



**HAL**  
open science

## A transient analysis of the two-node series Jackson network

François Baccelli, William A. Massey

► **To cite this version:**

| François Baccelli, William A. Massey. A transient analysis of the two-node series Jackson network.  
| [Research Report] RR-0852, INRIA. 1988. inria-00075701

**HAL Id: inria-00075701**

**<https://inria.hal.science/inria-00075701v1>**

Submitted on 24 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# INRIA

UNITÉ DE RECHERCHE  
INRIA-SOPHIA ANTIPOLIS

Institut National  
de Recherche  
en Informatique  
et en Automatique

Domaine de Voluceau  
Rocquencourt  
BP 105  
78153 Le Chesnay Cedex  
France

Tél. (1) 39 63 55 11

## Rapports de Recherche

N° 852

### A TRANSIENT ANALYSIS OF THE TWO-NODE SERIES JACKSON NETWORK

François BACCELLI  
William A. MASSEY

JUIN 1988



**ANALYSE TRANSITOIRE  
D'UN RESEAU DE JACKSON  
DE DEUX FILES D'ATTENTE EN TANDEM**

François BACCELLI<sup>1</sup> et William A. MASSEY<sup>2</sup>

<sup>1</sup>INRIA Sophia-Antipolis, Valbonne 06565, France

<sup>2</sup>AT&T BELL LABORATORIES, Murray Hill - NJ 07940, U.S.A.

**RESUME**

Cet article donne une représentation de la distribution transitoire de la taille des files d'attente dans un réseau de deux files en tandem au moyen de fonctions de Bessel Laticielles. On obtient par la même approche les distributions jointes des périodes d'activité des deux files d'attente et de l'état du réseau à la fin de ces périodes. Ces derniers résultats sont établis au moyen d'un théorème général sur les processus de Markov qui relie la distribution jointe des temps et lieu de sortie d'un domaine au comportement du processus avant l'atteinte de sa frontière. Ces résultats sont ensuite étendus au cas d'un réseau à  $N$  noeuds.

## **A Transient Analysis of the Two-Node Series Jackson Network**

*François Baccelli*

INRIA - Sophia  
06565 Valbonne (France)

*William A. Massey*

AT&T Bell Laboratories  
Murray Hill, New Jersey 07974

### **ABSTRACT**

In this paper, we give the exact solution for the transient queue length distribution of the two-node series Jackson network, in terms of lattice Bessel functions. Exact solutions are also derived for the transient busy period distribution of both queues prior to having a given node become idle. In terms of this solution, we can in turn solve for the associated joint distribution of the queues and the time that this idleness occurs. We obtain this by deriving a general solution to the problem of solving for the joint distribution of where a Markov process exits a region, and its time of departure, in terms of the behavior of the process prior to leaving the region. Finally, we generalize all of these results to the  $N$  node series Jackson network.

**Keywords:** Transient Analysis, Bessel Functions, Markov Processes, Busy Period, Exit Times

December 1, 1987

# A Transient Analysis of the Two-Node Series Jackson Network

*François Baccelli*

INRIA - Sophia  
06565 Valbonne (France)

*William A. Massey*

AT&T Bell Laboratories  
Murray Hill, New Jersey 07974

## 1. Introduction

Let  $\mathbf{Q}(t) = (Q_1(t), Q_2(t))$  denote the vector queue length process for a two-node series Jackson network. The first node has Poisson arrivals with rate  $\mu_0$ , and its service time is exponentially distributed with rate  $\mu_1$ . Upon completion of that service time, the customer then transfers to the second node whose service time is exponentially distributed with rate  $\mu_2$ . In this paper, we will concern ourselves with the solution for the following transient distributions of  $\mathbf{Q}(t)$ :

$$(a, a)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n}, T > t \mid \mathbf{Q}(0) = \mathbf{m}\} \quad (1.1.a)$$

$$(a, b)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n}, T_1 > t \mid \mathbf{Q}(0) = \mathbf{m}\} \quad (1.1.b)$$

$$(b, a)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n}, T_2 > t \mid \mathbf{Q}(0) = \mathbf{m}\} \quad (1.1.c)$$

$$(b, b)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n} \mid \mathbf{Q}(0) = \mathbf{m}\} \quad (1.1.d)$$

where  $T_1 = \inf\{t \mid Q_1(t) = 0\}$ ,  $T_2 = \inf\{t \mid Q_2(t) = 0\}$  and  $T = \min(T_1, T_2)$ . The solution to these four quantities can be given in terms of lattice Bessel functions of rank 2. These functions were introduced in [4] for an arbitrary rank  $N$ . For the case  $N=2$ , the generating function relation for these functions reduces to

$$e^{\frac{\gamma}{3} \left( x_1 + \frac{x_2}{x_1} + \frac{1}{x_2} \right)} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \mathbf{x}^{\mathbf{n}} I(\mathbf{n}, \gamma) \quad (1.2)$$

