A bidimensional electromagnetic problem

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Un Problème Electromagnétique en Dimension Deux

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Abstract : Electric alternating currents running in a system of infinite cylindrical parallel conductors generate an electromagnetic field which is defined by a potential $\varphi : \mathbb{R}^2 \to \mathbb{C}$. φ is harmonic in the exterior domain and satisfies a Helmholtz equation in the cross-sections of the conductors. We study the asymptotic properties of φ as the frequency tends to infinity, in particular the boundary layers due to the skin effect.

Résumé : On considère le champ électromagnétique induit par la circulation d'un courant électrique alternatif dans un système de conducteurs parallèles, cylindriques, de longueur infinie. Dans une coupe transversale, le potentiel correspondant $\varphi : \mathbb{R}^2 \to \mathbb{C}$ est harmonique à l'extérieur des conducteurs et vérifie à l'intérieur une équation d'Helmholtz. Nous étudions le comportement asymptotique de φ lorsque la fréquence tend vers l'infini et en particulier les couches limites engendrées par l'effet pelliculaire.

Key words : electromagnetic fields, asymptotic behaviour, boundary layer, skin effect, Helmholtz equation.

AMS (MOS) subject classification : 35J05, 35B25, 35B40.

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PAPIER RECUPÉRÉ ET RECYCLÉ

A BIDIMENSIONAL ELECTROMAGNETIC PROBLEM

1. INTRODUCTION.

This paper has been motivated by the numerical simulation of the electromagnetic casting process (EMC). When the ingot is sufficiently long, the electromagnetic part of the problem reduces to the search of a complex potential φ in $\mathbb{R}^2$, of class C^1, with logarithmic behaviour at infinity, satisfying the conditions

$$\Delta\varphi + 2i\alpha^2(\varphi + C_k) = 0 \text{ in } \Omega_k, 1 \leq k \leq N ; \Delta\varphi = 0 \text{ in } \overline{\Omega}^c ; \tag{1.1}$$

here $\Omega_k \subset \mathbb{R}^2, 1 \leq k \leq N$, are the cross-sections in the x_1, x_2 plane of cylindrical electric conductors in which run a current with angular frequency ω ; $2\alpha^2 = \mu_o\sigma\omega$ is a real constant where μ_o is the magnetic permeability of the air and σ the conductivity ; i is the imaginary unit and the $C_k's$ are given complex constants.

Our initial purpose was to find a fast, accurate and reliable numerical algorithm for solving Problem (1.1). We have been led to a theorical study of Problem (1.1), partly independent of our original aim, which is the subject of this paper. We shall report on the numerical aspects in a forthcoming paper.

Maybe because of its particular character, we have found few references in the literature relative to Problem (1.1). All of them ([2], [4]) are essentially concerned with numerical, practical or theoretical, questions.

In Section 2, we introduce some notations and recall a few basic mathematical tools and results.

We feel it was necessary to construct with some details our mathematical model ; this is the object of Section 3. In Section 4, because of their physical and mathematical relevance, we define two specific problems, closely related to (1.1), which are studied in the remaining parts of the paper.

For many applications, in particular for the EMC process, the coefficient α is large and one can observe a very pronounced skin effect in the conductors, i.e. the electric current $j = 2i\alpha^2(\varphi+C_k)$ almost vanishes in Ω_k except in a thin layer in the neighborhood of $\partial\Omega_k$. This is a serious difficulty for numerical computations, in particular since we are specially interested by the fields (current, induction, Laplace forces) in the conductors. For this reason we analyse in Sections 5, 6 and 7

the behaviour of φ for large α. Section 5 gives some global estimates for the rate of convergence of φ to a limit function φ_∞, as α tends to infinity. Section 6 contains more refined local estimates. We construct in Section 7 a simple and accurate boundary layer approximation of φ in the neighborhood of smooth parts of $\partial\Omega_k$. All these results are rather satisfactory and complete except in the neighborhoods of corners of $\partial\Omega_k$ for which many improvements remain to do ; in particular we have not succeeded in getting a simple picture of the singularity of φ at a corner for large α.

As a consequence of Proposition 7.2, it follows that, along regular parts of $\partial\Omega_k$, φ satisfies approximatively a Robin type boundary condition. We show in Section 8 how this property can be used, in the case where the $\partial\Omega_k$'s have no corner, for defining an approximation of φ which can be computed easily.

2. NOTATIONS AND SOME BASIC TOOLS.

An element of $\mathbb{R}^2$ is denoted by $x = (x_1,x_2)$; $|x|^2 = x_1^2 + x_2^2$. $B(x,\delta)$ is the open ball with center x and radius δ. ∂_1 and ∂_2 are the partial derivative with respect to x_1 and x_2 and $\vec{\nabla} = (\partial_1,\partial_2)$. For $m \geq 0$, ∂_1^m and ∂_2^m are the m-th derivatives with respect to x_1 and x_2. For $\Lambda \subset \mathbb{R}^2$, $\overline{\Lambda}$ is the closure of Λ, $\partial\Lambda$ is the boundary of Λ and χ_Λ is the caracteristic set of Λ. For a function $v : \Lambda \rightarrow \mathbb{C}$, $\overline{v}$ is the complex conjugate of v, and $\frac{\partial v}{\partial n}$ denotes the exterior normal derivative. For $\Lambda \subset \mathbb{R}^2$ open, the symbols $L^p(\Lambda)$, $H^m(\Lambda)$, $W^{m,p}(\Lambda)$ (Sobolev spaces), $C^m(\Lambda)$, $C_o^m(\Lambda)$, $C^{m,\beta}\Lambda)$ (the m-th derivates are β Hölder continuous, $0 < \beta < 1$) and the corresponding norms $\|.\|_{L^p(\Lambda)}$,...., have their usual meanings for complex valued functions. We shall also use the spaces $L_{loc}^p(\Lambda)$,... and $L^p(\partial\Lambda)$....

For an open set $\Lambda \subset \mathbb{R}^2$, $\partial\Lambda$ is piecewise C^∞ if $\partial\Lambda$ is composed of a finite number of closed arcs which are C^∞ and if for each interior angle γ at corners one has $0 < \gamma < 2\pi$; furthermore we impose the condition $\partial\Lambda = \partial\overline{\Lambda}$.

Let $\Lambda \subset \mathbb{R}^2$ be an open connected set such that Λ^c is bounded and let $\Phi \neq \emptyset$ be an open bounded subset of Λ. We define

$$W_0^1(\Lambda) = \{v : \Lambda \to \mathbb{C} \mid \frac{v}{(1+|x|)\ln(2+|x|)} \in L^2(\Lambda) : \partial_l v \in L^2(\Lambda), l=1,2\}, \quad (2.1)$$

$$\|v\|_{W_0^1(\Lambda)}^2 = \|v\|_{L^2(\Phi)}^2 + \|\partial_1 v\|_{L^2(\Lambda)}^2 + \|\partial_2 v\|_{L^2(\Lambda)}^2. \quad (2.2)$$

For a study of the space $W_0^1(\Lambda)$, one can consult for example [6]. In particular, one has :

Proposition 2.1.

a) $W_0^1(\Lambda)$ is a complete space for the norm $\|.\|_{W_0^1(\Lambda)}$.

b) $C_0^\infty(\mathbb{R}^2)$ is dense in $W_0^1(\mathbb{R}^2)$.

c) If $\partial\Lambda$ is piecewise C^∞, $W_0^1(\Lambda) = \{v_{|\Lambda} \mid v \in W_0^1(\mathbb{R}^2)\}$.

d) $1 \in W_0^1(\Lambda)$, $\ln(1+|x|) \notin W_0^1(\mathbb{R}^2)$.

The following result of potential theory is classical.

Proposition 2.2. Let v be an harmonic complex function on the open domain $\Lambda \subset \mathbb{R}^2$ with bounded complement. We suppose that $v(x) = 0(\ln|x|)$ as $|x| \to \infty$. Then there exists complex constants d and e such that

$$v(x) = d \ln|x| + e + 0 (|x|^{-1}),$$

$$\partial_1^k \partial_2^\ell v(x) = d \ \partial_1^k \partial_2^\ell \ln|x| + 0 (|x|^{-(k+\ell+1)}), k+\ell \geq 1.$$

Next trace estimate will play an important role in Section 5.

Proposition 2.3. Let $\Lambda \subset \mathbb{R}^2$ be an open domain such that $\partial\Lambda$ is piecewise C^∞. Then there exists a constant c independent of $v \in H^1(\Lambda)$ such that

$$\|v\|_{L^2(\partial\Lambda)}^2 \leq c \|v\|_{H^1(\Lambda)} \|v\|_{L^2(\Lambda)}. \quad (2.3)$$

<u>Proof</u>. By a classical imbedding theorem (see for example [1]), for any $w \in L^1(\Lambda)$ such that $\partial_\ell w \in L^1(\Lambda)$, $\ell = 1,2$, we have $\|w\|_{L^1(\partial\Lambda)} \leq c \, \{\|w\|_{L^1(\Lambda)} + \|\partial_1 w\|_{L^1(\Lambda)} + \|\partial_2 w\|_{L^1(\Lambda)}\}$, where c is some generic constant. We apply this result to $w = v^2$ and by Schwarz's inequality we obtain

$$(\|v\|_{L^2(\partial\Lambda)})^2 \leq c \, \{\|v\|_{L^2(\Lambda)}^2 + \|v\|_{L^2(\Lambda)} (\|\partial_1 v\|_{L^2(\Lambda)} + \|\partial_2 v\|_{L^2(\Lambda)})\}$$

$$\leq c \, \|v\|_{L^2(\Lambda)} \, \|v\|_{H^1(\Lambda)}.$$

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3. <u>MATHEMATICAL MODEL</u>.

