and the line which cross  $A_4$  and  $B_2$  has a common point with line 6. The coplanarity of 2-3-4 is given by:

$$(v_2y_3 - v_2y_1 + v_1x_3)(ya_4 - ya_1) + xa_2(v_5y_3 - v_5y_1 + v_4x_3) = 0$$

The other constraint yields to a second degree equation in  $z_0$ .

```
(c3) /* singularite 3d pour leMSSM: 2,3,4,5 coplanaires*/

/* les operations elementaires*/

dot(v1,v2):=block([p],
p:sum(v1[i]*v2[i],i,1,3),
return(p))$

(c4) cross(u1,u2,u3):=block(
u3[1]:u1[2]*u2[3]-u1[3]*u2[2],
u3[2]:u1[3]*u2[1]-u1[1]*u2[3],
u3[3]:u1[1]*u2[2]-u1[2]*u2[1]
)$
(c5) rot:matrix([v1,v2,v3],[v4,v5,v6],[v7,v8,v9])$
(c6) b1r:matrix([0],[y1],[0])$(c12) a1:matrix([-xa2],[ya1],[0])$
(c7) b2r:matrix([0],[y1],[0])$(c13) a2:matrix([xa2],[ya1],[0])$
(c8) b3r:matrix([x3],[y3],[0])$(c14) a3:matrix([xa2],[ya1],[0])$
(c9) b4r:matrix([x3],[y3],[0])$(c15) a4:matrix([0],[ya4],[0])$
(c10) b5r:matrix([-x3],[y3],[0])$(c16) a5:matrix([0],[ya4],[0])$
(c11) b6r:matrix([-x3],[y3],[0])$(c17) a6:matrix([-xa2],[ya1],[0])$

(c18) /* position of the center of the mobile */

cen:matrix([x0],[y0],[z0])$
(c19) /* articulation point of the mobile in absolute coordinates */

b1:cen+rot.b1r$(c25) a1b1:b1-a1$
(c20) b2:cen+rot.b2r$(c26) a2b2:b2-a2$
(c21) b3:cen+rot.b3r$(c27) a3b3:b3-a3$
(c22) b4:cen+rot.b4r$(c28) a4b4:b4-a4$
(c23) b5:cen+rot.b5r$(c29) a5b5:b5-a5$
(c24) b6:cen+rot.b6r$(c30) a6b6:b6-a6$

(c31) a2a4:a4-a2$
(c32) /* the projection of B2 on the base plane*/
b2proj:matrix([b2[1,1]],[b2[2,1]],[0])$

(c33) /* the projection of B4 on the base plane*/
b4proj:matrix([b4[1,1]],[b4[2,1]],[0])$

(c34) /* the projection of B2B4 on the base plane*/
b2b4proj:b4proj-b2proj$

(c35) /* 2,3,4 coplanar*/
eq1:b2b4proj[1,1]*a2a4[2,1]-b2b4proj[2,1]*a2a4[1,1];
(d35) (v2 y3 - v2 y1 + v1 x3) (ya4 - ya1) + xa2 (v5 y3 - v5 y1 + v4 x3)

(c36) /* a line D crossing A4 and B2*/
m:matrix([x],[y],[z])$
```

```

(c37) a4m:m-a4$
(c38) a4b2:b2-a4$
(c40) cross(a4m,a4b2,nn)$
(c41) /* line 6 */
a1m:m-a1$
(c42) a1b5:b5-a1$
(c44) cross(a1m,a1b5,nn1)$
(c45) eq2:nn[1,1]$
(c46) eq3:nn[2,1]$
(c47) eq4:nn1[1,1]$
(c48) linsolve([eq4,eq2,eq3],[x,y,z]),globalsolve:true$
(c49) eq5:nn1[2,1]$
(c52) eq5:ratsimp(num(%),z0)$
(d53)
((v2 y3-v2 y1- v1 x3) ya4 + (- v2 y3 + v2 y1 + v1 x3) ya1 - v5 xa2 y3
      2          2
+ v5 xa2 y1 + v4 x3 xa2) z0 + ((v2 v8 y3
+ (- v2 v8 y1 - v8 xa2 + (- v1 v8 - v2 v7) x3 - v8 x0) y3
      2
+ (v8 xa2 + (2 v2 v7 - v1 v8) x3 + v8 x0) y1 + v7 x3 xa2 + v1 v7 x3
      2
+ v7 x0 x3) ya4 + (- v2 v8 y3 + (v2 v8 y1 + (v1 v8 + v2 v7) x3 + v8 x0) y3
      2
+ ((v1 v8 - 2 v2 v7) x3 - v8 x0) y1 - v1 v7 x3 - v7 x0 x3) ya1
      2
- v5 v8 xa2 y3 + (v5 v8 xa2 y1 + v8 xa2 y0 + (v4 v8 + v5 v7) x3 xa2) y3
      2
+ ((v4 v8 - 2 v5 v7) x3 xa2 - v8 xa2 y0) y1 - v7 x3 xa2 y0 - v4 v7 x3 xa2) z0
      2          2          2          2
+ ((- v8 xa2 - v8 x0) y3 + ((v8 xa2 + (v2 v7 v8 - v1 v8) x3 + v8 x0) y1
+ 2 v7 v8 x3 xa2 + 2 v7 v8 x0 x3) y3 + (- v7 v8 x3 xa2
      2          2          2          2
+ (v1 v7 v8 - v2 v7) x3 - v7 v8 x0 x3) y1 - v7 x3 xa2 - v7 x0 x3) ya4
      2          2          2
+ (v8 x0 y3 + ((v1 v8 - v2 v7 v8) x3 - v8 x0) y1 - 2 v7 v8 x0 x3) y3
      2          2          2
+ ((v2 v7 - v1 v7 v8) x3 + v7 v8 x0 x3) y1 + v7 x0 x3) ya1
      2          2          2
+ v8 xa2 y0 y3 + ((v4 v8 - v5 v7 v8) x3 xa2 - v8 xa2 y0) y1
      2          2
- 2 v7 v8 x3 xa2 y0) y3 + (v7 v8 x3 xa2 y0 + (v5 v7 - v4 v7 v8) x3 xa2) y1
      2          2
+ v7 x3 xa2 y0

```

## 9 Appendix

## 3 ) The MSSM as a general complex: first case

We consider here a MSSM which is able to translate and rotate around the vertical axis we may get a general complex if the rotation angle is  $\frac{\pi}{2}$  or  $-\frac{\pi}{2}$  whatever is the position of its center.

We consider the pencils of lines spanned by 1-6,2-3 and 4-5. Then we consider in each of these pencils the line  $D_i$  which is coplanar to the base. If lines 1-2-3-4-5-6 spanned a general complex then the lines  $D_i$  must have a common point. The following REDUCE program computes the condition for which this is true. We get the following condition :

$$2\cos(\theta)z_0(-x_3ya_1 + x_3ya_4 + xa_1y_1 - xa_1y_3) = 0$$

which is true only for  $\theta = \frac{\pi}{2}$  or  $-\frac{\pi}{2}$

```

comment coordinates of the articulation points on the base$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((-xa1),(ya1),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((0),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((0),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((xa1),(ya1),(0))$b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$
comment : rotation matrix$
rot:=mat((v1,v2,0),(-v2,v1,0),(0,0,1))$

comment: the axis vector of the link$
a1b1:=cen+rot*b1r-a1$a2b2:=cen+rot*b2r-a2$
a3b3:=cen+rot*b3r-a3$a4b4:=cen+rot*b4r-a4$
a5b5:=cen+rot*b5r-a5$a6b6:=cen+rot*b6r-a6$

comment: the normal to the flat pencils 1-6,2-3,4-5$
n16:=mat((a1b1(2,1)*a6b6(3,1)-a6b6(2,1)*a1b1(3,1)),
(a1b1(3,1)*a6b6(1,1)-a6b6(3,1)*a1b1(1,1)),
(a1b1(1,1)*a6b6(2,1)-a6b6(1,1)*a1b1(2,1)))$
n23:=mat((a2b2(2,1)*a3b3(3,1)-a3b3(2,1)*a2b2(3,1)),
(a2b2(3,1)*a3b3(1,1)-a3b3(3,1)*a2b2(1,1)),
(a2b2(1,1)*a3b3(2,1)-a3b3(1,1)*a2b2(2,1)))$
n45:=mat((a4b4(2,1)*a5b5(3,1)-a5b5(2,1)*a4b4(3,1)),
(a4b4(3,1)*a5b5(1,1)-a5b5(3,1)*a4b4(1,1)),
(a4b4(1,1)*a5b5(2,1)-a5b5(1,1)*a4b4(2,1)))$

comment coordinates of the intersection point $
m:=mat((x),(y),(0))$a1m:=m-a1$a2m:=m-a2$a4m:=m-a4$

comment intersection condition $
eq1:=a1m(1,1)*n16(1,1)+a1m(2,1)*n16(2,1)$
eq2:=a2m(1,1)*n23(1,1)+a2m(2,1)*n23(2,1)$
eq3:=a4m(1,1)*n45(1,1)+a4m(2,1)*n45(2,1)$

comment the two first equations yield x and y$
solve(lst(eq1,eq2),x,y)$

comment the third equation yields the condition$
eq4:=num(eq3);
2*v1*z0*( - x3*ya1 + x3*ya4 + xa1*y1 - xa1*y3)

```



## 10 Appendix

4 ) The MSSM as a general complex: general case

This appendix find the condition on  $z_0$  such that the MSSM will be a general complex.

To get this condition we write that the line of the flat pencils spanned by 1-6,2-3,3-4 which lie on the plane of the base must intersect at the same point. This yields to a third degree equation in  $z_0$ .

```

comment coordinates of the articulation points on the base$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((-xa1),(ya1),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((0),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((0),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((xa1),(ya1),(0))$b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$
comment : rotation matrix$
rot:=mat((v1,v2,3),(v4,v5,v6),(v7,v8,v9))$

comment: the axis vector of the link$
b1:=cen+rot*b1r$a1b1:=b1-a1;b2:=cen+rot*b2r$a2b2:=b2-a2;
b3:=cen+rot*b3r$a3b3:=b3-a3;b4:=cen+rot*b4r$a4b4:=b4-a4;
b5:=cen+rot*b5r$a5b5:=b5-a5;b6:=cen+rot*b6r$a6b6:=b6-a6;

comment: the normal to the flat pencils 1-6,2-3,4-5$
n16:=mat((a1b1(2,1)*a6b6(3,1)-a6b6(2,1)*a1b1(3,1)),
(a1b1(3,1)*a6b6(1,1)-a6b6(3,1)*a1b1(1,1)),
(a1b1(1,1)*a6b6(2,1)-a6b6(1,1)*a1b1(2,1)))$
n23:=mat((a2b2(2,1)*a3b3(3,1)-a3b3(2,1)*a2b2(3,1)),
(a2b2(3,1)*a3b3(1,1)-a3b3(3,1)*a2b2(1,1)),
(a2b2(1,1)*a3b3(2,1)-a3b3(1,1)*a2b2(2,1)))$
n45:=mat((a4b4(2,1)*a5b5(3,1)-a5b5(2,1)*a4b4(3,1)),
(a4b4(3,1)*a5b5(1,1)-a5b5(3,1)*a4b4(1,1)),
(a4b4(1,1)*a5b5(2,1)-a5b5(1,1)*a4b4(2,1)))$

comment coordinates of the intersection point $
m:=mat((x),(y),(0))$a1m:=m-a1$a2m:=m-a2$a4m:=m-a4$

comment intersection condition $
eq1:=a1m(1,1)*n16(1,1)+a1m(2,1)*n16(2,1)$
eq2:=a2m(1,1)*n23(1,1)+a2m(2,1)*n23(2,1)$
eq3:=a4m(1,1)*n45(1,1)+a4m(2,1)*n45(2,1)$

comment the two first equations yield x and y$
solve(lst(eq1,eq2),x,y)$
eqfin:=num(eq3)$
3
2 z0 v9 x3 (v1 x3 y1 ya1 - v1 x3 y1 ya4 - v1 x3 y3 ya1 + v1 x3 y3 ya4
2 2 2
- v5 xa1 y1 + 2 v5 xa1 y1 y3 - v5 xa1 y3 ) + 2 z0 x3 (
2 2
- v1 v6 x3 y1 ya1 + v1 v6 x3 y1 ya1 ya4 + v1 v6 x3 y3 ya1 - v1
2 2
v6 x3 y3 ya1 ya4 + v2 v3 v5 xa1 y1 y3 - 2 v2 v3 v5 xa1 y1 y3 + v2
3 2 2
v3 v5 xa1 y3 + v2 v3 v9 x3 y1 ya1 - v2 v3 v9 x3 y1 ya4 - v2 v3
2