with  $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} x_2^{n_2}$ . These functions have an associated symmetry group  $G_2$  where

$$G_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \right\}. \quad (1.3)$$

For  $g \in G_2$ , we define  $g(\mathbf{m})$  to be matrix multiplication between  $g$  and  $\mathbf{m}$  where  $\mathbf{m}$  is viewed as a column vector. Generalizing the property of modified Bessel functions that  $I_n(\gamma) = I_{-n}(\gamma)$ , we have  $I(g(\mathbf{m}), \gamma) = I(\mathbf{m}, \gamma)$  for all  $g \in G_2$  and  $\mathbf{m} \in \mathbb{Z}^2$ . A special case of Theorem 1.2 in [4] gives a

solution for  $(a, a)_t[\mathbf{m}, \mathbf{n}]$ , namely

$$(a, a)_t[\mathbf{m}, \mathbf{n}] = e^{-3\alpha t} \beta^{\mathbf{n}-\mathbf{m}} \sum_{g \in G_2} \text{define } I(\mathbf{n} - g(\mathbf{m}), 3\gamma t) \quad (1.4)$$

where  $(-1)^g$  is the sign of  $g$ , viewed as a permutation,  $\alpha = \frac{\mu_0 + \mu_1 + \mu_2}{3}$ ,  $\gamma = (\mu_0\mu_1\mu_2)^{1/3}$ , and

$$\beta = \left[ \frac{\mu_0}{\gamma}, \frac{\mu_0\mu_1}{\gamma^2} \right].$$

We will now solve for the remaining three quantities by showing that they can

be solved in terms of  $(a, a)_t[\mathbf{m}, \mathbf{n}]$ . By (1.4), we will have then solved for  $(a, b)_t[\mathbf{m}, \mathbf{n}]$ ,  $(b, a)_t[\mathbf{m}, \mathbf{n}]$ , and  $(b, b)_t[\mathbf{m}, \mathbf{n}]$  in terms of lattice Bessel functions of rank 2.

**Theorem 1.1.** *For all  $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^2$  and  $t > 0$ , we have*

$$(a, b)_t[\mathbf{m}, \mathbf{n}] = \sum_{j=1}^{\infty} (a, a)_t[\mathbf{m} + j\mathbf{e}_2, \mathbf{n} + \mathbf{e}_2] - (a, a)_t[\mathbf{m} + j\mathbf{e}_2, \mathbf{n}]$$

$$(b, a)_t[\mathbf{m}, \mathbf{n}] = \sum_{j=1}^{\infty} \left( \frac{\mu_1}{\mu_0} \right)^{j-1} \left( (a, a)_t[\mathbf{m} + \mathbf{e}_1, \mathbf{n} + j\mathbf{e}_1] - \frac{\mu_1}{\mu_0} (a, a)_t[\mathbf{m}, \mathbf{n} + j\mathbf{e}_1] \right)$$

$$(b, b)_t[\mathbf{m}, \mathbf{n}] = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{\mu_1}{\mu_0} \right)^{k-1} \left( (a, a)_t[\mathbf{m} + \mathbf{e}_1 + j\mathbf{e}_2, \mathbf{n} + k\mathbf{e}_1 + \mathbf{e}_2] - (a, a)_t[\mathbf{m} + \mathbf{e}_1 + j\mathbf{e}_2, \mathbf{n} + k\mathbf{e}_1] \right. \\ \left. - \frac{\mu_1}{\mu_0} (a, a)_t[\mathbf{m} + j\mathbf{e}_2, \mathbf{n} + k\mathbf{e}_1 + \mathbf{e}_2] + \frac{\mu_1}{\mu_0} (a, a)_t[\mathbf{m} + j\mathbf{e}_2, \mathbf{n} + k\mathbf{e}_1] \right).$$

In Blanc [2], Riemann-Hilbert methods are employed to determine the Laplace transform of  $(b, b)_t[\mathbf{0}, \mathbf{0}]$ , which in turn is used to derive its asymptotic expansion and relaxation time parameter. We will treat the issue of asymptotic expansions for our distributions in a subsequent paper. In Section 2, we will prove our main theorem. In Section 3, we will solve for the transient distributions related to  $(a, b)_t[\mathbf{m}, \mathbf{n}]$  and  $(b, a)_t[\mathbf{m}, \mathbf{n}]$ . We can solve for the joint density of  $Q(T_1)$  and  $T_1$  in terms of the  $(a, b)_t[\mathbf{m}, \mathbf{n}]$  and similarly, the joint density of  $Q(T_2)$  and  $T_2$  in terms of the  $(b, a)_t[\mathbf{m}, \mathbf{n}]$ . We will do this by solving a related problem for general Markov processes. Using martingales, we can solve for the joint density of a Markov process evaluated at an exit time for leaving some region and the random time of departure, in terms of the stochastic behavior of the process prior to leaving the region.

