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### DYNAMIC, TRANSIENT AND STATIONARY BEHAVIOR OF THE M/GI/1 QUEUE VIA MARTINGALES

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# DYNAMIC, TRANSIENT AND STATIONARY BEHAVIOR OF THE $M/GI/1$ QUEUE VIA MARTINGALES

by

Fran çois Baccelli<sup>1</sup> and Armand M. Makowski<sup>2</sup>

## ABSTRACT

An exponential martingale is associated with the Markov chain of the number of customers in the  $M/GI/1$  queue. This together with renewal theory are shown to provide a unified probabilistic framework for deriving several well-known generating functions for the  $M/GI/1$  queue, including the Pollaczek-Khinchin formula, the transient generating function of the number of customers at departure epochs and the generating function of the number of customers served in a busy period.

## INTRODUCTION

An exponential martingale is associated with the Markov chain of the number of customers at departure epochs in the  $M/GI/1$  queue. Basic regularity properties of this martingale and standard arguments from Renewal Theory are shown to provide a unified probabilistic framework for deriving three well-known analytical formulae which respectively characterize the dynamic, transient and stationary behavior of this queue. The main theoretical interest of this new approach stems from a general equivalence relationship between the law of this embedded Markov chain and the law of the forward recurrence time of a discrete-time renewal process associated with this chain (Theorem 3). This equivalence produces several results which appear to be new, at least to the best of the authors' knowledge. For instance, a new probabilistic representation is established for the generating function of the Markov chain of the number of customers at the  $n^{\text{th}}$  departure epoch (Corollary 7). The usual representation of the transient generating function in terms of the double generating function of this quantity, both in time and space, is shown to be derivable from this new probabilistic representation.

The paper is organized as follows: The exponential martingale is introduced in Section 1, and its definition is followed by a summary of its key regularity properties, already established

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Mai 1988

# COMPORTEMENTS DYNAMIQUE, TRANSITOIRE ET STATIONNAIRE DE LA FILE M/GI/1 PAR LES MARTINGALES

par

François Baccelli<sup>1</sup> et Armand M. Makowski<sup>2</sup>

## RESUME

On associe une martingale exponentielle à la chaîne de Markov du nombre des clients dans la file d'attente  $M/GI/1$ . Utilisant cette martingale, on montre que la théorie du renouvellement constitue un cadre probabiliste unifié pour obtenir les fonctions génératrices classiques de cette file d'attente, comme la formule de Pollaczek-Khinchine ou les fonctions génératrices du nombre des clients à la date du  $n$ -ième départ et du nombre des clients servis pendant une période d'activité.

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by the authors in a previous paper [1]. Section 2 contains the derivation of the analytical results mentioned above, with some of the calculations collected in Section 3. The generating function of the number of customers served in a busy period is obtained in Section 2.1. The proof of this first result is very much in the spirit of the derivation of the Laplace transform of the length of the busy period via martingales, in [7]. The full power of the equivalence relationship mentioned above becomes apparent in Sections 2.2 and 2.3, where the Pollaczek-Khinchin formula and the transient generating function of the number of customers are shown to be mere rephrasings of basic Renewal Theory formulae.

It should be emphasized that the authors' aim was not to give here a comprehensive list of the applications of this new approach, but rather to illustrate the power of the approach. For instance, the results on the transient generating function are limited to the analysis of a simple particular case. Similarly, only the hitting time to the zero state (also known as busy period) was considered in the study of the dynamic properties of the chain. However, it is easy to see that most known formulae concerning the transient dynamic behavior can simply be obtained through similar arguments. These techniques are of independent interest and apply to other queueing systems, like for instance queues in random environments [2].

## 1. PRELIMINARIES

### 1.1. Notations

All the random variables (RV) and stochastic elements occurring in this paper are implicitly defined on a fixed probability triple  $(\Omega, \mathcal{F}, \mathcal{P})$ . Throughout, the characteristic function of any event  $A$  in  $\mathcal{F}$  is denoted by  $I[A]$ . The collection of all non-negative integers is denoted by  $\mathbb{N}$  and  $\mathbb{R}$  denotes the set of real numbers.