```

$$\begin{aligned}
& v_9 x_3 y_1 y_3 y_{a1} + v_2 v_3 v_9 x_3 y_1 y_3 y_{a4} - v_2 v_3 x_{a1} y_1 y_{a4} + 2 v_2 \\
& v_3 x_{a1} y_1 y_3 y_{a4} - v_2 v_3 x_{a1} y_3 y_{a4} - v_3 v_4 v_9 x_3 x_{a1} y_1 + v_3 v_4 \\
& v_9 x_3 x_{a1} y_3 - v_3 v_4 x_3 x_{a1} y_1 + v_3 v_4 x_3 x_{a1} y_3 + v_5 v_6 x_{a1} \\
& y_1 y_3 - 2 v_5 v_6 x_{a1} y_1 y_3 + v_5 v_6 x_{a1} y_3 - v_6 v_9 x_3 y_1 y_{a1} + \\
& v_6 v_9 x_3 y_1 y_{a4} + v_6 v_9 x_3 y_3 y_{a1} - v_6 v_9 x_3 y_3 y_{a4} - v_8 v_9 x_{a1} \\
& y_1 y_{a1} - v_8 v_9 x_{a1} y_1 y_{a4} + 2 v_8 v_9 x_{a1} y_1 y_3 y_{a1} + 2 v_8 v_9 x_{a1} \\
& y_1 y_3 y_{a4} - v_8 v_9 x_{a1} y_3 y_{a1} - v_8 v_9 x_{a1} y_3 y_{a4}) + 2 z_0 x_3 (v_2 v_3 \\
& v_6 x_3 y_1 y_3 y_{a1} - v_2 v_3 v_6 x_3 y_1 y_3 y_{a4} - v_2 v_3 v_6 x_3 y_1 y_3 y_{a1} \\
& + v_2 v_3 v_6 x_3 y_1 y_3 y_{a4} + v_2 v_3 v_8 x_{a1} y_1 y_3 y_{a1} - 2 v_2 v_3 v_8 \\
& x_{a1} y_1 y_3 y_{a1} + v_2 v_3 v_8 x_{a1} y_3 y_{a1} - v_2 v_6 v_7 x_3 y_1 y_{a1} + v_2 \\
& v_6 v_7 x_3 y_1 y_{a1} y_{a4} + v_2 v_6 v_7 x_3 y_1 y_3 y_{a1} - v_2 v_6 v_7 x_3 y_1 y_3 \\
& y_{a1} y_{a4} - v_3 v_9 x_3 x_{a1} y_1 + v_3 v_9 x_3 x_{a1} y_1 y_3 - v_3 x_3 x_{a1} \\
& y_1 + v_3 x_3 x_{a1} y_3 - v_3 v_5 v_7 x_3 x_{a1} y_1 + v_3 v_5 v_7 x_3 x_{a1} y_1 \\
& y_3 - v_3 v_7 v_9 x_3 x_{a1} y_1 y_{a1} + v_3 v_7 v_9 x_3 x_{a1} y_3 y_{a1} - v_3 v_7 x_3 \\
& x_{a1} y_1 y_{a4} + v_3 v_7 x_3 x_{a1} y_3 y_{a4} + v_4 v_6 v_7 x_3 x_{a1} y_1 y_{a4} - v_4 \\
& v_6 v_7 x_3 x_{a1} y_3 y_{a4} + v_5 v_6 x_3 y_1 y_3 y_{a1} - v_5 v_6 x_3 y_1 y_3 y_{a4} \\
& - v_5 v_6 x_3 y_1 y_3 y_{a1} + v_5 v_6 x_3 y_1 y_3 y_{a4} + v_5 v_6 v_8 x_{a1} y_1 \\
& y_3 y_{a1} + v_5 v_6 v_8 x_{a1} y_1 y_3 y_{a4} - 2 v_5 v_6 v_8 x_{a1} y_1 y_3 y_{a1} - 2 v_5 \\
& v_6 v_8 x_{a1} y_1 y_3 y_{a4} + v_5 v_6 v_8 x_{a1} y_3 y_{a1} + v_5 v_6 v_8 x_{a1} y_3 y_{a4} \\
& + v_6 x_3 y_1 y_{a1} - v_6 x_3 y_1 y_{a1} y_{a4} - v_6 x_3 y_3 y_{a1} + v_6 x_3 \\
& y_3 y_{a1} y_{a4} + v_6 v_8 x_{a1} y_1 y_{a1} y_{a4} - 2 v_6 v_8 x_{a1} y_1 y_3 y_{a1} y_{a4} + \\
& v_6 v_8 x_{a1} y_3 y_{a1} y_{a4}) + 2 x_3 (- v_3 v_8 x_3 x_{a1} y_1 y_3 + v_3 v_8 x_3 \\
& x_{a1} y_1 y_3 + v_3 v_6 v_7 x_3 x_{a1} y_1 y_{a4} - v_3 v_6 v_7 x_3 x_{a1} y_1 y_3 \\
& y_{a4} - v_3 v_7 v_8 x_3 x_{a1} y_1 y_{a4} + v_3 v_7 v_8 x_3 x_{a1} y_1 y_3 y_{a4} + v_6 \\
& v_8 x_3 y_1 y_3 y_{a1} - v_6 v_8 x_3 y_1 y_3 y_{a1} y_{a4} - v_6 v_8 x_3 y_1 y_3 \\
& y_{a1} + v_6 v_8 x_3 y_1 y_3 y_{a1} y_{a4} + v_6 v_7 x_3 x_{a1} y_1 y_{a1} y_{a4} - v_6 \\
& v_7 x_3 x_{a1} y_3 y_{a1} y_{a4} + v_6 v_8 x_{a1} y_1 y_3 y_{a1} y_{a4} - 2 v_6 v_8 x_{a1} \\
& y_1 y_3 y_{a1} y_{a4} + v_6 v_8 x_{a1} y_3 y_{a1} y_{a4})
\end{aligned}$$

## 11 Study of the TSSM

We will deal now with the case of the TSSM (Figure 26). Let us make a

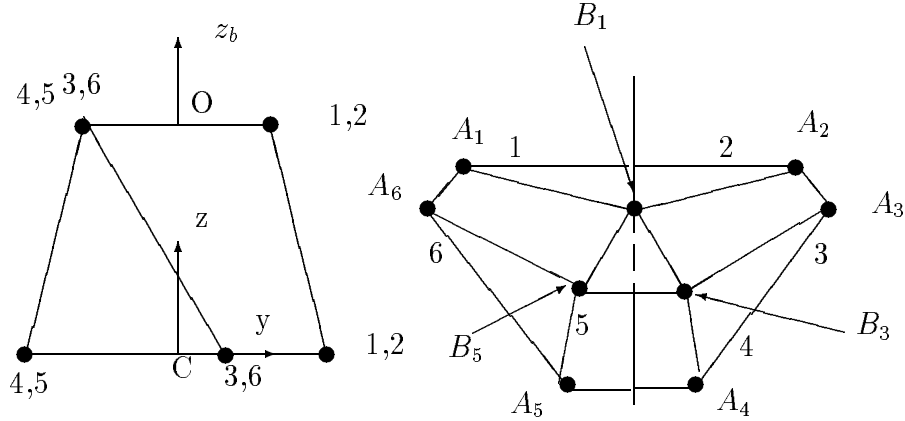


Figure 26: triangular simplified symmetric manipulator TSSM

preliminary remark: for the TSSM we may have at most 2 coplanar lines. Indeed we notice that there is at most two segments with collinear articulation points on the base.

### 11.1 Subset of 3 bars

Three segments must be coplanar : this is impossible.

### 11.2 Subset of 4 bars

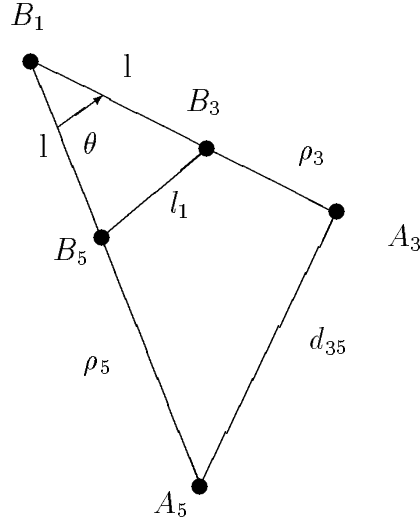
#### 11.2.1 Type 3d

Four segments must be coplanar : this is impossible.

#### 11.2.2 Type 3c

(Four lines cross the same point).

Among a set of four bars two have a common articulation point on the mobile. Thus this common point must be the common point to the four lines. We will assume that the two lines with a common point are 1,2. Lines 3,4 and 5,6 have a common point different from  $B_1$  and thus cannot have another one. Thus the only sets to be considered are  $(1,2,3,5)$ ,  $(1,2,3,6)$ ,  $(1,2,4,5)$  and  $(1,2,4,6)$ . The demonstration of the following result will be the same in each case and we will study only the case of the set  $1,2,3,5$ .

Figure 27: Four lines intersecting  $B_1$ 

If 3 crosses  $B_1$  then 3 is collinear to the edge  $B_1B_3$ . In the same manner if 5 crosses  $B_1$  then 5 is collinear to the edge  $B_1B_5$  and thus 3 and 5 are coplanar. Let us consider now the problem in this plane. For a given position of  $A_3, A_5$  and a geometry of the mobile is it possible to find the links lengths  $\rho_3$  and  $\rho_5$  such that 3,5 intersect  $B_1$  (Figure 27). Let  $d_{35}$  denotes the distance between  $A_3$  and  $A_5$ ,  $l$  and  $l_1$  the dimension of the mobile. We may write :

$$l_1^2 = 2l^2(1 - \cos\theta) \quad (13)$$

$$d_{35}^2 = (l + \rho_3)^2 + (l + \rho_5)^2 - 2(l + \rho_3)(l + \rho_5)\cos\theta \quad (14)$$

which yields :

$$l_1^2 = l^2 \frac{d_{35}^2 - (\rho_3 - \rho_5)^2}{(l + \rho_3)(l + \rho_5)} \quad (15)$$

This equation enable to test quickly if such a configuration is reachable for a given manipulator. We describe in the appendix 5 in section 13, page 44, the relations which describe such a singular configuration. Basically we must have :

$$\tan \psi = \frac{ya_3 - ya_4}{xa_3 + xa_4}$$

and we can compute  $x_0, y_0, z_0$  for a given  $\theta$ . Figure 28 shows such a case.

### 11.2.3 Type 3b

In this case three lines must be coplanar : this is impossible.

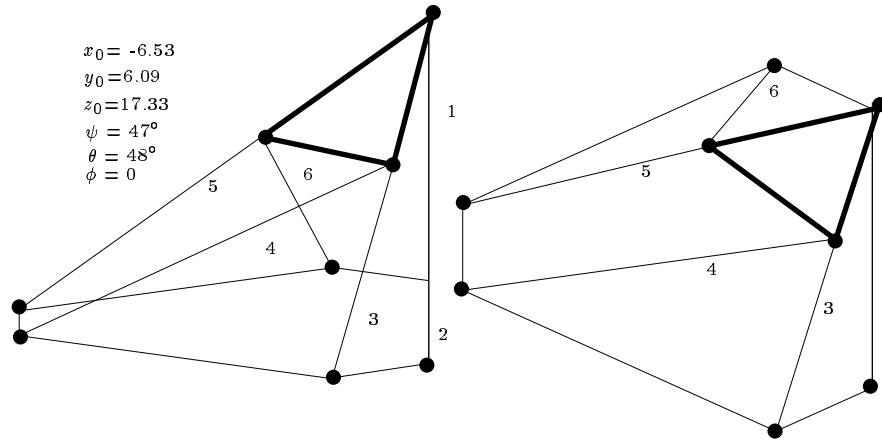


Figure 28: Perspective and top view of a singular configuration of type 3c

### 11.2.4 Type 3a

We will use the same method as for the MSSM. Let us suppose that line 1 belongs to the family (1) spanned by the regulus. Line 2 intersects line 1 and thus belongs to the family (2) spanned by the complementary regulus. For the same reason lines 3,4 and 5,6 cannot belong to the same family. Therefore 4 lines cannot belong to the same regulus.

## 11.3 Set of five bars

### 11.3.1 Configuration 4d

(all lines in a plane or crossing a point of this plane)

Let us remember that we have at most 2 coplanar lines. Among a set of 5 lines two pairs are coplanar and have a common point which is their articulation points on the mobile. These two points being different this imply that the plane to be considered is spanned by a pair of lines and that the common point to the three other lines is the articulation point of the second pair. For example if we consider lines 1,2,3,4,5 we will consider the plane spanned by 1,2, put the articulation point  $B_3$  common to 3,4 in this plane and look if 5 can intersect  $B_3$  (Figure 29).