In $\mathbb{R}^3$ we consider a system of N cylindrical conductors parallel to the x_3 axis in an alternating electromagnetic field. We denote by $\Omega_1, \Omega_2,...,\Omega_N \subset \mathbb{R}^2$ their cross-sections, that we suppose bounded in the x_1, x_2 plane and set $\Omega = \cup \, \Omega_k$.

Let $\vec{J}, \vec{E}$ and $\vec{H}$ be respectively the tridimensional complex current, electric and magnetic fields. We introduce the following assumptions :

a) $\vec{E}(x_1,x_2,x_3,t) = E(x_1,x_2) \, e^{-i\omega t} \, \vec{e_3}$ in the conductors, $\qquad$ (3.1)

b) $\vec{H}(x_1,x_2,x_3,t) = H_1(x_1,x_2) \, e^{-i\omega t} \, \vec{e_1} + H_2(x_1,x_2) \, e^{-i\omega t} \, \vec{e_2}$ in $\mathbb{R}^3$, $\qquad$ (3.2)

c) $\vec{J} = \sigma \vec{E}$ in the conductors, $\vec{J} = 0$ outside the conductors, $\qquad$ (3.3)

d) $\partial_t \vec{H} + 1/\mu_o \, \vec{\text{rot}} \, \vec{E} = 0$ in the conductors, $\qquad$ (3.4)

e) $\vec{\text{rot}} \, \vec{H} = \vec{J}$ in $\mathbb{R}^3$, $\qquad$ (3.5)

f) $\text{div} \, \vec{H} = 0$ in $\mathbb{R}^3$. $\qquad$ (3.6)

Let us precise and comment these hypotheses. i is the complex unit ; ω is the angular frequency ; t is the time ; μ_o is the magnetic permeability, in the vacuum ; $\vec{e_1}, \vec{e_2}, \vec{e_3}$ are the unit vectors along the coordinate axis. (3.1) is assumed to be valid only in the conductors ; supposing (3.1) valid in $\mathbb{R}^3$ is physically not realistic (see Sommerfeld [7]) and would lead to mathematical contradictions. In (3.3), we suppose that the electric conductivity σ is a same positive constant for all conductors ; in fact, all results contained in this paper can be extended without difficulty to the case where the conductivity is constant in each conductor but

different from one to another. (3.4), (3.5), (3.6) are standard Maxwell's equations where in (3.5), we have neglected the displacement currents which is legitimate for moderate frequencies. Finally, we observe that all fields are x_3 independent and can be considered are mapping from $\mathbb{R}^2$ into $\mathbb{C}$ or $\mathbb{C}^2$.

From (3.1), (3.3), there exists $j : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\vec{j}(x_1,x_2,x_3,t) = j(x_1,x_2)\, e^{-i\omega t}\vec{e_3} \qquad \text{and}$$

$$j(x) = \sigma\, E(x) \;\text{ for } x \in \Omega, \; j(x) = 0 \;\text{ for } x \in \Omega^c, \tag{3.7}$$

where $x = (x_1,x_2) \in \mathbb{R}^2$. From (3.6) and (3.2) follows the existence of a potential φ in $\mathbb{R}^2$ such that

$$(H_1(x),H_2(x)) = \overrightarrow{\text{rot}}\, \varphi(x) = (\partial_2\, \varphi(x), -\partial_1\, \varphi(x)), \, x \in \mathbb{R}^2 \tag{3.8}$$

where $\overrightarrow{\text{rot}}$ denotes here the bidimensional vector curl. By using in particular (3.1), (3.4) (3.7) and (3.8), we obtain

$$\overrightarrow{\text{rot}}\, (i\,\omega\,\mu_o\, \sigma\, \varphi - j) = 0 \qquad \text{in } \Omega_k, \, k = 1,2,...,N\,;$$

this implies the existence of constants $C_k \in \mathbb{C}$ such that

$$j = i\,\omega\,\mu_o\, \sigma\, (\varphi + C_k) \qquad \text{in } \Omega_k, \, k = 1,2,...,N. \tag{3.9}$$

By (3.2), (3.5), (3.7) and (3.8), we have

$$-\Delta\, \varphi = j \qquad \text{in } \mathbb{R}^2\,; \tag{3.10}$$

finally, with (3.7), (3.9), we obtain

$$-\Delta\, \varphi = i\,\omega\,\mu_o\, \sigma\, \chi\, \varphi + f \qquad \text{in } \mathbb{R}^2 \tag{3.11}$$

where χ is the characteristic function of Ω and $f = i\,\omega\,\mu_o\, \sigma\, C_k$ in $\Omega_k, f = 0$ in Ω^c.

For physical reasons, it is natural to impose to φ a behaviour at most logarithmic at infinity. By Proposition 2.2 there exist constants d and $e \in \mathbb{C}$ such that

$$\varphi(x) = d \ln|x| + e + O\,(|x|^{-1}) \;\text{ as } |x| \to \infty, \tag{3.12}$$

Without restriction of generality, we can furthermore impose the constraint $e = 0$ since the physical fields are independent of e.

On the basis of equations (3.11), (3.12) with the condition $e = 0$, we are now in the position of formulating the mathematical problem we shall study.

Let $\Omega = \cup_k \Omega_k \subset \mathbb{R}^2$; we suppose

a) Ω_k is a bounded connected open set, $1 \leq k \leq n$; $\qquad$ (3.13)

b) $\Omega_k \cap \Omega_\ell = \emptyset$ for $k \neq \ell$; $\partial\Omega$ is piecewise C^∞. $\qquad$ (3.14)

For given constants $C_k \in \mathbb{C}$, $k = 1,2,...,N$, the problem is to find $\varphi \in L^2_{loc}(\mathbb{R}^2)$ such that

$$-\Delta \varphi = 2i\,\alpha^2\,\chi_\Omega\,\varphi + f \quad \text{in distribution in } \mathbb{R}^2, \qquad (3.15)$$

$$\varphi(x) = d \ln|x| + 0\,(|x|^{-1}) \text{ as } |x| \to \infty, \qquad (3.16)$$

where χ_Ω is the characteristic function of Ω, $f = 2i\,\alpha^2 C_k$ in Ω_k, $f = 0$ in Ω^c and

$\alpha^2 = \omega\,\mu_o\,\sigma/2 > 0$; in (3.16) $d \in \mathbb{C}$ is an unknown constant.

From potential theory (see for example NEDELEC [6]), it follows that Problem (3.15), (3.16) is equivalent to the following integral problem : For given constants $C_k \in \mathbb{C}$, find $\varphi \in L^2(\Omega)$ such that

$$\varphi(x) = \frac{-i\alpha^2}{\pi} \int_\Omega \ln|\xi-x|\,\varphi(\xi)\,d\xi - \frac{1}{2\pi} \int_\Omega \ln|\xi-x|\,f(\xi)\,d\xi. \qquad (3.17)$$

Our first basic result is contained in the following.

<u>Proposition 3.1</u>. *Problem (3.15), (3.16) has one and only one solution. Furthermore* $\varphi \in W^{2,p}_{loc}(\mathbb{R}^2)$, *for any* $p \geq 1$, $\quad \varphi \in C^{1,\beta}(\mathbb{R}^2)$ *for any* $0 < \beta < 1$, $\quad \varphi \in C^\infty(\Omega \cup \overline{\Omega}^c)$.

<u>Proof</u>. For proving the existence and uniqueness of the solution, it suffices, by the Fredholm alternative, to verify that, for $f = 0$, (3.17) admits only the trivial solution ; we multiply (3.17) by $\overline{\varphi}(x)$ and integrate over Ω ; we then remark that the left member of the resulting equation is real whereas the right member is purely imaginary. Standard regularity results for the Laplacian operator show that $\varphi \in C^\infty(\Omega \cup \overline{\Omega}^c)$ and $\varphi \in W^{2,p}_{loc}(\mathbb{R}^2)$; therefore $\varphi \in C^{1,\beta}_{loc}(\mathbb{R}^2)$; Proposition 2.2 implies that, in fact, $\varphi \in C^{1,\beta}(\mathbb{R}^2)$, $0 < \beta < 1$.

For a solution of (2.15), (2.16), we introduce the notion of total current defined by

$$J = \int_\Omega j(x)\,dx. \qquad (3.18)$$

Since $2\alpha^2 = \omega \, \mu_o \, \sigma$, we obtain by (2.9)

$$J = 2\alpha^2 \, i \sum_{k=1}^{N} \int_{\Omega_k} (\varphi + C_k)\,(x)\,dx. \qquad (3.19)$$

Furthermore, we have the following relation :

<u>Proposition 3.2</u>. *Let φ be the solution of* (3.15), (3.16). *Then*

$$J = 2\pi\,d.$$

<u>Proof</u>. By Proposition 2.2, we have $\dfrac{d\varphi}{dn} = \dfrac{d}{R} + 0\,(R^{-2})$.

Integrating Relation (3.10) on $B(0,R)$, we obtain the desired result by using Green's formula and (3.18).

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4. TWO PARTICULAR CASES OF PHYSICAL IMPORTANCE.

