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ON POSITIVE OCCUR-CHECKS IN UNIFICATION

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On Positive Occur-Checks in Unification

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Abstract

First-order unification can fail either while simplifying two subterms with distinct head function symbols, or by a so-called positive occur-check for some variable. In this paper we address the problem of classifying these occur-checks. We introduce the notions of *elementary* and *derived* occur-checks. The finite basis of elementary occur-checks for a given unification problem is obtained by a *linearization* process of the input. We first establish that linearization gives unification problems that possess a *single* positive occur-check. Next, we establish the completeness of an equational deduction system well-suited for cyclic equations (= positive occur-checks). Finally, up to permutations, there exists a *minimum* equational deduction associated to an elementary positive occur-check. We give a deterministic algorithm computing this deduction. This problem was encountered while dealing with instances of higher-order unification problems. This technical analysis should also be of interest in symbolic debugging for systems where unification is involved, e.g., the programming languages ML, Prolog, proof-checkers...

Sur les Tests d'Occurrence de l'Unification

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Résumé

L'unification de deux termes peut échouer soit par un conflit entre symboles de fonction ou par un "occur-check" positif. Dans ce rapport nous classifions ces occur-checks en les divisant entre *élémentaires* et *dérivés*. La base finie des occur-checks élémentaires est obtenue en linéarisant les équations de l'input. Dans un premier temps, nous établissons que cette linéarisation permet d'obtenir des problèmes d'unification possédant un *unique* occur-check positif. Puis nous établissons la complétude d'un système de déduction équationnelle, bien adapté à la dérivation des équations cycliques représentant ces occur-checks positifs. Enfin, à permutation près de sous-déductions, il existe une déduction équationnelle minimum pour les occur-checks élémentaires. Nous donnons un algorithme déterministe qui calcule ces déductions. Ces problèmes ont été rencontrés lors de l'étude d'instances d'unification à l'ordre supérieur. Cette analyse technique peut s'appliquer au "debugging" de systèmes formels où intervient l'unification du premier ordre, tels que les langages ML, Prolog, ou des vérificateurs de théorèmes.

PAPIER RECUPERÉ ET RECYCLÉ

1 Introduction

Let $\mathcal{E}$ be a unification problem, i.e. a set of equations on some free algebra of terms. If the simplification step does not fail while the set $\mathcal{E}$ is not unifiable, the algorithm generates a positive occur-check or an equation of the form $x = C[x]$ where $C[\_]$ denotes a non-trivial context. We may incrementally remove such failure cases by *linearizing* the set $\mathcal{E}$: if the variable x possesses at least two distinct occurrences in $\mathcal{E}$, one among those can be replaced with a fresh variable. Also a good definition for an "elementary positive occur-check" is a set of equations that possesses at least one positive occur-check, but such that removing one equation or linearizing one variable yields a unifiable set of equations. The first result of this paper establishes that under this definition the positive occur-check is *unique*. We also address the problem of equational deductions for reasoning about these occur-checks. The principal result here is the existence of a minimum deduction associated to an elementary positive occur-check.

A word on the origin of the problem: it arose while studying instances of higher-order unification in connection with the full-fledged problem of type inference for Girard's higher-order polymorphic λ-calculus F_ω. Higher-order unification is undecidable already at order 2 [5,7]. However, the unification problems involved in type inference possess a shallow first-order structure, given by the constants and the head variables of a unification problem. Also, a natural idea is that a regular structure underlies this search tree, as positive occur-checks are closely related to regular trees. Omitting details, a first step towards this goal is the separability result of section 3. To such elementary occur-checks is canonically associated a finite automaton recognizing derived sets of equations, where the notion of derivation is borrowed from higher-order unification [7]. But, as is easily seen, in presence of several occur-checks, such a language can be non-context-free. The next step is to ensure that the first-order cyclic equations (with infinite solutions) are protected by type lifting: some first-order variables become functional, e.g., an equation $x = f(x, y)$ becomes $X(z) = f(X(t), y)$. Notice that this latter equation has a trivial solution $X = \lambda x.x$ and $z = f(t, y)$. Type lifting is handled through equational deductions: we establish soudness and completeness of an equational deduction system for cyclic equations. Further, for a given elementary cycle, there exists essentially a unique deduction of a cyclic equation associated to the cycle. Therefore, from this deduction, it is sufficient to check that the associated higher-order equation is protected in the way mentionned above.

This paper is organized as follows. Section 2 recalls the definitions and basic results on unification. Especially, we establish technical lemmas relating occurrences of variables in the "unification graph" to occurrences of variables in the input set of equations. We hope that these results will prove useful in other studies of unification. Section 3 establishes the separability result mentionned above. Section 4 proves the soudness and completeness of the equational system. We also define reduced forms for the deductions. As far as we know, a detailed study of this kind of equational reasoning does not appear in the literature. Section 5, the principal part of the paper, proves the unicity of the minimal deduction by providing a deterministic algorithm that finds such a deduction. We conclude by giving an example that will support the intuition of the reader through the technical proofs.

By structuring the set of positive occur-checks, the paper provides a (theoretical) means for debugging in symbolic system involving unification, such as PROLOG, ML or theorem-provers: the basis of elementary cyclic sets is the set of "bugs" introduced by the user in an attempt to prove some formula.

2 First-Order Unification

Throughout the paper, terms in $\mathcal{T}$ are built up over a single binary infix function symbol, denoted by an arrow $\rightarrow$ and a denumerable set of variables $\mathcal{S}$. By the well-known bijective correspondance between k-ary and binary trees, the general case is reducible to this one. We assume known the theory of regular trees and unifying substitutions [2,8]. We first remind a well-known fact about first-order unification of regular trees. These trees are finite or infinite labeled trees with a finite number of distinct subtrees. A unification problem is a set $\mathcal{E}$ of equations of the form $\phi = \tau$, $\phi \in \mathcal{S}$, $\tau \in \mathcal{T}$. The equation is strict when $\tau \in \mathcal{S}$. The set of variables that occur in $\mathcal{E}$ is noted $\mathcal{S}_{\mathcal{E}}$. The size of a term is its number of occurrences of the binary function symbol.

Lemma 2.1 *Any set of equations $\mathcal{E}$ has a most general unifier, mapping variables from $\mathcal{S}_{\mathcal{E}}$ to regular trees.*

Proof. Direct consequence of theorem 4.9.2 p.141 of [2]: if τ and τ' are unifiable regular trees, then their most general unifier is regular. It can be effectively computed and is unique up to a renaming of variables. Presently, we have only one function symbol. Hence the simplification step of unification never fails and two terms always are unifiable as regular trees. □
This theorem tells us nothing about the fine structure of the substitution. A useful tool here is Huet's version of first-order unification [8]. This algorithm computes an equivalence relation on terms, then it checks that the "subterm" relation computed is acyclic. The relation is represented by a graph $\mathcal{G}_{\mathcal{E}} = (V^{\mathcal{E}}, E^{\mathcal{E}})$, whose vertices are congruence classes of terms and egdes encode the subtree relation.

2.1 The Algorithm Ratio

We present the algorithm Ratio from [8]. The input is the set of equations $\mathcal{E}$. The graph $\mathcal{G}_{\mathcal{E}}$ is computed incrementally. To each non-variable subterm of the right-hand sides in $\mathcal{E}$ we associate a unique auxiliary variable from a denumerable set $\mathcal{R}$, disjoint from $\mathcal{S}$. In an obvious way, this defines two maps (subscripts are dropped):

$$\begin{array}{rcll} suc(w_1) & = & w_2 \rightarrow w_3, & w_1 \in \mathcal{R}, w_2, w_3 \in \mathcal{R} \cup \mathcal{S}, \\ val(w) & = & w, & w \in \mathcal{S}, \\ val(w_1) & = & val(w_2) \rightarrow val(w_3), & w_1 \in \mathcal{R},\ suc(w_1) = w_2 \rightarrow w_3. \end{array}$$

Occurrences are sequences of 0's and 1's. The concatenation of occurrences O_1 and O_2 is noted $O_1.O_2$, $|O|$ denotes the length of occurrence O. We use the context notation $C[\_]$ for terms. The occurrence of the hole is noted O_C. The contexts $C_1[\_]$ and $C_2[\_]$ are equivalent, noted $C_1[\_] \sim C_2[\_]$, iff $O_{C_1} = O_{C_2}$. When O_{C_1} is a prefix of O_{C_2}, we write $C_1[\_] \leq C_2[\_]$.

Definition 2.1 *A graph $\mathcal{G}$ is a pair of sets (V, E) such that:*

- *V is a finite set of finite subsets of $\mathcal{S} \cup \mathcal{R}$,*
- *E is a finite set of triples $e = (b, w, w')$ with $b = 0$ or 1, $w \in v \in V$, $w' \in v' \in V$, subject to the condition that if $(b, w, w') \in E$ then there exists $w'' \in v'' \in V$ such that $(\bar{b}, w, w'') \in E$ where $\bar{b} = 0$ (resp. 1) if $b = 1$ (resp. 0), and $(b, w, w'), (b, w, w'') \in E$ implies $w' = w''$.*

By the abstract graph associated to $\mathcal{G}$ we mean the usual underlying graph as a set of vertices together with a edge multiset of pairs of vertices. The vertex v of the variable w is noted $V(w)$. We also write $V(\tau)$ if $\tau = val(w)$. The source and target of an edge (i, w, w') are $V(w)$ and $V(w')$ respectively. The algorithm is first-order unification without occur-check. We set $V^0 = \{\{w\} | w \in \mathcal{S} \cup \mathcal{R}\}$ and $E^0 = \{(0, w, w'),\ (1, w, w'') \mid suc(w) = w' \rightarrow w''\}$. Each vertex has a representative, initialized to its unique variable. The representative of w is noted $\bar{w}$. We define $E_0 = \{w_1 = w_2 | w_1 \in \mathcal{S},\ w_2 \in \mathcal{S} \cup \mathcal{R},\ w_1 = val(w_2) \in \mathcal{E}\}$. Each equation e in E_0 has a level l_e, initially set to 0.

RATIO

Input E_0; V^0; E^0;
$i := 0$;

Step 1 If $E_i = \emptyset$
Then Stop;
Else $E_i := E_i - \{e : w_1 = w_2\}$ with $e \in E_i$, l_e maximal;
If $\bar{w}_1 = \bar{w}_2$
Then $V^{i+1} := V^i$; $E^{i+1} := E^i$; $E_{i+1} := E_i$;
$i := i + 1$; Return to Step 1;

Step 2 If $suc(\bar{w}_1) = w_3 \rightarrow w_4$ and $suc(\bar{w}_2) = w_5 \rightarrow w_6$
Then $E_{i+1} := E_i \cup \{e_1 : w_3 = w_5, e_2 : w_4 = w_6\}$;
$l_{e_1} := l_{e_2} := l_e + 1$;
$E^{i+1} := E^i - \{(0, \bar{w}_2, \bar{w}_5),\ (1, \bar{w}_2, \bar{w}_6)\}$;
Else $E^{i+1} := E^i$;
$E_{i+1} := E_i$;
$v := V(w_1) \cup V(w_2)$;
$V^{i+1} := (V^i - \{V(w_1), V(w_2)\}) \cup \{v\}$;
The representant $\bar{w}$ of v is $\bar{w}_1$ if $suc(\bar{w}_1)$ is defined, $\bar{w}_2$ otherwise;
Replace $\bar{w}_1$ and $\bar{w}_2$ by $\bar{w}$ in E^{i+1};
$i := i + 1$; Return to Step 1 □

The proofs of termination and correctness of Ratio can be found in [2,7]. The equations e_1 (e_2) possibly created in Step 2 will be refered to as the left (right) equation created at iteration i.

2.2 Ratio Properties

In the sequel N denotes the number of iterations of Ratio. The level l_i of the ith iteration is the level of the equation selected by this iteration, with $l_N = 0$. To an iteration $i < N$ we associate the first iteration $j > i$ such that $l_j \leq l_i$, this iteration is noted i^+. The graph at ith iteration is defined by $G^i = (V^i, E^i)$, the values of these sets are taken at the beginning of the ith iteration. Here are some immediate properties of these graphs:

1. $\forall v, v' \in V^i$, $v \neq \emptyset$; if $v \neq v'$ then $v \cap v' = \emptyset$;
2. $\forall v \in V^i$, $\exists! v' \in V^j$, $j \geq i$, $v \subseteq v'$;
3. $\forall (i, w_1, w_2) \in E^j$, $\exists! (i, w_1', w_2') \in E^k$, $k \geq j$, such that $V^j(w_1) \subseteq V^k(w_1')$ and, if $k \geq j^+$, $V^j(w_2) \subseteq V^k(w_2')$;

4. $|E^i| \leq |E^j|$, $|V^i| \leq |V^j|$ if $j \leq i$;

5. $\forall \phi \in \mathcal{S} \cup \mathcal{R}$, the vertex of ϕ in G^i is uniquely determined and noted $V^i(\phi)$; $\forall \phi, \psi$, $V^i(\phi) = V^i(\psi)$ implies $V^j(\phi) = V^j(\psi)$, $V^i(\phi) \subseteq V^j(\phi)$, $j \geq i$;

6. The outdegree of a vertex is 0 or 2; $\forall e_1 = (b_1, w, w_1)$, $e_2 = (b_2, w, w_2) \in E^i$, $b_1 = b_2$ implies $w_1 = w_2$. The edges have the form $(b, \bar{w}, \bar{w}')$, $b = 0, 1$;

7. if $v \in V^i$ possesses two successors, a left one v' and a right one v'', then there exists $w \in v$, $w' \in v'$, $w'' \in v''$ such that $suc(w) = w' \rightarrow w''$;

8. If $suc(w) = w_1 \rightarrow w_2$, $suc(w') = w_3 \rightarrow w_4$ and $V^N(w) = V^N(w')$, there are two equations e, e' in E_0 such that w (resp. w') occurs in the right-hand side of e (resp. e'). Let i be the first level 0 iteration such that e, e' do not belong to E_i, then $V^i(w) = V^i(w')$, $V^i(w_1) = V^i(w_3)$ and $V^i(w_2) = V^i(w_4)$.

We define $\mathcal{S}_v = \mathcal{S} \cap v$ (resp. $\mathcal{R}_v$). A path is a pair $p = (v_0, O)$, v_0 a vertex and O an occurrence $b_1 \cdots b_n$ such that if $v_i = V(w)$ and $b_{i+1} = k$, $0 \leq i < n$, then $suc(\bar{w})$ is defined and $v_{i+1} = V(w_k)$ where $suc(\bar{w}) = w_0 \rightarrow w_1$. The source (resp. target) of p is the vertex v_0 (resp. v_n). The length $|p|$ of the path is equal to n, its sets of variables are the unions $\mathcal{S}_p = \bigcup_{v_i} \mathcal{S}_{v_i}$, $\mathcal{R}_p = \bigcup_{v_i} \mathcal{R}_{v_i}$. We shall identify the path with its sequence of vertices $v_0, \ldots, v_n$. A (fundamental) cycle c is a path such that $v_0 = v_n$ and $v_i \neq v_j$, $0 \leq i < j < n$. Concatenation of paths p_1 and p_2 such that the final vertex of p_1 is the initial vertex of p_2 will be noted $p_1;p_2$. For each w in $\mathcal{R}_c$, we define the leaf occurrence $O(w, c)$ of $val(w)$ along c as $b'_1 \cdots b'_k$ where, if $w \in v_i$, $b'_j = b_{i+j}$ and the occurrence $b'_1 \cdots b'_j$ is an occurrence of $val(w)$. We also use the notation $O(\tau, c)$ if $\tau = val(w)$. Let $\mathcal{B}_\mathcal{E}$ be the set of fundamental cycles of $\mathcal{G}_\mathcal{E}$ [1]. For each cycle c in $\mathcal{B}_\mathcal{E}$, $\mathcal{S}_c$ is non-empty: the cycle c contains at least one vertex with a variable w. If w is in $\mathcal{S}$ we are done. Otherwise the term $val(w)/O(w, c)$ is an $\mathcal{S}$-variable and belongs to the cycle. This merely restates the fact that a cycle corresponds to a positive occur-check in first-order unification. A vertex v is a predecessor (resp. ancestor) of a vertex v' if there is an edge (resp. path) from v to v', resp. successor (resp. descendant). We also use the notation v/O meaning the final vertex of the path (v, O) if well-defined. We say that a vertex v is initial if v is not the target of any edge, terminal if v is not the source of any edge, and internal if it is neither initial nor terminal. Let $e = (b, \bar{w}, \bar{w}')$ be an edge in $\mathcal{G}$. We say that e is incident to the vertex $V(w')$.