In appendix 6 in section 14, page 45, we show how we may calculate the value of  $x_0, y_0, z_0$  according to the value of  $\psi, \theta, \phi$  to get such a singular configuration. An example is given in figure 30.

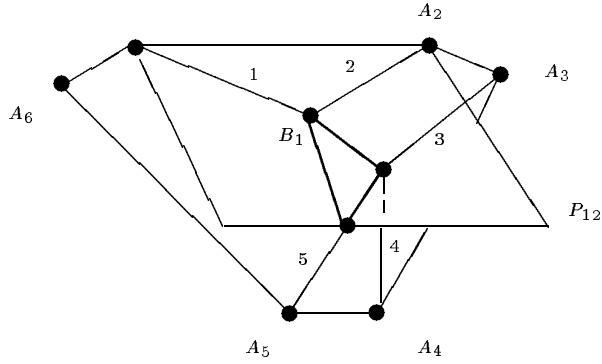
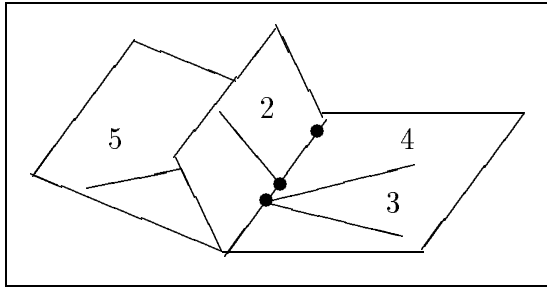


Figure 29: Singular configuration of type 4d

11.3.2 Configuration 4c



In this case we must have three coplanar lines: this is impossible.

11.3.3 Configuration 4b

Let us consider first lines 1,2,3,4. We have to find two skew lines  $D_1, D_2$  which intersect this 4 lines. We have 4 possibilities for a line  $D$  which intersects lines 1,2,3,4:

- $D \in P_{12}$  and  $D$  intersects  $B_3$
- $D \in P_{34}$  and  $D$  intersects  $B_1$
- $D = P_{12} \cap P_{34}$
- $D$  intersects both  $B_1$  and  $B_3$

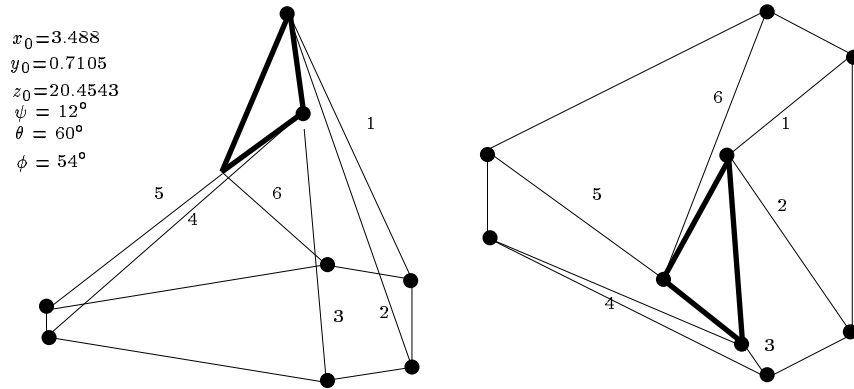


Figure 30: Perspective and top view of a singular configuration of type 4d

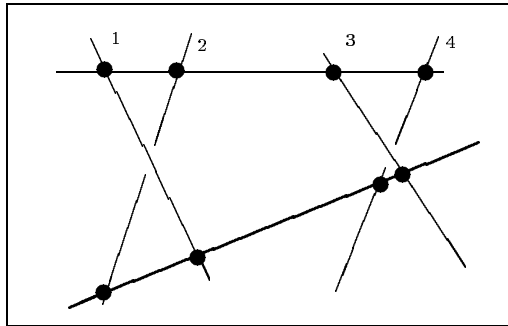


Figure 31: 3-dimensional Grassmann variety of type 4b

**11.3.4  $D_1 \in P_{12}$  and  $D_1$  intersects  $B_3$**

If  $D_2$  is skew to  $D_1$  then  $D_2 \notin P_{12}$  and  $D_2$  does not intersect  $B_3$ . Therefore  $D_2$  cannot be neither  $P_{12} \cap P_{34}$  nor  $B_1B_3$ . Thus the only remaining case is  $D_2 \in P_{34}$  and  $D_2$  intersects  $B_1$ . In this case we have :

$$B_1 \in P_{34} \quad B_1 \in P_{12} \Rightarrow B_1 \in P_{12} \cap P_{34}$$

$$B_3 \in P_{34} \quad B_3 \in P_{12} \Rightarrow B_3 \in P_{12} \cap P_{34}$$

Thus  $C_{23}, B_1, B_3$  belong to the same line. Let  $M_{12}$  be the intersection point of line 5 with  $P_{12}$  and  $M_{34}$  the intersection point of line 5 with  $P_{34}$ . If  $M_{12}$  is different from  $M_{34}$  then the lines  $B_3M_{12}$  and  $B_1M_{45}$  are skew and intersect the lines 1,2,3,4,5. This is then a singular configuration (Figure 32). In appendix 7 in section 15, page 47, we show how to get the value of  $x_0, y_0$  according to  $z_0, \psi, \theta, \phi$ .

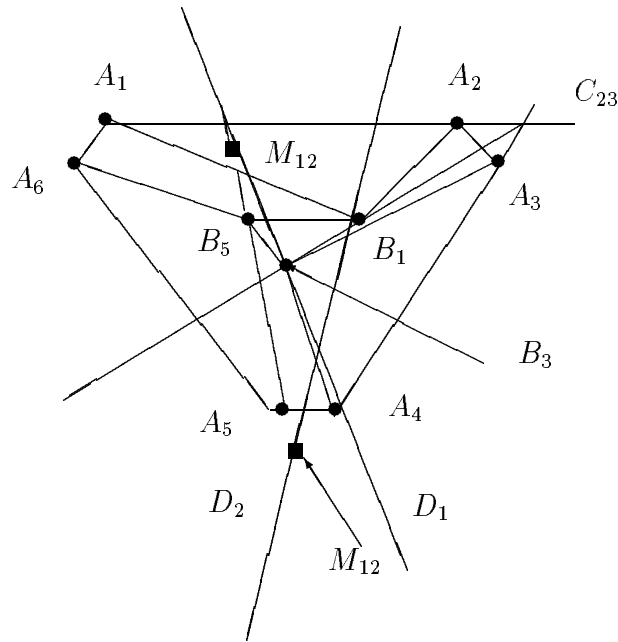


Figure 32: 2 skew lines intersecting 1,2,3,4,5

**11.3.5**  $D_1 \in P_{34}$  and  $B_1 \in D_1$

As in the previous part we get:

$$D_2 \in P_{12} \quad B_3 \in D_2$$

which is the case we investigated above.

**11.3.6**  $D_1 = P_{12} \cap P_{34}$

If  $D_2$  is not to be coplanar with  $D_1$  then we must have:

$$D_2 = B_1B_3$$

But if line 5 intersects  $D_2$  then line 5 and  $B_1B_3$  are coplanar and thus the mobile and line 5 are coplanar. We must then investigate if line 5 intersects  $D_1$ .

Appendix 8 in section 16, page 48, shows how we can verify if we are in such a configuration. Basically the coplanarity of line 5 and the mobile yields:

$$y_0 = -\frac{v9z0 - v6ya4 + v3xa4 + v3x0}{v6}$$



If we write that line 5 intersects  $D_1$  we get four linear equations in term of the intersection point. A numerical resolution of the three first equations give the position of the intersection point and then we have to verify if the fourth equation is satisfied.

### 11.4 Set of six bars

#### 11.4.1 Configuration 5a

In this case the variety spanned by the 6 lines is a general complex. We make the same analysis as for the MSSM. We consider the lines  $D_i$  belonging to the flat pencils spanned by lines 1-2, 3-4, and 5-6 and lying in the mobile plane. We get a general complex if and only if these 3 lines intersects the same point.

We will consider first the case where we have only rotation around the vertical axis. The appendix 9 in section 17, page 50 shows that we get then a singular configuration if we have  $\psi = \pm \frac{\pi}{2}$   $\theta = \phi = 0$  whatever is the position of the center of the mobile.

In a second part we consider the general case. The appendix 10 in section 18, page 52, shows that we must have then either  $\psi = \phi$  or  $\theta = \pm \frac{\pi}{2}$ . In both case we found that  $z_0$  is then solution of a third degree equation. Figure 33 shows a configuration in the first case and Figure 34 shows a configuration in the later case

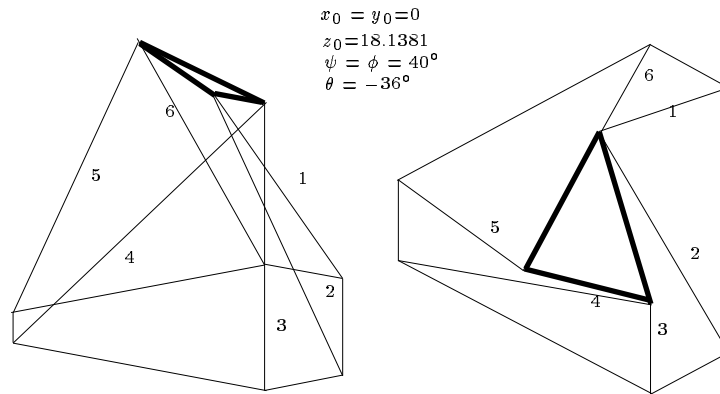


Figure 33: Perspective and top view of a singular configuration of type 5a ( $\psi = \phi$ )

#### 11.4.2 Configuration 5b

We have to consider the case where the 6 segments cross the same line. Let us consider lines 1,2,3,4.

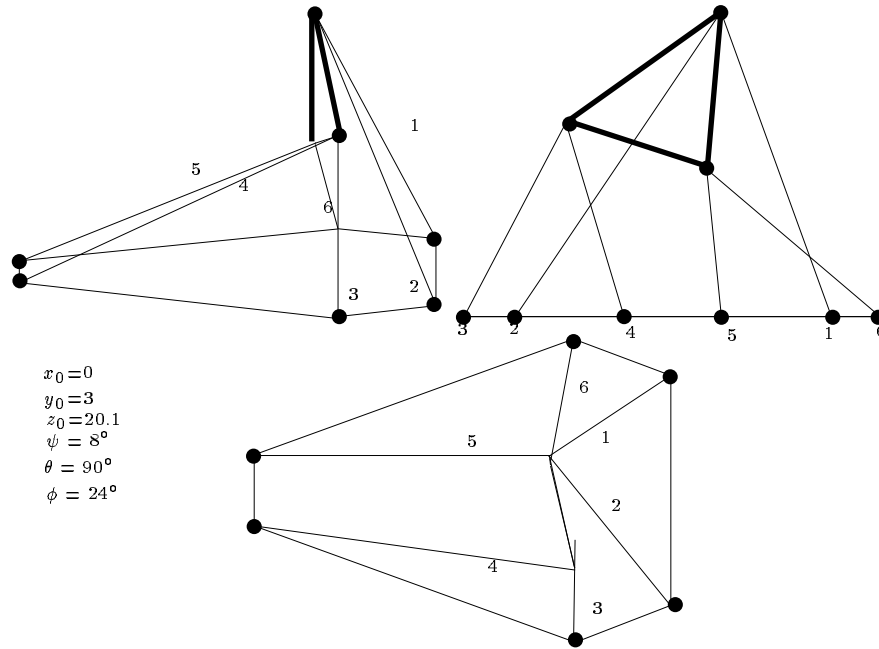


Figure 34: Perspective,side and top view of a singular configuration of type 5a ( $\theta = \pm \frac{\pi}{2}$ )

We have 4 possibilities for line  $D$  to intersect 1,2,3,4:

- $D = P_{12} \cap P_{34}$
- $D$  intersects both  $B_1$  and  $B_3$
- $D \in P_{12}$  and  $D$  intersects  $B_3$
- $D \in P_{34}$  and  $D$  intersects  $B_1$

Let us now consider lines 5,6 in each of these cases.

### 11.5 $D = P_{12} \cap P_{34}$

We may have:

- $B_5 \in P_{12} \cap P_{34}$
- $D = P_{56} \cap P_{12} \cap P_{34}$

In the first case we may deduce from the preliminary remark that line  $D$  intersects both  $B_5$  and  $C_{23}$ . Appendix 11 in section 19, page 54, gives the condition on  $x_0, y_0$  according to  $z_0, \psi, \theta, \phi$  to realize the first case. Figure 36 shows such a singular configuration.