In simulating the industrial process of the electromagnetic casting (EMC), we have encountered the following particular case of the situation described in the preceding section. For each $1 \le k \le N$, there exists $1 \le \ell \le N$ such that Ω_k and Ω_ℓ are symmetric with respect to the axis Ox_2 ; furthermore $C_k = -C_\ell$; k may be equal to ℓ in which case $C_k = 0$. The constants C_k are obtained from the specifications of the EMC installation. By uniqueness of the solution of Problem (3.15), (3.16), it follows that $\varphi(-x_1,x_2) = -\varphi(x_1,x_2)$ for all $(x_1,x_2) \in \mathbb{R}^2$ which implies that $d = 0$ in (3.16). As a consequence of Propositions 2.1, 2.2 and 3.2, $\varphi \in W_0^1(\mathbb{R}^2)$ so that by (3.15) φ satisfies the variational equation

$$\int_{\mathbb{R}^2} \vec{\nabla}\varphi.\vec{\nabla}\,\overline{v} - 2i\,\alpha^2 \int_{\Omega} \varphi\,\overline{v} = \int_{\Omega} f\,\overline{v} \qquad \forall v \in W_0^1(\mathbb{R}^2). \qquad (4.1)$$

For mathematical convenience, we generalize somewhat the above situation and consider the following problem. For given sets Ω_k satisfying (3.13), (3.14) and for given constants $C_k \in \mathbb{C}, 1 \le k \le N$, find $\varphi \in W_0^1(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} \vec{\nabla}\varphi.\vec{\nabla v} - 2i\,\alpha^2 \int_{\Omega} \varphi\,\overline{v} = \int_{\Omega} f\,\overline{v} \qquad \forall v \in W_0^1(\mathbb{R}^2), \qquad (4.2)$$

where $\Omega = \bigcup_{k=1}^{N} \Omega_k$, $f = 2i\,\alpha^2\,C_k$ on Ω_k, $f = 0$ in Ω^c.

We shall study Problem (4.2) in details in the next sections. We however immediatly remark that, by Propositions 2.1 and 2.2, a solution φ of (4.2) will behave at infinity as $e + 0\,(|x|^{-1})$ and consequently will be a solution of the original Problem (3.15), (3.16) if and only if the constant $e = 0$.

Another situation of interest is the case of a single conductor so that $\Omega = \Omega_1$. In this context, however, the total current J defined in (3.13) appears as a data and the constant C_1 as an unknown. By Proposition 3.2, the relevant problem then reads : for given $J \in \mathbb{C}$, find $\varphi \in L^2_{\ell oc}(\mathbb{R}^2)$ and $\xi \in \mathbb{C}$ such that

$$-\Delta\varphi = 2i\,\alpha^2\,\chi_\Omega(\varphi + \xi) \qquad \text{in distribution in } \mathbb{R}^2, \tag{4.3}$$

$$\varphi(x) = \frac{J}{2\pi}\,\ln|x| + 0\,(|x|^{-1}) \qquad \text{as } |x| \to \infty. \tag{4.4}$$

In (4.4), the constant J is immaterial ; furthermore, from a mathematical point of view, the number of conductors is a little importance. In the next sections we shall consider the following problem : for given Ω_k, $1 \le k \le N$ satisfying (3.13), (3.14) find $\varphi \in L^2_{\ell oc}(\mathbb{R}^2)$ and $\xi \in \mathbb{C}$ such that

$$-\Delta\varphi = 2i\,\alpha^2\,\chi_\Omega(\varphi + \xi) \qquad \text{in distribution in } \mathbb{R}^2, \tag{4.5}$$

$$\varphi(x) = \ln|x| + 0(|x|^{-1}) \qquad \text{as } |x| \to \infty. \tag{4.6}$$

We conclude this section by two remarks.

<u>Remark 4.1</u>. The single conductor problem (4.3), (4.4) can be generalized in a natural way for the case of several conductors : we prescribe the current J_k for each of them. By (3.5), this leads to the following problem :

for given $J_k \in \mathbb{C}$, $1 \le k \le N$, find $\varphi \in L^2_{loc}(\mathbb{R}^2)$ and the constants $C_k \in \mathbb{C}$, $1 \le k \le N$, such that (3.15), (3.16) are satisfied under the constraint

$$2i\,\alpha^2 \int_{\Omega_k}(\varphi + C_k)\,(x)\,dx = J_k\,, \qquad 1 \le k \le N.$$

Clearly, by (3.15), the total current J will be the sum of the J_k. This problem which is mentionned in the literature [2] possesses one and only one solution. To a large extent, its study, in particular the convergence when α tends to infinity, can

be reduced to the ones of Problems (4.2) and (4.5),(4.6). For the sake of simplicity, we shall not consider it in the following.

Remark 4.2. One can object that the problem introduced in Remark 4:1 is not physical if the total current J does not vanish ; in particular, this will be always the case for the situation of the single conductor. The following argument shows that this difficulty can be overcome. To this end, we add to the system an auxiliary conductor Ω_{N+1} for which we prescribe a current equal to $-J$; we then translate in a given direction. Let b denote the distance between the origin and the center of gravity of Ω_{N+1}, φ_b and φ be the solutions of the auxiliary and of the original problems respectively. It is then possible to show that for any fixed α and for any fixed bounded domain Λ we have $\|\varphi_b - \varphi\|_{H^1(\Lambda)} = 0(1/b)$.

5. GLOBAL ASYMPTOTIC ESTIMATES.

As mentionned in the Introduction, we are interested in situations with pronounced skin effect in the conductors. It is then natural to consider what happens if we let the angular frequency ω tend to infinity ; since $2\alpha^2 = \mu_o \sigma \omega$, this amounts to let the parameter α tend to infinity. Unfortunately for Problem (3.15), (3.16), it may happen for exceptional domains Ω that d becomes unbounded as α tends to infinity ; see Remark 5.2 below. For this reason, we shall treat Problems (4.2) and (4.5),(4.6). In order to insist on the dependence on α, we shall use sometimes the notation φ_α, ξ_α,... instead of φ, ξ .... We begin with Problem (4.2).

Proposition 5.1. *Problem (4.2) has one and only one solution. Furthermore* $\varphi \in C^{1,\beta}(\mathbb{R}^2) \cap W_{loc}^{2,p}(\mathbb{R}^2)$, $0 < \beta < 1, p \geq 1,$ *and there exists a constant* $e \in \mathbb{C}$ *such that*

$$\varphi(x) = e + 0 \, (|x|^{-1}) \text{ as } |x| \to \infty.$$

Proof. Multiplying (4.2) by $(1+i)$, we see that the left member then defines a continuous and coercive form on $W_0^1(\mathbb{R}^2)$ defined in Section 2 ; existence and uniqueness follows by Lax-Milgram's Lemma. Regularity follows from the fact that φ satisfies (3.15). The asymptotic relation as $|x| \to \infty$ can be deduced from Propositions 2.1 and 2.2.

We now introduce the function φ_∞, candidate for the limit of φ_α when $\alpha \to \infty$.

Lemma 5.1. *In connection with Problem (4.2), there exists a unique function $\varphi_\infty \in W_0^1(\mathbb{R}^2)$ such that $\varphi_\infty = -C_k$ in Ω_k and $\Delta\varphi_\infty = 0$ in $\overline{\Omega}^c$. Furthermore $d\varphi_\infty/dn \in L^2(\partial\Omega)$, where $d\varphi_\infty/dn$ is the normal derivative, exterior to Ω, of the restriction of φ to $\overline{\Omega}^c$.*

Proof. It suffices to find $\varphi \in W_0^1(\overline{\Omega}^c)$ such that $\Delta\varphi = 0$ and $\varphi = C_k$ on $\partial\Omega_k$. This problem can be reduced to a problem of the form : find $\sigma \in G$ such that $\int_{\overline{\Omega}^c} \nabla\sigma.\nabla\overline{v} = \int_{\overline{\Omega}^c} g\,\overline{v}$ for all $v \in G$, where g is a bounded function with bounded support and $G = \{v \in W_0^1(\overline{\Omega}^c) \,|\, v = 0 \text{ on } \partial\Omega\}$; one remarks that the Dirichlet form is coercive on G and apply Lax-Milgram's Lemma. That $d\varphi/dn \in L^2(\partial\Omega)$ is a consequence of the fact that $\partial\Omega$ is piecewise C^∞; see GRISVARD [3]. ∎

Proposition 5.2. *Let φ_α be the solution of Problem (4.2) and φ_∞ be defined by Lemma 5.1. Then asymptotically, as $\alpha \to \infty$, we have :*

a) $\|\varphi_\alpha - \varphi_\infty\|_{W_0^1(\mathbb{R}^2)} = 0\,(\alpha^{-1/2})$,   b) $\|\varphi_\alpha - \varphi_\infty\|_{L^2(\Omega)} = 0\,(\alpha^{-3/2})$,

c) $\|\varphi_\alpha - \varphi_\infty\|_{L^2(\partial\Omega)} = 0\,(\alpha^{-1})$,   d) $\|\nabla\varphi_\alpha\|_{L^2(\partial\Omega)} = 0\,(1)$.