Lemma 2.2 *Let v be a vertex of G^i such that $w_1 \neq w_2$ are in v. There exists an iteraton $j < i$ that selects an equation $w = w'$ with $V^j(w) = V^j(w_1)$ and $V^j(w') = V^j(w_2)$. Assume that $suc(w_1) = w_3 \rightarrow w_4$ and $suc(w_2) = w_5 \rightarrow w_6$, then either (i) $V^i(w_3) = V^i(w_5)$ or (ii) there exists $w'_3 \in V(w_3)$ and $w'_5 \in V(w_5)$ such that $w'_3 = w'_5$ belongs to E_i, with a positive level (resp. for w_4 and w_6).*

Proof. Let j be the first iteration such that $V^j(w_1) = V^j(w_2)$. We have $j > 0$. There exists w'_1 and w'_2 such that $w'_1 = w'_2 \in E_{j-1}$ and $V^{j-1}(w'_i) = V^{j-1}(w_i)$, $i = 1, 2$. If both $suc(w_1)$ and $suc(w_2)$ are defined, we have necessarily $suc(\bar{w}'_1) = w'_3 \rightarrow w'_4$ and $suc(\bar{w}'_2) = w'_5 \rightarrow w'_6$. Then E_{j+1} contains $w'_3 = w'_5$ of positive level. Let k be the iteration that selects this equation. If $j \leq i \leq k$, we are in case (ii), if $k < i$ we are in case (i), the case $i < j$ is impossible. □

Definition 2.2 *Let $G = (V, E)$ be a graph and v be in V. The graph $G \downarrow v$ below v defined by (V_v, E_v) where $V_v = \{v' \in V \mid \exists p = v, \ldots, v'\}$ and $E_v = \{(i, w, w') \in E \mid V(w), V(w') \in V_v\}$.*

The graph $G \uparrow v = (V^v, E^v)$ above v is defined by $V^v = (V - V_v) \cup \{v\}$ and $E^v = \{(i, w, w') \in E \mid V(w), V(w') \in V^v\}$.

We will now establish a technical result relating the local structure of the graph $\mathcal{G}_\mathcal{E}$ to the occurrences of variables in the equations E. Notice that the representant $\bar{w}$ of w may change according to the iterations. However, due to the context, the notation is non-ambiguous.

Definition 2.3 *The two edges $e_i = (b_i, w_i, w_i^l) \in E^{j_i}$, $i = 1, 2$, are congruent, noted $e_1 = e_2$, iff $b_1 = b_2$ and the source and target of one edge are included in respectively the source and target of the other edge.*

Every edge in E^{i+1} possesses a congruent edge in E^i. The following observation will be useful throughout the paper. If the level 0 equation $w_1 = w_2$ is selected at iteration i, then the graph $G \downarrow V^i(w_2)$ "starts" with the "tree" $val(w_2)$. The non-terminal vertices of the tree are $\mathcal{R}$-singletons, the internal ones possess a unique predecessor, and the vertex $V^i(w_2)$ is initial if $w_2 \in \mathcal{R}$. The equality between $\mathcal{S}$-variables generated by the strict equations of the input set $\mathcal{E}$ is noted by $=_s$.

Lemma 2.3 *If the equation $e : w_1 = w_2$ is created at iteration j and selected at iteration i, then any equation created at iteration k such that $j < k < i$ is selected at iteration l such that $k < l < i$. Reciprocally, any equation selected at iteration l such that $j < l < i$ is created at iteration k such that $j < k < l$.*

Proof. If e' is created at iteration k, $e \in E^k$ implies $l_{e'} > l_e$. Hence e' is selected before e. If e' is selected at iteration l, then $l_{e'} > l_e$. But when e was created, it was of maximal level in E_{j+1}, also e' is created after iteration j. □

The non-set variables of `Ratio` will be superscripted by iterations, thus denoting their value at this iteration.

Lemma 2.4 *Assume that the equation $e : w_1^i = w_2^i$ is created at iteration j and selected at iteration i, then $V^i(w_1^i)$ possesses $V^i(w_1^j) = V^i(w_2^j)$ as predecessor.*

Proof. If $i = j + 1$ or if the edge $e_1 = (b, \bar{w}_1^j, \bar{w}_1^i)$ is not deleted between iterations j and i, this is immediate. Otherwise, if the edge e_1 is deleted at iteration l, $j < l < i$, $suc(\bar{w}_1^l)$ contains w, $suc(\bar{w}_2^l)$ contains w', and $e_l : w = w'$ is created at iteration l. Further, $V^l(w_1^i) = V^l(w')$ and $V^l(w_1^j) = V^l(w_2^l)$. We have $V^{l+1}(w_1^l) = V^{l+1}(w_1^j) = V^{l+1}(w_2^l)$ and $e_2 = (b, \bar{w}_1^j, \bar{w}) \in E^{l+1}$. The equation e_l is selected strictly before iteration i by Lemma 2.3. Also, there exists an iteration k, $l < k \leq i$, such that $V^k(w_1^i) = V^k(w') = V^k(w)$ possesses an incident edge, which is $e_3 = (b, \bar{w}_1^j, \bar{w}_1^i) \in E^k$. That is $e_1 = e_2 = e_3$. In turn, if this last edge is not deleted we get the result. Otherwise the same reasoning applies and some edge $e_n = (b, \bar{w}, \bar{w}_1^i)$, congruent to e_1, is incident to $V^i(w_1^i)$. □

Corollary 2.5 *Assume that the equation $e_1 : w_1^i = w_2^i$ is created at iteration j and selected at iteration i, then the vertex $V^{i+1}(w_1^i) = V^{i+1}(w_2^i)$ possesses as predecessor the vertex $V^{i+1}(w_1^j) = V^{i+1}(w_2^j)$.*

Proof. Consequence of Lemma 2.4, the last two vertices are merged at iteration $j+1$, the first two ones at $i+1$. □

Lemma 2.6 *Assume that the equation $e_1 : w_1^i = w_2^i$ is created at iteration j, selected at iteration i, and that $V^j(w_2^i)$ possesses a unique incident edge, then $V^j(w_2^i) = V^i(w_2^i)$.*

Proof. The edge $(b, \bar{w}_2^j, \bar{w}_2^i)$ is suppressed at iteration j, and is by hypothesis the only edge incident to $V^j(w_2^i)$. If $i = j+1$, the result is true. Otherwise, at iteration $j+1$, the other equation created at iteration j is selected. Assume that $V^j(w_2^i) = V^{j+1}(w_2^i) \neq V^i(w_2^i)$ and let k be the first iteration such that $V^j(w_2^i) \neq V^k(w_2^i)$. The equation $e : w = w'$ selected at iteration $k-1$ is such that $V^{k-1}(w)$ or $V^{k-1}(w')$ is $V^{k-1}(w_2^i) = V^j(w_2^i)$. But equation e has been created at iteration l such that $j < l < k-1$ by Lemma 2.3. Also, at iteration l, $V^l(w_2^i)$ is non-initial. But $V^j(w_2^i)$ is not. This is possible only if $V^l(w_2^i) \neq V^j(w_2^i)$, which contradicts the minimality of k. □

Corollary 2.7 *Assume that the equation $e : w_1^i = w_2^i$ is created at iteration j and selected at iteration i, then every edge incident to $V^j(w_l^i)$ is congruent to an edge incident to $V^i(w_l^i)$, $l = 1, 2$, with the exception of the edge $(b, V^j(w_2^j), V^j(w_2^i))$ that creates the equation e.*

Proof. Consequence of Lemmas 2.3, 2.4 and 2.6. □

Lemma 2.8 *Let v be an initial vertex of V^i, then $\forall \phi, \psi \in S_v$, $\phi =_s \psi$.*

Proof. By induction on `Ratio`'s iterations. It is true initially as $S_v \neq \emptyset$ implies that this set is singleton. Otherwise, let v be in V^i, $i > 0$, if this set is empty or singleton the result is trivially true. It still is true by induction hypothesis if there exists v' in V^{i-1} such that $\phi, \psi \in v' \subseteq v$. Finally, we have two distinct vertices $V^{i-1}(\phi)$ and $V^{i-1}(\psi)$ that are merged in V^i. By Lemma 2.4 the equation selected at iteration $i-1$ is of level 0. By the observation preceding Lemma 2.3 on the selection of level 0 equations, this equation is strict. The result follows by transitivity and induction hypothesis. □

Lemma 2.9 *Let v be a non-initial vertex of V^i, then $\forall \phi \in S_v\ \exists \psi \in S_v$, $\phi =_s \psi$ and ψ occurs by some edge incident to v.*

Proof. By induction on the iterations. Initially, the lemma is true for variables that occur in some non-strict right-hand side. Assume it true at iterations $j < i+1$. Let $v \in V^{i+1}$ and $\phi \in S_v$. If v also belongs to V^i or if $v = V^i(w_1^i) \cup V^i(w_1^i)$, where $e : w_1^i = w_2^i$ is the equation selected at iteration i, and if ϕ belongs to a non-initial member of this union, we apply the induction hypothesis. Hence remain the non-trivial cases:

1. $\phi \in V^i(w_1^i)$, this vertex is initial but $V^i(w_2^i)$ is not.
2. $\phi \in V^i(w_2^i)$, this vertex is initial but $V^i(w_1^i)$ is not.

In case 1, the equation e is of level 0 by Lemma 2.4. It is strict as $V^i(w_2^i)$ is non-initial (cf. observations before Lemma 2.3). Hence we apply the induction hypothesis to this vertex and the S-variable w_1^i: there exists ψ in $V^i(w_2^i)$ that occurs by some edge and $\psi =_s w_2^i =_s w_1^i$. By Lemma 2.8 we have $\phi =_s w_1^i$. Hence $\phi =_s \psi$.

In case 2, if e is of level 0, it is strict as $\phi \in V^i(w_2^i)$. Applying the induction hypothesis to the vertex $V^i(w_1^i)$ and $w_1^i \in S$ gives $\psi =_s w_1^i =_s w_2^i$. But $\phi =_s w_2^i$ by Lemma 2.8. Otherwise let j be the iteration that creates e. By Corollary 2.7 $\phi \in V^j(w_2^i)$. But this vertex is non-initial. By induction hypothesis and Corollary 2.5 we get the result at iteration $i+1$. □

Let $e = (b, w, w')$ be an edge incident to a vertex v. We say that $\phi \in S_v$ occurs by e iff there exists $w'' \in V(w) \cap \mathcal{R}$ such that $suc(w'') = \phi \rightarrow w'''$ and $b = 0$, or $suc(w'') = w''' \rightarrow \phi$ and $b = 1$. A vertex is shared iff it is the target of two distinct edges. Notice that if v is shared, ϕ can occur by distinct edges. The following Proposition introduces chains of variables, these are sequences of variables that possess multiple occurrences and belong to the same vertex. A weak consequence of the second part of this Proposition can be termed the unique incident edge property for S-free vertices.

Proposition 2.10 *Let E be a set of equations, input of* `Ratio`*, and i be some iteration of* `Ratio`*, then:*

1. *For all edges $e = (b, w, w')$ in E^i such that $V^i(w') \cap S \neq \emptyset$, there exists a variable ϕ in this set that occurs by e.*
2. *For all vertices v in V^i and for all pairs (e, e') of distinct edges in E^i incident to v, there exists a sequence of pairs (ϕ_j, ψ_j) of variables in S_v, $j = 0, \ldots, n-1$, and a sequence of edges (e_k) incident to v, $k = 0, \ldots, n$, $e_0 = e$, $e_n = e'$, such that $\phi_j =_s \psi_j$, ϕ_j occurs by e_j and ψ_j occurs by e_{j+1}.*

Proof. The two propositions are simultaneously proved by induction on the iterations. Their truth at iteration 0 follows from: non S-free vertices are terminal, such vertices are non-initial iff their single S-variable occurs in some non-strict right-hand side, and a terminal vertex is shared iff its S-variable possesses as many occurrences in E as the number of distinct incident edges.

Assume the two propositions true at iterations $j < i+1$. We first prove 1 at iteration $i+1$. Let $e = (b, w, w')$ in E^{i+1} such that $V^{i+1}(w')$ is not S-free.

If $e \in E^i$ and $V^i(w') \cap S \neq \emptyset$, we apply the induction hypothesis.

Otherwise, there is no creation of edges. Hence either $v^i = V^{i+1}(w)$ or $v^i = V^{i+1}(w')$ (where v^i is the value of v in Step 2 of `Ratio`), in both cases there exists an associated edge in E^i.

In the former case, we have at least one of $suc(\bar{w}_1^i)$, $suc(\bar{w}_2^i)$ well-defined. We apply the induction hypothesis to an edge of E^i: $(b, \bar{w}_2^i, w')$ if $suc(\bar{w}_2^i)$ is defined and $suc(\bar{w}_1^i)$ is undefined, $(b, \bar{w}_1^i, w')$ otherwise, in both cases $V^i(w') = V^{i+1}(w')$ is not S-free.

In the latter case, when $(b, w, \bar{w}_l^i) \in E^i$ and $V(w_l^i) \cap S$ is non-empty, $l = 1$ or 2, we apply the induction hypothesis. Otherwise we are led to the non-trivial cases:

1. $(b, w, \bar{w}_1^i) \in E^i$, $V^i(w_1^i)$ is S-free, $V^i(w_2^i)$ is not S-free.
2. $(b, w, \bar{w}_2^i) \in E^i$, $V^i(w_2^i)$ is S-free, $V^i(w_1^i)$ is not S-free.

Let j, if it exists, be the iteration that creates the equation $e_1 : w_1^i = w_2^i$ selected at iteration i.

In case 1, the equation e_1 cannot be of level 0, for this implies $w_1^i \in S$. Also, we claim that $V^j(w_2^i) \cap S \neq \emptyset$. Otherwise, by 2 true at j by induction hypothesis, this vertex possesses a unique incident edge. By Lemma 2.6, we have $V^j(w_2^i) = V^i(w_2^i)$ which is S-free, in contradiction with the hypotheses of case 1. Hence we may apply 1 to the edge $e' = (b', \bar{w}_2^j, \bar{w}_2^i) \in E^j$ with $V^j(w_2^i) \cap S \neq \emptyset$. There exists ϕ in $V^j(w_2^i)$, w'' in $V^j(w_2^j) \cap \mathcal{R}$ such that ϕ occurs in $suc(w'')$, according to e'. Next we

claim that the edges e and e' are congruent. This follows from Lemma 2.4 and the unique incident edge property as $V^i(w_1^i)$ is S-free. We conclude this case by Corollary 2.5.