Let us consider now the second case. The three planes must have a line in common. Let us consider the intersection line of plane  $P_{34}, P_{56}$ . We know that  $C_{45}$  belongs to this line. If the intersection line lie also in the plane  $P_{12}$  then

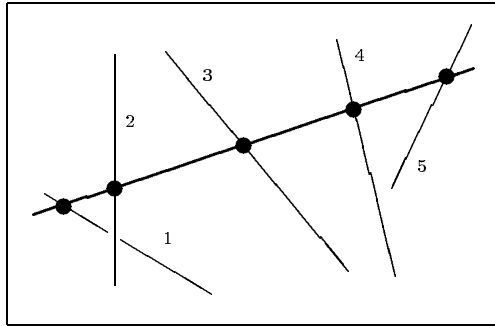


Figure 35: 5-dimensional Grassmann variety of type 5b

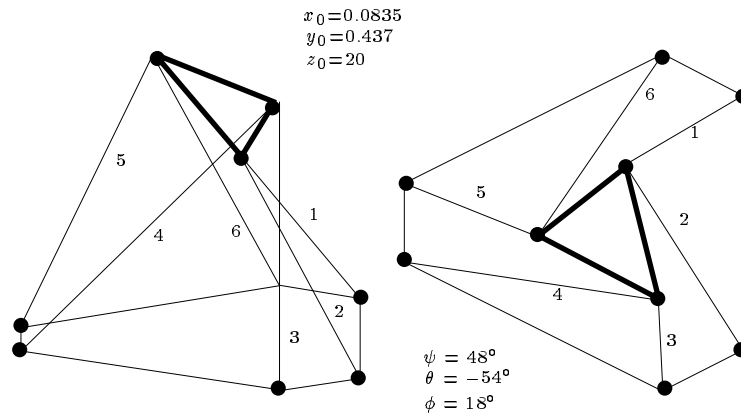


Figure 36: Perspective and top view of a singular configuration of type 5b

$C_{45}$  must also lie in this plane. This is impossible under our assumption and therefore the three planes cannot have a line in common.

### 11.6 $D$ intersects both $B_1$ and $B_3$

Thus the line common to the 6 segments is the edge  $B_1B_3$  of the mobile. If lines 5,6 intersect both this edge this mean that the edge is coplanar to  $P_{56}$ . This is the classical Hunt's singular configuration (Figure 37).

### 11.7 $D \in P_{12}$ and $D$ intersects $B_3$

Thus  $B_3$  belongs to  $P_{12}$ . If  $D$  intersects also 5,6 we may have two possibilities.

- $B_5 \in P_{12}$
- the intersection line  $P_{12} \cap P_{56}$  intersects  $B_3$

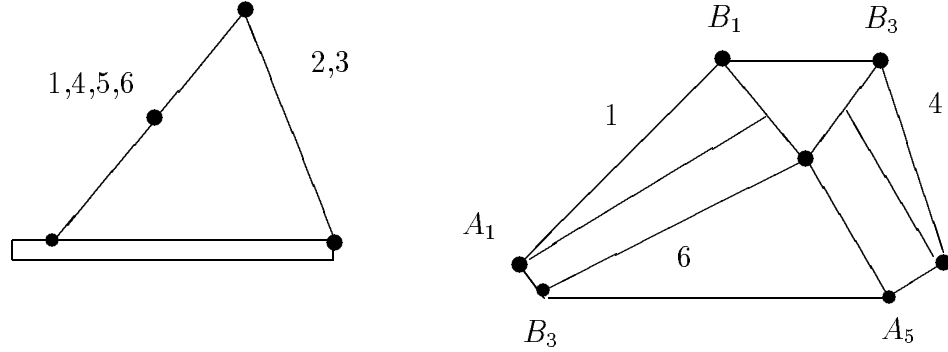


Figure 37: Hunt's singular configuration for the TSSM

In the first case that  $D$  is the edge  $B_3B_5$  of the mobile and two of the segments are coplanar to the mobile. This is again Hunt's singular configuration.

In the second case we may deduce that the intersection line must be the line joining  $C_{16}$  and  $B_3$ . We have dealt with a similar problem in a previous part ( $D = P_{12} \cap P_{34}$  and  $B_5 \in D$ ).

**11.8  $D \in P_{34}$  and  $D$  intersects  $B_1$**

This case is similar to the previous one.

**12 Summary of the results**

The table below summarizes the results.

3c	$\tan \psi = (ya_3 - ya_4)/(xa_3 + xa_4)$ $x_0 = A(\psi, \theta, \phi), y_0 = B(\psi, \theta, \phi), z_0 = C(\psi, \theta, \phi)$
degenerate congruence	$x_0 = A(\psi, \theta, \phi), y_0 = B(\psi, \theta, \phi), z_0 = C(\psi, \theta, \phi)$
hyperbolic congruence (first case)	$x_0 = A(z_0, \psi, \theta, \phi), y_0 = B(z_0, \psi, \theta, \phi)$
hyperbolic congruence (second case)	$y_0 = A(x_0, z_0, \psi, \theta, \phi), F(x_0, z_0, \psi, \theta, \phi) = 0$
general complex (first case)	$\theta = \phi = 0 \quad \psi = \pm \frac{\pi}{2}$
general complex (second case)	$\theta = \pm \frac{\pi}{2}$ or $\psi = \phi$ $A(x_0, y_0, \psi, \theta, \phi)z_0^3 + B(x_0, y_0, \psi, \theta, \phi)z_0^2 + C(x_0, y_0, \psi, \theta, \phi)z_0 + D(x_0, y_0, \psi, \theta, \phi) = 0$
special complex	$x_0 = A(z_0, \psi, \theta, \phi), y_0 = B(z_0, \psi, \theta, \phi)$

## 13 Appendix

5 ) Configuration 3c for the TSSM

In this case line 3 and 5 are coplanar. This yields to the conditions:

$$\mathbf{A}_5 \mathbf{B}_5 \wedge \mathbf{B}_1 \mathbf{B}_5 = 0$$

$$\mathbf{A}_3 \mathbf{B}_3 \wedge \mathbf{B}_1 \mathbf{B}_3 = 0$$

which in turn yield four independent equations. Among these equations (which are linear in  $x_0, y_0, z_0$ ) we choose three of them to find  $x_0, y_0, z_0$ . The last equation give the condition to be fulfilled by the orientation parameters. This last equation is:

$$-(y_1 - y_3)(-v_3 x a_3 - v_3 x a_4 - v_6 y a_3 + v_6 y a_4)(v_7 x_3 + v_8 y_1 - v_8 y_3)(v_7 x_3 - v_8 y_1 + v_8 y_3) = 0$$

The last three terms can be equal to zero. However it can be shown that the second term must be equal to zero.

In term of the Euler's angles this is equivalent to :

$$\sin\theta(-(x a_3 + x a_4)\sin\psi - (y a_4 - y a_3)\cos\psi) = 0$$

If  $\theta = 0$  we get  $z_0 = 0$ , the mobile and the base are coplanar. Otherwise we found :

$$\tan\psi = \frac{y a_3 - y a_4}{x a_3 + x a_4}$$

The values of  $x_0, y_0, z_0$  can be then computed if we fixed  $\theta$ .

comment: line 3,5 are coplanar with the mobile

comment coordinates of the articulation points on the base and the mobile\$

```
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((xa3),(ya3),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((xa4),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((-xa4),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((-xa3),(ya3),(0))$b6r:=mat((-x3),(y3),(0))$
```

comment position of the center of the mobile\$

```
cen:=mat((x0),(y0),(z0))$
```

comment : rotation matrix\$

```
rot:=mat((v1,v2,v3),(v4,v5,v6),(v7,v8,v9))$
```

```
b1:=cen+rot*b1r$a1b1:=b1-a1$
```

```
b2:=cen+rot*b2r$a2b2:=b2-a2$
```

```
b3:=cen+rot*b3r$a3b3:=b3-a3$
```

```
b4:=cen+rot*b4r$a4b4:=b4-a4$
```

```
b5:=cen+rot*b5r$a5b5:=b5-a5$
```

```
b6:=cen+rot*b6r$a6b6:=b6-a6$
```

```
b1b3:=b3-b1$b1b5:=b5-b1$
```

```
nn:=mat((a3b3(2,1)*b1b3(3,1)-b1b3(2,1)*a3b3(3,1)),
(a3b3(3,1)*b1b3(1,1)-b1b3(3,1)*a3b3(1,1)),
(a3b3(1,1)*b1b3(2,1)-b1b3(1,1)*a3b3(2,1)))$
```

```
nn1:=mat((a5b5(2,1)*b1b5(3,1)-b1b5(2,1)*a5b5(3,1)),
(a5b5(3,1)*b1b5(1,1)-b1b5(3,1)*a5b5(1,1)),
(a5b5(1,1)*b1b5(2,1)-b1b5(1,1)*a5b5(2,1)))$
```

```

eq1:=nn(1,1)$
eq2:=nn(2,1)$
eq3:=nn1(1,1)$
eq4:=nn1(2,1)$
solve(lst(eq1,eq2,eq3),x0,y0,z0)$
let x0=soln(1,1)$
let y0=soln(1,2)$
let z0=soln(1,3)$
eqfin:=num(eq4)$
array w(10)$
factorize(eqfin,w)$
let v1*v8-v2*v7=-v6$
let v4*v8-v5*v7=v3$
w(0);
(-1)
w(1);
- v3*xa3 - v3*xa4 - v6*ya3 + v6*ya4
w(2);
v7*x3 + v8*y1 - v8*y3
w(3);
v7*x3 - v8*y1 + v8*y3
w(4);
y1 - y3

let w(1)=0$
x0;
( - v1 v7 x3 ya3 + v1 v7 x3 ya4 - 2 v2 v3 x3 y1 + 2 v2 v3 x3 y1 y3
+ v2 v8 y1 ya3 - v2 v8 y1 ya4 - 2 v2 v8 y1 y3 ya3 + 2 v2 v8 y1 y3
ya4 + v2 v8 y3 ya3 - v2 v8 y3 ya4 + 2 v3 x3 xa3 y1 - 2 v3 x3 xa3 y3
+ v6 x3 y1 ya3 - v6 x3 y1 ya4 - v6 x3 y3 ya3 + v6 x3 y3 ya4)/(2 v3
x3 (y1 - y3))
y0;
( - 2 v3 v5 x3 y1 + 2 v3 v5 x3 y1 y3 + v3 x3 y1 ya3 + v3 x3 y1 ya4 -
v3 x3 y3 ya3 - v3 x3 y3 ya4 - v4 v7 x3 ya3 + v4 v7 x3 ya4 + v5 v8
y1 ya3 - v5 v8 y1 ya4 - 2 v5 v8 y1 y3 ya3 + 2 v5 v8 y1 y3 ya4 + v5
v8 y3 ya3 - v5 v8 y3 ya4)/(2 v3 x3 (y1 - y3))
z0;
( - 2 v3 v8 x3 y1 + 2 v3 v8 x3 y1 y3 - v7 x3 ya3 + v7 x3 ya4 +
v8 y1 ya3 - v8 y1 ya4 - 2 v8 y1 y3 ya3 + 2 v8 y1 y3 ya4 + v8
y3 ya3 - v8 y3 ya4)/(2 v3 x3 (y1 - y3))
end;

```

## 14 Appendix

6 ) Singularity of type 4d for the TSSM
---

We assume here that  $B_3$  lie in the plane spanned by lines 1,2 and that line 5 intersects  $B_3$ . If  $B_3$  lie in the plane  $P_{12}$  then :

$$\mathbf{A}_1 \mathbf{B}_3 \cdot (\mathbf{A}_1 \mathbf{B}_1 \wedge \mathbf{A}_2 \mathbf{B}_1) = 0 \quad (16)$$