Proof. Set $\eta = \varphi_\alpha - \varphi_\infty$. We have

$$-\Delta\eta - 2i\alpha^2\,\eta = 0 \quad \text{in } \Omega, \quad -\Delta\eta = 0 \quad \text{in } \overline{\Omega}^c . \tag{5.1}$$

By Proposition 5.1 and Lemma 5.1, $\eta \in W_0^1(\mathbb{R}^2)$ but its normal derivative on $\partial\Omega$ has a jump which is equal to $d\varphi_\infty/dn$. We multiply both equations of (5.1) by $\overline{\eta}$, integrate respectively on Ω and $\overline{\Omega}^c$, use Green's formula and add the resulting relations ; we get

$$\int_{\mathbb{R}^2} |\nabla\eta|^2 - 2i\,\alpha^2 \int_\Omega |\eta|^2 = -\int_{\partial\Omega} d\varphi_\infty/dn\,\overline{\eta} . \tag{5.2}$$

Let us take the square of the modulus in (5.2). The left member is bounded from above by

$$(\|\nabla\eta\|^2_{L^2(\mathbb{R}^2)} + 2\,\alpha^2\|\eta\|^2_{L^2(\Omega)})/2 \geq (\|\eta\|^2_{W^1_0(\mathbb{R}^2)} + \|\alpha\eta\|^2_{L^2(\Omega)})/2 \quad \text{if } \alpha \geq 1$$

and if we set $\Lambda = \mathbb{R}^2$, $\Phi = \Omega$ in (2.2). As far as the right member is concerned, we can write by Propositions 2.3 and Lemma 5.1 :

$$\Big|\int_{\partial\Omega} \frac{d\varphi_\infty}{dn}\,\overline{\eta}\,\Big|^2 \leq \|\frac{d\varphi_\infty}{dn}\|^2_{L^2(\partial\Omega)}\,\|\eta\|^2_{L^2(\partial\Omega)} \leq c\,\|\eta\|_{H^1(\Omega)}\,\|\eta\|_{L^2(\Omega)}$$

$$\leq \frac{c}{\alpha}\,(\|\eta\|^2_{W^1_0(\mathbb{R}^2)} + \|\alpha\,\eta\|^2_{L^2(\Omega)})\,,$$

where c is a generic constant independent of α ; parts a and b follow immediatly. By Proposition 2.1, part c is a consequence of parts a and b. By (5.1) and part b we obtain that $\|\Delta\eta\|_{L^2(\Omega)} = 0(\alpha^{1/2})$ and consequently $\|\Delta\varphi_\alpha\|_{L^2(\mathbb{R}^2)} = 0(\alpha^{1/2})$; since (part a) $\|\varphi_\alpha\|_{W^1_0(\mathbb{R}^2)} = 0(1)$, we conclude by standard elliptic regularity results that $\|\varphi_\alpha\|_{H^2(\Omega)} = 0(\alpha^{1/2})$. Consider a particular Ω_k and set $\zeta = \varphi_\alpha - C_k$; from the above results we have $\|\zeta\|_{H^2(\Omega_k)} = 0(\alpha^{1/2})$ and $\|\zeta\|_{H^1(\Omega_k)} = 0(\alpha^{-1/2})$; applying Proposition (2.3) to $\partial_\ell\zeta$ we obtain $\|\partial_\ell\zeta\|_{L^2(\partial\Omega)} = 0(1)$, $\ell = 1,2$; this proves part d of Proposition 5.2. ●

Next we consider Problem (4.5), (4.6). Let $\zeta \in C^\infty(\mathbb{R}^2)$ such that $\zeta(x) = 0$ for $|x| \leq a$ and $\zeta(x) = 1$ for $|x| \geq a + 1$; a is a fixed number chosen in such a way that $\Omega \subset B(0,a-1)$. $g(x) = \Delta(\zeta(x)\ln|x|)$ is then a $C^\infty(\mathbb{R}^2)$ function, with bounded support and vanishing on Ω. By using variational methods as in the proof of Proposition 5.1 and of Lemma 5.1 we define uniquely w_α and $w_\infty \in W^1_0(\mathbb{R}^2)$ by the requirements

$$-\Delta w_\alpha = 2i\,\alpha^2\,\chi_\Omega\,w_\alpha + g \qquad \text{in distribution in } \mathbb{R}^2, \tag{5.3}$$

$$w_\infty = 0 \text{ in } \Omega, \quad -\Delta w_\infty = g \qquad \text{in distribution in } \overline{\Omega}^c\,. \tag{5.4}$$

By Propositions 2.1 and 2.2, there exist ξ_α and $\xi_\infty \in \mathbb{C}$ such that

$$w_\alpha(x) = \xi_\alpha + 0(|x|^{-1}), \qquad w_\infty(x) = \xi_\infty + 0(|x|^{-1}). \qquad (5.5)$$

Set now

$$\varphi_\alpha = \zeta \ln |x| + w_\alpha - \xi_\alpha, \qquad \varphi_\infty = \zeta \ln |x| + w_\infty - \xi_\infty. \qquad (5.6)$$

<u>Proposition 5.3.</u> a) *Problem* (4.5), (4.6) *possesses a unique solution* $(\varphi_\alpha, \xi_\alpha)$; φ_α

belongs to $C^{1,\beta}(\mathbb{R}^2) \cap W^{2,p}_{loc}(\mathbb{R}^2)$, *for any* β, p *with* $0 < \beta < 1, 1 \leq p < +\infty$.

b) *There exist a unique function* φ_∞ *and a unique constant* $\xi_\infty \in \mathbb{C}$

satisfying the properties : $\varphi_\infty \in H^1_{loc}(\mathbb{R}^2)$, $\varphi_\infty = -\xi_\infty$ *on* Ω, φ_∞ *is harmonic on*

$\overline{\Omega}^c$, $\varphi_\infty(x) = \ln |x| + 0 (|x|^{-1})$ *as* $|x| \to \infty$.

c) $(\varphi_\alpha, \xi_\alpha)$ *and* $(\varphi_\infty, \xi_\infty)$ *are given by* (5.5), (5.6).

<u>Proof</u>. We only check part a. Direct calculations show that $(\varphi_\alpha, \xi_\alpha)$, as defined by (5.5), (5.6), is solution of Problem (4.5), (4.6). Suppose it has two solutions and denote their difference by (v, η). Then, by Propositions 2.1 and 2.2, $v \in W^1_0(\mathbb{R}^2)$ and satisfies the equation $-\Delta v = 2i \, \alpha^2(v + \eta)$ which implies $v = -\eta$ on $\mathbb{R}^2$; since $v(x) = 0(|x|^{-1})$ as $|x| \to \infty$, we get $\eta = 0$. Regularity results follow as in the proof of Proposition 3.1.

We now study the convergence of φ_α towards φ_∞. The arguments of the proof of Proposition 5.2 apply to w_α and w_∞ ; in fact, as easily checked, Proposition 5.2 is valid when we replace φ_α and φ_∞ by w_α and w_∞ respectively. By (5.6), it remains to estimate the term $(\xi_\alpha - \xi_\infty)$. To this aim, consider the domain $B(0,r) \setminus \overline{\Omega}$ for r large enough. By Green's formula, we obtain

$$\int_{\partial\Omega} \{(w_\alpha - w_\infty) \frac{d\varphi_\infty}{dn} - \varphi_\infty \frac{d(w_\alpha - w_\infty)}{dn} \} = \int_{\partial B(0,r)} \{(w_\alpha - w_\infty) \frac{d\varphi_\infty}{dn} - \varphi_\infty \frac{d(w_\alpha - w_\infty)}{dn} \}, \qquad (5.7)$$

where the normal is exterior to Ω and $B(0,r)$. We use the asymptotic behaviours of φ_∞, w_α and w_∞ as $|x| \to \infty$. Standard arguments show that

$\int_{\partial\Omega} \varphi_\infty \, d(w_\alpha - w_\infty)/dn = -\xi_\infty \int_{\partial\Omega} d(w_\alpha - w_\infty)/dn = 0$; furthermore, as $r \to \infty$, the right hand-side of (5.7) converges to $2\pi \, (\xi_\alpha - \xi_\infty)$. It follows from above that

$$\left| \xi_\alpha - \xi_\infty \right| = \left| \frac{1}{2\pi} \int_{\partial\Omega} (w_\alpha - w_\infty) \frac{d\varphi_\infty}{dn} \right| \leq \frac{1}{2\pi} \left\| \frac{d\varphi_\infty}{dn} \right\|_{L^2(\partial\Omega)} \left\| w_\alpha - w_\infty \right\|_{L^2(\partial\Omega)} = 0 \, (\alpha^{-1}).$$

Summarizing this discussion, we have got :

<u>Proposition 5.4</u>. *Let* $(\varphi_\alpha, \xi_\alpha)$ *and* $(\varphi_\infty, \xi_\infty)$ *be given by Proposition 5.3. Then :*

a) $\| \varphi_\alpha - \varphi_\infty \|_{W_o^1(\mathbb{R}^2)} = 0 \, (\alpha^{-1/2})$,

b) $\| \varphi_\alpha - \varphi_\infty + (\xi_\alpha - \xi_\infty) \|_{L^2(\Omega)} = 0 \, (\alpha^{1/2})$,

c) $\| \varphi_\alpha - \varphi_\infty \|_{L^2(\partial\Omega)} = 0 \, (\alpha^{-1})$,

d) $\| \vec{\nabla} \varphi_\alpha \|_{L^2(\partial\Omega)} = 0 \, (1)$,

e) $\left| \xi_\alpha - \xi_\infty \right| = 0 \, (\alpha^{-1})$.