## 2. Proving the Main Theorem

Before we begin the proof, the following conventions will be used. Let  $\mathbb{Z}_+^2$  denote the quadrant of non-negative integer ordered pairs. Typical elements will be denoted as  $\mathbf{m} = (m_1, m_2)$  or  $\mathbf{n} = (n_1, n_2)$ . The Banach space  $l_1(\mathbb{Z}_+^2)$  is the set of absolutely summable real valued functions on  $\mathbb{Z}_+^2$ . We will also view it as  $l_1(\mathbb{Z}_+) \otimes l_1(\mathbb{Z}_+)$  which is the tensor product of absolutely summable real valued functions on  $\mathbb{Z}_+$  with itself. Let  $(\mathbf{a}, \mathbf{b})_t$  denote the operator that maps  $l_1(\mathbb{Z}_+^2)$  into itself where  $(\mathbf{a}, \mathbf{b})_t[\mathbf{m}, \mathbf{n}]$  is the  $\mathbf{e}_{n_1} \otimes \mathbf{e}_{n_2}$  entry for the action of  $(\mathbf{a}, \mathbf{b})_t$  on  $\mathbf{e}_{m_1} \otimes \mathbf{e}_{m_2}$ . Since  $(\mathbf{a}, \mathbf{b})_t[\mathbf{m}, \mathbf{n}]$  is a transition probability for a sub-Markov process,  $(\mathbf{a}, \mathbf{b})_t$  is an operator semigroup. We will let  $(\mathbf{a}, \mathbf{b})$  denote its infinitesimal generator. Using the tensor formalism of [3], we can decompose the infinitesimal generators of our four processes in terms of right and left shift operators as follows:

$$\mathbf{R}_1 \mathbf{R}_2(\mathbf{a}, \mathbf{a}) \mathbf{L}_2 \mathbf{L}_1 = \mu_0 \mathbf{R}_1 + \mu_1 \mathbf{L}_1 \mathbf{R}_2 + \mu_2 \mathbf{L}_2 - (\mu_0 + \mu_1 + \mu_2) \mathbf{I} \quad (2.1.a)$$

$$\mathbf{R}_1(\mathbf{a}, \mathbf{b}) \mathbf{L}_1 = \mu_0 \mathbf{R}_1 + \mu_1 \mathbf{L}_1 \mathbf{R}_2 + \mu_2 \mathbf{L}_2 - \mu_0 \mathbf{I} - \mu_1 \mathbf{I} - \mu_2 \mathbf{L}_2 \mathbf{R}_2 \quad (2.1.b)$$

$$\mathbf{R}_2(\mathbf{b}, \mathbf{a}) \mathbf{L}_2 = \mu_0 \mathbf{R}_1 + \mu_1 \mathbf{L}_1 \mathbf{R}_2 + \mu_2 \mathbf{L}_2 - \mu_0 \mathbf{I} - \mu_1 \mathbf{L}_1 \mathbf{R}_1 - \mu_2 \mathbf{I} \quad (2.1.c)$$

$$(\mathbf{b}, \mathbf{b}) = \mu_0 \mathbf{R}_1 + \mu_1 \mathbf{L}_1 \mathbf{R}_2 + \mu_2 \mathbf{L}_2 - \mu_0 \mathbf{I} - \mu_1 \mathbf{L}_1 \mathbf{R}_1 - \mu_2 \mathbf{L}_2 \mathbf{R}_2. \quad (2.1.d)$$

The last expression for  $(\mathbf{b}, \mathbf{b})$  can be recognized as a special case for the operator decomposition of the generator for a Jackson network as introduced in [3]. We will motivate the decomposition for  $\mathbf{R}_1(\mathbf{a}, \mathbf{b}) \mathbf{L}_1$  as given by (2.1.b), and remark that the formulas for  $\mathbf{R}_1 \mathbf{R}_2(\mathbf{a}, \mathbf{a}) \mathbf{L}_2 \mathbf{L}_1$  and  $\mathbf{R}_2(\mathbf{b}, \mathbf{a}) \mathbf{L}_2$  are achieved in a similar fashion.