If line 5 intersects  $B_3$  then :

$$\mathbf{A}_5 \mathbf{B}_5 \wedge \mathbf{A}_5 \mathbf{B}_3 = 0 \quad (17)$$

Equation 16 is a linear equation in term of  $x_0, y_0, z_0$ . Equation 17 yields two independent equations which are linear in term of  $x_0, y_0, z_0$ . Thus we get a system of three linear equations which permit to find  $x_0, y_0, z_0$  according to  $\psi, \theta, \phi$ .

```

comment coordinates of the articulation points on the base and the mobile$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((xa3),(ya3),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((xa4),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((-xa4),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((-xa3),(ya3),(0))$b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$

comment : rotation matrix$
rot:=mat((v1,v2,v3),(v4,v5,v6),(v7,v8,v9))$
let v4*v8-v5*v7=v3$
let v2*v7-v1*v8=v6$
let v1*v5-v2*v4=v9$
let v1*v3+v4*v6+v3*v9=0$

b1:=cen+rot*b1r$a1b1:=b1-a1$
b2:=cen+rot*b2r$a2b2:=b2-a2$
b3:=cen+rot*b3r$a3b3:=b3-a3$
b4:=cen+rot*b4r$a4b4:=b4-a4$
b5:=cen+rot*b5r$a5b5:=b5-a5$
b6:=cen+rot*b6r$a6b6:=b6-a6$

b1b3:=b3-b1$
b1b5:=b5-b1$

comment: the normal to the flat pencils 1-2,3-4,5-6$
n12:=mat((a1b1(2,1)*a2b2(3,1)-a2b2(2,1)*a1b1(3,1)),
(a1b1(3,1)*a2b2(1,1)-a2b2(3,1)*a1b1(1,1)),
(a1b1(1,1)*a2b2(2,1)-a2b2(1,1)*a1b1(2,1)))$
a5b3:=b3-a5$
a1b3:=b3-a1$
eq1:=n12(1,1)*a1b3(1,1)+n12(2,1)*a1b3(2,1)+n12(3,1)*a1b3(3,1);
nn:=mat((a5b5(2,1)*a5b3(3,1)-a5b3(2,1)*a5b5(3,1)),
(a5b5(3,1)*a5b3(1,1)-a5b3(3,1)*a5b5(1,1)),
(a5b5(1,1)*a5b3(2,1)-a5b3(1,1)*a5b5(2,1)))$
eq2:=nn(1,1);
eq3:=nn(2,1);
solve(lst(eq1,eq2,eq3),x0,y0,z0)$
on gcd;
let x0=soln(1,1)$
let y0=soln(1,2)$
let z0=soln(1,3)$
x0;
y0;
z0;
2
2
2
2

```

$$\begin{aligned} & (-v1*v7*x3*ya1 + v1*v7*x3*ya4 + v2*v7*y1*ya1 - v2*v7*y1*ya4 - v2 \\ & \quad v7^2*y3*ya1 + v2*v7^2*y3*ya4 + v3*v7*v9*x3*y1 - v3*v7*v9*x3*y3 - v3*v7 \\ & \quad *xa4*y1 + v3*v7*xa4*y3 + v3*v8*v9*y1*y3 - v3*v8*v9*y3^2 + v4*v6*v7*x3* \\ & \quad y1 - v4*v6*v7*x3*y3 + v5*v6*v7*y1*y3 - v5*v6*v7*y3^2 - v6*v7*y1*ya1 + \\ & \quad v6*v7*y1*ya4 + v6*v7*y3*ya1 - v6*v7*y3*ya4)/(v3*v7*(y1 - y3)) \end{aligned}$$

$$\begin{aligned} & (-v3*v4*x3*y1 + v3*v4*x3*y3 - v3*v5*y1*y3 + v3*v5*y3^2 + v3*y1*ya1 - \\ & \quad v3*y3*ya1 - v4*v7*x3*ya1 + v4*v7*x3*ya4 + v5*v7*y1*ya1 - v5*v7*y1*ya4 \\ & \quad - v5*v7*y3*ya1 + v5*v7*y3*ya4)/(v3*(y1 - y3)) \end{aligned}$$

$$\begin{aligned} & (-v3*v7*x3*y1 + v3*v7*x3*y3 - v3*v8*y1*y3 + v3*v8*y3^2 - v7^2*x3*ya1 \\ & \quad + v7^2*x3*ya4 + v7*v8*y1*ya1 - v7*v8*y1*ya4 - v7*v8*y3*ya1 + v7*v8*y3 \\ & \quad *ya4)/(v3*(y1 - y3)) \end{aligned}$$

## 15 Appendix

7 Singularity of type 4b for the TSSM, first case

In this case  $B_3$  belongs to  $P_{12}$  and  $B_1$  to  $P_{34}$ . This yields two linear equations which enable to calculate  $z_0, y_0$  according to  $z_0, \psi, \theta, \phi$ . We get:

$$y_0 = \frac{(v5y3 - v5y1 + v4x3)z_0 + (v8y3 - v8y1 + v7x3)ya1 + v3x3y1}{v8y3 - v8y1 + v7x3}$$

and

$$\begin{aligned} x_0 = & ((v2y3 - v2y1 + v1x3)(ya4 - ya3)z_0 \\ & + (v8y3 - v8y1 + v7x3)(xa3ya4 - xa4ya3) + x3y1(v6ya3 - v6ya4) \\ & + (xa4 - xa3)(v8y3 - v8y1 + v7x3)ya1) \\ & /((v8y3 - v8y1 + v7x3)(ya4 - ya3)) \end{aligned}$$

(c3) /\* solve the problem of singularities of type 4b  
for the TSSM.

a line D1 belongs to plane 1,2 and

B3 belongs to D1

a line D2 belongs to plane 3,4 and

B1 belongs to D2

\*/

/\* rotation matrix of the mobile \*/



```

rot:matrix([v1,v2,v3],[v4,v5,v6],[v7,v8,v9])$
(c16) /* position of the center of the mobile */
cen:matrix([x0],[y0],[z0])$
(c31) /*u12: normal to plane 1-2*/
(c32) cross(a1b1,a2b2,u12)$
(c34) /*u34: normal to plane 3-4*/
(c35) cross(a3b3,a4b4,u34)$
(c37) /* a point M belongs to P12 if A1M.u12=0*/
m:matrix([x],[y],[z])$
(c39) eq1:expand(dot(a1m,u12))$
(c41) /*a point M belongs to P34 if A3M.u34=0*/
(c42) eq2:expand(dot(a3m,u34))$
(c44) eq3:ev(eq1,x=b3[1],y=b3[2],z=b3[3])$
(c46) eq4:ev(eq2,x=b1[1],y=b1[2],z=b1[3])$
linsolve([eq3[1][1],eq4[1][1]],[x0,y0]),globalsolve:true$
(c58) y0;
      (v5 y3 - v5 y1 + v4 x3) z0 + (v8 y3 - v8 y1 + v7 x3) ya1 + v3 x3 y1
(d58) -----
              v8 y3 - v8 y1 + v7 x3
x0;
(d123) ((v2 y3 - v2 y1 + v1 x3) (ya4 - ya3) z0
+ (v8 y3 - v8 y1 + v7 x3) (xa3 ya4 - xa4 ya3) + x3 y1 (v6 ya3 - v6 ya4)
+ (xa4 - xa3) (v8 y3 - v8 y1 + v7 x3) ya1)
/((v8 y3 - v8 y1 + v7 x3) (ya4 - ya3))

```

## 16 Appendix

8 ) Singularity 4b for the TSSM, second case

In this case we have :

$$D_1 = P_{12} \cap P_{34}$$

and

$$D_2 = B_1 B_3$$

This last condition means that line 5 and the mobile are coplanar. We get:

$$y_0 = -\frac{b_9 z_0 - b_6 y a_4 + b_3 x a_4 + b_3 x_0}{b_6}$$

The first condition yields four linear equations in term of the coordinates  $x, y, z$  of the intersection point of  $D_1$  and line 5. These equations are denoted  $eq1, eq2, eq3, eq4$  in the following MACSYMA program A numerical procedure may be used to get  $x, y, z$  and verify the last equation.

```
(c3) /* solve the problem of singularities of type 4b
for the TSSM.
D1=P12 inter P34
D2=B1B3 D1,D2 intersects line 5
*/

/* rotation matrix of the mobile */
rot:matrix([v1,v2,v3],[v4,v5,v6],[v7,v8,v9])$

(c16) /* position of the center of the mobile */
cen:matrix([x0],[y0],[z0])$

(c31) /*u12: normal to plane 1-2*/
(c32) cross(a1b1,a2b2,u12)$
(c34) /*u34: normal to plane 3-4*/
(c35) cross(a3b3,a4b4,u34)$

(c37) /*the mobile and A5B5 are coplanar
=> A5B5.(B1B3~B1B5)=0*/
(c40) cross(b1b5,b1b3,pvec1)$
(c41) eq6:dot(pvec1,a5b5)$
(c49) eq6:apply1(eq6,ru1,ru2,ru9,ru10,ru6,ru5)$
(c50) linsolve(eq6,y0),globalsolve:true$
(c51) /* equation of D1*/
m:matrix([x],[y],[z])$
(c53) eq1:ratsimp(dot(a1m,u12))$
(c56) eq2:ratsimp(dot(a4m,u34))$

(c58) /*equation of A5B5*/
(c59) cross(a5b5,a6b6,pvec3)$
(c61) eq3:ratsimp(dot(a5m,pvec3))$
(c62) eq4:ratsimp(dot(a5m,b1b3))$
(c64) y0;
(d64) 
$$-\frac{v9 z0 - v6 ya4 + v3 xa4 + v3 x0}{v6}$$

```

(c65) eq1;

$$(d65) [z (- \frac{2 \text{ xa2} (\text{v9} \text{ z0} - \text{v6} \text{ ya4} + \text{v3} \text{ xa4} + \text{v3} \text{ x0})}{\text{v6}} - 2 \text{ xa2} \text{ ya1} + 2 \text{ v5} \text{ xa2} \text{ y1})$$

$$+ (2 \text{ xa2} \text{ ya1} - 2 \text{ xa2} \text{ y}) \text{ z0} + 2 \text{ v8} \text{ xa2} \text{ y1} \text{ ya1} - 2 \text{ v8} \text{ xa2} \text{ y} \text{ y1}]$$

(c66) eq2;

$$(d66) [z (- \frac{(\text{xa4} - \text{xa3}) (\text{v9} \text{ z0} - \text{v6} \text{ ya4} + \text{v3} \text{ xa4} + \text{v3} \text{ x0})}{\text{v6}}$$

$$+ (- \text{v2} \text{ y3} + \text{xa3} - \text{v1} \text{ x3} - \text{x0}) \text{ ya4} + (\text{v2} \text{ y3} - \text{xa4} + \text{v1} \text{ x3} + \text{x0}) \text{ ya3}$$

$$+ (\text{v5} \text{ xa4} - \text{v5} \text{ xa3}) \text{ y3} + \text{v4} \text{ x3} \text{ xa4} - \text{v4} \text{ x3} \text{ xa3})$$

$$+ ((\text{x} - \text{xa3}) \text{ ya4} + (\text{xa4} - \text{x}) \text{ ya3} + (\text{xa3} - \text{xa4}) \text{ y}) \text{ z0}$$

$$+ ((\text{v8} \text{ x} - \text{v8} \text{ xa3}) \text{ y3} - \text{v7} \text{ x3} \text{ xa3} + \text{v7} \text{ x} \text{ x3}) \text{ ya4}$$

$$+ ((\text{v8} \text{ xa4} - \text{v8} \text{ x}) \text{ y3} + \text{v7} \text{ x3} \text{ xa4} - \text{v7} \text{ x} \text{ x3}) \text{ ya3} + (\text{v8} \text{ xa3} - \text{v8} \text{ xa4}) \text{ y} \text{ y3}$$

$$+ (\text{v7} \text{ x3} \text{ xa3} - \text{v7} \text{ x3} \text{ xa4}) \text{ y}]$$

(c68) eq3;

$$(d68) [((- \text{xa3} - \text{x}) \text{ ya4} + (\text{xa4} + \text{x}) \text{ ya3} + (\text{xa3} - \text{xa4}) \text{ y}) \text{ z0}$$

$$+ ((\text{v2} \text{ y3} + \text{xa3} - \text{v1} \text{ x3} + \text{x0}) \text{ ya4} + (- \text{v2} \text{ y3} - \text{xa4} + \text{v1} \text{ x3} - \text{x0}) \text{ ya3}$$

$$+ (\text{v5} \text{ xa4} - \text{v5} \text{ xa3}) \text{ y3} + (\text{xa4} - \text{xa3}) \text{ y0} - \text{v4} \text{ x3} \text{ xa4} + \text{v4} \text{ x3} \text{ xa3}) \text{ z}$$

$$+ ((- \text{v8} \text{ xa3} - \text{v8} \text{ x}) \text{ y3} + \text{v7} \text{ x3} \text{ xa3} + \text{v7} \text{ x} \text{ x3}) \text{ ya4}$$

$$+ ((\text{v8} \text{ xa4} + \text{v8} \text{ x}) \text{ y3} - \text{v7} \text{ x3} \text{ xa4} - \text{v7} \text{ x} \text{ x3}) \text{ ya3} + (\text{v8} \text{ xa3} - \text{v8} \text{ xa4}) \text{ y} \text{ y3}$$

$$+ (\text{v7} \text{ x3} \text{ xa4} - \text{v7} \text{ x3} \text{ xa3}) \text{ y}]$$

(c70) eq4;

$$(d70) [(\text{v8} \text{ y3} - \text{v8} \text{ y1} + \text{v7} \text{ x3}) \text{ z} + (- \text{v5} \text{ y3} + \text{v5} \text{ y1} - \text{v4} \text{ x3}) \text{ ya4}$$

$$+ (\text{v5} \text{ y} + \text{v2} \text{ xa4} + \text{v2} \text{ x}) \text{ y3} + (- \text{v5} \text{ y} - \text{v2} \text{ xa4} - \text{v2} \text{ x}) \text{ y1} + \text{v4} \text{ x3} \text{ y}$$

$$+ \text{v1} \text{ x3} \text{ xa4} + \text{v1} \text{ x} \text{ x3}]$$

## 17 Appendix

9 ) The TSSM as a general complex,first case

3 lines belonging to the the flat pencils spanned by 1-2,4-3,5-6 lying on the mobile must intersect the same point m if the 6 lines belong to a general complex. We assume there is only rotation around the z axis.