<u>Remark 5.1</u>. The estimates of Propositions 5.2 and 5.4 are identical except for part b.

The next result, which is probably not optimal, will not be used later ; we quote it without proof.

<u>Proposition 5.5</u>. *Let* φ_α *and* φ_∞ *be defined either by Proposition 5.2 or Proposition 5.3. Then*

$$\| \varphi_\alpha - \varphi_\infty \|_{L^\infty(\mathbb{R}^2)} = 0 \, (\alpha^{-1/2}).$$

The following estimate, difficult and certainly not optimal, concerns the convergence of the normal derivative as α tends to infinity. We shall prove it at the end of Section 6.

<u>Proposition 5.6</u>. *Let* φ_α *and* φ_∞ *be defined either by Proposition 5.2 or Proposition 5.3. Then* $\| \vec{\nabla} (\varphi_\alpha - \varphi_\infty) \|_{L^1(\partial\Omega)} = 0((\alpha/\ln\alpha)^{-1/2})$.

We conclude this section by an example and a remark.

Example 5.1. The system consists of one conductor with circular section i.e. $\Omega = B(0,R)$. For the non physical problem (4.2) we obtain the trivially solution $\varphi_\alpha = \varphi_\infty = - C_1$ on $\mathbb{R}^2$. More interesting is Problem (4.5), (4.6) we consider now. Here $\varphi_\alpha(x) = \varphi_\infty(x) = \ln(x)$ for $|x| \geq R$ and $\varphi_\infty(x) = - \xi_\infty = \ln R$ for $|x| \leq R$, whereas, for $|x| \leq R$, φ_α is of the form

$$\varphi_\alpha(x) = - \xi_\alpha + a_\alpha \; J_0((1+i) \; \alpha|x|). \tag{5.8}$$

In (5.8), J_0 denotes the Bessel function of order 0 ; the complex constants ξ_α and a_α are determined by imposing the continuity of φ_α and of its normal derivative on $\partial\Omega$. Explicit calculations using the asymptotic expansions of Bessel functions show that all estimates of Proposition 5.4 are optimal.

Remark 5.2. For $\Omega = B(0,R)$, let $(\varphi_\alpha, \xi_\alpha)$ be the solution of Problem (4.5), (4.6) and $\tilde{\varphi}_\alpha$ be the solution of the initial problem (3.15), (3.16) with $C_1 = 1$. Clearly $\tilde{\varphi}_\alpha = (1/\xi_\alpha) \varphi_\alpha$. By Proposition 5.4 and the above discussion of Example 5.1, $\lim_{\alpha\to\infty} \xi_\alpha = \xi_\infty = \ln R$ and $\tilde{\varphi}_\alpha(x) = (1/\xi_\alpha) \ln(x)$ $\alpha|x| \geq R$. If $R = 1$, there is no possible convergence $\tilde{\varphi}_\alpha$ as $\alpha \to \infty$ on any subset of $\overline{\Omega}^c$. In fact, for any Ω, there exists one and only one domain homothetic to Ω for which the same phenomenon occurs with $C_k = 1$, $k = 1,2,...,N$.

6. SOME LOCAL ESTIMATES.

In this section φ_α and φ_∞ will denote the functions defined either in Proposition 5.2 or in Proposition 5.3.

We first note that if $\Lambda \subset \mathbb{R}^2$ is a bounded open set such that $\overline{\Lambda}$ does not contain any corner of $\partial\Omega$, then for any m we have : φ_α and φ_∞ belong to $H^m(\Lambda \cup \Omega)$ and to $H^m(\Lambda \cap \overline{\Omega}^c)$. Loosely said, the restrictions of φ_α and φ_∞ to $\overline{\Omega}$ or to Ω^c are C^∞ except at the corners of $\partial\Omega$. These properties are consequences of standard regularity

results for elliptic equations. In order to investigate properties in the neighborhood of $\partial\Omega$, we introduce the following local transformation of coordinates. Let $\Gamma \subset \partial\Omega$ be a smooth arc. One of its end point, say P, can be a corner of $\partial\Omega$. Γ admits the parametrization $(\gamma_1(t), \gamma_2(t))$ where t is the arc length parameter such that P corresponds to $t = 0$. With $\vec{n} = (n_1(t), n_2(t))$ denoting as usual the unit normal derivative exterior to Ω, we set

$$(x_1, x_2) = (\gamma_1(\xi_2), \gamma_2(\xi_2)) + \xi_1(n_1(\xi_2), n_2(\xi_2)) \tag{6.1}$$

As well-known, (ξ_1, ξ_2) defines a local orthogonal curvilinear system of coordinates,
the metric of which is given by the quadratic form

$$d\xi_1^2 + s^2(\xi_1, \xi_2)\, d\xi_2^2 \quad , \qquad s(\xi_1, \xi_2) = 1 + \frac{\xi_1}{R(\xi_2)} \; ; \tag{6.2}$$

$R(\xi_2)$ is the radius of curvature of Γ, where $R(\xi_2) > 0$ if Ω is "convex" at the point of Γ with parameter $t = \xi_2$. We shall denote par D_1^l and D_2^l the $l{-}th$ partial derivatives with respect to ξ_1 and ξ_2 respectively, so that, for example, the scalar product of two gradients has for expression

$$\vec{\nabla}u \cdot \vec{\nabla}v = D_1 u \cdot D_1 v + D_2 u \cdot D_2 v / s^2. \tag{6.3}$$

To P, end point of Γ or to any Q interior point of Γ, we associate an open rectangle T as shown in Figure 6.1 and 6.2. In particular we suppose :

a) the system of coordinates (ξ_1, ξ_2) is defined and regular in $\overline{T}$,

b) $T \cap \Omega = \{x \in T \mid \xi_2 < 0\}$ and $T \cap \overline{\Omega}^c = \{x \in T \mid \xi_2 > 0\}$ are non empty and satisfy the cone condition,
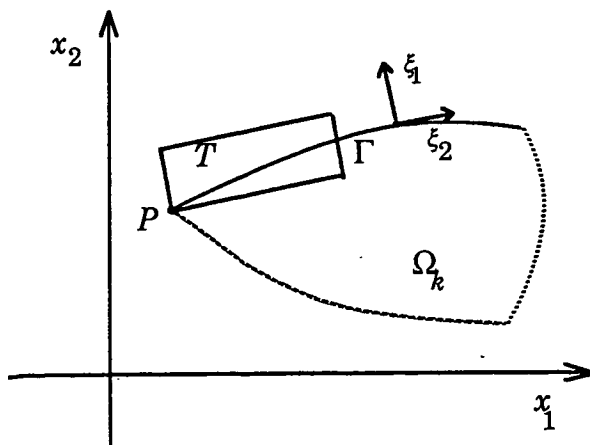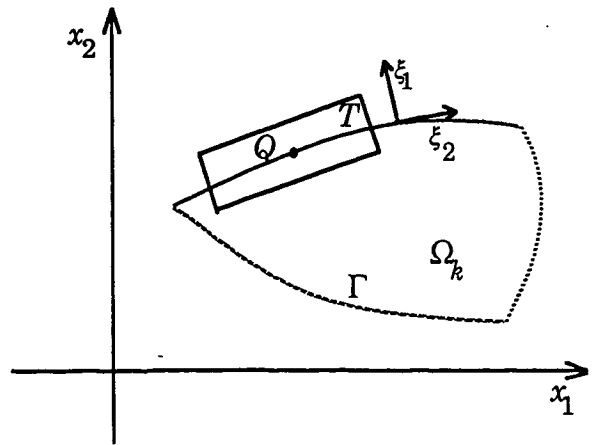
c) $P \in \overline{T}$ or $Q \in T$.

Figure 6.1

Figure 6.2

<u>Lemma 6.1.</u> *Let T be a rectangle as defined above and let* $\theta : \mathbb{R}^2 \to \mathbb{R}_+$ *be a function such that* $\theta, \partial_1 \theta$ *and* $\partial_2 \theta$ *belong to* $L^\infty(\mathbb{R}^2)$ *and* $\theta = 0$ *in* T^c. *Then for any* $m = 1,2,3,...,$ *we have as* α *tends to infinity*

a) $\|\theta^m D_2^m(\varphi_\alpha - \varphi_\infty)\|_{W_o^1 (\mathbb{R}^2)} = 0 \ (\alpha^{-1/2})$,

b) $\|\theta^m D_2^m(\varphi_\alpha - \varphi_\infty)\|_{L^2(\Omega)} = 0 \ (\alpha^{-3/2})$,

c) $\|\theta^m D_2^m(\varphi_\alpha - \varphi_\infty)\|_{L^2(\partial\Omega)} = 0 \ (\alpha^{-1})$.