Consider an  $M/GI/1$  queue with Poissonian arrival pattern of intensity  $\lambda$ . The consecutive service times form a sequence of i.i.d.  $\mathbb{R}$ -valued RV's  $\{S_n\}_1^\infty$  assumed independent of the arrival process. Throughout, the common probability distribution of the service times and its Laplace-Stieltjes transform are denoted by  $S$  and  $S^*$ , respectively. The initial queue size is given by an  $\mathbb{N}$ -valued RV  $\Xi$ .

### 1.2. The embedding

At time  $t = 0$ , a dummy customer is assumed to complete service and by leaving the system, it generates the  $0^{th}$  departure. For  $n = 0, 1, \dots$ , let  $X_n$  denote the number of customers in the system as seen by the  $n^{th}$  departing customer. For  $n = 1, 2, \dots$ , let  $A_n$  denote the number of arrivals during the  $n^{th}$  service period. From these definitions, the queue size sequence  $\{X_n\}_0^\infty$  is readily seen to satisfy the recursion

$$X_{n+1} = X_n + A_{n+1} - I[X_n \neq 0], \quad n = 0, 1, \dots \quad (1.1)$$

with random initial condition  $X_0 = \Xi$ . Under the enforced assumptions, the RV's  $\{A_{n+1}\}_0^\infty$  are i.i.d. RV's independent of the initial queue size  $\Xi$ , and the  $\mathbb{N}$ -valued process  $\{X_n\}_0^\infty$  is an *irreducible* Markov chain with countable state space [3,5].

### 1.3. The martingale

For all  $n = 0, 1, \dots$ , the RV's  $\{\Xi, A_k, 0 < k \leq n\}$  generate the  $\sigma$ -field  $\mathcal{F}_n$ ; in view of (1.1) the RV's  $\{X_k\}_0^n$  are all  $\mathcal{F}_n$ -measurable. Consider an arbitrary  $\mathcal{F}_n$ -stopping time  $\sigma$  and define the  $\mathbb{N}$ -valued RV  $\nu(\sigma)$  by

$$\nu(\sigma) = \begin{cases} \inf\{n \geq 1 : X_{\sigma+n} = 0\} & \text{if } \sigma < \infty \text{ and this set is non empty;} \\ \infty & \text{otherwise.} \end{cases} \quad (1.2)$$

Let  $a$  denote the generating function of the i.i.d. sequence  $\{A_{n+1}\}_0^\infty$ , namely

$$a(z) = E[z^{A_n}] = S^*(\lambda(1-z)), \quad 0 \leq z \leq 1. \quad n = 0, 1, \dots \quad (1.3)$$

For  $0 < z \leq 1$ , the  $\mathcal{F}_n$ -measurable  $\mathbb{R}_+$ -valued RV's  $\{M_n(z)\}_0^\infty$  are defined by

$$M_n(z) = \begin{cases} z^{X_0} & \text{if } n = 0; \\ z^{X_n} \frac{\sum_{k=0}^{n-1} I_{\{X_k \neq 0\}}}{a(z)^n} & \text{for } n = 1, 2, \dots \end{cases} \quad (1.4)$$

The following result was established in [1, Thm. 2 pp. 181,186].

**Theorem 1.** For all  $0 < z < 1$ , the RV's  $\{M_n(z)\}_0^\infty$  form a positive and integrable  $\mathcal{F}_n$ -martingale. If  $\rho \leq 1$ , the stopping time  $\nu(\sigma)$  is regular for this martingale and the relation

$$E[I\{\sigma < \infty, \nu(\sigma) < \infty\} \left[ \frac{z}{a(z)} \right]^{\nu(\sigma)} | \mathcal{F}_\sigma] = I\{\sigma < \infty\} z^{I\{X_\sigma=0\}} X_\sigma \quad \text{a.s.} \quad (1.5)$$

holds for all  $0 < z \leq 1$ .