Let  $\mathbf{Q}^*(t) = (Q_1^*(t), Q_2^*(t))$  be the queue length process for a two node series Jackson network with the following modifications. At  $t=0$ , an additional tagged customer is added to the first queue. The discipline is such that all the other customers have a preemptive priority over this special customer. Moreover, we let the event of this tagged customer completing service be an absorbing state. Therefore, the process  $\mathbf{Q}^*(t)$  "stops" after this event occurs. If  $\mathbf{A}^*$  equals the infinitesimal generator of  $\mathbf{Q}^*(t)$ , we can write it as

$$\mathbf{A}^* = \mu_0 \mathbf{R}_1 + \mu_1 \mathbf{L}_1 \mathbf{R}_2 + \mu_2 \mathbf{L}_2 - (\mu_0 + \mu_1) \mathbf{I} - \mu_2 \mathbf{L}_2 \mathbf{R}_2. \quad (2.2)$$

This follows since  $\mathbf{Q}^*(t)$  has the same transitions as  $\mathbf{Q}(t)$  with one exception. Only when

$Q_1^\#(t) = 0$  can an exponential service with rate  $\mu_1$  occur for the tagged customer. Consequently, we replace  $\mu_1 L_1 R_1$  in (b, b) with  $\mu_1 I$ .

Let  $A_t^\# = \{A_t^\#[m, n] \mid m, n \in \mathbb{Z}_+^2\}$  denote the semigroup of transition probabilities of  $Q^\#(t)$ . Observing that the transitions of  $Q^\#(t)$  are identical to those of  $Q(t) + e_1$  prior to the stopping time  $T_1$ , we have

$$A_t^\#[m, n] = (a, b)_t[m+e_1, n+e_1].$$

In terms of operators, this becomes

$$A_t^\# = R_1(a, b)_t L_1.$$

Taking derivatives on both sides, and setting  $t=0$  gives us  $A^\# = R_1(a, b)L_1$ , and so (2.2) gives us (2.1.b). Let  $\sigma$  be an arbitrary scalar. Since  $RL = I$ , we have the following identities:

$$(I - \sigma L)R = [R + \sigma(I - LR)](I - \sigma L) \quad (2.3.a)$$

$$L(I - \sigma R) = (I - \sigma R)[L + \sigma(I - LR)]. \quad (2.3.b)$$

To proceed further, we must redefine the topology for our operators. Let  $\rho = (\rho_1, \rho_2)$  where  $\rho_1$  and  $\rho_2$  are positive real numbers. We define  $l_1(\mathbb{Z}_+^2, \rho)$  to be the Banach space of real valued functions  $f$  on  $\mathbb{Z}_+^2$  such that

$$\|f\|_\rho = \sum_{n \in \mathbb{Z}_+^2} |f(n)| \cdot \rho^n < \infty.$$

Letting  $\|\cdot\|_\rho$  also denote the induced norm for operators acting on  $l_1(\mathbb{Z}_+^2, \rho)$ , we have  $\|R_i\|_\rho = \rho_i$  and  $\|L_i\|_\rho = \frac{1}{\rho_i}$  for  $i = 1, 2$ . This means that all operators which are in the algebraic closure of  $R_i$  and  $L_i$  are still bounded operators with respect to  $\|\cdot\|_\rho$ . As a consequence, the set of bounded operators on  $l_1(\mathbb{Z}_+^2, \rho)$  includes the various generators that we are considering and their semigroups. Moreover,  $\|e_n\|_\rho = \rho^n$  for all  $n \in \mathbb{Z}_+^2$ , so all vectors of interest to us still belong to  $l_1(\mathbb{Z}_+^2, \rho)$ .

The advantage of using this weighted Banach space becomes clear if we set  $\rho_1 < \frac{\mu_0}{\mu_1}$  and  $\rho_2 < 1$ .

Now  $\left(I - \frac{\mu_1}{\mu_0} L_1\right)^{-1}$  and  $(I - R_2)^{-1}$  are bounded operators for  $l_1(\mathbb{Z}_+^2, \rho)$ . Now using (2.3.a) and



(2.3.b), we get the resulting set of operator identities:

$$\left( \mathbf{I} - \frac{\mu_1}{\mu_0} \mathbf{L}_1 \right) \mu_0 \mathbf{R}_1 \left( \mathbf{I} - \frac{\mu_1}{\mu_0} \mathbf{L}_1 \right)^{-1} = \mu_0 \mathbf{R}_1 + \mu_1 (\mathbf{I} - \mathbf{L}_1 \mathbf{R}_1) \quad (2.4.a)$$

$$(\mathbf{I} - \mathbf{R}_2)^{-1} \mathbf{L}_2 (\mathbf{I} - \mathbf{R}_2) = \mathbf{L}_2 + \mathbf{I} - \mathbf{L}_2 \mathbf{R}_2 \quad (2.4.b)$$

Using our  $\| \cdot \|_p$  norm, as well as (2.4.a) and (2.4.b), we get the following set of "similarity" transformations from (a, a) to (a, b), (b, a), and (b, b).