<u>Proof</u>. We shall prove Lemma 6.1 only for $m = 1$ and for φ_α, φ_∞ defined by Proposition 5.2. The estimates relative to $m \geq 2$ can be obtained with similar arguments by an induction process. The situation corresponding to Proposition 5.3 is treated easily by considering the function $\varphi_\alpha - \varphi_\infty + \xi_\alpha - \xi_\infty$; see Proposition 5.4.b. We start as in the proof of Proposition 5.2 with $\eta = \varphi_\alpha - \varphi_\infty$. Multiplying (5.1) by $v \in C_0^\infty(T)$, we obtain after integration by parts

$$\int_{\mathbb{R}^2} \vec{\nabla}\eta \cdot \vec{\nabla}v - 2i \alpha^2 \int_\Omega \eta v = - \int_{\partial\Omega} d\varphi_\infty / dn \ v. \tag{6.4}$$

In the coordinates (ξ_1, ξ_2), by using in particular (6.3), (6.4) becomes

$$\int_{\mathbb{R}^2} (s \ D_1\eta D_1 v + s^{-1} D_2 \eta D_2 v) - 2i \alpha^2 \int_{\tilde{\Omega}} s \eta v = - \int_{\mathbb{R}} D_1\varphi_\infty(0,\xi_2) v(0,\xi_2) d\xi_2 \tag{6.5}$$

where $\tilde{\Omega}$ is the image of $T \cap \Omega$ in the (ξ_1,ξ_2) plane. We set $v = -D_2 w$ in (6.5), where $w \in C_0^\infty(\tilde{T})$ and $\tilde{T}$ is the image of T in the (ξ_1,ξ_2) plane ; after integration by parts, we obtain

$$\int_{\mathbb{R}^2} (s \ D_1 D_2\eta D_1 w + s^{-1} D_2^2 \eta D_2 w) - 2i \alpha^2 \int_{\tilde{\Omega}} s D_2\eta w =$$

$$- \int_{\mathbb{R}} D_1 D_2\varphi_\infty(0,\xi_2) w(0,\xi_2) d\xi_2$$

$$- \int_{\mathbb{R}^2} (D_2 s \ D_1\eta D_1 w + D_2 (s^{-1}) D_2\eta D_2 w) + 2i \alpha^2 \int_{\tilde{\Omega}} D_2 s \eta w . \tag{6.6}$$

In a first step, we assume that $\theta \in C_0^\infty(T)$, therefore we can set $w = \theta z$ with $z \in C_0^\infty(\tilde{T})$; after some elementary calculations, we get with $\Psi = D_2\eta = D_2(\varphi_\alpha - \varphi_\infty)$:

$$\int_{\mathbb{R}^2}(s \ D_1(\theta\Psi)D_1 z + s^{-1} D_2(\theta\Psi)D_2 z) - 2i \alpha^2 \int_{\tilde{\Omega}} s \theta \Psi z =$$

$$- \int_{\mathbb{R}} \theta \ D_1 D_2\varphi_\infty(0,\xi_2) z(0,\xi_2) d\xi_2$$

$$- \int_{\mathbb{R}^2} (D_2 s \ D_1\eta \ D_1(\theta z) + D_2(s^{-1}) D_2\eta \ D_2(\theta z)) + 2i \alpha^2 \int_{\tilde{\Omega}} D_2 s \eta \theta z$$

$$+ \int_{\mathbb{R}^2} \{s \ D_1\theta \ (\Psi D_1 z - z \ D_1\Psi) + s^{-1} D_2\theta \ (\Psi D_2 z - z \ D_2\Psi)\}. \tag{6.7}$$

From classical results of regularity $\theta\, D_1 D_2 \varphi_\infty(0,\xi_2) \in C^\infty(\partial\Omega)$, therefore it follows from (6.7) that

$$\mathcal{L}(\theta\,\Psi) \equiv - [D_1(s\, D_1) + D_2(s^{-1}\, D_2)]\,(\theta\,\Psi) \in H^{-1}(\widetilde{T}) ;$$

since $\theta\Psi$ has a compact support included in T, we deduce from the hypoellipticity of $\mathcal{L}$ that $\theta\Psi \in H_0^1(T)$. Then by density we can set $z = \theta\,\overline{\Psi}$ in (6.7). In the last term of the right member, we replace $\theta\,\overline{\Psi}\,D\Psi$ by $\overline{\Psi}\,D(\theta\Psi) - \Psi\,\overline{\Psi}\,D\theta$; we use Proposition 2.3 and Proposition 5.2 parts a, b and obtain the estimate

$$\left| \|\theta\Psi\|^2_{W_o^1(\mathbb{R}^2)} - 2i\,\alpha^2\,\|\theta\Psi\|^2_{L^2(\Omega)} \right| \leq C\,\|\theta\, D_1 D_2\,\varphi_\infty\|_{L^2(\partial\Omega)}\, \left(\|\theta\Psi\|_{L^2(\Omega)}\,\|\theta\Psi\|_{H^1(\Omega)}\right)^{1/2}$$

$$+ C\,\alpha^{-1/2}\,\|\theta^2\,\Psi\|_{W_o^2(\mathbb{R}^2)} + C\,\alpha^{1/2}\,\|\theta^2\,\Psi\|_{L^2(\Omega)}$$

$$+ C\,\|\vec{\nabla}\theta\|_{L^\infty(\mathbb{R}^2)}\,(\alpha^{-1/2}\,\|\theta\Psi\|_{W_o^1(\mathbb{R}^2)} + \alpha^{-1}\,\|\vec{\nabla}\theta\|_{L^\infty(\mathbb{R}^2)}) \qquad (6.8)$$

where C is a generic constant independent of α and θ.

From results of GRISVARD [3], $\delta\, D_1 D_2\varphi_\infty \in L^2(\partial\Omega)$, where $\delta(x)$ denotes the distance from x to the corners of $\partial\Omega$; clearly $|\theta(x)| \leq \delta(x)\,\|\vec{\nabla}\theta\|_{L^\infty(\mathbb{R}^2)}$ and $\|\theta\, D_1 D_2\varphi_\infty\|_{L^2(\partial\Omega)} \leq C\,\|\vec{\nabla}\theta\|_{L^\infty(\mathbb{R}^2)}$. After a convenient treatment of the terms $\theta^2\Psi$ in (6.8) and by using classical inequalities, we obtain for C independent of α and θ

$$\|\theta\vec{\nabla}\Psi\|_{L^2(\mathbb{R}^2)} \leq C\,\alpha^{-1/2}\,\|\vec{\nabla}\theta\|_{L^\infty(\mathbb{R}^2)}, \quad \|\theta\Psi\|_{L^2(\Omega)} \leq C\,\alpha^{-3/2}\,\|\vec{\nabla}\theta\|_{L^\infty(\mathbb{R}^2)}, \qquad (6.9)$$

this proves parts a and b of Lemma 6.1 when $\theta \in C_0^\infty(T)$; parts c follows from Proposition 2.3.

Now we consider the case where $\theta(x) = g(x) =$ distance from x to ∂T if $x \in T$ and $\theta(x) = 0$ if $x \in T^c$; then we can find a sequence of $\theta_n \in C_0^\infty(T)$ such that

$$\theta_n \text{ converges to } g \text{ everywhere,}$$

and $\|\vec{\nabla}\theta_n\|_{L^\infty(\mathbb{R}^2)} \leq (1 + \tfrac{1}{n})\,\|\vec{\nabla}g\|_{L^\infty(\mathbb{R}^2)}$.

We obtain (6.9) for $\theta = g$ as the limit for $\theta = \theta_n$ by using the Lebesgue Theorem. The general situation is treated easily by noticing that $|\theta(x)| \leq g(x)\,\|\vec{\nabla}\theta\|_{L^\infty(\mathbb{R}^2)}$.

Lemma 6.2. *Let* $\Lambda \subset \overline{\Omega}^c$ *be an open bounded set. Then*

$$\|\varphi_\alpha - \varphi_\infty\|_{L^2(\Lambda)} = 0 \; (\alpha^{-1}).$$

Proof. Set $\eta = \varphi_\alpha - \varphi_\infty$ and let $w \in W_0^1(\overline{\Omega}^c)$ satisfy the relations

$$\Delta w = \eta \, \chi_\Lambda \text{ in } \overline{\Omega}^c \;\;, \;\; w = 0 \text{ on } \partial\Omega, \tag{6.10}$$

where χ_Λ is the caracteristic function of Λ. By Grisvard's regularity results [3] , there exists a constant c , independent of η such that $\|\frac{dw}{dn}\|_{L^2(\partial\Omega)} \leq c \; \|\eta\|_{L^2(\Lambda)}$. Let r_0 be such that $\Omega \cup \Lambda \subset B(0,r_0)$. By Green's formula, we have for $r > r_0$:

$$\|\eta\|^2_{L^2(\Lambda)} = \int_{\Omega^c \cap B(0,r)} \eta \Delta w = -\int_{\partial\Omega} \eta \, \frac{dw}{dn} + \int_{\partial B(0,r)} (\eta \, \frac{dw}{dn} - w \, \frac{d\eta}{dn}). \tag{6.11}$$

η and w belongs to $W_0^1(\mathbb{R}^2)$ and are harmonic in $(\overline{B}(0,r_0))^c$. By Propositions 2.1 and 2.2, these functions and their normal derivatives behave respectively like as $0(1)$ and $0(|x|^{-2})$ as $|x|$ tends to infinity ; we conclude that the second term of the right hand side of (6.11) vanishes. Lemma 6.2 then follows from (6.11) and from Proposition 5.2c or Proposition 5.4c.

Proposition 6.1. *Let* $\Lambda \subset \overline{\Omega}^c$ *be an open bounded set such that* $\overline{\Lambda}$ *contains no corner of* $\partial\Omega$. *Then for any* $m = 0,1,2,...$ *we have*

$$\|\varphi_\alpha - \varphi_\infty\|_{H^m(\Lambda)} = 0 \; (\alpha^{-1}).$$

Proof. We remark that by Lemma 6.1c we have the estimate $\|\varphi_\alpha - \varphi_\infty\|_{H^m(\Gamma)} = 0(\alpha^{-1})$ for any $m = 0,1,2,...$ and any $\Gamma \subset \partial\Omega$ such that $\overline{\Gamma}$ contains no corner of $\partial\Omega$. Together with standard regularity results and Lemma 6.2, this implies Proposition 6.1.