The relation (1.5) is a simple consequence of the regularity of the stopping time  $\sigma$  and of Doob's Optional Sampling Theorem [6 Cor. IV-2-6, p. 67]. Moreover,  $z < a(z)$  for  $0 < z < 1$  under the condition  $\rho \leq 1$  (Takàcs' Lemma [8, p. 46], and letting  $z \uparrow 1$  in (1.5) yields

$$P\{\sigma < \infty, \nu(\sigma) < \infty | \mathcal{F}_\sigma\} = I\{\sigma < \infty\} \quad \text{a.s.} \quad (1.6)$$

as an immediate consequence of the Bounded Convergence Theorem.

## 2. TRANSFORMS

Let  $\{\tau_n\}_0^\infty$  be the sequence of  $\mathcal{F}_n$ -stopping times defined by the recursion

$$\tau_{n+1} = \tau_n + \nu(\tau_n) \quad n = 0, 1, \dots \quad (2.1)$$

with  $\tau_0 = 0$ . With  $\sigma = 0$ , (1.6) specializes to

$$P\{\tau_1 < \infty | \mathcal{F}_0\} = 1 \quad \text{a.s.} \quad (2.2)$$

**Lemma 2.** If  $\rho \leq 1$ , the RV's  $\{\tau_n\}_0^\infty$  form a possibly delayed recurrent renewal process.

**Proof.** It is plain from (1.5) that the RV  $\nu(\tau_n)$  and the  $\sigma$ -field  $\mathcal{F}_{\nu(\tau_n)}$  are independent for all  $n = 0, 1, \dots$ . Consequently the RV's  $\{\tau_n\}_0^\infty$  form a renewal process if  $\Xi = 0$  a.s. and a delayed renewal process otherwise. For  $\rho \leq 1$ , the recurrence property is immediately obtained from (1.6).  $\square$

The forward recurrence times  $\{\mu(n)\}_0^\infty$  of the recurrent renewal process  $\{\tau_n\}_0^\infty$  are defined by

$$\mu(n) = \begin{cases} \inf\{m \geq 0 : X_{n+m} = 0\} & \text{if this set is non empty;} \\ \infty & \text{otherwise.} \end{cases} \quad n = 0, 1, \dots \quad (2.3)$$

It turns out that the generating function of the number  $X_n$  of customers at service completion is very simply related to the generating function of the forward recurrence time  $\mu(n)$ . The

key relationship is provided in the next theorem. For every  $0 < z \leq 1$ , it is convenient to introduce the quantity  $\xi(z)$  as the ratio

$$\xi(z) = \frac{z}{a(z)}. \quad (2.4)$$

**Theorem 3.** Assume  $\rho < 1$ . For all  $0 < z \leq 1$ , the relation

$$E[z^{X_n}] = E[\xi(z)^{\mu(n)}] \quad n = 0, 1, \dots (2.5)$$

holds true.

**Proof.** Note from (1.2) and (2.3) the easy facts

$$[X_n = 0] = [\mu(n) = 0] \quad n = 0, 1, \dots (2.6)$$

and

$$\nu(n) = \mu(n) \quad \text{on} \quad [X_n \neq 0]. \quad n = 0, 1, \dots (2.7)$$

It is then plain that for each  $y > 0$ , the relations

$$y^{\mu(n)} I[\mu(n) \neq 0] = y^{\mu(n)} I[X_n \neq 0] = y^{\nu(n)} I[X_n \neq 0] \quad n = 0, 1, \dots (2.8)$$

hold true.