In case 2, the equation e_1 cannot be of level 0. For, by the observations preceding Lemma 2.3, this equation would be strict as $V^i(w_2^i)$ is non-initial. But this strictness should imply that this vertex is not S-free, contradiction. Thus e_1 has been created at iteration j. We have $V^j(w_2^i)$ is S-free. Hence by 2 and Lemma 2.6, $V^j(w_2^i) = V^i(w_2^i)$. Consequently the vertex $(b, w, \bar{w}_2^i) \in E^i$ also belongs to E^j, perhaps with w replaced by some w'' such that $V^j(w'') \subseteq V^i(w)$. Thus, $V^j(w_2^i)$ possesses two distinct incident edges. By 2, this vertex is not S-free, which is again a contradiction.

We prove 2 at iteration $i+1$. Let $v \in V^{i+1}$ and the two distinct incident edges $e_1 = (b_1, w_1, w)$ and $e_2 = (b_2, w_2, w)$ be in E^{i+1}, $w \in v$. If both edges are in E^i or if e_1', e_2' belong to E^i that differ from e_1, e_2 by w_1 or w_2 such that $V^{i+1}(w_1) = V^{i+1}(w_1')$ or $V^{i+1}(w_2) = V^{i+1}(w_2')$, we apply the induction hypothesis. Otherwise v is the vertex in Step 2 of `Ratio` at iteration i. Let $e : w_1^i = w_2^i$ be the equation selected at iteration i. If the edges $(b_1, w_1, \bar{w}_l^i)$ and $(b_2, w_2, \bar{w}_l^i)$ for $l = 1$ or 2 exist in E^i, we apply the induction hypothesis. Otherwise, we have:

$$e_1' = (b_1, w_1, \bar{w}_1^i) \in E^i,$$

$$e_2' = (b_2, w_2, \bar{w}_2^i) \in E^i.$$

If the equation e is of level 0, then it is strict as $V^i(w_2^i)$ is non-initial. We apply 1 to the edges e_1' and e_2', this gives us ϕ and ψ that occur by e_1' and e_2' respectively. By Lemma 2.9 applied to the two non S-free vertices $V^i(w_1^i)$, $V^i(w_1^i)$, and to the S-variables w_1^i, w_2^i, we get $\phi_1 =_s w_1^i$ and $\psi_1 =_s w_2^i$, where ϕ_1 occurs by e_1'' and ψ_1 by e_2''. By induction hypothesis applied to the pairs of edges (e_1', e_1'') and (e_2'', e_2'), we have two sequences of edges and pairs of variables according to 2. They can be appended by transitivity of $=_s$ and give the result for iteration $i+1$.

Otherwise the equation e is not of level 0. Let j be the iteration that creates the equation e. We know that all edges incident to $V^j(w_l^i)$ are also incident to $V^i(w_l^i)$, $l = 1, 2$ by Corollary 2.7.

If both e_1' and e_2' are such edges, we apply the induction hypothesis at iteration j to the two vertices $V^j(w_1^i)$ and $V^j(w_1^i)$, with respectively the pairs of edges (e_1', e_3), $e_3 = (b, \bar{w}_1^j, \bar{w}_1^i)$, and (e_4, e_2'), $e_4 = (b, \bar{w}_2^j, \bar{w}_2^i)$). The sequences thus obtained give the sequences for iteration $i+1$, by Corollary 2.5 and the congruences $e_1 = e_1' = e_2'$, $e_3 = e_4$, true at $i+1$.

Otherwise, we apply a first time the induction hypothesis to the first iteration $k < i+1$ such that e_1' is incident to $V^k(w_1^i)$ and to the pair (e_1', e_3), notice that $e_3 \in E^k$. The only trouble comes from the other vertex $V(w_2^i)$. Let l_0 be the first iteration such that $e_2' \in E^{l_0}$ is incident to $V^{l_0}(w_2^i)$, $j < l_0 < i+1$. The equation selected at iteration $l_0 - 1$ has been created at iteration k_0 such that $j < k_0 < l_0$ by Lemma 2.3. If the edge in E^{k_0}, incident to $V^{k_0}(w_2^i)$, is not congruent to some edge in E^j incident to $V^j(w_2^i)$, then we consider the first iteration l_1 so that such an edge is incident to $V^{l_1}(w_2^i)$, $j < l_1 < k_0$, and so on.

This gives a sequence (k_m), $m = 0, \ldots, n$, $k_n = j$. At iteration k_m, $0 \leq m < j$, the new vertex that will increase $V^{l_m}(w_2^i)$ and add a new edge incident to this last vertex, possesses two incident edges d_{m+1}' and d_m. The edges d_{m+1} and d_{m+1}' are congruent and identified at iteration k_m. This creates the equation selected at iteration l_m. Hence we apply 2 to these new vertices at iterations k_m, $m = 0, \ldots, n-1$ and to the pairs (d_{m+1}', d_m). The resulting sequences can be appended by the identification of edges. We also apply 2 to the vertex $V^j(w_2^i)$ and the pair (e_4, d_n). Finally we append the sequences to get the result at iteration $i+1$. □

3 Elementary Cyclic Sets of Equations

To a set $\mathcal{E}$ of equations we associate its directed graph $\mathcal{G}_{\mathcal{E}} = (V^{\mathcal{E}}, E^{\mathcal{E}})$ computed by `Ratio` with input $\mathcal{E}$. We shall consider "subgraphs" $\mathcal{G}_E$ for $E \subseteq \mathcal{E}$ that separate the cycles in $\mathcal{B}_{\mathcal{E}}$.

Definition 3.1 *The set E' is a one-step linearization of a set E of equations iff E' is equal to E where some occurrence of a variable in $\mathcal{S}_E$ that occurs at least twice in E has been uniquely renamed. The set E' is a linearization of E iff there is a non-void sequence of one-step linearizations from E to E'.*

An elementary cyclic set E of a set of equations $\mathcal{E}$ is a linearization of some subset of $\mathcal{E}$ such that

1. *$\mathcal{G}_E$ contains at least one cycle,*
2. *the graph $\mathcal{G}_{E'}$ is cycle-free for every linearization E' of E,*
3. *the graph $\mathcal{G}_{E'}$ is cycle-free for every proper subset E' of E.*

A variable $\alpha \in \mathcal{S}$ is needed iff α possesses at least two distinct occurrences in E.
A variable $\alpha \in \mathcal{S}$ is cyclic iff α is needed and $V(\alpha)$ belongs to some cycle.
A variable $w \in \mathcal{R}$ is needed iff $val(w)$ contains at least one occurrence of a needed $\mathcal{S}$-variable.
A vertex v is needed iff it contains at least one needed $\mathcal{S}$-variable.

If $\mathcal{G}_E$ is a dag, so is $\mathcal{G}_{E'}$ for E' a linearization of E. Linearization stepwise removes the cycles. We establish a separability lemma for positive occur-checks.

Theorem 3.1 *Let E be an elementary cyclic set of equations, then $\mathcal{B}_E$ contains a unique cycle. Let c be the cycle of $\mathcal{B}_E$, then each vertex of c contains a unique needed $\mathcal{R}$-variable.*

Proof. Without loss of generality, we assume that the set E does not contain strict equations. We consider two cases: either $\mathcal{G}_E$ contains some initial vertex or it does not.

In the latter case, we establish that each equation in E possesses exactly one marked occurrence in its (non-strict) right-hand side. We select a non $\mathcal{S}$-empty vertex v_0. By Lemma 2.9, there exists some variable ϕ_0 in v_0 that occurs in some non-strict right-hand side of an equation e_0. Let v_1 be the vertex of the left-hand side ϕ_1 of this equation. As we do not have strict equations, by the same Lemma the variable ϕ_1 occurs in some non-strict right-hand side of equation e_1, and so on. The number of equations being finite, let j, n be the smallest integers so that $e_j = e_{j+n}$. This subsequence defines a cycle. By definition of E, each equation occurs exactly once in this subsequence and each left-hand side has exactly two occurrences: one as left-hand side of e_i and one in the right-hand side of e_{i+1}. No other variable occurs twice in E, and this cycle is obviously the only one in $\mathcal{G}_E$.

In the former case, by the preceding observations, for any cycle, there exists some cyclic vertex that possesses at least two incident edges. Such vertices are non $\mathcal{S}$-empty by Proposition 2.10. There exists at least one such vertex v such that some $\mathcal{S}$-variable $\psi \in v$ occurs by an edge not belonging to any cycle. To see this, let v be some initial vertex. The set E being elementary, there exists at least one path from v to some cyclic vertex. Let $p = v, \ldots, v'$ be such a path, minimal in the sense that for no subpath $p' = v, \ldots, v''$ of p we have v'' cyclic. Then v' is shared and is the required vertex. Let ψ be a cyclic variable in this vertex, so that ψ occurs by an edge not belonging

to the cycle. This occurrence belongs to the right-hand side of equation $e : \phi = C[\psi]$. We may further assume that ψ possesses at least two distinct occurrences by Proposition 2.10. Without loss of generality we assume the equation e has a single variable occurrence in its right-hand side whose variable possesses multiple occurrences, namely ψ. If not so the equation e can be replaced by two equations without altering the abstracts graphs of the set E. The resulting set still is elementary cyclic. Such an equation will be said linearized. By definition of an elementary cyclic set, the graph $\mathcal{G}_{E-\{e\}}$ is a dag. We assume that the equation e is the last level 0 equation to be selected by `Ratio`, say at iteration n. Notice that the occurrence O_C is needed: if E is linearized at O_C, then $\mathcal{G}_E$ is acyclic.

Finally, concerning the existence of cycles in the graphs of `Ratio`, we note that the graph G^{i+1} is cyclic while all G^j are dags for $j \leq i$ iff there exists some path $V^i(w_1^i), \ldots, V^i(w_2^i)$ or $V^i(w_2^i), \ldots, V^i(w_1^i)$. In order to complete the proof, we need some technical lemmas on the existence of paths in the graphs G_i.

Lemma 3.2 *Let $i < j < i^+$ be three iterations of* `Ratio`. *There exists two paths $p_1 = V^i(w_k^i), \ldots, V^i(w_1^j)$, $k = 1$ or 2, and $p_2 = V^i(w_l^i), \ldots, V^i(w_2^j)$, $l = 1$ or 2.*

Proof. By induction on $n = j - i$. If $n = 1$ the equation selected at iteration j has been created at iteration i. The two paths are defined by the function *succ*.

Assume the lemma true for $m \leq n-1$. Let $w_1^j = w_2^j$ be the equation selected at iteration j such that $j - i = n$. This equation has been created at some iteration k_0 with $i \leq k_0 < j$. Without loss of generality we assume that this equation is the left one. By induction hypothesis, we have two paths p_l^i ending in $V^i(w_l^{k_0})$, $l = 1, 2$. At iteration k_0 these two vertices possess successors. If each vertex $V^i(w_l^{k_0})$ also possesses a left successor that include the variable w_l^j, the result is immediate. Otherwise, one of the vertices, say $V^i(w_1^{k_0})$ does not possess w_1^j in its (possibly non-existent) left successor. Let k_1 be the first iteration such that w_1^j is in the left successor of $V^{k_1+1}(w_1^{k_0})$. We apply the induction hypothesis to iteration k_1 and consider the path ending in the vertex $V^{k_1+1}(w_l^{k_1})$ that does not contain $w_1^{k_0}$ but possesses two successors, the left one containing w_1^j. If $V^i(w_l^{k_1})$ possesses a left successor containing w_1^j, we get the result. Otherwise we iterate the construction. This halts as $i \leq \cdots < k_1 < k_0 < j$. □

Lemma 3.3 *Let $i < j < i^+$ be three iterations of* `Ratio` *and assume there exists a path $p_j = V^j(w_k^j), \ldots, V^j(w)$, $k = 1$ or 2. There exists a path $p_i = V^i(w_l^i), \ldots, V^i(w)$, $l = 1$ or 2.*

Proof. By Lemma 3.2 there exists a path $p_1 = V^i(w_l^i), \ldots, V^i(w_k^j)$, $l = 1$ or 2. If there exists a path $p_2 = V^i(w_k^j), \ldots, V^i(w)$, we take $p_i = p_1; p_2$. Otherwise, let k_0, $i < k_0 < j$ be the first iteration so that there exists a path $V^{k_0+1}(w_k^j), \ldots, V^{k_0+1}(w)$. By Lemma 3.2, there exists a path $p_1 = V^i(w_l^i), \ldots, V^i(w_m^{k_0})$, $m = 1$ or 2, together with a path $V^{k_0}(w_m^{k_0}), \ldots, V^{k_0}(w)$. If there exists a path $p_2 = V^i(w_m^{k_0}), \ldots, V^i(w)$, we take $p_i = p_1; p_2$. Otherwise we iterate the construction. This halts as $i \leq \cdots < k_1 < k_0 < j$. □

Lemma 3.4 *Let $e : \phi = C[\psi]$ be as in the proof of Theorem 3.1, then there exists an occurrence O, prefix of O_C, and a path $p = v_0, \ldots, v_0'$ in G^n, n the iteration that selects e, with $v_0 = V^n(w)/O$ and $v_0' = V^n(\phi)/O$, or $v_0 = V^n(\phi)/O$ and $v_0' = V^n(w)/O$, where $w \in \mathcal{R}$ is associated to the (non-strict) right-hand side of e.*

Proof. Assume the lemma false. At iterations following the nth one, we assume that equations not along O_C in the "tree" $val(w)$ are first selected by Ratio. By hypothesis, $V^N(\psi)$ is cyclic, while G^n is a dag. If the vertex $V^n(\phi)/O_C$ is undefined, or if one of the vertices $V^n(\phi)/O_C$ and $V^n(\psi) = V^n(w)/O_C$ does not possess successors, then G^N still is acyclic. This is so as 1) the occurrence of ψ in e is needed in the elementary cyclic set E, 2) the lemma is assumed to be false and 3) e is linearized.

Hence, let i be the iteration that selects $w_1^i = w_2^i$ such that $i > n$, no equation selected at iteration j, $n < j < i$, satisfies $V^j(w_1^j) = V^j(w')$ and $V^j(w_2^j) = V^j(\psi)$, w' some variable in $V^n(\phi)/O_C$. We have $V^i(w_1^i) = V^i(w')$ and $V^i(w_2^i) = V^i(\psi)$. The lemma being assumed false, G^i is acyclic. Also, let j, $i < j < i^+ = N$ be the first iteration so that $V^{j+1}(\psi)$ is cyclic. There exists a path p between the vertices of w_1^j and w_2^j, e.g. $p = V^j(w_1^j), \ldots, V^j(w_2^j)$, such that $V^j(\psi) \in p$. By Lemma 3.3 we have a path $p_i = V^i(w_l^i), \ldots, V^i(\psi)$. Necessarily $w_l^i = w_1^i$ as G^i is acyclic. From the observation preceding Lemma 2.3 and the fact that e is linearized, p_i belongs to G^n. This contradicts the assumed falsity of the lemma. □

Lemma 3.5 *Let O be the smallest occurrence satisfying the conditions of Lemma 3.4, then there exists a unique path from v_0 to v_0'.*

Proof. First of all, notice that there does not exist a path from v_0' to v_0 as G^n is acyclic. Further, notice that there does not exist a path from $V^n(\psi)$ to $V^n(\phi)$ by our choice of equation e. Otherwise ϕ would be cyclic. Consequently, $|O| > 0$. Let p_1 be a path from v_0 to v_0'. By O's minimality, the edge of p_1 incident to v_0' is distinct from the edge incident to v_0' along the path $V^n(\phi), \ldots, v_0'$ or $V^n(w), \ldots, v_0'$, i.e. along O_C. Otherwise, we have a path between the predecessors of v_0 and v_0' along O_C, which contradicts O's minimality. Therefore, v_0' posseses two distinct incident edges and is non S-empty by Proposition 2.10. Let p_2 be another path from v_0 to v_0'. Let v be the first vertex above v_0' so that the path $p_3 = v, \ldots, v_0'$ is the maximal suffix path common to both p_1 and p_2. Then v possesses two distinct incident edges e_1 and e_2, $e_1 \in p_1$, $e_2 \in p_2$. Let $\phi_0 \in S_{v_0'}$ that occurs by the common path p_3. Let $e_0 : \psi_0 = C_0[\phi_0]$ be the associated equation. We have two cases: either the path $V^n(\psi_0), \ldots, V^n(\phi_0)$ along O_{C_0} includes p_3 or it does not. In the second case, by Proposition 2.10 and Lemma 2.9, we have a sequence of multiple occurring variables starting with ψ_0 and ending with a variable ϕ_1 that occurs by p_3. Hence this second case ends up in the first one after a sequence of multiple occurring variables along p_3.