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{I} - \mathbf{R}_2)^{-1} \mathbf{R}_2 (\mathbf{a}, \mathbf{a}) \mathbf{L}_2 (\mathbf{I} - \mathbf{R}_2) \quad (2.5.a)$$

$$(\mathbf{b}, \mathbf{a}) = \left( \mathbf{I} - \frac{\mu_1}{\mu_0} \mathbf{L}_1 \right) \mathbf{R}_1 (\mathbf{a}, \mathbf{a}) \mathbf{L}_1 \left( \mathbf{I} - \frac{\mu_1}{\mu_0} \mathbf{L}_1 \right)^{-1} \quad (2.5.b)$$

$$(\mathbf{b}, \mathbf{b}) = \left( \mathbf{I} - \frac{\mu_1}{\mu_0} \mathbf{L}_1 \right) (\mathbf{I} - \mathbf{R}_2)^{-1} \mathbf{R}_1 \mathbf{R}_2 (\mathbf{a}, \mathbf{a}) \mathbf{L}_2 \mathbf{L}_1 (\mathbf{I} - \mathbf{R}_2) \left( \mathbf{I} - \frac{\mu_1}{\mu_0} \mathbf{L}_1 \right)^{-1} \quad (2.5.c)$$

We will prove (2.5.a). Verifying the other two identities can be done in a similar manner. By (2.1.a), (2.1.b) and (2.4.b), we can show that

$$\mathbf{R}_1 (\mathbf{a}, \mathbf{b}) \mathbf{L}_1 = (\mathbf{I} - \mathbf{R}_2)^{-1} \mathbf{R}_1 \mathbf{R}_2 (\mathbf{a}, \mathbf{a}) \mathbf{L}_1 \mathbf{L}_2 (\mathbf{I} - \mathbf{R}_2).$$

Multiplying both sides on the right by  $\mathbf{R}_1$  and on the left by  $\mathbf{L}_1$  give us

$$\mathbf{L}_1 \mathbf{R}_1 (\mathbf{a}, \mathbf{b}) \mathbf{L}_1 \mathbf{R}_1 = (\mathbf{I} - \mathbf{R}_2)^{-1} \mathbf{R}_2 \cdot \mathbf{L}_1 \mathbf{R}_1 (\mathbf{a}, \mathbf{a}) \mathbf{L}_1 \mathbf{R}_1 \cdot \mathbf{L}_2 (\mathbf{I} - \mathbf{R}_2). \quad (2.6)$$

To derive (2.5.a) from (2.6), it only remains to establish the following two identities:

$$\mathbf{L}_1 \mathbf{R}_1 (\mathbf{a}, \mathbf{a}) = (\mathbf{a}, \mathbf{a}) \mathbf{L}_1 \mathbf{R}_1 = (\mathbf{a}, \mathbf{a}) \quad (2.7.a)$$

$$\mathbf{L}_1 \mathbf{R}_1 (\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{b}) \mathbf{L}_1 \mathbf{R}_1 = (\mathbf{a}, \mathbf{b}) \quad (2.7.b)$$

They follow from showing that (2.7.a) and (2.7.b) also hold if (a, a) and (a, b) are replaced respectively by  $(\mathbf{a}, \mathbf{a})_t$  and  $(\mathbf{a}, \mathbf{b})_t$ . The resulting identities would then be a simple consequence of the fact that  $(\mathbf{a}, \mathbf{a})_t[m, n] = (\mathbf{a}, \mathbf{b})_t[m, n] = 0$  whenever  $m_1$  or  $n_1$  equals zero. Similar arguments of this type are made by noting that (a, a) and (b, a) have analogous commutivity relations with  $\mathbf{L}_2 \mathbf{R}_2$ .

After verifying (2.5.a), (2.5.b), and (2.5.c), we note that all of these identities still hold when the generators are replaced by their semigroups. Consider for example,  $(\mathbf{a}, \mathbf{b})_t$ . Using (2.5.a), we

have

$$\frac{d}{dt} (\mathbf{I} - \mathbf{R}_2)(\mathbf{a}, \mathbf{b})_t = (\mathbf{I} - \mathbf{R}_2)(\mathbf{a}, \mathbf{b}) \cdot (\mathbf{a}, \mathbf{b})_t = \mathbf{R}_2(\mathbf{a}, \mathbf{a})\mathbf{L}_2 \cdot (\mathbf{I} - \mathbf{R}_2)(\mathbf{a}, \mathbf{b})_t. \quad (2.8)$$

By the uniqueness of this Cauchy problem, we get

$$(\mathbf{I} - \mathbf{R}_2)(\mathbf{a}, \mathbf{b})_t = \exp(t \mathbf{R}_2(\mathbf{a}, \mathbf{a})\mathbf{L}_2)(\mathbf{I} - \mathbf{R}_2). \quad (2.9)$$

Using  $\mathbf{A}_t^\#$ , we see that  $\exp(t \mathbf{R}_2(\mathbf{a}, \mathbf{a})\mathbf{L}_2) = \mathbf{R}_2(\mathbf{a}, \mathbf{a})_t \mathbf{L}_2$ . This fact coupled with the application of  $(\mathbf{I} - \mathbf{R}_2)^{-1}$  to both sides of (2.9) gives us

$$(\mathbf{a}, \mathbf{b})_t = (\mathbf{I} - \mathbf{R}_2)^{-1} \mathbf{R}_2(\mathbf{a}, \mathbf{a})_t \mathbf{L}_2 (\mathbf{I} - \mathbf{R}_2).$$

The semigroup versions of identities (2.5.b) and (2.5.c) can be proved in a similar manner.