We get the condition:

$$2v1z0(-x3ya3 + x3ya4 - xa3y1 + xa3y3 + xa4y1 - xa4y3) = 0$$

which is true if  $v1 = \cos\psi = 0$  and therefore  $\psi = \pm \frac{\pi}{2}$  (the second term will be zero if the edge of the mobile and the base were parallel).

comment 3 lines belonging to the the flat pencils spanned by 1-2,4-3,5-6 lying on the mobile must intersect the same point m

if the 6 lines belong to a general complex. We assume there  
is only rotation around the z axis\$

comment coordinates of the articulation points on the base and the mobile\$

```
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((xa3),(ya3),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((xa4),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((-xa4),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((-xa3),(ya3),(0))$b6r:=mat((-x3),(y3),(0))$
```

comment position of the center of the mobile\$

```
cen:=mat((x0),(y0),(z0))$
```

comment : rotation matrix\$

```
rot:=mat((v1,v2,0),(-v2,v1,0),(0,0,0))$
let v4*v8-v5*v7=v3$
let v2*v7-v1*v8=v6$
let v1*v5-v2*v4=v9$
let v1*v3+v4*v6+v3*v9=0$
```

comment: the axis vector of the link\$

```
b1:=cen+rot*b1r$a1b1:=b1-a1$
b2:=cen+rot*b2r$a2b2:=b2-a2$
b3:=cen+rot*b3r$a3b3:=b3-a3$
b4:=cen+rot*b4r$a4b4:=b4-a4$
b5:=cen+rot*b5r$a5b5:=b5-a5$
b6:=cen+rot*b6r$a6b6:=b6-a6$
```

```
b1b3:=b3-b1$
```

```
b1b5:=b5-b1$
```

comment: the normal to the flat pencils 1-2,3-4,5-6\$

```
n12:=mat((a1b1(2,1)*a2b2(3,1)-a2b2(2,1)*a1b1(3,1)),
(a1b1(3,1)*a2b2(1,1)-a2b2(3,1)*a1b1(1,1)),
(a1b1(1,1)*a2b2(2,1)-a2b2(1,1)*a1b1(2,1)))$
n34:=mat((a4b4(2,1)*a3b3(3,1)-a3b3(2,1)*a4b4(3,1)),
(a4b4(3,1)*a3b3(1,1)-a3b3(3,1)*a4b4(1,1)),
(a4b4(1,1)*a3b3(2,1)-a3b3(1,1)*a4b4(2,1)))$
n56:=mat((a6b6(2,1)*a5b5(3,1)-a5b5(2,1)*a6b6(3,1)),
(a6b6(3,1)*a5b5(1,1)-a5b5(3,1)*a6b6(1,1)),
(a6b6(1,1)*a5b5(2,1)-a5b5(1,1)*a6b6(2,1)))$
```

comment: normal to the mobile\$

```
normal:=mat((b1b3(2,1)*b1b5(3,1)-b1b5(2,1)*b1b3(3,1)),
(b1b3(3,1)*b1b5(1,1)-b1b5(3,1)*b1b3(1,1)),
(b1b3(1,1)*b1b5(2,1)-b1b5(1,1)*b1b3(2,1)))$
```

comment coordinates of the intersection point \$

```
m:=mat((x),(y),(z0))$
```

```
b1m:=m-b1$
```

```
b3m:=m-b3$
```

```
b5m:=m-b5$
```

comment intersection condition \$

```
eq1:=b1m(1,1)*n12(1,1)+b1m(2,1)*n12(2,1)+b1m(3,1)*n12(3,1)$
eq2:=b3m(1,1)*n34(1,1)+b3m(2,1)*n34(2,1)+b3m(3,1)*n34(3,1)$
```

```

eq3:=b5m(1,1)*n56(1,1)+b5m(2,1)*n56(2,1)+b5m(3,1)*n56(3,1)$
comment: the two equations eq1,eq2 enable to find the coordinates x,y
of the intersection point$
solve(lst(eq1,eq2),x,y)$
let x=soln(1,1)$
let y=soln(1,2)$
comment : intersection condition$
eq3;
2*v1*z0*( - x3*ya3 + x3*ya4 - xa3*y1 + xa3*y3 + xa4*y1 - xa4*y3)
end;

```

## 18 Appendix

10 ) The TSSM as a general complex,second case

3 lines belonging to the the flat pencils spanned by 1-2,4-3,5-6 lying on the mobile must intersect the same point m if the 6 lines belong to a general complex.

We write the following equations :

$$eq1 = B_1 M \wedge N_{12} = 0 \quad (18)$$

$$eq2 = B_1 M \wedge N_m = 0 \quad (19)$$

$$eq3 = B_3 M \wedge N_{34} = 0 \quad (20)$$

$$eq4 = B_3 M \wedge N_m = 0 \quad (21)$$

$$eq5 = B_5 M \wedge N_{56} = 0 \quad (22)$$

$$eq6 = B_4 M \wedge N_m = 0 \quad (23)$$

If we combine equations 19 and 21 we get :

$$v_9 x_3 (v_3 - v_7) = 0$$

which is true if :

$$\theta = 0 \quad \theta = \pm \frac{\pi}{2} \quad \psi = \phi$$

The first case has been treated in the previous appendix.

```

comment coordinates of the articulation points on the base and the mobile$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((xa3),(ya3),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((xa4),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((-xa4),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((-xa3),(ya3),(0))$b6r:=mat((-x3),(y3),(0))$

comment coordinates of the articulation points on the mobile$
b1r:=mat((0),(y1),(0))$
b2r:=mat((0),(y1),(0))$
b3r:=mat((x3),(y3),(0))$
b4r:=mat((x3),(y3),(0))$
b5r:=mat((-x3),(y3),(0))$
b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$

comment : rotation matrix$
rot:=mat((v1,v2,v3),(v4,v5,v6),(v7,v8,v9))$
let v4*v8-v5*v7=v3$

```

```

let v2*v7-v1*v8=v6$
let v1*v5-v2*v4=v9$
let v1*v3+v4*v6+v3*v9=0$

comment: the axis vector of the link$

b1:=cen+rot*b1r$a1b1:=b1-a1$
b2:=cen+rot*b2r$a2b2:=b2-a2$
b3:=cen+rot*b3r$a3b3:=b3-a3$
b4:=cen+rot*b4r$a4b4:=b4-a4$
b5:=cen+rot*b5r$a5b5:=b5-a5$
b6:=cen+rot*b6r$a6b6:=b6-a6$

b1b3:=b3-b1$
b1b5:=b5-b1$

comment: the normal to the flat pencils 1-2,3-4,5-6$
n12:=mat((a1b1(2,1)*a2b2(3,1)-a2b2(2,1)*a1b1(3,1)),
(a1b1(3,1)*a2b2(1,1)-a2b2(3,1)*a1b1(1,1)),
(a1b1(1,1)*a2b2(2,1)-a2b2(1,1)*a1b1(2,1)))$
n34:=mat((a4b4(2,1)*a3b3(3,1)-a3b3(2,1)*a4b4(3,1)),
(a4b4(3,1)*a3b3(1,1)-a3b3(3,1)*a4b4(1,1)),
(a4b4(1,1)*a3b3(2,1)-a3b3(1,1)*a4b4(2,1)))$
n56:=mat((a6b6(2,1)*a5b5(3,1)-a5b5(2,1)*a6b6(3,1)),
(a6b6(3,1)*a5b5(1,1)-a5b5(3,1)*a6b6(1,1)),
(a6b6(1,1)*a5b5(2,1)-a5b5(1,1)*a6b6(2,1)))$

comment: normal to the mobile$

normal:=mat((b1b3(2,1)*b1b5(3,1)-b1b5(2,1)*b1b3(3,1)),
(b1b3(3,1)*b1b5(1,1)-b1b5(3,1)*b1b3(1,1)),
(b1b3(1,1)*b1b5(2,1)-b1b5(1,1)*b1b3(2,1)))$

comment coordinates of the intersection point $
m:=mat((x),(y),(z))$

b1m:=m-b1$
b3m:=m-b3$
b5m:=m-b5$

comment intersection condition $
eq1:=b1m(1,1)*n12(1,1)+b1m(2,1)*n12(2,1)+b1m(3,1)*n12(3,1)$
eq2:=b1m(1,1)*normal(1,1)+b1m(2,1)*normal(2,1)+b1m(3,1)*normal(3,1)$
eq3:=b3m(1,1)*n34(1,1)+b3m(2,1)*n34(2,1)+b3m(3,1)*n34(3,1)$
eq4:=b3m(1,1)*normal(1,1)+b3m(2,1)*normal(2,1)+b3m(3,1)*normal(3,1)$
eq5:=b5m(1,1)*n56(1,1)+b5m(2,1)*n56(2,1)+b5m(3,1)*n56(3,1)$
eq6:=b5m(1,1)*normal(1,1)+b5m(2,1)*normal(2,1)+b5m(3,1)*normal(3,1)$

let v2*v3+v5*v6+v3*v9=0$

eq1;

2*xa1*(-v5*y1*z - v8*y1*ya1 + v8*y1*y - y0*z - ya1*z0 + ya1*z + y*z0)

eq2;
v3*x0 - v3*x + v6*y0 - v6*y + v9*z0 - v9*z

eq3;
-v1*x3*ya3*z + v1*x3*ya4*z - v2*y3*ya3*z + v2*y3*ya4*z + v4*x3*xa3*z

```

```

- v4*x3*xa4*z + v5*xa3*y3*z - v5*xa4*y3*z + v7*x3*xa3*ya4 - v7*x3*xa3
*y - v7*x3*xa4*ya3 + v7*x3*xa4*y + v7*x3*x*ya3 - v7*x3*x*ya4 + v8*xa3*
y3*ya4 - v8*xa3*y3*y - v8*xa4*y3*ya3 + v8*xa4*y3*y + v8*x*y3*ya3 - v8*
x*y3*ya4 - x0*ya3*z + x0*ya4*z + xa3*y0*z + xa3*ya4*z0 - xa3*ya4*z -
xa3*y*z0 - xa4*y0*z - xa4*ya3*z0 + xa4*ya3*z + xa4*y*z0 + x*ya3*z0 - x
*ya4*z0
eq4;
v3*v9*x3 - v3*x0 + v3*x - v6*y0 + v6*y - v7*v9*x3 - v9*z0 + v9*z
eq5;
- v1*x3*ya3*z + v1*x3*ya4*z + v2*y3*ya3*z - v2*y3*ya4*z - v4*x3*xa3*z
+ v4*x3*xa4*z + v5*xa3*y3*z - v5*xa4*y3*z - v7*x3*xa3*ya4 + v7*x3*xa3
*y + v7*x3*xa4*ya3 - v7*x3*xa4*y + v7*x3*x*ya3 - v7*x3*x*ya4 + v8*xa3*
y3*ya4 - v8*xa3*y3*y - v8*xa4*y3*ya3 + v8*xa4*y3*y - v8*x*y3*ya3 + v8*
x*y3*ya4 + x0*ya3*z - x0*ya4*z + xa3*y0*z + xa3*ya4*z0 - xa3*ya4*z -
xa3*y*z0 - xa4*y0*z - xa4*ya3*z0 + xa4*ya3*z + xa4*y*z0 - x*ya3*z0 + x
*ya4*z0
eq2+eq4;
v9*x3*(v3 - v7)
end;

```

## 19 Appendix

### 11 ) Singularity of type 5b for the TSSM

We suppose here that a line  $D$  intersects the 6 segments. This line is the intersection of the planes spanned by 1,2 and 3,4 and  $B_5$  belongs to this line.