The last equation so construed $\psi_m = C_m[\phi_m]$ defines a path $V^n(\psi_m), \ldots, V^n(\phi_m)$ including one of e_1 or e_2, e.g. e_1. Then one occurrence, given by Proposition 2.10, of a variable occurring by e_2 can be linearized. The resulting set still is cyclic as the path p_1 always exists in G^n (details left out, the equations involving multiple occurring variables found along p_3 implies that the path still exists).

If the last equation is such that $V^n(\psi_m) = v$, then one variable occurrence either by e_1 or e_2 can be linearized without destroying one of the two paths p_1 or p_2.

In both cases we get a contradiction with E an elementary cyclic set. Hence there exists a unique path from v_0 to v_0'. □

We conclude the proof of Theorem 3.1. By the previous Lemmas, there exists an occurrence O such that we have a unique path between $V^n(\phi)/O$ and $V^n(w)/O$. The two graphs $G^n \downarrow v_0$ and $G^n \downarrow v_0'$ are trees. Otherwise they are dags such that at least one vertex, distinct from both v_0 and

v_0', has multiple predecessors. Linearizing one variable in this vertex does not modify the unique path between v_0 and v_0', contradiction. The same observation implies that the trees are disjoint. Therefore $\mathcal{G}_E$ contains a unique cycle as the iteration that merges v_0 and v_0' creates a unique cycle, and afterwards two disjoint trees are merged. This does not create any new cycle.

The second part of the Theorem follows from the unicity of the path $p = v_0, \ldots, v_0'$. We first build a chain of needed variables as follows. As v_0' is shared, we may choose some needed $\mathcal{S}$-variable ϕ_0 that occurs in v_0' by the last edge e_0 of p by Proposition 2.10. There exists an equation $e^0 : \psi_0 = C_0[\phi_0]$ such that the last edge of the path $p_0 = (V(\psi_0), O_{C_0})$ is e_0. Let V_1 be the initial vertex of the maximal common suffix path of p and p_0. Either $V(\psi_0) = V_1$ or not, in this latter case V_1 is shared. In both cases V_1 contains at least one needed variable and we can iterate the process with (ψ_1, ϕ_1) by Proposition 2.10, first part. This stop as soon as V_n is above v_0. The sequence selects for each vertex in p a single needed $\mathcal{R}$-variable. The sequence (ψ_i, ϕ_i) so constructed is rigid in the sense that there exists a chain as in Proposition 2.10 that links ψ_{i-1} to ϕ_i.

Assume that some vertex in p contains two distinct needed $\mathcal{R}$-variables w_1 and w_2. One of them, say w_1, is not captured by the above sequence. Then the needed occurrence in $val(w_1)$ is necessarily the occurrence of a variable ϕ such that $V(\phi) \in p$ by the unicity of the cycle in G^N. Linearizing this occurrence of ϕ preserves the sequence (ψ_i, ϕ_i) hence the unique path p, and finally the resulting graph still is cyclic, contradiction. □

Let $\mathcal{E}$ be a unification problem. As the powerset of $\mathcal{E}$ is finite and there is a finite number of possible linearizations from a given set, the effectiveness of the base of elementary cyclic sets is trivial.

4 Equational Deductions

Under the *axioms* $\mathcal{E}$ and equational inference rules, an equational deduction $\mathcal{D} \vdash val(w) = val(w')$ will be associated to each pair of distinct variables w, w' in a vertex of $\mathcal{G}_\mathcal{E}$. To any cycle c will be associated a set of equational deductions: $\mathcal{D} \vdash \phi = C[\phi]$, $C[\_]$ a non-trivial context, $\phi \in \mathcal{S}_c$, $O_C = O(C[\phi], c)$. The set of hypotheses or axioms of $\mathcal{D}$ is noted $\mathcal{A}(\mathcal{D})$. The form $\phi = \tau$ of the given equations will be important for the second inference system that we present.

4.1 Inference Rules, Completeness

We introduce inference rules of symmetry, transitivity, simplification and substitution, ϕ, ψ denote variables, the other greek letters denote terms.

$$(s)\ \frac{\tau = \rho}{\rho = \tau} \qquad (t)\ \frac{\tau = \phi \quad \phi = \rho}{\tau = \rho}$$

$$(sl)\ \frac{\rho \to \sigma = \tau \to \upsilon}{\rho = \tau} \qquad (sr)\ \frac{\rho \to \sigma = \tau \to \upsilon}{\sigma = \upsilon}$$

$$(su)\ \frac{\phi = \tau \quad \psi = C[\phi]}{\psi = C[\tau]}$$

The following derived rules will be used:

$$(dl)\ \frac{\phi = \sigma \to \tau \quad \phi = \upsilon \to \chi}{\sigma = \upsilon} \qquad (dr)\ \frac{\phi = \sigma \to \tau \quad \phi = \upsilon \to \chi}{\tau = \chi}$$

Notice that this set of inference rules is not complete for equational reasoning. We do not need reflexivity nor congruence (or equivalently the full substitution rule).

Lemma 4.1 *Let $w_1 \neq w_2$ be two variables in $S \cup \mathcal{R}$, then $\mathcal{D} \vdash val(w_1) = val(w_2)$ where $\mathcal{D}$ is (su)-free, iff $V(w_1) = V(w_2)$ in $\mathcal{G}_{\mathcal{A}(\mathcal{D})}$.*

Proof. The proof of adequation is straighforward by structural induction on deductions. If $\mathcal{D}$ is restricted to an axiom, it is immediate. Otherwise, by cases on the last inference of $\mathcal{D}$. If this rule is (s), the result is immediate by induction hypothesis. If this rule is (t), the result is immediate by using twice the induction hypothesis. If the rule is (sl) with premisses $\phi = \tau_1 \to \tau_2$ and $\phi = \tau_3 \to \tau_4$, then by induction hypothesis we have $V(\tau_1 \to \tau_2) = V(\tau_3 \to \tau_4)$. By observation 8 of section 2.2, we have $V(\tau_1) = V(\tau_3)$. The cases of other rules are similar.

The proof of completeness is by induction on the cardinality of $\mathcal{E} = \mathcal{A}(\mathcal{D})$. With a single equation $e : \phi = \tau$, the only non-singleton vertex is $V(\phi)$, the deduction reduces to an axiom. Assume the lemma true for $\mathcal{E}$, we add an equation $e : \phi = \tau$. If ϕ does not occur in $\mathcal{G}_{\mathcal{E}}$, there is nothing to prove. Otherwise, in $V(\phi)$ we have $\mathcal{D} \vdash val(w_1) = val(w_2)$ for all pairs of distinct variables w_1 and w_2 by induction hypothesis. Especially, $\mathcal{D} \vdash val(w_1) = \phi$, which gives the deduction $\mathcal{D}'$:

$$(t)\ \frac{\begin{array}{c}\mathcal{D}\\ val(w_1) = \phi \quad \phi = \tau\end{array}}{val(w_1) = \tau}$$

and $\mathcal{D}'' \vdash \tau = val(w_1)$ with an instance of the symmetry rule. If $\tau \in S$ or if the vertex $V(\phi)$ is not predecessor, there is nothing more to prove.

Otherwise, $\tau = \tau_1 \to \tau_2$ and $V^{\mathcal{E}}(\phi)$ has two successors. We construct new deductions, say for the left successor v and the term τ_1.

If $S_v \neq \emptyset$, by Lemma 2.10 there exists $w \in \mathcal{R} \cap \mathcal{V}^{\mathcal{E}}(\phi)$ and $\psi \in S_v$ such that $suc(w) = \psi \to w'$ for some w'. By induction hypothesis there exists a deduction $\mathcal{D} \vdash \phi = \psi \to val(w')$. We have a deduction $\mathcal{D}'$:

$$(t)\ \frac{\begin{array}{c}\mathcal{D}\\ \phi = \psi \to val(w') \quad \phi = \tau_1 \to \tau_2\end{array}}{\psi = \tau_1}$$

Now, for all w'' in v, we have deductions for the equations $val(w'') = \tau_1$ and $\tau_1 = val(w'')$, with $\mathcal{D}'$, transitivity and symmetry.

Otherwise, S_v is empty. By Lemma 2.10, the vertex v possesses a unique incident edge. Consequently, for all w_1 in v, w_1 is in $\mathcal{R}$ and there exists w_0 in $\mathcal{R} \cap \mathcal{V}^{\mathcal{E}}(\phi)$ such that $suc(w_0) = w_1 \to w_2$ for some w_2. We apply the rule (sl) as above.

In turn, if v is not predecessor or if $\tau_1 \in S$, there is nothing more to prove. Otherwise the same proof is carried on. □

Lemma 4.2 *Let p be a path such that $S_{v_i} = \emptyset$ for $0 < i < n = |p|$, $S_{v_0} \neq \emptyset$ and $S_{v_n} \neq \emptyset$, then*

$$\forall \phi \in S_{v_0}, \forall \psi \in S_{v_n}, \exists w \in \mathcal{R}_{v_0}, \exists \mathcal{D} \vdash \phi = val(w),\ val(w) = C[\psi],\ O_C = O(w,p),\ |O_C| = n.$$

Proof. By Lemma 2.10 there exists w_1 in $\mathcal{R}_{v_{n-1}}$ and ψ in S_{v_n} such that $suc(w_1) = \psi \to w'$ or $suc(w_1) = w' \to \psi$. By the unique incident edge property for S-free vertices, there exists w in v_0 such that $val(w) = C[\psi]$, $O_C = O(p,w)$. The conclusion follows from Lemma 4.1. □

Corollary 4.3 *Let $p = (v_0, O)$ be a path such that both S_{v_0} and S_{v_n} are non-empty, $n = |p|$, then for all $\phi \in S_{v_0}$ and for all $\psi \in S_{v_n}$, there exists a deduction $\mathcal{D} \vdash \phi = C[\psi]$ with $O_C = O$. Reciprocally, for all deductions $\mathcal{D} \vdash \phi = C[\psi]$, there exists a path from $V^{\mathcal{E}}(\phi)$ to $V^{\mathcal{E}}(\psi)$, where $\mathcal{E} = \mathcal{A}(\mathcal{D})$.*

Proof. By successive applications of Lemma 4.2, using the substitution rule. Let v be the vertex of c containing the variable ϕ. The inference tree has the following form:

$$(su)\ \dfrac{\begin{array}{c}\mathcal{D}_2\\ \phi_2 = val(\bar{\phi}_2)\end{array} \quad (su)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \phi_1 = val(\bar{\phi}_1)\end{array} \quad \begin{array}{c}\mathcal{D}_0\\ \phi = val(\bar{\phi})(= C_0[\phi_1])\end{array}}{\phi = C_0[val(\bar{\phi}_1)](= C_1[\phi_2])}}{\phi = C_1[val(\bar{\phi}_2)](= \cdots)}$$

$$\vdots$$

$$(su)\ \dfrac{\begin{array}{c}\mathcal{D}_{n+1}\\ \phi_{n+1} = val(w)\end{array} \quad \phi = C_{n-1}[val(\bar{\phi}_n)](= C_n[\phi_{n+1}])}{\phi = C_n[val(w)](= C[\phi])}$$

This inference successively substitutes vertex representatives for variables. The contexts are determined by the occurrence of the path p. The intermediate equations are the conclusions of the inference step (su). The new context $C_{i+1}[\_]$ is defined by the occurrence $O_{C_i}.O(c, \phi_{i+1})$ in $C_i[val(\bar{\phi}_{i+1})]$. □
The deductions $\mathcal{D}_i, 0 \leq i \leq n+1$, are the *auxiliary* deductions of this cycle deduction.

Corollary 4.4 *For each cycle c in $\mathcal{B}_S$ and each ϕ in S_c, there exists a deduction of a cyclic equation $\mathcal{D} \vdash \phi = C[\phi]$ with $|O_C| = |c|$, $O_C = O(c, C[\phi])$, and reciprocally.*

Proof. Immediate consequence of the previous Corollary. □

4.2 Reduced Deductions

We introduce reduced forms for deductions. In fact, we only need deductions of marked equations, an equation being marked iff one and only one among its right-hand side occurrences is marked. Such equations have the form $\phi = C[\psi]$. The reductions may replace the context $C[\_]$ by an equivalent one. They introduce a simpler inference system.

Proposition 4.5 *The following inference system is sound and complete for path deductions, where ϕ, ψ denote S-variables:*

$$(s)\ \frac{\phi = \psi}{\psi = \phi} \quad (t)\ \frac{\phi = \psi \quad \psi = \tau}{\phi = \tau}$$

$$(d)\ \frac{\phi = C_1[\psi] \quad \phi = C_2[\tau] \quad C_1[\_] \sim C_2[\_]}{\psi = \tau} \quad (su)\ \frac{\phi = \tau \quad \psi = C[\phi]}{\psi = C[\tau]}$$

Proof. Soudness is immediate. For completeness, we establish that any deduction under the first inference system can be transformed into a deduction under the above inference system according to the following reductions (we omit reductions involving the derived rules (dl) and (dr)).

Instances of the symmetry rule can be lifted up to axioms. Two successive applications of (s) can be removed and we have the following reductions for (t), (sl) (resp. (sr)):

$$(s)\ \dfrac{(t)\ \dfrac{\overset{\mathcal{D}}{\tau=\phi}\quad \overset{\mathcal{D}'}{\phi=\rho}}{\tau=\rho}}{\rho=\tau} \quad\Longrightarrow\quad (t)\ \dfrac{(s)\ \dfrac{\overset{\mathcal{D}'}{\phi=\rho}}{\rho=\phi}\quad (s)\ \dfrac{\overset{\mathcal{D}}{\tau=\phi}}{\phi=\tau}}{\rho=\tau}$$

$$(s)\ \dfrac{(sl)\ \dfrac{\overset{\mathcal{D}}{\tau\to\upsilon=\rho\to\sigma}}{\tau=\rho}}{\rho=\tau} \quad\Longrightarrow\quad (sl)\ \dfrac{(s)\ \dfrac{\overset{\mathcal{D}}{\tau\to\upsilon=\rho\to\sigma}}{\rho\to\sigma=\tau\to\upsilon}}{\rho=\tau}$$

We assume that all deductions are in this (s)-reduced form.