Finally, writing out in detail such matrix expressions gives us the theorem.

### 3. Solving for Hitting Time Distributions in Terms of Exit Time Distributions

With  $(a, a)_t[\mathbf{m}, \mathbf{n}]$ ,  $(a, b)_t[\mathbf{m}, \mathbf{n}]$ , and  $(b, a)_t[\mathbf{m}, \mathbf{n}]$ , we now associate the following set of hitting time distributions:

$$(a^\dagger, a^\dagger)_t[\mathbf{m}, \mathbf{n}] = P_{\mathbf{m}}(Q(T) = \mathbf{n}, T = t) \quad (3.1.a)$$

$$(a^\dagger, b)_t[\mathbf{m}, \mathbf{n}] = P_{\mathbf{m}}(Q(T_1) = \mathbf{n}, T_1 = t) \quad (3.1.b)$$

$$(b, a^\dagger)_t[\mathbf{m}, \mathbf{n}] = P_{\mathbf{m}}(Q(T_2) = \mathbf{n}, T_2 = t). \quad (3.1.c)$$

The right-hand side of (3.1.a) is shorthand for the density or derivative of the distribution  $P_{\mathbf{m}}(Q(T) = \mathbf{n}, T \leq t)$ . The others are defined similarly. A special case of Theorem 1 in [1] gives us the solution for (3.1.a),

$$(a^\dagger, a^\dagger)_t[\mathbf{m}, \mathbf{n}] = \begin{cases} \mu_1 \cdot (a, a)_t[\mathbf{m}, \mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2] & \mathbf{n} = (0, n_2), \\ \mu_2 \cdot (a, a)_t[\mathbf{m}, \mathbf{n} + \mathbf{e}_2] & \mathbf{n} = (n_1, 0), \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

Results of this type suggest a deeper relation that we will state and prove here.

**Lemma 3.1.** *Let  $X_t$  be a Feller-Dynkin process (see Williams [5]), with Polish state space  $E$ , generator  $A$ , and domain  $D(A)$ . Moreover, let  $f \in D(A)$  be a continuous function on  $E$  such that both*

$f$  and  $Af$  are bounded. If  $U$  is a Borel-measurable subset of  $E$  and disjoint from the support of  $f$ , then

$$E_x(f(X_T); T \leq t) = \int_0^t E_x(Af(X_s); T > s) ds$$

for all  $x \in U$ , where  $T = \inf\{t \mid X_t \notin U\}$ .

**Proof:** By Williams [5], page 129, we know that the process  $M_t = f(X_t) - \int_0^t Af(X_s) ds$  is a martingale. Now  $T$  is a stopping time for our Markov process  $X_t$ , so by the optional sampling theorem,  $M_{T \wedge t}$  is a martingale as well. This means that  $E_x(M_{T \wedge t}) = f(x) = 0$ , and so

$$E_x(f(X_{T \wedge t})) = E_x\left(\int_0^{T \wedge t} Af(X_s) ds\right) = \int_0^t E_x(Af(X_s); T > s) ds$$

by Fubini's theorem. Finally, we note that  $E_x(f(X_{T \wedge t})) = E_x(f(X_T); T \leq t)$ , since  $f(X_t) = 0$  for all  $t < T$  by hypothesis. ■

We can alternatively write the result of Lemma 3.1 as

$$E_x(f(X_T); T = t) = E_x(Af(X_t); T > t).$$

Note that this result is *only* useful for Markov processes with pure jumps or discontinuous sample paths. The relation is vacuous for diffusions. Equation (3.2) and Theorem 1 of [1] can be viewed as simple consequences of Lemma 3.1 and we can now solve for our remaining hitting time probabilities:

$$(a^\dagger, b)_i[\mathbf{m}, \mathbf{n}] = \begin{cases} \mu_1 \cdot (a, b)_i[\mathbf{m}, \mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2] & \mathbf{n} = (0, n_2), \\ 0 & \text{otherwise.} \end{cases}$$

$$(b, a^\dagger)_i[\mathbf{m}, \mathbf{n}] = \begin{cases} \mu_2 \cdot (b, a)_i[\mathbf{m}, \mathbf{n} + \mathbf{e}_2] & \mathbf{n} = (n_1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