Specialize (1.5) to  $\sigma = n$  and multiply both sides of the resulting equation by  $I[X_n \neq 0]$ . For all  $0 < z \leq 1$ , the relation

$$E[I[\nu(n) < \infty, X_n \neq 0] \xi(z)^{\nu(n)} | \mathcal{F}_n] = I[X_n \neq 0] z^{X_n} \quad \text{a.s.} \quad (2.9)$$

readily follows since the RV  $X_n$  is  $\mathcal{F}_n$ -measurable, and taking the mathematical expectation of both sides of (2.9) implies

$$E[I[\nu(n) < \infty, X_n \neq 0] \xi(z)^{\nu(n)}] = E[I[X_n \neq 0] z^{X_n}]. \quad (2.10)$$

Starting with a simple decomposition argument, it is now plain that for all  $n = 0, 1, \dots$ , the relations

$$\begin{aligned} E[z^{X_n}] &= P[X_n = 0] + E[z^{X_n} I[X_n \neq 0]] \\ &= P[\mu(n) = 0] + E[I[\nu(n) < \infty, X_n \neq 0] \xi(z)^{\nu(n)}] \\ &= P[\mu(n) = 0] + E[I[\nu(n) < \infty, \mu(n) \neq 0] \xi(z)^{\mu(n)}] \end{aligned} \quad (2.11)$$

hold true. In this chain of equalities, the second one is a consequence of (2.6) and (2.10), while the last equality follows from (2.8). Under the foregoing assumptions,  $P[\nu(n) < \infty | \mathcal{F}_n] = 1$  a.s. by virtue of (1.6), whence  $\mu(n) < \infty$  a.s. since  $0 \leq \mu(n) \leq \nu(n)$  as a consequence of (2.6) and (2.7). This remark readily validates the passage from (2.11) to (2.5).  $\square$

The aim of the remainder of the section is to recover various known transforms from (1.5) and (2.5), thus avoiding the usual analytical calculations.

### 2.1. Generating function of the number of customers served in a busy period

Denote by  $\{f(n)\}_0^\infty$  the point mass distribution function of the number of customers served in a busy period of the  $M/GI/1$  queue. On the event  $[X_0 = 0]$ , the number of customers served in the first busy period coincides with  $\nu(0)$ , so that the generating function of  $\{f(n)\}_0^\infty$ , denoted by  $F^*$ , is given by the relation

$$F^*(y) = E[y^{\nu(0)} | X_0 = 0], \quad 0 \leq y \leq 1. \quad (2.12)$$

In other words,  $F^*$  is also the generating function of the interevent times of the discrete-time renewal process  $\{\tau_n\}_0^\infty$ . It is now shown that  $F^*$  can be obtained as an immediate consequence of (1.5).

**Lemma 4.** Assume  $\rho \leq 1$ . For each  $0 < \xi \leq 1$ , the equation in the unknown variable  $z$

$$z = \xi a(z) \quad (2.13)$$

has a unique root in the interval  $[0, 1]$ , denoted by  $Z(\xi)$ . The generating function  $F^*$  of the number of customers served in a busy period, or equivalently of the interevent times of the discrete-time renewal process  $\{\tau_n\}_0^\infty$ , is given by

$$F^*(y) = Z(y), \quad 0 \leq y \leq 1, \quad (2.14)$$

and the mean value  $m$  of this distribution function is given by

$$m = E[\nu(0) | X_0 = 0] = \begin{cases} \frac{1}{1-\rho} & \text{if } \rho < 1; \\ \infty & \text{if } \rho = 1. \end{cases} \quad (2.15)$$

**Proof.** The first statement concerning the roots of (2.13) follows from classical convexity arguments [8, Lemma 1, p. 47] and its proof is therefore omitted for the sake of brevity. Upon specializing again (1.5) to  $\sigma = 0$ , it is plain that for all  $0 < z \leq 1$ ,

$$E[\xi(z)^{\nu(0)} | X_0 = 0] = z \quad \text{a.s.} \quad (2.16)$$

Let  $\xi$  be a real number such that  $0 \leq \xi \leq 1$  and let  $Z(\xi)$  denote the root corresponding to (2.13). It is now immediate from (2.16) and from the definition of  $Z(\xi)$  that

$$F^*(\xi) = E[\xi^{\nu(0)} | X_0 = 0] = Z(\xi) \quad \text{a.s.}, \quad (2.17)$$

and (2.14) is obtained. Equation (2.15) is now obtained by differentiating (2.16) with respect to  $z$  in a left neighborhood of 1 and by letting  $z$  go to 1 in the resulting expression. The differentiation step is validated by well-known properties of uniform convergence for generating functions.  $\square$

## 2.2 Generating function of the number of customers at steady state

The next step consists in establishing the Pollaczek-Khinchin transform.