As an immediate consequence of the important Proposition 6.1, we have for example the following.

Corollary 6.1 *For* $\Gamma \subset \partial\Omega$ *such that* $\overline{\Gamma}$ *contains no corner of* $\partial\Omega$ *and for any* $m = 0,1,2,...,$ *we have*

$$\|\vec{\nabla}(\varphi_\alpha - \varphi_\infty)\|_{H^m(\Gamma)} = 0 \; (\alpha^{-1}).$$

We conclude this section by the proof of Proposition 5.6. By Corollary 6.1, it suffices to prove Proposition 5.6 when we replace $\partial\Omega$ by a small arc Γ one end point of which is a corner ; we can assume that $\Gamma \subset T \cap \partial\Omega$ where T is a rectangle as shown in Figure 6.1. We use Proposition 5.2a, Proposition 5.4a and Lemma 6.1a with $m = 1$ and $\theta = g$, where $g(x) = 0$ if $x \in T^c$ and $g(x)$ is the distance from x to ∂T if $x \in T$. We deduce immediately the relations $\|g\ D_1\ D_2\ \eta\|_{L^2(T)} = 0\ (\alpha^{-1/2})$ and $\|g\ D_2^2\ \eta\|_{L^2(T)} = 0\ (\alpha^{-1/2})$; from this last estimate and the fact that η is harmonic in $\overline{\Omega}^c$, we obtain that $\|g\ D_1^2 \eta\|_{L^2(T \cap \overline{\Omega}^c)} = 0\ (\alpha^{-1/2})$; then we can conclude that

$$\|g\ \partial_\ell\ \eta\|_{H^1(T \cap \overline{\Omega}^c)} = 0(\alpha^{-1/2})\ , \quad \ell = 1,2. \tag{6.12}$$

Setting $w = \partial_\ell\ \eta$ and $\Lambda = T \cap \overline{\Omega}^c$, we have by a classical imbedding theorem and Schwarz's inequality

$$\|g\,|w|^2\|_{L^1(\partial\Lambda)} \le C\ \{\ \|g\,|w|^2\|_{L^1(\Lambda)} + \sum_{j=1}^{2}\ \|\partial_j\ (g\,|w|^2)\|_{L^1(\Lambda)}\}$$

$$\le C\ \{\|g\|_{L^\infty(\mathbb{R}^2)}\|w\|^2_{L^2(\Lambda)} + \sum_{j=1}^{2}(2\|\partial_j(gw)\|_{L^2(\Lambda)}\|w\|_{L^2(\Lambda)} + \|\partial_j g\|_{L^\infty(\Lambda)}\|w\|^2_{L^2(\Lambda)}\}\ . \tag{6.13}$$

Since $\|w\|_{L^2(\Lambda)} = 0(\alpha^{-1/2})$, it follows by (6.12) that the right hand side of (6.13) is $0\ (\alpha^{-1})$; furthermore we observe that along Γ, the arc length parameter ξ_2 is $0(g)$; consequently we have obtained the estimate

$$\|\xi_2^{1/2}\ w\|_{L^2(\Gamma)} = 0\ (\alpha^{-1/2}). \tag{6.14}$$

For $0 < \gamma < 1$, we can write $|w| = \xi_2^{(\gamma-1)/2}\ [\xi_2^{1/2}\ |w|]^{1-\gamma}\ |w|^\gamma$ so that by Hölder's inequality we obtain

$$\|w\|_{L^1(\Gamma)} \le \|\xi_2^{(\gamma-1)/2}\|_{L^2(\Gamma)}\ (\|\xi_2^{1/2}\ w\ \|_{L^2(\Gamma)})^{1-\gamma}\ \|w\|_{L^2(\Gamma)}^\gamma\ . \tag{6.15}$$

By Proposition 5.2d and 5.4d, we know that $\|w\|_{L^2(\Gamma)}$ is uniformly bounded with respect to α ; from (6.15) with $\gamma = 1/\ln\alpha$, we readily obtain the final estimate

$$\|\partial_\ell(\varphi_\alpha - \varphi_\infty)\|_{L^1(\Gamma)} = \|w\|_{L^1(\Gamma)} = 0\ ((\alpha\ /\ \ln\alpha)^{-1/2}),\quad \ell = 1,2.$$

$\bullet$

7. A BOUNDARY LAYER APPROXIMATION.

As in the previous section, φ_α will denote here the function defined either in Proposition 5.2 or in Proposition 5.3. Our purpose is to study the behaviour of φ_α

for large α in a conductor section Ω_k. We fix k, $1 \leq k \leq N$ and set $\Lambda = \Omega_k$. We furthermore set $\eta_\alpha = \eta = \varphi + C_k$ for Problem (4.2) and $\eta_\alpha = \eta = \varphi + \xi$ for Problem (4.5), (4.6).

We first remark that η_α satisfies in Λ the following Helmholtz homogeneous equation

$$\Delta \eta_\alpha + 2i\, \alpha^2\, \eta_\alpha = 0 \qquad \text{in } \Lambda. \tag{7.1}$$

It is well-known that $F(x,y) = \frac{1}{4i} H_0^1((1+i)\alpha\, |x-y|)$ is a fundamental solution of this Helmholtz equation where H_0^1 is a Hankel function. It follows that η_α admits for $x \in \Lambda$ the representation

$$\eta_\alpha(x) = \int_{\partial\Lambda} \{\eta_\alpha(y) \frac{d}{dn_y} F(x,y) - F(x,y) \frac{d\eta_\alpha(y)}{dn_y}\}\, d\mu(y). \tag{7.2}$$

Then, by Proposition 5.2, 5.4 and by the asymptotic expansion formulae of the Hankel functions (see for example [5]), we immediately deduce the following

Proposition 7.1. *Let* $x \in \Lambda$ *and* d *be the distance of* x *to* $\partial\Lambda$. *There exists a constant* C_{mn} *such that for* $\alpha \geq 1/d$ *we have*

$$\left|\partial_1^m \partial_2^n \eta_\alpha(x)\right| \leq C_{mn}\, \alpha^{m+n}\, \frac{e^{-\alpha d}}{\sqrt{\alpha d}}, \qquad m,n = 0,1,2,\dots .$$

Consider now a smooth arc $\Gamma \subset \partial\Lambda$ such that the end points of Γ are not corners of $\partial\Lambda$. With the system of coordinates (ξ_1, ξ_2) introduced by (6.1), Γ is the set of points $(0, \xi_2)$, $0 \leq \xi_2 \leq \xi_{02}$. Furthermore there exists $\delta > 0$ such that $V = \{x \mid -\delta < \xi_1 < 0,\ 0 < \xi_2 < \xi_{02}\} \subset \Lambda$ and such that the distance of the points of ∂V corresponding to $\xi_1 = -\delta$ to $\partial\Lambda$ is equal to δ. We defined in $\overline{V}$ an approximate u_α of η_α by the relation

$$u_\alpha(\xi_1, \xi_2) = \frac{1}{\sqrt{s(\xi_1, \xi_2)}}\, e^{(1-i)\alpha \xi_1}\, \eta_\alpha(0, \xi_2), \tag{7.3}$$

where s is given by (6.2). We remark that $u_\alpha = \eta_\alpha$ on Γ.

As in Section 6, D_1^ℓ and D_2^ℓ denote the partial derivative of order ℓ with respect to ξ_1 and ξ_2, whereas $R = R(\xi_2)$ is the radius of curvature of Γ.

Proposition 7.2. *For any* $m = 0,1,2,...$ *we have*

a) $\|D_2^m(\eta_\alpha - u_\alpha)\|_{L^\infty(\overline{V})} = 0\ (\alpha^{-3})$

b) $\|D_2^m D_1(\eta_\alpha - u_\alpha)\|_{L^\infty(\overline{V})} = 0\ (\alpha^{-2})$.