Next, instances of (t) such that the leftmost term of the premisses is of positive size cannot be the last inference of a path deduction. Further, it can only be followed by (t), (sl) or (sr). This allows us to define a reduction on deductions that contain such an instance of (t). We give three reductions, the others are deduced by symmetry. In each rule τ_1 is of positive size.

$$(t)\ \dfrac{(t)\ \dfrac{\overset{\mathcal{D}_1}{\tau_1=\phi}\quad \overset{\mathcal{D}_2}{\phi=\psi}}{\tau_1=\psi}\quad \overset{\mathcal{D}_3}{\psi=\tau_2}}{\tau_1=\tau_2} \quad\Longrightarrow\quad (t)\ \dfrac{\overset{\mathcal{D}_1}{\tau_1=\phi}\quad (t)\ \dfrac{\overset{\mathcal{D}_2}{\phi=\psi}\quad \overset{\mathcal{D}_3}{\psi=\tau_2}}{\phi=\tau_2}}{\tau_1=\tau_2}$$

$$(sl)\ \dfrac{(t)\ \dfrac{(s)\ \dfrac{\overset{\mathcal{D}_1}{\phi=\tau_1}}{\tau_1=\phi}\quad \overset{\mathcal{D}_2}{\phi=\tau_2}}{\tau_1=\tau_2}}{\tau_1^i=\tau_2^i} \quad\Longrightarrow\quad (dl)\ \dfrac{\overset{\mathcal{D}_1}{\phi=\tau_1}\quad \overset{\mathcal{D}_2}{\phi=\tau_2}}{\tau_1^i=\tau_2^i}$$

$$(sl)\ \dfrac{(t)\ \dfrac{(sl)\ \dfrac{\overset{\mathcal{D}_1}{\tau_1\to\tau_3=\phi\to\tau_4}}{\tau_1=\phi}\quad \overset{\mathcal{D}_2}{\phi=\tau_2}}{\tau_1=\tau_2}}{\tau_1^i=\tau_2^i} \quad\Longrightarrow\quad (dl)\ \dfrac{(sl)\ \dfrac{(s)\ \dfrac{\overset{\mathcal{D}_1}{\tau_1\to\tau_3=\phi\to\tau_4}}{\phi\to\tau_4=\tau_1\to\tau_3}}{\phi=\tau_1}\quad \overset{\mathcal{D}_2}{\phi=\tau_2}}{\tau_1^i=\tau_2^i}$$

In the last two reductions, the existence of an inference above the left premiss of (t) exists due to the form of axioms. If the left premiss of (t) were the conclusion of an instance of (t), the first reduction would apply. We used the derived rules. Redexes containing them are easily deduced from the above reductions. This reduction is well-defined as the number of (t)-rules whose left premiss is of positive size strictly decreases.

By inspecting reduced deductions, we see that the (s)-instances applied to non-strict axioms occur as intances of the second rule left-hand side. Also, they cannot occur. In turn, instances of (t) are as in the second system. Then, any subdeduction involving only rules (dl), (dr), (sl) and (sr) must end up in an equation of the form $\phi=\tau$ for $\phi\in S$. Further, such a subdeduction starts necessarily by an instance of (dl) or (dr). Hence, such deductions are instances of the rule (d). The substitution rule remains unchanged. This establishes the completeness of the above system. □

The size of an instance of (t) is the size of the right-hand side of its conclusion, the size of an instance of (su) is the size of the right-hand side of its left premiss. We introduce some other reductions. Their effect is:

1. Localization of the inference of strict equations (variable equals another variable, rule 1).
2. Diminution of the number of substitutions (rules 2 and 3).
3. Cancellation of internal 0-sized subsitutions.

$$(t)\ \dfrac{\begin{array}{c}\mathcal{D}_3\\ \phi=\psi\end{array}\quad (t)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \psi=\theta\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \theta=\tau\end{array}}{\psi=\tau}}{\phi=\tau}\quad\Longrightarrow\quad (t)\ \dfrac{(t)\ \dfrac{\begin{array}{c}\mathcal{D}_3\\ \phi=\psi\end{array}\quad\begin{array}{c}\mathcal{D}_1\\ \psi=\theta\end{array}}{\phi=\theta}\quad\begin{array}{c}\mathcal{D}_2\\ \theta=\tau\end{array}}{\phi=\tau}$$

$$(su)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \psi=\tau\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \phi=\psi\end{array}}{\phi=\tau}\quad\Longrightarrow\quad (t)\ \dfrac{\begin{array}{c}\mathcal{D}_2\\ \phi=\psi\end{array}\quad\begin{array}{c}\mathcal{D}_1\\ \psi=\tau\end{array}}{\phi=\tau}$$

$$(su)\ \dfrac{\begin{array}{c}\mathcal{D}_3\\ \psi=\tau\end{array}\quad (su)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \phi=\psi\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \theta=C[\phi]\end{array}}{\theta=C[\psi]}}{\theta=C[\tau]}\quad\Longrightarrow\quad (su)\ \dfrac{(t)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \phi=\psi\end{array}\quad\begin{array}{c}\mathcal{D}_3\\ \psi=\tau\end{array}}{\phi=\tau}\quad\begin{array}{c}\mathcal{D}_2\\ \theta=C[\phi]\end{array}}{\theta=C[\tau]}$$

Hence, left branches of length greater than one in auxiliary deductions are sequences of (d)-rules. Observe that the number of (su) decreases, and if constant, the sum over the deduction of the size of the inferences decreases.

Finally, we can remove 0-sized instances of (d), and succçessive instances of (t) and (d) such that the conclusion of (t) is the rightmost premiss of (d).

$$(d)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \phi=\psi\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \phi=\tau\end{array}}{\psi=\tau}\quad\Longrightarrow\quad (t)\ \dfrac{(s)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \phi=\psi\end{array}}{\psi=\phi}\quad\begin{array}{c}\mathcal{D}_2\\ \phi=\tau\end{array}}{\psi=\tau}$$

$$(d)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \phi=C_1[\theta]\end{array}\quad (t)\ \dfrac{\begin{array}{c}\mathcal{D}_2\\ \phi=\psi\end{array}\quad\begin{array}{c}\mathcal{D}_3\\ \psi=C_2[\tau]\end{array}}{\phi=C_2[\tau]}}{\theta=\tau}\quad\Longrightarrow\quad (d)\ \dfrac{(t)\ \dfrac{(s)\ \dfrac{\begin{array}{c}\mathcal{D}_2\\ \phi=\psi\end{array}}{\psi=\phi}\quad\begin{array}{c}\mathcal{D}_1\\ \phi=C_1[\theta]\end{array}}{\psi=C_1[\theta]}\quad\begin{array}{c}\mathcal{D}_3\\ \psi=C_2[\tau]\end{array}}{\theta=\tau}$$

For (su)-free deductions, we have a subformula property: both sides of the conclusion are subterms of the axioms.

Definition 4.1 *Let $\mathcal{D}\vdash\phi=C[\psi]$ be a path deduction. We define marked occurrences in terms occurring in $\mathcal{D}$:*

- *The occurrence O_C of the conclusion and the left-hand side are marked.*
- *If in the conclusions, displayed occurrences are marked, then in the premisses:*

$$(s)\ \dfrac{\psi=\phi}{\phi=\psi}\quad \psi \text{ and } \phi \text{ are marked;}$$

$$(t)\ \frac{\phi=\theta \quad \theta=C[\psi]}{\phi=C[\psi]} \qquad \phi \text{ and } \theta,\ \theta \text{ and } O_C \text{ are marked;}$$

$$(d)\ \frac{\theta=C_1[\psi] \quad \theta=C_2[C[\psi]]}{\psi=C[\psi]} \qquad \tau \text{ and } O_{C_1},\ \tau \text{ and } O_{C_2C} \text{ are marked;}$$

$$(su)\ \frac{\theta=C_1[\psi] \quad \phi=C_2[\theta]}{\phi=C_2[C_1[\psi]]} \qquad \tau \text{ and } O_{C_1},\ \phi \text{ and } O_{C_2} \text{ are marked.}$$

Therefore, in any path deduction $\mathcal{D}$, every equation $\phi = \tau$ is such that ϕ and one and only one occurrence of τ are marked.

Lemma 4.6 *Let $\mathcal{D}$ be a path deduction of $\phi = C[\psi]$, then for any S-variable θ that possesses some marked occurrence in $\mathcal{D}$, there exists a path from the vertex $V(\theta)$ to some vertex of the path $V(\phi),\ldots,V(\psi)$ in the graph of $\mathcal{A}(\mathcal{D})$.*

Proof. Immediate by structural induction on $\mathcal{D}$. □

We conclude this section by a description of reduced deductions. Such a deduction has the form of the deduction in the proof of Corollary 4.3, where for all (su)-rules, except possibly the last one, the right-hand side of their left premiss is of positive size. Further the auxilliary deductions are such that the right-hand sides of the premisses of (d)-rules are of positive size, and the right premiss of both (t)- and (d)-rules is not the conclusion of a (t)-rule. Consequently, the following observation will be useful in the next section: each inference rule (except the symmetry rule) eliminates a variable. If one of the two occurrences of the eliminated variable is extracted from a non-strict right-hand side in the axioms, then this extraction is performed by a sequence (right branch) of consecutive instances of (d)-rules.

5 Minimum Deductions are Deterministic

We now address the problem of effectively finding an equational deduction of an equation $\phi = C[\phi]$ given an elementary cyclic set S so that its cycle is equal to $(V(\phi), O_C)$. We first establish that in a minimal deduction, the auxiliary deductions do not eliminate cyclic variables. Next, we prove unicity properties for chains in vertices of the graph $\mathcal{G}_S$. These two results imply the correctness of a deterministic algorithm finding a minimal deduction. In turn the existence of this algorithm proves that, up to permutation, there exists a minimum deduction.

5.1 Auxiliary Deductions are Cycle-Free

We first establish that a minimal deduction doest not involve unneeded variables. The left-hand side of the conclusion of a cyclic inference is its main variable, the variables that are eliminated by substitutions are its proper variables.

Lemma 5.1 *Let $\mathcal{D}$ be a minimal reduced cyclic deduction from an elementary cyclic set S. All marked variables of $\mathcal{D}$ are needed variables of S.*

Proof. Firstly, any left-hand side is marked in $\mathcal{D}$. Otherwise, some equation is redundant, which contradicts S an elementary cyclic set. Secondly, we will prove that if the variable α is eliminated by some inference rule in $\mathcal{D}$ and the two occurrences of α in the premisses are equal occurrences in some right-hand side of S, then $\mathcal{D}$ is not minimal. This establishes the result but for the main variable of $\mathcal{D}$ (the same question for left-hand sides is trivial). This in turn follows by the same technique. These claims are proved in the three following (technical) lemmas. □

Lemma 5.2 *Let $\mathcal{D}$ and $\mathcal{E}$ be the two following deductions:*

$$(d)\ \dfrac{\begin{array}{c}\mathcal{D}_k\\ \theta_{k-1} = C_k^1[\theta_k]\end{array} \qquad \begin{array}{c}(d)\ \dfrac{\begin{array}{c}\mathcal{D}_2\\ \theta_1 = C_2^1[\theta_2]\end{array} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \theta_0 = C_1^1[\theta_1]\end{array} \quad \begin{array}{c}\mathcal{D}_0\\ \theta_0 = C_1\cdots C_k[\tau]\end{array}}{\theta_1 = C_2\cdots C_k[\tau]}}{\theta_2 = C_3\cdots C_k[\tau]}\\ \vdots\\ \theta_{k-1} = C_k[\tau]\end{array}}{\theta_k = \tau}$$

$$(d)\ \dfrac{\begin{array}{c}\mathcal{E}_l\\ \psi_{l-1} = D_l^1[\psi_l]\end{array} \qquad \begin{array}{c}(d)\ \dfrac{\begin{array}{c}\mathcal{E}_2\\ \psi_1 = D_2^1[\psi_2]\end{array} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{E}_1\\ \psi_0 = D_1^1[\psi_1]\end{array} \quad \begin{array}{c}\mathcal{E}_0\\ \psi_0 = D_1\cdots D_l[\rho]\end{array}}{\psi_1 = D_2\cdots D_l[\rho]}}{\psi_2 = D_3\cdots D_l[\rho]}\\ \vdots\\ \psi_{l-1} = D_l[\rho]\end{array}}{\psi_l = \rho}$$

such that the two equations $\psi_0 = D_1\cdots D_l[\rho]$ and $\theta_0 = C_1\cdots C_k[\tau]$ are the same axiom (up to strict equality of the left-hand sides). Assume that $D_1\cdots D_l \leq C_1\cdots C_k$. There exists a deduction $\mathcal{D} \wedge \mathcal{E} \vdash \psi_l = C_i^3[\theta_i]$, where i is such that $C_1\cdots C_{i-1} \leq D_1\cdots D_l \leq C_1\cdots C_i$ and $C_i^1 = C_i^2 C_i^3$. Further this deduction does not contain $\mathcal{D}_0$ nor $\mathcal{E}_0$ as subdeductions.

Proof. Without loss of generality, we assume that $\psi_0 \equiv \theta_0$ and that both $\mathcal{D}_0$ and $\mathcal{E}_0$ are axioms (in the general case we have an axiom $\omega = D_1\cdots D_l[\rho] = C_1\cdots C_k[\tau]$ and $\mathcal{D}_0$, $\mathcal{E}_0$ decompose in (t)-proofs $\theta_0 = \omega$ and $\psi_0 = \omega$, completed by this axiom). The proof is by induction on l. Notice that when deductions are reduced, if some right-hand side occurrence is eliminated, there exists a subdeduction of the form $\mathcal{D}$ or $\mathcal{E}$, i.e. a right branch of (d)-rules.

For $l = 1$, we have $D_1^1 = E_1 \cdots E_i$ with $E_j \sim C_j^1$, $j = 1, \ldots, i-1$ and $C_i^1 = C_i^2 C_i^3$ with $C_i^2 \sim E_i$. We build the deduction:

$$
(d)\ \dfrac{(d)\ \dfrac{\begin{array}{c}\mathcal{D}_2\\ \theta_1 = C_2^1[\theta_2]\end{array} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \theta_0 = C_1^1[\theta_1]\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \psi_0 = E_1 \cdots E_i[\psi_1]\end{array}}{\theta_1 = E_2 \cdots E_i[\psi_1]}}{\theta_2 = E_3 \cdots E_i[\psi_1]}
$$

$$
(d)\ \dfrac{(d)\ \dfrac{\begin{array}{c}\mathcal{D}_{i-1}\\ \theta_{i-2} = C_{i-1}^1[\theta_{i-1}]\end{array} \quad \begin{array}{c}\vdots\\ \theta_{i-2} = E_{i-1}E_i[\psi_1]\end{array}}{\theta_{i-1} = E_i[\psi_1]} \quad \begin{array}{c}\mathcal{D}_i\\ \theta_{i-1} = C_i^2 C_i^3[\theta_i]\end{array}}{\psi_1 = C_1^3[\theta_i]}
$$

Assuming the lemma true for $l-1$, let i be such that $C_1 \cdots C_{i-1} \leq D_1 \cdots D_{l-1} \leq C_1 \cdots C_i$. By induction hypothesis, we have $C_i^1 = C_i^2 C_i^3$ and a proof $\mathcal{D}' \vdash \psi_{l-1} = C_i^3[\theta_i]$ without using $\mathcal{D}_0$ nor $\mathcal{E}_0$. There exists j such that $C_1 \cdots C_{j-1} \leq D_1 \cdots D_l \leq C_1 \cdots C_j$. We have $D_l^1 = E_0 \cdots E_{j-i}$ with $E_0 \sim C_i^3$, $E_k \sim C_{i+k}^1$, $k = 1, \ldots j-i-1$, and $C_j^i = C_j^2 C_j^3$ with $C_j^2 \sim E_{j-i}$. We build the deduction:

$$
(d)\ \dfrac{(d)\ \dfrac{\begin{array}{c}\mathcal{D}_{j-1}\\ \theta_{j-2} = C_{j-1}^1[\theta_{j-1}]\end{array} \quad \begin{array}{c}\vdots\\ \theta_{j-2} = E_{j-i-1}E_{j-i}[\psi_l]\end{array}}{\theta_{j-i} = E_{j-i}[\psi_l]} \quad \begin{array}{c}\mathcal{D}_j\\ \theta_{j-1} = C_j^2 C_j^3[\theta_j]\end{array}}{\psi_l = C_j^3[\theta_j]}
$$

The reader may check that the presence of (t)-proofs merely complicates the above argument. □

Lemma 5.3 *Let S be an elementary cyclic set. Assume that in some cyclic deduction $\mathcal{D}$, a rule eliminates the variable ϕ and that the two occurrences of ϕ in the premisses come from the same occurrence in some right-hand side of S. Then the deduction $\mathcal{D}$ is not minimal among cyclic deductions for S.*

Proof. Given such a deduction, we reduce it according to the kind of the eliminating rule. In each case we have two left (d)-branches, defining two subdeductions $\mathcal{D}$ and $\mathcal{E}$ as in Lemma 5.2. Also, we use this Lemma to get a deduction $\mathcal{D} \wedge \mathcal{E} \vdash \psi_l = C_j^3[\theta_j]$. As usual, the possible presence of (t)-subproofs does not invalidate the argument.