**Corollary 5.** Assume  $\rho < 1$ . For all  $0 < z \leq 1$ ,  $\lim_n E[z^{X_n}]$  exists (when  $n$  goes to  $\infty$ ) and is given by

$$\lim_n E[z^{X_n}] = (1 - \rho) \frac{(1 - z)a(z)}{a(z) - z}. \quad (2.18)$$



**Proof.** From the Key Renewal Theorem [9, Thm. 2.3, p. 18], the forward recurrence times  $\{\mu(n)\}_0^\infty$  of a discrete-time renewal process with interevent generating function  $F^*$  and finite mean  $m$  satisfy the convergence property

$$\lim_n E[y^{\mu(n)}] = \frac{1 - F^*(y)}{m(1 - y)}, \quad (2.19)$$

and this independently of the initial condition. Using this fact and (2.5), it is now plain that  $\lim_n E[z^{X_n}]$  exists, is independent of the initial condition and is given by

$$\lim_n E[z^{X_n}] = \lim_n E[\xi(z)^{\mu(n)}] = (1 - \rho) \frac{1 - Z(\xi(z))}{1 - \xi(z)}. \quad (2.20)$$

Equation (2.18) follows immediately from (2.20) since  $Z(\xi(z)) = z$ .  $\square$

### 2.3 Transient generating function for the number of customers

For the sake of brevity, the discussion is limited to the case of a stable queue, namely  $\rho < 1$ , when the initial queue size is zero, namely  $\Xi = 0$  a.s. The proposed approach extends easily to the other cases by following arguments very similar to the ones reported below. Details are left to the interested reader.

The renewal kernel  $R : \mathbb{N} \rightarrow \mathbb{R}$  associated with the discrete-time, non-delayed renewal process  $\{\tau_n\}_0^\infty$  is given by

$$R(n) = \sum_{k=0}^{\infty} P[\tau_k = n]. \quad n = 0, 1, \dots \quad (2.21)$$

Moreover, let the function  $g : [0, 1] \times \mathbb{N} \rightarrow \mathbb{R}$  be defined by

$$g(y, n) = E[y^{\mu(n)} I[\tau_1 > n]] = \begin{cases} E[y^{\tau_1 - n} I[\tau_1 > n]] & \text{if } n \geq 1; \\ 1 & \text{if } n = 0. \end{cases} \quad (2.22)$$

As indicated in the next corollary, the *transient* behavior of the queue size process can be fully characterized by means of these two functions.

**Corollary 7.** Assume  $\rho < 1$  and  $\Xi = 0$ . For all  $0 < z \leq 1$ , the relation

$$E[z^{X_n}] = \sum_{m=0}^n R(m)g(\xi(z), n - m) \quad n = 0, 1, \dots \quad (2.23)$$

holds, with  $\xi(z)$  defined by (2.4).