#### 4. Generalizing to the $N$ -node Series Jackson Network

We now give a sketch of how these results can be extended to  $N$  dimensions, using lattice Bessel functions of rank  $N$ . Let  $\mathbf{Q}(t) = (Q_1(t), \dots, Q_N(t))$  be the vector queue length process for an  $N$ -node series Jackson network, whose state space is  $\mathbb{Z}_+^N$ , the non-negative orthant of the  $N$

dimensional integer lattice. If  $J$  is an arbitrary subset of  $\{1, \dots, N\}$ , we define  $(a_j)_t[\mathbf{m}, \mathbf{n}]$  for all  $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^N$  to be the transition probabilities for the following stopped process

$$(a_j)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n}, T_j > t \mid \mathbf{Q}(0) = \mathbf{m}\},$$

where  $T_j = \inf\{t \mid Q_j(t) = 0 \text{ for some } j \in J\}$ . We also can define the following hitting time probabilities

$$(a_j^*)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n}, T_j = t \mid \mathbf{Q}(0) = \mathbf{m}\}$$

where  $n_j = 0$  for some  $j \in J$ . By Lemma 3.1, we can solve for these in terms of the  $(a_j)_t[\mathbf{m}, \mathbf{n}]$

$$(a_j^*)_t[\mathbf{m}, \mathbf{n}] = \begin{cases} \mu_j \cdot (a_j)_t[\mathbf{m}, \mathbf{n} - \mathbf{v}_j] & n_j = 0, n_{j+1} > 0, \text{ and } j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

As before, the  $(a_j)_t[\mathbf{m}, \mathbf{n}]$  are the entries for an operator semigroup  $(\mathbf{a}_j)_t$  with generator  $(\mathbf{a}_j)$ . They are both bounded operators on the Banach space  $l_1(\mathbb{Z}_+^N)$ . Since  $(a_j)_t[\mathbf{m}, \mathbf{n}] = 0$  if any  $m_j = 0$  or  $n_j = 0$  for some  $j \in J$ , this translates into operators as

$$(\mathbf{a}_j)_t \cdot \mathbf{L}_j \mathbf{R}_j = \mathbf{L}_j \mathbf{R}_j \cdot (\mathbf{a}_j)_t = (\mathbf{a}_j)_t$$

for all  $j \in J$ , and consequently

$$(\mathbf{a}_j) \cdot \mathbf{L}_j \mathbf{R}_j = \mathbf{L}_j \mathbf{R}_j \cdot (\mathbf{a}_j) = (\mathbf{a}_j). \quad (4.1)$$

Now let  $\mathbf{R}_J = \prod_{j \in J} \mathbf{R}_j$  and  $\mathbf{L}_J = \prod_{j \in J} \mathbf{L}_j$ . Constructing a process similar to  $\mathbf{Q}^*(t)$  in Section 2, adding tagged customers to every node indexed by  $J$ , we can deduce that

$$\mathbf{R}_J(\mathbf{a}_J)\mathbf{L}_J = \mu_0 \mathbf{R}_1 + \sum_{j=1}^N \mu_j \mathbf{L}_j \mathbf{R}_j + \mu_N \mathbf{L}_N - \sum_{j \in \{0\} \cup J} \mu_j \mathbf{I} - \sum_{j \notin J} \mu_j \mathbf{L}_j \mathbf{R}_j. \quad (4.2)$$

Moreover, from (4.1) we can show that

$$\exp(t \mathbf{R}_J(\mathbf{a}_J)\mathbf{L}_J) = \mathbf{R}_J(\mathbf{a}_J)_t \mathbf{L}_J.$$

From now on, we will only consider  $J$  to be the following sets:

1.  $\{1, \dots, N\}$

2.  $(1) = \{1\}^c$
3.  $(N) = \{N\}^c$
4.  $(1, N) = \{1, N\}^c$ ,

where  $c$  denotes the complement of the indicated set. Setting  $(a_*) = (a_{\{1, \dots, N\}})$ ,  $R_* = \prod_{j=1}^N R_j$ , and

$L_* = \prod_{j=1}^N L_j$ , we get from (4.2),

$$R_*(a_*)L_* = \mu_0 R_1 + \sum_{j=1}^{N-1} \mu_j L_j R_{j+1} + \mu_N L_N - \sum_{j=0}^N \mu_j I$$

$$R_{(1)}(a_{(1)})L_{(1)} = \mu_0 R_1 + \sum_{j=1}^{N-1} \mu_j L_j R_{j+1} + \mu_N L_N - \mu_0 I - \mu_1 L_1 R_1 - \sum_{j=2}^N \mu_j I$$

$$R_{(N)}(a_{(N)})L_{(N)} = \mu_0 R_1 + \sum_{j=1}^{N-1} \mu_j L_j R_{j+1} + \mu_N L_N - \sum_{j=0}^{N-1} \mu_j I - \mu_N L_N R_N$$

$$R_{(1,N)}(a_{(1,N)})L_{(1,N)} = \mu_0 R_1 + \sum_{j=1}^{N-1} \mu_j L_j R_{j+1} + \mu_N L_N - \mu_0 I - \mu_1 L_1 R_1 - \sum_{j=2}^{N-1} \mu_j I - \mu_N L_N R_N.$$