Case of (d)-rule. Without loss of generality the deduction is:

$$
(d)\ \dfrac{(d)\ \dfrac{\begin{array}{c}\mathcal{D}\\ \theta_k = E_1[\phi]\end{array} \quad \begin{array}{c}\mathcal{D}_1\\ \theta_k = E_2E_3[\psi]\end{array}}{\phi = E_3[\psi]} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{E}\\ \psi_l = E_4[\phi]\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \psi_l = E_5E_6E_7[\tau]\end{array}}{\phi = E_6E_7[\tau]}}{\psi = E_7[\tau]}
$$

We have the following equivalences of contexts, as both $E_1[\phi]$ and $E_4[\phi]$ come from the same axiom by assumption (with the notations of Lemma 5.2):

$$D_1 \cdots D_l E_4 = C_1 \cdots C_k E_1.$$

But $D_1 \cdots D_l = C_1 \cdots C_{j-1} C_j^2$, $C_j^2 C_j^3 = C_j$, hence $E_4 \sim E_5 \sim C_j^3 C_{j+1} \cdots C_k E_1$. With abuse of notations with respect to contexts, we build the deduction:

$$(d)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \theta_k = E_2E_3[\psi]\end{array} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{D}_k\\ \theta_{k-1} = C_k^1[\theta_k]\end{array} \quad \begin{array}{c}\vdots\\ \theta_{k-1} = C_kE_1E_6E_7[\tau]\end{array}}{\theta_k = E_1E_6E_7[\tau]}}{\psi = E_7[\tau]}$$

$$\begin{array}{c}\mathcal{D}_{j+1}\\ \theta_j = C_{j+1}^1[\theta_{j+1}]\end{array} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{D}'\\ \psi_l = C_j^3[\theta_j]\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \psi_l = E_5E_6E_7[\tau]\end{array}}{\theta_j = C_{j+1}\cdots C_kE_1E_6E_7[\tau]}$$

$$(d)\ \dfrac{\qquad}{\theta_{j+1} = C_{j+2}\cdots C_kE_1E_6E_7[\tau]}$$

Case of (t)-rule. Without loss of generality we have the configuration:

$$(t)\ \dfrac{(d)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \theta_k = E_2[\alpha]\end{array} \quad \begin{array}{c}\mathcal{D}\\ \theta_k = E_1[\phi]\end{array}}{\alpha = \phi} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{E}\\ \psi_l = E_4[\phi]\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \psi_l = E_3[\tau]\end{array}}{\phi = \tau}}{\alpha = \tau}$$

We have $\mathcal{D}' \vdash \psi_l = C_j^3[\theta_j]$ and $E_3 \sim C_j^3 C_{j+1} \cdots C_k E_1$. We build:

$$(d)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \theta_k = E_2[\alpha]\end{array} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{D}_k\\ \theta_{k-1} = C_k^1[\theta_k]\end{array} \quad \begin{array}{c}\vdots\\ \theta_{k-1} = C_kE_1[\tau]\end{array}}{\theta_k = E_1[\tau]}}{\alpha = \tau}$$

$$\begin{array}{c}\mathcal{D}_{j+1}\\ \theta_j = C_{j+1}^1[\theta_{j+1}]\end{array} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{D}'\\ \psi_l = C_j^3[\theta_j]\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \psi_l = E_3[\tau]\end{array}}{\theta_j = C_{j+1}\cdots C_kE_1[\tau]}$$

$$(d)\ \dfrac{\qquad}{\theta_{j+1} = C_{j+2}\cdots C_kE_1[\tau]}$$

Case of (su)-rule. We have four subcases. First,

$$(su)\ \dfrac{(d)\ \dfrac{\begin{array}{c}\mathcal{D}\\ \theta_k = E_1[\phi]\end{array} \quad \begin{array}{c}\mathcal{D}_1\\ \theta_k = E_2[\tau]\end{array}}{\phi = \tau} \quad \begin{array}{c}\mathcal{E}\\ \psi_l = E_4[\phi]\end{array}}{\psi_l = E_4[\tau]}$$

With $\mathcal{D}' \vdash \psi_l = C_j^3[\theta_j]$ and $E_4 \sim C_j^3 C_{j+1} \cdots C_k E_2$, we build:

$$(su)\ \dfrac{\begin{array}{c}\mathcal{D}_{j+1}\\ \theta_j = C_{j+1}^1[\theta_{j+1}]\end{array} \quad \begin{array}{c}\mathcal{D}'\\ \psi_l = C_j^3[\theta_j]\end{array}}{\psi_l = C_j^3 C_{j+1}^1[\theta_{j+1}]}$$

$$\vdots$$

$$(su)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \theta_k = E_2[\tau]\end{array} \quad (su)\ \dfrac{\begin{array}{c}\mathcal{D}_k\\ \theta_{k-1} = C_k^1[\theta_k]\end{array} \quad \begin{array}{c}\vdots\\ \psi_l = C_j^3 C_{j+1}^1 \cdots C_{k-1}^1[\theta_{k-1}]\end{array}}{\psi_l = C_j^3 C_{j+1}^1 \cdots C_k^1[\theta_k]}}{\psi_l = C_j^3 C_{j+1}^1 \cdots C_k^1 E_2[\tau]}$$

Second,

$$(su)\ \dfrac{(d)\ \dfrac{\begin{array}{c}\mathcal{E}\\ \psi_l = E_4[\phi]\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \psi_l = E_3[\tau]\end{array}}{\phi = \tau} \quad \begin{array}{c}\mathcal{D}\\ \theta_k = E_1[\phi]\end{array}}{\theta_k = E_1[\tau]}$$

With $\mathcal{D}' \vdash \psi_l = C_j^3[\theta_j]$ and $E_3 \sim C_j^3 C_{j+1} \cdots C_k E_1$, we build

$$(d)\ \dfrac{\begin{array}{c}\mathcal{D}_{j+1}\\ \theta_j = C_{j+1}^1[\theta_{j+1}]\end{array} \quad (d)\ \dfrac{\begin{array}{c}\mathcal{D}'\\ \psi_l = C_j^3[\theta_j]\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \psi_l = E_3[\tau]\end{array}}{\theta_j = C_{j+1} \cdots C_k E_1[\tau]}}{\theta_{j+1} = C_{j+2} \cdots C_k E_1[\tau]}$$

$$\vdots$$

$$\theta_k = E_1[\tau]$$

Third,

$$(su)\ \dfrac{(d)\ \dfrac{\begin{array}{c}\mathcal{D}\\ \theta_k = E_1[\phi]\end{array} \quad \begin{array}{c}\mathcal{D}_1\\ \theta_k = E_2[\tau]\end{array}}{\phi = \tau} \quad (su)\ \dfrac{\begin{array}{c}\mathcal{E}\\ \psi_l = E_4[\phi]\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \alpha = E_3[\psi_l]\end{array}}{\alpha = E_3 E_4[\phi]}}{\alpha = E_3 E_4[\tau]}$$

With $\mathcal{D}' \vdash \psi_l = C_j^3[\theta_j]$ and $E_4 \sim C_j^3 C_{j+1} \cdots C_k E_2$, we build

$$(su)\ \dfrac{\begin{array}{c}\mathcal{D}_{j+1}\\ \theta_j = C_{j+1}^1[\theta_{j+1}]\end{array} \quad (su)\ \dfrac{\begin{array}{c}\mathcal{D}'\\ \psi_l = C_j^3[\theta_j]\end{array} \quad \begin{array}{c}\mathcal{E}_1\\ \alpha = E_3[\psi_l]\end{array}}{\alpha = E_3 C_j^3[\theta_j]}}{\alpha = E_3 C_j^3 C_{j+1}^1[\theta_{j+1}]}$$

$$\vdots$$

$$(su)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \theta_k = E_2[\tau]\end{array} \quad (su)\ \dfrac{\begin{array}{c}\mathcal{D}_k\\ \theta_{k-1} = C_k^1[\theta_k]\end{array} \quad \begin{array}{c}\vdots\\ \alpha = E_3 C_j^3 C_{j+1}^1 \cdots C_{k-1}^1[\theta_{k-1}]\end{array}}{\alpha = E_3 C_j^3 C_{j+1}^1 \cdots C_k^1[\theta_{k-1}]}}{\alpha = E_3 C_j^3 C_{j+1}^1 \cdots C_k^1 E_2[\tau]}$$

Fourth,

$$(su)\ \dfrac{(d)\ \dfrac{\overset{\mathcal{E}}{\psi_l = E_4[\phi]} \quad \overset{\mathcal{E}_1}{\psi_l = E_3[\tau]}}{\phi = \tau} \quad (su)\ \dfrac{\overset{\mathcal{D}}{\theta_k = E_1[\phi]} \quad \overset{\mathcal{D}_1}{\alpha = E_2[\theta_k]}}{\alpha = E_2E_1[\phi]}}{\alpha = E_2E_1[\tau]}$$

With $\mathcal{D}' \vdash \psi_l = C_j^3[\theta_j]$ and $E_3 \sim C_j^3C_{j+1}\cdots C_kE_1$, we build

$$(su)\ \dfrac{\dfrac{\overset{\mathcal{D}'}{\psi_l = C_j^3[\theta_j]} \quad \overset{\mathcal{E}_1}{\psi_l = E_3[\tau]}}{(d)\ \theta_j = C_{j+1}\cdots C_kE_1[\tau]} \quad \overset{\vdots}{\theta_k = E_1[\tau]} \qquad \overset{\mathcal{D}_1}{\alpha = E_2[\theta_k]}}{\alpha = E_2E_1[\tau]}$$

Notice that new deductions are reduced. □

Lemma 5.4 *Let S be an elementary cyclic set. Let $\mathcal{D} \vdash \phi = C[\phi]$ be a cyclic deduction such that both occurrences of ϕ are the same in S. Then $\mathcal{D}$ is not minimal among cyclic deductions from S.*

Proof. The proof is similar to the previous one. Once more, if ϕ comes from a single left-hand side, the result is trivial. We have two cases. First,

$$(su)\ \dfrac{\overset{\mathcal{D}}{\theta_k = E_1[\phi]} \quad (su)\ \dfrac{\overset{\mathcal{F}_m}{\omega_{m-1} = F_m[\theta_k]} \quad \overset{\vdots}{\phi = F_0\cdots F_{m-1}[\omega_{m-1}]}}{\phi = F_0\cdots F_m[\theta_k]} \quad (su)\ \dfrac{\overset{\mathcal{F}_1}{\omega_0 = F_1[\omega_1]} \quad (d)\ \dfrac{\overset{\mathcal{E}}{\psi_l = E_4[\phi]} \quad \overset{\mathcal{E}_1}{\psi_l = E_3F_0[\omega_0]}}{\phi = F_0[\omega_0]}}{\phi = F_0F_1[\omega_1]}}{\phi = F_0\cdots F_mE_1[\phi]}$$

With the previous notations we have $E_3 \sim C_j^3C_{j+1}\cdots C_kE_1$ and $\mathcal{D}' \vdash \psi_l = C_j^3[\theta_j]$. We build

$$(su)\ \dfrac{\overset{\mathcal{F}_m}{\omega_{m-1} = F_m[\theta_k]} \quad \overset{\vdots}{\theta_k = E_1F_0\cdots F_{m-1}[\omega_{m-1}]}}{\theta_k = E_1F_0\cdots F_m[\theta_k]} \quad \overset{\vdots}{\theta_k = E_1F_0[\omega_0]} \quad (su)\ \dfrac{\overset{\mathcal{F}_1}{\omega_0 = F_1[\omega_1]} \quad (d)\ \dfrac{\dfrac{\overset{\mathcal{D}_{j+1}}{\theta_j = C_{j+1}^1[\theta_{j+1}]} \quad (d)\ \dfrac{\overset{\mathcal{D}'}{\psi_l = C_j^3[\theta_j]} \quad \overset{\mathcal{E}_1}{\psi_l = E_3F_0[\omega_0]}}{\theta_j = C_{j+1}\cdots C_kE_1F_0[\omega_0]}}{\theta_{j+1} = C_{j+1}\cdots C_kE_1F_0[\omega_0]}}{\theta_k = E_1F_0F_1[\omega_1]}}{\theta_k = E_1F_0\cdots F_m[\theta_k]}$$

Second,

$$(su)\ \dfrac{\overset{\mathcal{F}_1}{\omega_0 = F_1[\omega_1]} \quad (d)\ \dfrac{\overset{\mathcal{D}}{\theta_k = E_1[\phi]} \quad \overset{\mathcal{D}_1}{\theta_k = E_2F_0[\omega_0]}}{\phi = F_0[\omega_0]}}{\phi = F_0F_1[\omega_1]}$$

$$\vdots$$

$$(su)\ \dfrac{\overset{\mathcal{E}}{\psi_l = E_4[\phi]} \quad (su)\ \dfrac{\overset{\mathcal{F}_m}{\omega_{m-1} = F_m[\psi_l]} \quad \phi = F_0\cdots F_{m-1}[\omega_{m-1}]}{\phi = F_0\cdots F_m[\psi_l]}}{\phi = F_0\cdots F_mE_4[\phi]}$$

As usual $E_4 \sim C_j^3C_{j+1}\cdots C_kE_1$ and $\mathcal{D}' \vdash \psi_l = C_j^3[\theta_j]$. We build

$$(su)\ \dfrac{\overset{\mathcal{F}_1}{\omega_0 = F_1[\omega_1]} \quad \overset{\mathcal{D}_1}{\theta_k = E_2F_0[\omega_0]}}{\theta_k = E_2F_0F_1[\omega_1]}$$

$$\vdots$$

$$(su)\ \dfrac{\overset{\mathcal{D}_{j+1}}{\theta_j = C_{j+1}^1[\theta_{j+1}]} \quad (su)\ \dfrac{\overset{\mathcal{D}'}{\psi_l = C_j^3[\theta_j]} \quad (su)\ \dfrac{\overset{\mathcal{F}_m}{\omega_{m-1} = F_m[\psi_l]} \quad \theta_k = E_2F_0\cdots F_{m-1}[\omega_{m-1}]}{\theta_k = E_2F_0\cdots F_m[\psi_l]}}{\theta_k = E_2F_0\cdots F_mC_j^3[\theta_j]}}{\theta_k = E_2F_0\cdots F_mC_j^3C_{j+1}^1[\theta_j]}$$

$$\vdots$$

$$\theta_k = E_2F_0\cdots F_mC_j^3C_{j+1}^1\cdots C_k^1[\theta_k]$$

Notice that new deductions are reduced. The conclusions are not preserved in these two reductions. But the new deductions still are cyclic, and smaller than the previous ones. □