The point  $B_5$  belongs to the plane spanned by 1,2 if:

$$\mathbf{A}_1 \mathbf{B}_5 \cdot (\mathbf{A}_1 \mathbf{B}_1 \wedge \mathbf{A}_2 \mathbf{B}_2) = 0 \quad (24)$$

In a similar way  $B_5$  belongs to the plane spanned by 3,4 if:

$$\mathbf{A}_3 \mathbf{B}_5 \cdot (\mathbf{A}_3 \mathbf{B}_3 \wedge \mathbf{A}_4 \mathbf{B}_4) = 0 \quad (25)$$

This system may be solved to give the expression of  $x_0, y_0$  according to  $z_0, \psi, \theta, \phi$ . We get:

$$y_0 = \frac{(b5y_3 - b5y_1 - b4x_3)z_0 + (b8y_3 - b8y_1 - b7x_3)ya_1 - b3x_3y_1}{b8y_3 - b8y_1 - b7x_3} \quad (26)$$

The expression yielding to  $x_0$  is a little bit more complicated :

$$\begin{aligned}
x_0 = & ((v1(v8y_3 - v8y_1 - v7x_3)(ya_4 - ya_3) - 2v3(xa_4 - xa_3)y_3)z_0 \\
& -(v8y_3 - v8y_1 - v7x_3)(v7(-xa_3ya_4 + xa_4ya_3 + (xa_3 - xa_4)ya_1) \\
& + v6y_3(ya_4 - ya_3)) - v3(xa_4 - xa_3)(y_3 - y_1)(v8y_3 - v7x_3)) \\
& / (v7(v8y_3 - v8y_1 - v7x_3)(ya_4 - ya_3))
\end{aligned}$$

```

(c3) /* solve the problem of singularities of type 5b
for the TSSM.
a line D is the intersection of plane 1,2 and
plane 3,4 and D5 intersects B5
*/

/* rotation matrix of the mobile */

rot:matrix([v1,v2,v3],[v4,v5,v6],[v7,v8,v9])$

cen:matrix([x0],[y0],[z0])$

(c31) /*u12: normal to plane 1-2*/

(c32) cross(a1b1,a2b2,u12)$

(c34) /*u34: normal to plane 3-4*/

(c35) cross(a3b3,a4b4,u34)$

(c37) /* a point M belongs to D if A1M.u12=0 and
A3M.u34=0 */

m:matrix([x],[y],[z])$

(c39) eq1:expand(dot(a1m,u12))$

(c41) eq2:expand(dot(a3m,u34))$

(c56) linsolve(eq1[1][1],y0),globalsolve:true$

(c63) linsolve(eq2[1][1],x0),globalsolve:true$

(c65) x0;
(d65)
((v1 (v8 y3 - v8 y1 - v7 x3) (ya4 - ya3) - 2 v3 (xa4 - xa3) y3) z0
-(v8 y3 - v8 y1 - v7 x3) (v7 (-xa3 ya4 + xa4 ya3 + (xa3 - xa4) ya1)
+v6 y3 (ya4 - ya3)) - v3 (xa4 - xa3) (y3 - y1) (v8 y3 - v7 x3))
/(v7 (v8 y3 - v8 y1 - v7 x3) (ya4 - ya3))

(c66) y0;
      (v5 y3 - v5 y1 - v4 x3) z0 + (v8 y3 - v8 y1 - v7 x3) ya1 - v3 x3 y1
(d66) -----
              v8 y3 - v8 y1 - v7 x3

```

## 20 Study of the SSM

We will deal now with the case of the SSM (Figure 38)



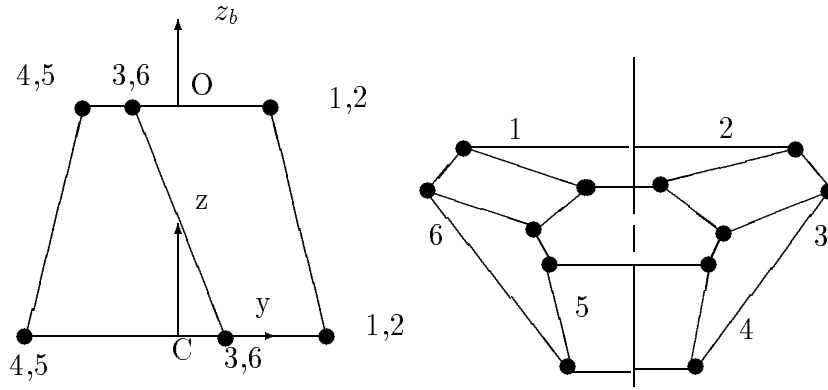


Figure 38: the simplified symmetric manipulator (SSM)

**20.1 Subsets of 2,3 bars**

**20.1.1 Subset of 2 bars**

In the case of 2 bars we have only to count if there is 6 bars in our mechanism. Thus this verification is trivial.

**20.1.2 Subset of 3 bars**

There is no set of three lines which have collinear articulation points. Thus the only case where three segments are in the same plane is obtained when the base and the mobile are in the same plane. In this case all the lines are in this plane.

**20.2 Subsets of 4 bars**

**20.2.1 Degeneracy of type 3d**

We have seen in the preceding part that 4 segments will lie in a plane if and only if the mobile and the base are in the same plane. Thus we have a singular configuration if the mobile and the base are in the same plane.

**20.2.2 Degeneracy of type 3c**

Let us say to begin with that lines 1,2,3,4 have a common point M. In this case we have:

$$\mathbf{A}_1\mathbf{M} = \lambda_1\mathbf{n}_1 \quad \mathbf{A}_2\mathbf{M} = \lambda_2\mathbf{n}_2 \quad (27)$$

where  $\lambda_1$  denotes the distance from the articulation points  $A_1$  to the intersection point M. This yield to :

$$\mathbf{A}_1\mathbf{A}_2 \wedge \mathbf{n}_2 = \lambda_1\mathbf{n}_1 \wedge \mathbf{n}_2 \quad (28)$$

where  $\wedge$  denotes the cross product. In the same manner we have:

$$\mathbf{A}_1\mathbf{A}_3 \wedge \mathbf{n}_3 = \lambda_1\mathbf{n}_1 \wedge \mathbf{n}_3 \quad \mathbf{A}_1\mathbf{A}_4 \wedge \mathbf{n}_4 = \lambda_1\mathbf{n}_1 \wedge \mathbf{n}_4 \quad (29)$$

If  $\lambda'_1$  denote the distance between M and the articulation point  $B_1$  we get in the same manner:

$$\mathbf{B}_1\mathbf{B}_2 \wedge \mathbf{n}_2 = \lambda'_1\mathbf{n}_1 \wedge \mathbf{n}_2 \quad \mathbf{B}_1\mathbf{B}_3 \wedge \mathbf{n}_3 = \lambda'_1\mathbf{n}_1 \wedge \mathbf{n}_3 \quad (30)$$

$$\mathbf{B}_1\mathbf{B}_4 \wedge \mathbf{n}_4 = \lambda'_1\mathbf{n}_1 \wedge \mathbf{n}_4 \quad (31)$$

We calculate  $\lambda_1, \lambda'_1$  by:

$$\lambda_1 = \frac{\|\mathbf{A}_1\mathbf{A}_2 \wedge \mathbf{n}_2\|}{\|\mathbf{n}_1 \wedge \mathbf{n}_2\|} \quad \lambda'_1 = \frac{\|\mathbf{B}_1\mathbf{B}_2 \wedge \mathbf{n}_2\|}{\|\mathbf{n}_1 \wedge \mathbf{n}_2\|} \quad (32)$$

where  $\|\cdot\|$  denotes the euclidian norm of a vector. Let us calculate now  $\mathbf{A}_1\mathbf{B}_1$ :

$$\mathbf{A}_1\mathbf{B}_1 = \rho_1\mathbf{n}_1 = \mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_1 \quad (33)$$

which yield to:

$$\rho_1\mathbf{n}_1 \wedge \mathbf{n}_2 = (\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{B}_2\mathbf{B}_1) \wedge \mathbf{n}_2 = (\mathbf{A}_1\mathbf{A}_2 + \mathbf{B}_2\mathbf{B}_1) \wedge \mathbf{n}_2 \quad (34)$$

and therefore

$$\rho_1 = \frac{\|(\mathbf{A}_1\mathbf{A}_2 + \mathbf{B}_2\mathbf{B}_1) \wedge \mathbf{n}_2\|}{\|\mathbf{n}_1 \wedge \mathbf{n}_2\|} \quad (35)$$

but  $\rho_1 = \lambda_1 - \lambda'_1$  and therefore a necessary condition for the intersection of the segment 1 and 2 is:

$$\|(\mathbf{A}_1\mathbf{A}_2 + \mathbf{B}_2\mathbf{B}_1) \wedge \mathbf{n}_2\| = \|\mathbf{A}_1\mathbf{A}_2 \wedge \mathbf{n}_2\| + \|\mathbf{B}_2\mathbf{B}_1 \wedge \mathbf{n}_2\| \quad (36)$$

This mean that either  $A_1A_2$  is parallel to  $B_1B_2$  or  $B_1B_2$  is collinear to either  $A_1B_1$  or  $A_2B_2$ . In the former case M is either  $B_2$  or  $B_1$ . The same reasoning can be made for point 3 and 4 and therefore  $A_3A_4$  is either parallel to  $B_3B_4$  or  $B_3B_4$  is collinear to either  $A_3B_3$  or  $A_4B_4$ .

$\alpha)$   $B_3B_4$  parallel to  $A_3A_4$

This yields to the necessary condition :

$$\psi = \theta = \phi = 0 \quad (37)$$

which mean that the base and the mobile are parallel. From this point we develop equations (2)(3) we find the necessary and sufficient conditions:

$$if \quad \psi = \theta = \phi = 0 \quad xa_2 = xa_4 \quad 2ya_3 - ya_1 - ya_4 = 0 \quad (38)$$

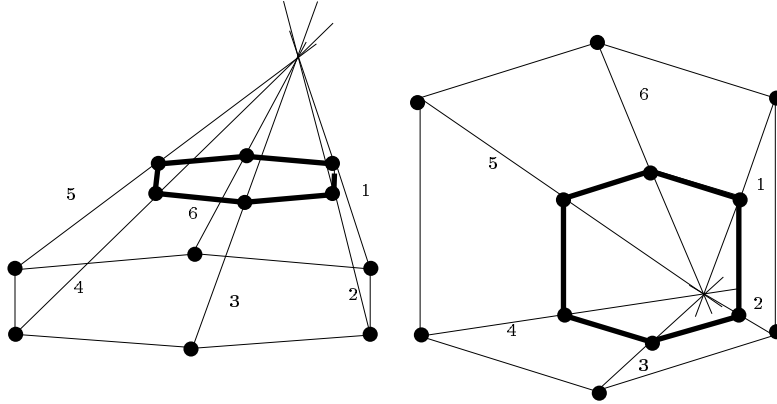


Figure 39: Perspective and top view of a singular configuration of dimension 4

then for every  $x_o, y_o, z_o$  we have a singular configuration. Appendix 12 in section 21, page 21, gives the details of the calculation which is valid whatever is the choice of points. This yields to rule 1.

**Rule 1: The articulation points of a SSM much satisfy the relation  $xa_2 \neq xa_4$  or  $2ya_3 - ya_1 - ya_4 \neq 0$**

It is easy to design a manipulator where these relations are not satisfied. Figure 39 gives an example of such configuration. We may notice that rule 1 is true if the articulation points are equally distributed on the circles. It may be noted that, for symmetry reasons, if 4 lines have a common intersection point then all 6 lines have also this common intersection point. In this latter case we can find rather easily the relation between the links . Let us calculate the difference between  $\rho_1^2$  and  $\rho_4^2$  . We get:

$$\begin{aligned} \|\mathbf{A}_1\mathbf{B}_1\|^2 &= \|\mathbf{A}_1\mathbf{C}\|^2 + \|\mathbf{CO}\|^2 + \|\mathbf{MOB}_{1r}\|^2 + \\ &2\mathbf{CO}.\mathbf{A}_1\mathbf{C} + 2\mathbf{CO}.\mathbf{MOB}_{1r} \end{aligned} \quad (39)$$

$$+ 2\mathbf{A}_1\mathbf{C}.\mathbf{MOB}_{1r} \quad (40)$$

which yields to:

$$\rho_1^2 - \rho_4^2 = 2\mathbf{CO}.\mathbf{A}_1\mathbf{A}_4 + 2\mathbf{CO}.\mathbf{MB}_4\mathbf{B}_1 \quad (41)$$

In the same manner we get:

$$\rho_3^2 - \rho_6^2 = 2\mathbf{CO}.\mathbf{A}_3\mathbf{A}_6 + 2\mathbf{CO}.\mathbf{MB}_6\mathbf{B}_3 \quad (42)$$

Due to the symmetry of the position of the articulation point we have:

$$\mathbf{B}_1\mathbf{B}_4 + \mathbf{B}_3\mathbf{B}_6 = \mathbf{B}_2\mathbf{B}_5 \quad \mathbf{A}_1\mathbf{A}_4 + \mathbf{A}_3\mathbf{A}_6 = \mathbf{A}_2\mathbf{A}_5 \quad (43)$$

which yield to

$$\rho_1^2 - \rho_4^2 + \rho_3^2 - \rho_6^2 = \rho_2^2 - \rho_5^2 \quad (44)$$

Or:

$$\rho_1^2 + \rho_3^2 + \rho_5^2 = \rho_2^2 + \rho_4^2 + \rho_6^2 \quad (45)$$

$\beta$ )  $B_3B_4$  collinear to either  $A_3B_3$  or  $A_4B_4$

In this case  $M$  is either  $B_4$  or  $B_3$ . If the lines  $A_1B_1, A_2B_2$  pass through this point,  $B_1, B_2, B_3, B_4$  being coplanar then all the lines are coplanar. Thus the base and the mobile are in the same plane and we know that this is a singular configuration.