Constructing the appropriately weighted Banach space, and following arguments like those in Section 2, we get

$$(a_{(1)}) = \left( I - \frac{\mu_1}{\mu_0} L_1 \right) R_1(a_*)L_1 \left( I - \frac{\mu_1}{\mu_0} L_1 \right)^{-1} \quad (4.3.a)$$

$$(a_{(N)}) = (I - R_N)^{-1} R_N(a_*)L_N(I - R_N) \quad (4.3.b)$$

$$(a_{(1,N)}) = \left( I - \frac{\mu_1}{\mu_0} L_1 \right) (I - R_N)^{-1} R_1 R_N(a_*)L_N L_1 (I - R_N) \left( I - \frac{\mu_1}{\mu_0} L_1 \right)^{-1}. \quad (4.3.c)$$

Now by (4.3), we have

$$(a_*)_i[m, n] = e^{-(N+1)\alpha t} \cdot \beta^{n-m} \cdot \sum_{g \in G_N} (-1)^g I(n-g(m), (N+1)\gamma t). \quad (4.4)$$

From (4.3.a), (4.3.b), and (4.3.c), we have

$$(a_{(N)})_t[\mathbf{m}, \mathbf{n}] = \sum_{j=1}^{\infty} (a_{\bullet})_t[\mathbf{m} + j\mathbf{e}_N, \mathbf{n} + \mathbf{e}_N] - (a_{\bullet})_t[\mathbf{m} + j\mathbf{e}_N, \mathbf{n}]$$

$$(a_{(1)})_t[\mathbf{m}, \mathbf{n}] = \sum_{j=1}^{\infty} \left( \frac{\mu_1}{\mu_0} \right)^{j-1} \left( (a_{\bullet})_t[\mathbf{m} + \mathbf{e}_1, \mathbf{n} + j\mathbf{e}_1] - \frac{\mu_1}{\mu_0} (a_{\bullet})_t[\mathbf{m}, \mathbf{n} + j\mathbf{e}_1] \right)$$

$$(a_{(1,N)})_t[\mathbf{m}, \mathbf{n}] = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \frac{\mu_1}{\mu_0} \right)^{k-1} \left( (a_{\bullet})_t[\mathbf{m} + \mathbf{e}_1 + j\mathbf{e}_N, \mathbf{n} + k\mathbf{e}_1 + \mathbf{e}_N] - (a_{\bullet})_t[\mathbf{m} + \mathbf{e}_1 + j\mathbf{e}_N, \mathbf{n} + k\mathbf{e}_1] \right. \\ \left. - \frac{\mu_1}{\mu_0} (a_{\bullet})_t[\mathbf{m} + j\mathbf{e}_N, \mathbf{n} + k\mathbf{e}_1 + \mathbf{e}_N] + \frac{\mu_1}{\mu_0} (a_{\bullet})_t[\mathbf{m} + j\mathbf{e}_N, \mathbf{n} + k\mathbf{e}_1] \right).$$

By (4.4) we have a solution for  $(a_N)_t[\mathbf{m}, \mathbf{n}]$ ,  $(a_{(1)})_t[\mathbf{m}, \mathbf{n}]$ , and  $(a_{(1,N)})_t[\mathbf{m}, \mathbf{n}]$  in terms of lattice Bessel functions of rank  $N$ .

#### ACKNOWLEDGMENTS

This work was developed between mutual visits of each author to the institution of the other. We thank INRIA and AT&T Bell Laboratories for their hospitality.

## BIBLIOGRAPHY

1. Baccelli, F. and Massey, W. A., *On the Busy Period of Certain Classes of Queueing Networks*, Proceedings of the Second International MCPR Workshop, University of Rome, May 25-29, 1987.
2. Blanc, J. P. C., *The Relaxation Time of Two Queueing Systems in Series*, Comm. Statist. Stochastic Models 1, no. 1, pp. 1-16 (1985).
3. Massey, W. A., *An Operator Analytic Approach to the Jackson Network*, J. Appl. Prob., **21**, 379-393 (1984).
4. Massey, W. A., *Calculating Exit Times for Series Jackson Networks*, J. Appl. Prob., **24**, 226-234 (1987).
5. Williams, D., *Diffusions, Markov Processes, and Martingales, Vol. I.*, John Wiley & Sons (1979).