Proof. We consider only the case $m = 0$ since the treatment of the general situation needs essentially the same tools. We use the expression of the Laplace operator in the (ξ_1, ξ_2) system of coordinates and write (7.1) in $\tilde{\Lambda}$, image of Λ in the (ξ_1, ξ_2) plane. Setting $w = \eta_\alpha - u_\alpha$, we obtain after some calculations

$$D_1(s\,D_1 w) + 2i\,\alpha^2\,s\,w = z, \tag{7.4}$$

$$z(\xi_1, \xi_2) = - D_2(\frac{1}{s} D_2\,\eta)\,(\xi_1, \xi_2) - \eta(0, \xi_2)\,\frac{e^{(1-i)\alpha\xi_1}}{4s^{3/2}(\xi_1, \xi_2)\,R^2(\xi_2)}\ . \tag{7.5}$$

From Lemma 6.1b and Proposition 6.1, we easily deduce that we have uniformly with respect to $0 \le \xi_2 \le \xi_{02}$

$$\|D_2^\ell\,\eta(., \xi_2)\|_{L^2(-\delta, 0)} = 0\ (\alpha^{-3/2}),\ \ell = 1, 2\ ;\ \ |\eta(0, \xi_2)| = 0\ (\alpha^{-1}). \tag{7.6}$$

Then (7.5) and (7.6) imply that

$$\|z(., \xi_2)\|_{L^2(-\delta, 0)} = 0\ (\alpha^{-3/2}) \quad \text{uniformly for } 0 \le \xi_2 \le \xi_{02}. \tag{7.7}$$

For ξ_2 fixed, we multiply (7.4) by $\overline{w}$; after an integration by parts, we obtain since $w\,(0, \xi_2) = 0$:

$$\left| -\int_{-\delta}^0 s\,|D_1 w|^2\,d\xi_1 + 2i\,\alpha^2 \int_{-\delta}^0 s|w|^2 d\xi_1 \right| \le$$

$$\|z(\cdot, \xi_2)\|_{L^2(-\delta, 0)}\,\|w(\cdot, \xi_2)\|_{L^2(-\delta, 0)} + \left| s\,w\,D_1 w(-\delta, \xi_2) \right|\ ; \tag{7.8}$$

by (7.3), (7.6) and Proposition 7.1, the second term of the right hand member of (7.8) is $(0(e^{-\alpha\delta})$. Then (7.7) and (7.8) imply the estimate $\|w\|_{L^2(-\delta, 0)} = 0(\alpha^{-7/2})$ and $\|D_1 w\|_{L^2(-\delta, 0)} = 0(\alpha^{-5/2})$; with (7.4) and (7.7) we have furthermore $\|D_1^2 w\|_{L^2(-\delta, 0)} = 0(\alpha^{-3/2})$. We conclude by standard arguments.

$\bullet$

Remark 7.1. Results similar to those of Proposition 7.1 can be derived by using directly the Helmholz equation (7.1) instead of the integral representation (7.2).

This method is more complicated but can be generalized to the situation where α is variable, i.e. the conductivity is not constant.

Remark 7.2. Suppose that in the definition (7.3) of u_α we replace $s(\xi_1,\xi_2)$ by 1 ; then in Proposition 7.2 we loose one order i.e. $0(\alpha^{-3})$ and $0(\alpha^{-2})$ are replaced respectively by $0(\alpha^{-2})$ and $0(\alpha^{-1})$.

Remark 7.3. Because of the exponential decay of η_α and u_α in the boundary layer, L^2 estimates are better than the L^∞ ones. From the proof of Proposition 7.2 we obtain for example

$$\|\eta_\alpha - u_\alpha\|_{L^2(V)} = 0(\alpha^{-7/2}) \quad , \quad \|\eta_\alpha - u_\alpha\|_{H^1(V)} = 0(\alpha^{-5/2}). \tag{7.9}$$

8. AN APPROXIMATION OF φ_α IN $\overline{\Omega}^c$ SATISFYING A ROBIN BOUNDARY CONDITION.

Our purpose is to define a "cheap" approximation of φ_α. For the sake of briefness, we shall consider only Problem (4.2) ; Problem (4.5), (4.6) can be treated in a similar way. In order to insure the validity of this approximation we must introduce a severe restriction on the regularity of Ω ; in fact we shall suppose that

$$\partial\Omega \text{ is of class } C^\infty. \tag{8.1}$$

Let φ_α be the solution of Problem (4.2). Suppose we know φ_α or an approximation Ψ_α of φ_α on $\partial\Omega$. Then, because of Hypothesis (8.1), the results of Section 7 allow to define an explicit and simple approximation of φ_α in Ω.

Consider Proposition 7.2b with $m = 1$ and $\overline{V}$ being replaced by $\overline{V} \cap \partial\Omega$. Since $\eta_\alpha = \varphi_\alpha + C_k$ we obtain immediately by (7.3) :

Proposition 8.1. Let φ_α be the solution of Problem (4.2) and assume Hypothesis (8.1). Then

$$\left\| \frac{d\varphi_\alpha}{dn} - z_\alpha(\varphi_\alpha + C_k) \right\|_{L^\infty(\partial\Omega_k)} = 0(\alpha^{-2}) \quad ,1 \le k \le N,$$

where $z_\alpha = (1-i)\,\alpha - \frac{1}{2R}$ and R is the radius of curvature of $\partial\Omega_k$.

Proposition 8.1 shows that φ_α satisfies approximatively a Robin boundary condition on $\partial\Omega$. This leads us to introduce the following exterior problem :

find $\Psi_\alpha \in W_0^1(\overline{\Omega}^c)$ such that

$$\Delta\Psi_\alpha = 0 \text{ in } \overline{\Omega}^c \ ; \ \frac{d\Psi_\alpha}{dn} = z_\alpha(\Psi_\alpha + C_k) \text{ on } \partial\Omega_k, \ 1 \le k \le N, \qquad (8.2)$$

where the unit normal $\vec{n}$ on $\partial\Omega$ is exterior to Ω.

<u>Proposition 8.2</u>. *Let φ_α be the solution of Problem* (4.2) *and assume Hypothesis* (8.1). *Then*

  a) *Problem* (8.2) *has one and only one solution*,

  b) $\|\varphi_\alpha - \Psi_\alpha\|_{W_0^1(\overline{\Omega}^c)} = 0\,(\alpha^{-5/2})$.

  c) $\|\varphi_\alpha - \Psi_\alpha\|_{L^2(\partial\Omega)} = 0\,(\alpha^{-3})$.

<u>Proof</u>. Problem (8.2) admits the following variational formulation

$$\int_{\Omega^c}\vec{\nabla}\Psi_\alpha\cdot\vec{\nabla}v + \int_{\partial\Omega} z_\alpha\Psi_\alpha v = -\sum_{k=1}^{N} C_k \int_{\partial\Omega_k} z_\alpha v, \ \forall v \in W_0^1(\overline{\Omega}^c). \qquad (8.3)$$

Point a follows from (8.3) as an application of Lax-Milgram's Lemma. Set

$w_\alpha = \Psi_\alpha - \varphi_\alpha$. Since $\varphi_\alpha \in W_0^1(\overline{\Omega}^c)$ and is harmonic in $\overline{\Omega}^c$, we obtain from 8.3 :

$$\int_{\Omega^c}\vec{\nabla}w_\alpha\cdot\vec{\nabla}v + \int_{\partial\Omega} z_\alpha w_\alpha v = \sum_{k=1}^{N}\int_{\partial\Omega_k}(\frac{d\varphi_\alpha}{dn} - z_\alpha(\varphi_\alpha + C_k))v, \ \forall v \in W_0^1(\overline{\Omega}^c). \qquad (8.4)$$

We set $v = \overline{w}_\alpha$ in (8.4) and recall that $z_\alpha = (1-i)\,\alpha - 1/(2R)$; we take the imaginary part of this relation and obtain the estimate c by Proposition 8.1 ; replacing in (8.4) with $v = \overline{w}_\alpha$ we conclude that $\|\vec{\nabla}w_\alpha\|_{L^2(\overline{\Omega}^c)} = 0\,(\alpha^{-5/2})$; together with part c, this proves part b.

<u>Remark 8.1</u>. Besides those shown in Proposition 8.2, other estimates can be obtained . For example, with some extra calculations, one can prove that

$$\|\varphi_\alpha - \Psi_\alpha\|_{L^\infty(\partial\Omega)} = 0\,(\alpha^{-3}). \qquad (8.5)$$

Furthermore, it is possible to extend φ_α to $\mathbb{R}^2$ by the boundary layer approximation introduced in Section 7 and produce estimates relative to Ω. We shall not pursue this way. On one side these estimates are direct corollary of the preceeding one or can be obtained by using the same tools ; on the other side due to the very restrictive Hypothesis (8.1), they are of limited interest.

Remark 8.2. Numerical tests show that for many practical applications, Ψ_α gives a very satisfactory approximation of φ_α if $\partial\Omega$ is regular. If $\partial\Omega$ has corners, the definition (8.2) of Ψ_α is still meaningful ; however we get only very poor theorical error estimates which are confirmed by numerical experiments.

Remark 8.3. If $\partial\Omega$ has corners, Proposition 8.1 is still valid when we replace $L^2(\partial\Omega_k)$ by $L^\infty(\Gamma)$ where Γ is a closed part of $\partial\Omega_k$ without singularity. In a forthcoming paper, we shall present successful numerical computations obtained with a method which takes advantage of this fact.

REFERENCES.

[1]   R. ADAMS, Sobolev Spaces, Academic Press, 1975.

[2]   BOSSAVIT, Le concept de "rigidité" dans la méthode des sous-structures ; application à l'électronique. Numerical Methods for Engineering II, GAMNI II. Editeurs Absi, Glowinski, Lascaux, Weysseyre, Dunod - Bordas, Paris 1980.

[3]   P. GRISVARD, Elliptic Problems in Nonsmooth Domains. Pitman Advanced Publ. Program, 1985.

[4]   S.I. HARIHARAN, E. STEPHAN, A Boundary Element Method for a Two-Dimensional Interface Problem in Electromagnetics. Num. Math. 42, 311-322, 1983.

[5]   JAHNKE-EMDE-LOSCH, Tables of Higher Functions. B.G. Teubner Verlags gesellschaft, Stuttgart, 1960.

[6]   J.C. NEDELEC, Approximation des Equations Intégrales en Mécanique et en Physique. Rapport Centre de Mathématiques Appliquées, Ecole Polytechnique, 1977.

[7]   A. SOMMERFELD, Electrodynamics. Academic Press, New York, 1952.