It is clear that the other cases are equivalent to the preceding three cases.

### 20.2.3 Degeneracy of type 3b

In this case 4 lines constitute two flat pencils having a line in common but lying in distinct planes.

We will consider lines 1,2,3,4 without loss of generality. Among these four lines three of them must be coplanar. But there is not three articulation points which are co-linear. Thus to have three coplanar lines the mobile and the base must be coplanar and this a classical singular configuration.

**Degeneracy of type 3a** The problem is to find 4 lines which are on the same regulus.

We will consider without loss of generality the regulus defined by lines 1,2,3 and use the homogeneous coordinates with the reference frames X,Y as lines 1,2 and the origin be the point  $A_3$ . The four homogeneous coordinates  $\lambda_3, \lambda_1, \lambda_2, 1$  of a point M are such that:

$$\mathbf{A}_3\mathbf{M} = \lambda_1\mathbf{A}_3\mathbf{A}_1 + \lambda_2\mathbf{A}_3\mathbf{B}_1 + \lambda_3\mathbf{A}_3\mathbf{A}_2 + 1.\mathbf{A}_3\mathbf{B}_2 \quad (46)$$

For any point  $M$  which belongs to line 3 it exists  $\alpha$  such that:

$$\mathbf{A}_3\mathbf{M} = \alpha\mathbf{A}_3\mathbf{B}_3 \quad (47)$$

Therefore we get:

$$\begin{aligned} \lambda_1 &= \alpha\lambda_1(B3) \\ \lambda_2 &= \alpha\lambda_2(B3) \\ \lambda_3 &= \alpha\lambda_3(B3) \end{aligned} \quad (48)$$

Using the matrix notation we may write:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_{1B_3}}{\lambda_{3B_3}} & 0 \\ \frac{\lambda_{2B_3}}{\lambda_{3B_3}} & 0 \end{pmatrix} \begin{pmatrix} \lambda_3 \\ 1 \end{pmatrix}$$

or:

$$y = A_3x \quad (49)$$

For any point  $M$  which belongs to line 4 it exists  $\beta$  such that:

$$\mathbf{A}_4\mathbf{M} = \beta\mathbf{A}_4\mathbf{B}_4 \Leftrightarrow \mathbf{A}_3\mathbf{M} = \beta\mathbf{A}_4\mathbf{B}_4 + \mathbf{A}_3\mathbf{A}_4 \quad (50)$$

Therefore we get:

$$\begin{aligned} \lambda_1 &= \beta(\lambda_1(B_4) - \lambda_1(A_4)) + \lambda_1(A_4) \\ \lambda_2 &= \beta(\lambda_2(B_4) - \lambda_2(A_4)) + \lambda_2(A_4) \\ \lambda_3 &= \beta(\lambda_3(B_4) - \lambda_3(A_4)) + \lambda_3(A_4) \end{aligned} \quad (51)$$

Using the matrix notation we may write:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \lambda_3 \\ 1 \end{pmatrix}$$

with

$$\begin{aligned} P &= \frac{\lambda_1(B_4) - \lambda_1(A_4)}{\lambda_3(B_4) - \lambda_3(A_4)} \\ Q &= -\lambda_3(A_4)(\lambda_1(B_4) - \lambda_1(A_4)) + \lambda_1(A_4) \\ R &= \frac{\lambda_2(B_4) - \lambda_2(A_4)}{\lambda_3(B_4) - \lambda_3(A_4)} \\ S &= -\lambda_3(A_4)(\lambda_2(B_4) - \lambda_2(A_4)) + \lambda_2(A_4) \end{aligned}$$

We have seen that any generator of the regulus defined by the line 1,2,3 has a matrix equation:

$$y = \gamma A_3 x \quad (52)$$

Thus we may write a necessary condition for a line  $i, i \neq 1, 2, 3$  to belong to the regulus by:

$$P = \gamma \frac{\lambda_{1B_3}}{\lambda_{3B_3}} \quad (53)$$

$$Q = 0 \quad (54)$$

$$R = \gamma \frac{\lambda_{2B_3}}{\lambda_{3B_3}} \quad (55)$$

$$S = 0 \quad (56)$$

From this relation we can get three independent constraint equations in term of the position parameters. These equations are third degree polynomials in term of  $z_0$ . We calculate these equations with the help of REDUCE but the resolution of these equations seems to be rather difficult.

## 20.3 Subsets of 5 bars

### 20.3.1 configuration 4d

All the five lines are in a plane or pass through one point of this plane. Thus we need to have at least three coplanar lines and this not possible for the SSM.

### 20.3.2 configuration 4b

In this case three lines must constitute a flat pencil of lines and therefore this configuration is not to be considered.

### 20.3.3 configuration 4c

Five lines must pass through two skew lines. It is clear that this is a difficult geometric problem in the most general case. However if the orientation angles are equal to zero we have seen that lines 1-2,2-3,3-4,4-5,5-6 have an intersection point and therefore are coplanar. Thus we can apply the reasoning made for the MSSM to state that there is no two skew lines which intersect five segments.

When the base and the mobile are not parallel this problem has not been solved yet.

## 20.4 Subsets of 6 bars

It is clear that we found again Hunt's singular configuration as a special complex. However in the most general case two lines of the SSM being in general no more coplanar the reasoning we use for the TSSM or the MSSM cannot be used here. Thus these configurations are difficult to study.

## 21 Appendix

12 ) Singular configuration of type 3d

SINGULARITY OF RANK 3 FOR THE SSM  
(4 lines pass through the same point)

```
(c61) /* singularity of rank 4: links 1,2,3,4
have a common point.
It has been shown that a necessary condition for this
configuration is that the two plate have parallel sides.
Necessary and sufficient conditions are :
A1A2^n2=lambda1 n1^n2,A1A3^n3=lambda1 n1^n3
,A1A4^n2= lambda1 n1^n4
where n denote the unit vector of the links
,lambda1 a constant
and Ai the articulation point on the fixed plate
l is the homotetic factor between the base
and the mobile*/
```

```
/* rotation matrix */
```

```
rot: matrix([1,0,0],[0,1,0],[0,0,1])$
```

```
(c62) /* relative coordinates of the
articulation point of the
mobile plate ;the parallelism of the side
is expressed by the
fact that these coordinates are homotetic
```

```

to those of the fixed
plate*/

b4r: matrix([ xa2*1],[-ya1*1],[0])$
(c63) b5r: matrix([-xa2*1],[-ya1*1],[0])$
v(c64) b6r: matrix([-xa3*1],[-ya3*1],[0])$
(c65) b1r: matrix([-xa4*1],[-ya4*1],[0])$
(c66) b2r: matrix([ xa4*1],[-ya4*1],[0])$
(c67) b3r: matrix([ xa3*1],[-ya3*1],[0])$

(c68) /* absolute coordinates of the articulation
point on the fixed
plate */

a1: matrix([-xa2],[ya1],[0])$
(c69) a2: matrix([xa2],[ya1],[0])$
(c70) a3: matrix([xa3],[ya3],[0])$
(c71) a4: matrix([xa4],[ya4],[0])$
(c72) a5: matrix([-xa4],[ya4],[0])$
(c73) a6: matrix([-xa3],[ya3],[0])$

(c74) /* unit vector of the link 1,2,3,4 */

matrix([x0-a1[1]],[y0-a1[2]],[z0])$
(c75) n1:(%+rot.b1r)/r1$

(c76) matrix([x0-a2[1]],[y0-a2[2]],[z0])$
(c77) n2:(%+rot.b2r)/r2$

(c78) matrix([x0-a3[1]],[y0-a3[2]],[z0])$
(c79) n3:(%+rot.b3r)/r3$

(c80) matrix([x0-a4[1]],[y0-a4[2]],[z0])$
(c81) n4:(%+rot.b4r)/r4$

(c82) /* the three first equations */

a1a2: matrix([a2[1]-a1[1]],[a2[2]-a1[2]],[a2[3]-a1[3]])$

(c83) vect31: matrix([a1a2[2]*n2[3]-a1a2[3]*n2[2]],
[a1a2[3]*n2[1]-
a1a2[1]*n2[3]],[a1a2[1]*n2[2]-a1a2[2]*n2[1]])$

(c84) matrix([n1[2]*n2[3]-n1[3]*n2[2]],[n1[3]*n2[1]-
n1[1]*n2[3]],[n1[1]*n2[2]-n1[2]*n2[1]])$

(c85) vect32:lambda1*%$

(c86) /* 1st equation*/

vect31[1]= ratsimp (expand(vect32[1]),x0,y0,z0);
(d86)          [[[0]]] = [[[0]]]

/* 2nd equation */
(c87) vect31[2]= ratsimp (expand(vect32[2]),x0,y0,z0);
          2 xa2 z0          2 xa2 z0
(d87)          [[[- -----]]] = [[[- -----]]]
                r2                r2

```

```

/* 3thd equation */

(c88) t3:vect31[3]= ratsimp (expand(ev(vect32[3])),x0,y0,z0);
      2 xa2 (- 1 ya4 - ya1 + y0)
(d88) [[[------]]] =
      r2

      - 2 l xa2 ya4 - 2 xa2 ya1 + 2 xa2 y0
      [[[------]]]
      r2

(c89) /* the former equation give lambda1 */

/* the first 3 equations are always satisfied and enable
to calculate lambda1 */

lambda1: -xa2*r1/(1*xa4-xa2)$

(c90) /* verification of the third equation */

ratsimp (ev(t3));
      2 l xa2 ya4 + 2 xa2 ya1 - 2 xa2 y0
(d90) [[[- -----]]] =
      r2

      2 l xa2 ya4 + 2 xa2 ya1 - 2 xa2 y0
      [[[- -----]]]
      r2

(c91) a1a3: matrix([a3[1]-a1[1]],[a3[2]-a1[2]],
[a3[3]-a1[3]])$

(c92) a1a4: matrix([a4[1]-a1[1]],[a4[2]-a1[2]],
[a4[3]-a1[3]])$

(c93) vect31: matrix([a1a3[2]*n3[3]-a1a3[3]*n3[2]],
[a1a3[3]*n3[1]-
a1a3[1]*n3[3]],[a1a3[1]*n3[2]-a1a3[2]*n3[1]])$

(c94) matrix([n1[2]*n3[3]-n1[3]*n3[2]],[n1[3]*n3[1]-
n1[1]*n3[3]],[n1[1]*n3[2]-n1[2]*n3[1]])$

(c95) vect32:lambda1*%$

(c96) vect41: matrix([a1a4[2]*n4[3]-a1a4[3]*n4[2]],
[a1a4[3]*n4[1]-
a1a4[1]*n4[3]],[a1a4[1]*n4[2]-a1a4[2]*n4[1]])$

(c97) matrix([n1[2]*n4[3]-n1[3]*n4[2]],[n1[3]*n4[1]-
n1[1]*n4[3]],[n1[1]*n4[2]-n1[2]*n4[1]])$

(c98) vect42:lambda1*%$

(c99) /* we get the 6 equations */

vect31[1]-vect32[1]=0$

(c100) ev(%)$

```





$$(1 \ x a^4 \ - \ 1 \ x a^2 \ ) \ y^0 / (1 \ r^4 \ x a^4 \ - \ r^4 \ x a^2) \ ] \ ] = 0$$

## 22 Conclusion

We have proposed a systematic method to find all the singular configurations for a parallel manipulator. This geometrical approach, very different from the classical jacobian analysis, yields to very interesting results.

All the singular configurations for the MSSM and the TSSM are known. We plan to developed a software which will determine, for a given design, if the singular configurations are in the working area. Such tool will be useful for the design and the command of this kind of parallel manipulator.

For the SSM manipulator we found very difficult problems. It remain basically the following geometrical problems :

- 4 lines on a regulus
- the hyperbolic congruence
- the special complex
- the general complex

Our current prototype is a SSM. But this study shows that the use of TSSM seems to be more appropriate. We have some other reasons which justify the use of a TSSM like the possible determination of the jacobian matrix. Therefore our new prototype will be a TSSM.

The use of Grassmann geometry can be generalized to the design of other closed loop mechanisms such as robotics hands.

The last remark is that we use extensively symbolic computation tools in this paper, after the theoretical work. The classical tools like MACSYMA and REDUCE are not very convenient to deal with geometrical objects like lines. We hope that a geometric-oriented symbolic computation program will be available in the near future.

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