Lemma 5.5 *Let $\mathcal{D} \vdash \phi = C[\phi]$ be a minimal reduced deduction from some elementary cyclic set S. Then $\mathcal{D}$ does not eliminate cyclic variables above some (d)-rule (in its auxiliary deductions).*

Proof. The deduction $\mathcal{D}$ being minimal, all marked variables are needed by Lemma 5.1. If the variable ψ is eliminated by a (t)-rule above some (d)-rule, the variable eliminated by the first (d)-rule below this (t)-rule is also cyclic. Also, assume that the cyclic variable ψ is eliminated by a (d)-rule:

$$(d)\ \dfrac{\psi = C_1[\alpha] \quad \psi = C_2C_3[\beta]}{\alpha = C_3[\beta]}$$

The deduction being reduced, the contexts C_1 and C_2 are non-trivial and, if C_3 is trivial then $\alpha \not\equiv \beta$, as deductions are reflexivity-free. This means that the cyclic vertex $V(\psi)$ contains two needed $\mathcal{R}$-variables, namely w_1 such that $val(w_1) = C_1[\alpha]$ and w_2 such that $val(w_2) = C_2C_3[\beta]$. By the unicity of $\mathcal{R}$-variables occurrences, $w_1 \not\equiv w_2$. But this contradicts the second part of Theorem 3.1. □

Therefore, we know the "external" structure of a minimal deduction. This is a reduced deduction as described at the end of section 4.2. Further, let $(v_0, \ldots, v_n)$ be the sequence of needed vertices of the unique cycle c of the elementary cyclic set S, v_{i+1} being the first needed vertex above v_i along the cycle. By Proposition 2.10, for each vertex v_i there exists a variable ϕ_i that occurs by the cycle in v_i, this variable may be chosen needed. By Theorem 3.1, this variable is the unique needed variable occurrence of $val(w_{i+1})$, w_{i+1} the unique needed $\mathcal{R}$-variable that belongs to v_{i+1}. Hence the deductions must prove $\phi_i = val(w_i)$ for each vertex v_i. Further, the auxiliary deductions are "out of" the cycles. Also there is some hope here to find a deterministic algorithm that searches a proof.

5.2 Chains in Elementary Cyclic Sets

In this section, all variables are assumed to be needed. Let ϕ be some needed variable of an elementary cyclic set S, and assume that ϕ occurs by an edge e in $\mathcal{G}_S$. Then an occurrence O of ϕ in some non-strict right-hand side $e : \psi = C[\phi]$ is said to be associated to e iff the edge e is the last edge of the path $p = (V(\psi), O_C)$.

Lemma 5.6 *Let S be an elementary cyclic set. Let α_1 and α_2 be two needed variables that occur by the edge e in $V(\alpha_1) = V(\alpha_2)$ according to two distinct equations $e_1 : \psi_1 = C_1[\alpha_1]$ and $e_2 : \psi_2 = C_2[\alpha_2]$. Assume that S is linearized into S' at occurrence O_{C_1} by substituting α_1' to α_1. Then, in $\mathcal{G}_{S'}$ we have $V(\alpha_1') = V(\alpha_2)$.*

Proof. Notice first that the edge e is not cyclic by Theorem 3.1. Let $v = V(\alpha_1) = V(\alpha_2)$ and v' be the first needed vertex above v along e. This vertex is well-defined by (i) the unique incident edge property for $\mathcal{S}$-free vertices and (ii) if v'' is non $\mathcal{S}$-empty and unneeded, then v'' also has a unique incident edge by Proposition 2.10.

Then $V(\psi_1)$, $V(\psi_2) \in \mathcal{G}_S \uparrow v'$ as these two vertices are needed. Further $\mathcal{G}_S \uparrow v' = \mathcal{G}_{S'} \uparrow v'$ as the graph above the source of e is a dag and we rename a variable whose vertex is the target of e. Hence $V(\psi_1)$, $V(\psi_2) \in \mathcal{G}_{S'} \uparrow v'$ and $v' = V(\psi_1)/(O_{C_1}/O) = V(\psi_2)/(O_{C_2}/O)$ in $\mathcal{G}_{S'}$ where O is the occurrence defined by the path $p = v' \ldots, v = (v', O)$. This establishes the result. □

For any $\mathcal{R}$-variable, there exists a unique associated equation. This fact is also true for $\mathcal{S}$-variables when we require that the variable occurs in its vertex by some edge.

Lemma 5.7 *Let S be an elementary cyclic set and ϕ be some needed variable. If ϕ occurs in $V(\phi)$ of $\mathcal{G}_S$ by an edge e, then there exists a unique associated equation $e' : \psi = C[\phi]$ such that e is the last edge of the path $(V(\psi), O_C)$. The occurrence $O(e, \phi)$ will denote O_C.*

Proof. This is true if the edge e is cyclic by Theorem 3.1. Assume that we have two distinct equations so that ϕ occurs in $V(\phi)$ by e according to these two equations. By Lemma 5.6, we can linearize S by substituting ϕ' to one of these occurrences so that $V(\phi') = V(\phi)$ in the new graph.

If $V(\phi)$ had a unique incident edge, this is also true in the new graph. Hence for every variable ψ in $V(\phi)$ from the old graph we have $\psi =_s \theta$, θ occurs by e, by Lemma 2.9. Either $\theta \equiv \phi$ or not. In each case, we have $V(\phi) = V(\psi)$ in the new graph. This establishes that the two abstract graphs are equal, contradicting S an elementary cyclic set.

Otherwise, by considering an edge e' distinct from e and a chain between e and e' we also have that $V(\psi) = V(\phi)$ in the new graph for all variables $\psi \in V(\phi)$ in the old graph. Once more this

gives a contradiction. □
We precise the notion of chain, with the same notations of Proposition 2.10:

Definition 5.1 *A chain as defined in Proposition 2.10 is an open chain. A closed chain is a triple (α, c, β) where c is an open chain, α occurs by e_0 and β occurs by e_n. We also define in an obvious way left (resp. right) closed and right (resp. left) open chains.*

By extension, a closed chain of length 0 is a pair (α, β) of variables such that $\alpha =_s \beta$. A closed chain of length 1 is a triple (α, e, β) such that both α and β occur by the edge e. Such chains will be used in proving strict equations. Given two consecutive edges of a chain, the variable pairs (α_i, β_i) of a chain are unique:

Lemma 5.8 *Let S be an elementary cyclic set and v be some vertex of $\mathcal{G}_S$ such that e_1 and e_2 are two distinct edges both incident to v. Assume that α, β occur in v by e_1 and α', β' occur in v by e_2. Then $\alpha =_s \alpha'$ and $\beta =_s \beta'$ implies $\alpha \equiv \beta$ and $\alpha' \equiv \beta'$.*

Proof. We establish $\alpha \equiv \beta$ by contradiction. Assume $\alpha \not\equiv \beta$. By Lemma 5.6, if we linearize S into S' at $O(e, \alpha)$ by substituting α'' for α, we have $V(\alpha'') = V(\beta)$ in the new graph. But $\beta =_s \beta'$ implies $V(\beta) = V(\beta')$. Further $\alpha =_s \alpha'$ and α' still occurs by e_2 implies

$$V(\alpha) = V(\alpha') = V(\beta') = V(\beta) = V(\alpha'').$$

Hence the vertices of α and α'' are equal in the new graph. This means that the abstract graphs underlying $\mathcal{G}_S$ and $\mathcal{G}_{S'}$ are equal, which contradicts S an elementary cyclic set. □
The argument detailed in the above proof will be frequently used in the following lemmas.

Corollary 5.9 *Assume that in $\mathcal{G}_S$, S and elementary cyclic set, we have two open chains in a vertex v that share their sequence of edges, then the chains are equal.*

Proof. Direct consequence of Lemma 5.8. □

Definition 5.2 *Let E be a set of equations. Let v be some vertex in $\mathcal{G}_E$. A block of v is a set B containing at least three edges incident to v so that there exists a S-variable ϕ that occurs by e, for all e in B.*

Let c be a chain of v, then if c contains $(e_1, \ldots, e_n)$ that forms a block, any permutation of $(e_2, \ldots, e_{n-1})$, any subsequence $(e_1, e_{i_1}, \ldots, e_{i_k}, e_n)$ also defines a chain. We are interested in minimal chains. Notice first that such a chain does not repeat any edge. Further a minimal chain does not contain any subsequence of edges defining a block. According to the equality that is to be proven, the chain will be open, closed or left-open and right-closed. These situations are detailed in the following lemmas.

Lemma 5.10 *Let c_1 and c_2 be two minimal open chains between the edges e_1 and e_2 both incident to v. If their sets of edges are equal, then their sequences of edges also are equal.*

Proof. Let $(e_0, \ldots, e_{n+1})$ and $(e'_0, \ldots, e'_{n+1})$ be the two sequences of edges. Let i be the first index so that $e_i \neq e'_i$ and j, k be the indices such that $e'_j = e_i$ and $e_k = e'_i$. The two chains are

$$c_1 = (\ldots, \alpha_{i-1}, \beta_{i-1}, e_i, \alpha_i, \ldots, \alpha_{k-1}, \beta_{k-1}, e_k, \alpha_k, \beta_k, \ldots),$$

$$c_2 = (\ldots, \alpha_{i-1}, \beta'_{i-1}, e'_i, \alpha'_i, \ldots, \alpha'_{j-1}, \beta'_{j-1}, e'_j, \alpha'_j, \beta'_j, \ldots).$$

They share a common prefix, including the same variables, up to α_{i-1}, by Lemma 5.6. We have

(i) $\alpha_i \not\equiv \alpha'_j$,

(ii) $\beta_{i-1} \not\equiv \beta'_{j-1}$,

(iii) $\alpha_l \not\equiv \beta_{l-1}$, $l = 1, \ldots, n$,

Otherwise, each one of (i), (ii) implies that a chain is not minimal, e.g., $\alpha'_j \equiv \alpha_i$ implies that the chain

$$(\ldots, \beta_{i-1}, e_i, \alpha_i, \beta'_j, e'_{j+1}, \alpha'_{j+1}, \ldots)$$

does not contain the edge e'_i, while (iii) implies the existence of some block in a minimal chain.

Next, we claim that $\beta'_{j-1} \equiv \alpha_i$ implies $\alpha'_j \equiv \beta_{i-1}$. Otherwise, $\alpha_i \not\equiv \alpha'_j$ by (i), $\beta_{i-1} \not\equiv \beta'_{j-1}$ by (iii) and $\beta'_{j-1} \equiv \alpha_i$ (contradictory hypothesis), and $\beta_{i-1} \not\equiv \alpha'_j$ by contradictory hypothesis. Hence we have three distinct variables, namely α_i, α'_j and β_{i-1}, that occur in v by e_i. Next, we have the strict equalities

1. $\alpha_i =_s \beta_i$ that occurs by e_{i+1},
2. $\alpha'_j =_s \beta'_j$ that occurs by e'_{j+1},
3. $\beta_{i-1} =_s \alpha_{i-1}$ that occurs by e_{i-1}.

By the absence of repetitions in chains, we have $e_{i+1} \neq e_{i-1}$ and $e'_{j+1} \neq e_{i-1}$. If $e_{i+1} = e'_{j+1}$, by Lemma 5.6 applied to 1 and 2, we have $\alpha_i \equiv \alpha'_j$ which contradicts (i). Hence these three vertices are pairwise distinct. We claim that, among the three variable occurrences by e_i, one can be linearized without modification of the abstract graph. The edge e_i is not cyclic by Theorem 3.1, and we can apply Lemma 5.6 to see that this does not modify the abstract graphs (details left out, we have a vertex v with four incident edges, one among them is associated to three distinct variables, each other one is associated to a variable strictly equal to one of the preceding ones).

We have proved that either $\alpha_i \not\equiv \beta'_{j-1}$ or, $\alpha_i \equiv \beta'_{j-1}$ and $\alpha'_j \equiv \beta_{i-1}$. We conclude the proof by a case analysis.

In the former case, as $\alpha_i \not\equiv \beta'_{j-1}$ and $\beta_{i-1} \not\equiv \beta'_{j-1}$ by (ii), the chain c_1 still exists as e_1 is not cyclic (same proof as previous Lemma).

In the latter case, we get a chain c_3 without e_k: $c3 = (\ldots, \alpha_{i-1}, \beta_j, e'_{j+1}, \ldots)$ as $\alpha_{i-1} =_s \beta_{i-1} \equiv \alpha'_j =_s \beta'_j$. This concludes the proof. □

Corollary 5.11 *Two minimal chains c_1 and c_2 with the same sets of edges are equal.*

Proof. By application of Corollary 5.9 and Lemma 5.10. □

Lemma 5.12 *Let c_1 and c_2 be two minimal chains between e_1 and e_2 incident to v. Then they have the same edges.*

Proof. Assume that c_1 and c_2 differ by say $e \in c_1$, $e \notin c_2$. Both c_1 and c_2 being minimal, β_i and α_{i+1} that occur by e in v are distinct. Hence e is not cyclic, we can linearize S at the occurrence of β_i or α_{i+1} by Lemmas 5.6 and 5.7 without modifying the abstract graph. □

Proposition 5.13 *Let S be an elementary cyclic set and v be a shared vertex of $\mathcal{G}_S$. For all pairs (e_1, e_2) of distinct edges incident to v there exists a minimum open chain between e_1 and e_2.*

Proof. Consequence of Lemma 5.10 and Corollaries 5.9 and 5.11. □

Lemma 5.14 *Let S be an elementary cyclic set and α, β be two distinct variables. If $\alpha =_s \beta$ there exist a minimum (t)-proof of this equality.*

Proof. A reduced (t)-proof is a sequence of variables α_i, $i = 0, \ldots, n$ with $\alpha_0 \equiv \alpha$, $\alpha_n \equiv \beta$ and $\alpha_i = \alpha_{i+1} \in S$. Assume that there exists two distinct sequences (α_i) and (α'_i). Let j be the first index such that $\alpha_j \not\equiv \alpha'_j$. Then as $\alpha =_s \alpha_j$, $\alpha =_s \alpha'_j$, $\beta =_s \alpha_j$, $\beta =_s \alpha'_j$ we can remove, say the equation $\alpha_{j-1} = \alpha_j$ without modifying the abstract graph, contradiction. □

Proposition 5.15 *Let S be some elementary cyclic set and α, β be two variables with $V(\alpha) = V(\beta)$. Then there exists a minimum closed chain between α and β.*

Proof. We first establishes the existence of such chains. If $\alpha =_s \beta$, there exists a (t)-proof of this equation by definition of $=_s$. Otherwise $v = V(\alpha) = V(\beta)$ is non-initial. By Lemma 2.9, there exists γ and δ such that $\gamma =_s \alpha$, γ occurs by some edge e_1 in v, $\delta =_s \beta$, δ occurs by some edge e_2 in v. By Proposition 5.13, there exists a minimal chain between e_1 and e_2. This establishes the existence of a chain proving $\gamma = \delta$, hence $\alpha = \beta$.

We now establishes the unicity. Assume $\alpha =_s \beta$. By Lemma 5.14, there exists a minimum (t)-proof of $\alpha =_s \beta$. Otherwise, we firstly assume that in minimal chains only one edge is involved and that (1) γ and δ occur by e, with $\gamma \neq_s \delta$, $\alpha =_s \gamma$ and $\beta =_s \delta$, (2) γ' and δ' occur by e', with $\gamma' \neq_s \delta'$, $\alpha =_s \gamma'$ and $\beta =_s \delta'$. If $e = e'$, but say $\gamma \not\equiv \gamma'$, we can linearize e.g. γ by e, as $\gamma =_s \gamma'$. But $e \neq e'$ is impossible by Lemma 5.8, as $\gamma =_s \gamma'$ and $\delta =_s \delta'$.

Secondly, assume that $n+1$ edges, $n > 0$, are involved in minimal chains and that we have two closed minimal distinct chains proving the equality $\alpha = \beta$:

$$c = (\gamma, e_0, \alpha_0, \beta_0, \ldots, \alpha_{n-1}, \beta_{n-1}, e_n, \delta),$$

$$c' = (\gamma', e'_0, \alpha'_0, \beta'_0, \ldots, \alpha'_{n-1}, \beta'_{n-1}, e'_n, \delta').$$

Then if $e_0 = e'_0$ and $e_n = e'_n$, by Proposition 5.13, we have $\gamma \not\equiv \gamma'$ or $\delta \not\equiv \delta'$, say $\gamma \not\equiv \gamma'$. But $\gamma =_s \gamma'$ and we may linearize say the occurrence of γ by e_0. Otherwise $e_0 \neq e'_0$. Then we have $\gamma =_s \gamma'$, $\alpha_0 \neq_s \gamma$ and $\alpha'_0 \neq_s \gamma'$ by minimality of the chains. As γ, α_0 occur by e_0 and γ', α'_0 occur by e'_0, once more we may linearize. □

Proposition 5.16 *Let S be an elementary cyclic set. Let $\alpha \in v$ and e being incident to v. Then there exists a minimum left-closed and right-open chain that starts with α and ends with β, for some β that occurs in v by e.*

Proof. Existence: by Proposition 2.10, v being non S-empty, there exists at least one variable β that occurs by e in v. For every such β there exists a minimum closed chain that proves $\alpha = \beta$ by Proposition 5.15.

Unicity: if α occurs by e the unicity follows from Lemma 5.7. Next, assume firstly that α is strictly equal to some variable that occurs by e. If there exists two such variables β and β', then $\alpha \not\equiv \beta$, $\alpha \not\equiv \beta'$ and $\beta \not\equiv \beta'$ imply that we may suppress some equation involved in the (t)-proof of, e.g., $\alpha =_s \beta$, contradiction. Secondly, assume that we have two minimal left-closed right-open distinct chains:

$$c = (\gamma, e_0, \alpha_0, \beta_0, \ldots, \alpha_{n-1}, \beta_{n-1}, e_n),$$
$$c' = (\gamma', e'_0, \alpha'_0, \beta'_0, \ldots, \alpha'_{n-1}, \beta'_{n-1}, e'_n).$$

If $e_0 = e'_0$ then the open chains are equal by Proposition 5.13. We are in the previous case: γ, γ' occur by e, both are strictly equal to α, we may suppress some equation of S involved in e.g. $\alpha =_s \gamma$. Finally, if $e_0 \neq e_0$, then we may also suppress the same strict equation, the abstract graph still are equal by Lemma 5.6 and the existence in the new graph of the two open chains associated to c and c'. □

These three propositions detail the three cases where we will need the unicity property in the next section. In addition we have

Proposition 5.17 *Let v be some cyclic needed vertex in $\mathcal{G}_S$, S some elementary cyclic set. There exists a maximum open chain in v.*

Proof. Direct consequence of Corollary 5.11, as we do not have any block in a needed cyclic vertex. Hence all open chains are minimal and there trivially exists at least one maximal chain. □

5.3 Computing the Minimum Deduction

Definition 5.3 *A cyclic variable ϕ of the vertex v is the proper variable of v iff ϕ is the (unique) variable that occurs in v by the edge incident to v belonging to the cycle.*

Theorem 5.18 *Let ϕ be some proper variable of the elementary cyclic set S. There exists a minimum deduction $\mathcal{D} \vdash \phi = C[\phi]$ of the cyclic equation associated to ϕ. The minimum deductions for other proper variables are obtained from $\mathcal{D}$ by a cyclic permutation of the auxiliary deductions of $\mathcal{D}$.*

Proof. We establish the result by giving a deterministic algorithm that searches such a proof. The inference rules are denoted by function symbols of arity 2: SU for the substitution, T for the transitivity and D for the simplification rule. The algorithm includes a main body and three recursive procedures. The first one *Connect* returns a proof $\phi_{i+1} = C_i[\phi_i]$ where ϕ_{i+1} (resp. ϕ_i) is the proper variable of the cyclic needed vertex v_{i+1} (resp. v_i). The procedure *Chain* takes an open chain from α_0 to β_{n-1} (with the usual notations) and returns a proof of $\alpha_0 = \beta_{n-1}$. Finally,

the procedure *Edge* takes an edge and two (distinct) variables that occur by this edge (either $\mathcal{R}$- or $\mathcal{S}$-variables) and returns a proof of their equality. The intuition behind this last procedure is: the equations associated to these variables are unique. Also, we have three distinct cases according to the relative position of the paths defined by these two equations: they are equal, one is suffix of the other, or they diverge at some vertex. One of the three Propositions 5.13, 5.15 or 5.16 applies to each case.

CYCLE DEDUCTION

Input: an elementary cyclic set S, its graph $\mathcal{G}_S$, some needed cyclic vertex v_0;
Let $v_0, \ldots, v_n$ be the sequence of needed cyclic vertices of $\mathcal{G}_S$,
v_{i+1} the first needed cyclic vertex above v_i along the cycle, with $v_{n+1} = v_0$;
For $i = 0, \ldots, n$ Let $T_i = Connect(v_i, v_{i+1})$;
Return $(S(T_0, S(\cdots S(T_{n-1}, T_n)\cdots)))$.

Procedure $Connect(v_1, v_2)$
If v_2 possesses a unique incident edge
Then Let $e : \alpha = C[\beta]$ be the unique non-strict equation of v_2;
Let γ be the variable of v_2 that possesses an occurrence by the cycle;
Let $\mathcal{D}$ be the minimum (t)-proof of $\gamma =_s \alpha$;
Return$(T(\mathcal{D}, e)$ or e if $\mathcal{D}$ is void);
Else Let $c = (e_0, \alpha_0, \beta_0, e_1, \ldots, e_n, \alpha_{n-1}, \beta_{n-1}, e_n)$ be the maximum chain of v_2
such that w, the unique needed $\mathcal{R}$-variable of v_2 occurs by e_n,
and e_0 is the cyclic edge incident to v_2;
Let $\mathcal{D}_1 = Chain(v_2, c)$;
Let $\mathcal{D}_2 = Edge(\beta_{n-1}, e, w)$;
Return$(T(\mathcal{D}_1, \mathcal{D}_2))$.

Procedure $Chain(e_0, \alpha_0, \beta_0, e_1, \ldots, e_n, \alpha_{n-1}, \beta_{n-1}, e_n)$
For $i = 0, \ldots, n-1$ Let $\mathcal{D}_i^1$ be the minimum (t)-proof of $\alpha_i =_s \beta_i$;
For $i = 0, \ldots, n-1$ Let $\mathcal{D}_i^2 = Edge(v, \beta_i, e_{i+1}, \alpha_{i+1})$;
For $i = 0, \ldots, n-1$ If $\mathcal{D}_i^1$ is void Then $\mathcal{D}_i^3 = \mathcal{D}_i^2$ Else $\mathcal{D}_i^3 = T(\mathcal{D}_i^1, \mathcal{D}_i^2)$;
Let $\mathcal{D} = \mathcal{D}_1^3$;
For $i = 2, \ldots, n-1$ Let $\mathcal{D} = T(\mathcal{D}, \mathcal{D}_i^3)$;
Return$(T(\mathcal{D}, \mathcal{D}_{n-1}^1)$ or $\mathcal{D}$ if $\mathcal{D}_{n-1}^1$ is void).

Procedure $Edge(\alpha, e, \beta)$
Let $e_1 : \phi = C[\alpha]$ and $e_2 : \psi = D[\beta]$ be associated to $O(\alpha, e)$ and $O(\beta, e)$;
Let p be the maximal common suffix of $p_1 = (V(\phi), O_C)$ and $p_2 = (V(\psi), O_D)$;
If $p = p_1 = p_2$
Then If $\phi \equiv \psi$
Then Return$(D(e_1, e_2))$;
Else Let $c = (\psi, e_0, \alpha_0, \ldots, \beta_{n-1}, e_n, \phi)$ be
the minimum closed chain between ψ and ϕ;
Let $\mathcal{D}_1 = Edge(\psi, e_0, \alpha_0)$;

Let $\mathcal{D}_2 = Chain(e_0, \alpha_0, \ldots, \beta_{n-1}, e_n)$;
Let $\mathcal{D}_3 = Edge(\beta_{n-1}, e_n, \psi)$;
Let $Tr = T(T(\mathcal{D}_1, \mathcal{D}_2), \mathcal{D}_3)$;
Return$(D(T(Tr, e_1), e_2))$;

If $p = p_1$
Then Let e_0 be the edge of p_2 incident to $V(\phi)$;
Let w be the $\mathcal{R}$-variable associated to $D[\beta]$ that occurs by e_0 in $V(\phi)$;
Let $c = (e_0, \alpha_0, \ldots, \beta_{n-1}, e_n, \phi)$ be
the minimum left-open right-closed chain between e_0 and ϕ;
Let $\mathcal{D}_1 = Chain(e_0, \alpha_0, \ldots, \beta_{n-1}, e_n)$;
Let $\mathcal{D}_2 = Edge(\beta_{n-1}, e_n, \phi)$;
Let $\mathcal{D}_3 = Edge(\alpha_0, e_0, w)$;
Let $Tr = T(\mathcal{D}_1, \mathcal{D}_2)$;
Return$(D(T(Tr, e_1), \mathcal{D}_3))$;

If $p = p_2$
Then Let e_n be the edge of p_1 incident to $V(\psi)$;
Let w be the $\mathcal{R}$-variable associated to $C[\alpha]$ that occurs by e_n in $V(\psi)$;
Let $c = (\psi, e_0, \alpha_0, \ldots, \beta_{n-1}, e_n)$ be
the minimum left-closed right-open chain between ψ and e_n;
Let $\mathcal{D}_1 = Edge(\psi, e_0, \alpha_0)$;
Let $\mathcal{D}_2 = Chain(e_0, \alpha_0, \ldots, \beta_{n-1}, e_n)$;
Let $\mathcal{D}_3 = Edge(\beta_{n-1}, e_n, w)$;
Let $Tr = T(\mathcal{D}_1, \mathcal{D}_2)$;
Return$(D(T(Tr, \mathcal{D}_3), e_2))$;

Else Let v' be the source of the path p;
Let e, e' be the two edges of the paths p_1 and p_2 that occur in v';
Let w_0, w_1 be the two corresponding $\mathcal{R}$-variables;
Let $c = (e_0, \alpha_0, \ldots, \beta_{n-1}, e_n)$ be the minimum open chain between e' and e;
Let $\mathcal{D}_1 = Edge(\alpha_0, e_0, w_1)$;
Let $\mathcal{D}_2 = Chain(e_0, \alpha_0, \ldots, \beta_{n-1}, e_n)$;
Let $\mathcal{D}_3 = Edge(\beta_{n-1}, e_n, w_0)$;
Return$(D(T(\mathcal{D}_2, \mathcal{D}_3), \mathcal{D}_1))$.

The correction of the algorithm follows from the Lemmas in section 5.1, i.e. the algorithm terminates and computes a cyclic deduction. The computed deduction is minimum by Propositions 5.13, 5.15, 5.16 and 5.17. □

We conclude by giving an exemple of an elementary cyclic set S, the needed variables are represented by greek letters, the other ones by latin letters:

$$\psi = \mu \to a, \quad \psi = \nu \to b, \quad \psi = (c \to ((\beta \to d) \to e)) \to f$$

$$\theta = \nu \to g, \quad \theta = \lambda \to h, \quad \theta = (i \to (j \to \gamma)) \to k$$

$$\omega = \phi \to l, \quad \omega = \lambda \to m$$

$$\phi = \beta \to n, \quad \mu = (o \to \gamma) \to p, \quad \lambda = q \to (\alpha \to \alpha)$$

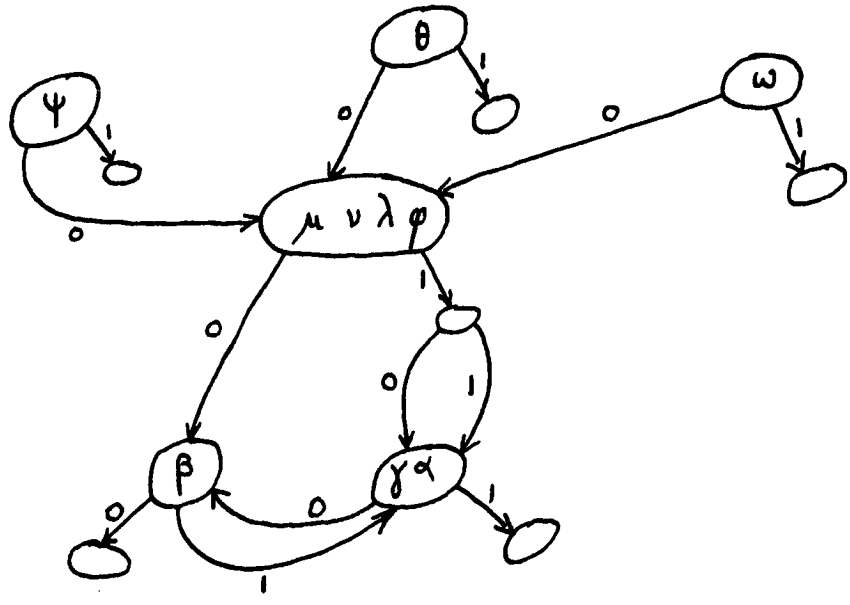

Figure 1: Graph $\mathcal{G}_S$

The graph is represented in Figure 1. The minimum cyclic deductions for the two variables β and γ that occur by the cycle in their vertices. are given by the following two auxiliary deductions:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\psi=\mu\to a \quad \psi=\nu\to b}{\mu=\nu} \quad \dfrac{\theta=\nu\to g \quad \theta=\lambda\to h}{\nu=\lambda}}{\mu=\lambda} \quad \dfrac{\omega=\lambda\to m \quad \omega=\phi\to l}{\lambda=\phi}}{\mu=\phi} \quad \phi=\beta\to n}{\mu=\beta\to n} \quad \mu=(o\to\gamma)\to p}{\beta=o\to\gamma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\theta=\lambda\to h \quad \theta=(i\to(j\to\gamma))\to k}{\lambda=i\to(j\to\gamma)} \quad \lambda=q\to(\alpha\to\alpha)}{\gamma=\alpha} \quad \dfrac{\lambda=q\to(\alpha\to\alpha) \quad \dfrac{\dfrac{\theta=\lambda\to h \quad \theta=\nu\to g}{\lambda=\nu} \quad \dfrac{\psi=\nu\to b \quad \psi=(c\to((\beta\to d)\to e))\to f}{\nu=c\to((\beta\to d)\to e)}}{\lambda=c\to((\beta\to d)\to e)}}{\alpha=\beta\to d}}{\gamma=\beta\to d}$$

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