**Proof.** The forward recurrence times process  $\{\mu(n)\}_0^\infty$  of the discrete-time, non-delayed renewal process  $\{\tau_n\}_0^\infty$  is also a discrete-time  $\mathbb{N}$ -valued regenerative process [4, Chap. 9]. As well known [4, Chap. 9], this regeneration property translates into the following Renewal Equation

$$E[y^{\mu(n)}] = E[y^{\mu(n)} I[\tau_1 > n]] + \sum_{m=0}^n f(m) E[y^{\mu(n-m)}] \quad (2.24)$$

valid for all  $0 \leq y \leq 1$ , where  $\{f(n)\}_0^\infty$  denotes the point mass distribution characterizing the non-delayed renewal process  $\{\tau_n\}_0^\infty$ . The solution of the Renewal Equation (2.24) is given by [4, Thm. 2.3, p. 294]

$$E[y^{\mu(n)}] = \sum_{m=0}^n R(m)g(y, n - m) \quad n = 0, 1, \dots \quad (2.25)$$

and the representation (2.23) follows immediately from (2.5) and (2.25).  $\square$

Classically [8, Eqn. (59), p. 70], the transient behavior of the queue is given in terms of the double transform  $N^*$  given by

$$N^*(z, t) = \sum_{n=0}^{\infty} t^n E[z^{X_n}], \quad 0 \leq z \leq 1, \quad 0 \leq t < 1. \quad (2.26)$$

In order to recover the classical expression for this function from (2.23), it is convenient to introduce the generating functions

$$R^*(t) = \sum_{n=0}^{\infty} R(n)t^n \quad (2.27)$$

and

$$G^*(y, t) = \sum_{n=0}^{\infty} g(y, n)t^n \quad (2.28)$$

These generating functions are expressed in term of the root function  $\xi \rightarrow Z(\xi)$  of Lemma 4 through the relations

$$R^*(t) = \frac{1}{1 - Z(t)} \quad (2.29)$$

and

$$G^*(y, t) = 1 + \frac{tZ(y) - yZ(t)}{y - t} \quad (2.30)$$

defined for  $0 \leq t < 1$ . The derivation of (2.29) is immediate from (2.21), while the derivation of (2.30) from (2.22) is established in Section 3. The well-known formula for  $N^*(z, t)$  given in [8, Eqn. (59), p. 70] is now directly recovered from these relations.

**Corollary 8.** Assume  $\rho < 1$  and  $\Xi = 0$ . For all  $0 < z \leq 1$ , the relation

$$N^*(z, t) = \frac{z(1 - Z(t)) - (1 - z)ta(z)}{(1 - Z(t))(z - ta(z))} \quad (2.31)$$

holds with  $Z$  as defined in Lemma 4.

**Proof.** Taking the generating function in  $n$  of both sides of Equation (2.23) readily yields the relation

$$N^*(z, t) = R^*(t)G^*(\xi(z), t) \quad n = 0, 1, \dots \quad (2.32)$$

and (2.31) is now a direct consequence of (2.29) and (2.30).  $\square$

### 3. Derivation of (2.30)

It is plain from (2.22) that the relation

$$g(y, n) = E[y^{\tau_1 - n} I[\tau_1 > n]] = \sum_{k=n+1}^{\infty} f(k)y^{k-n} \quad (3.1)$$

holds for all  $n = 1, 2, \dots$  and  $0 < y \leq 1$ . With  $C$  denoting the unit circle in the complex plane, the integral representation

$$f(k) = \frac{1}{2i\pi} \int_C \frac{F^*(u)}{u^{k+1}} du \quad k = 0, 1, \dots \quad (3.2)$$

can be used in (3.1) to readily yield the relation

$$g(y, n) = \frac{y}{2i\pi} \int_C \frac{F^*(u)}{u-y} \frac{du}{u^{n+1}} \quad n = 1, 2, \dots (3.3)$$

after standard algebraic manipulations. It is now plain from (2.22), (2.28) and (3.3) that

$$\begin{aligned} G^*(y, t) &= 1 + \frac{y}{2i\pi} \int_C \frac{F^*(u)}{u-y} \sum_{n=1}^{\infty} \frac{t^n}{u} \frac{du}{u} \\ &= 1 + \frac{yt}{2i\pi} \int_C \frac{F^*(u)}{(u-y)u(u-t)} du \end{aligned} \quad (3.4)$$

where the passage from the first to the second equality follows from standard facts on geometric series. Finally, substitute the decomposition

$$\frac{1}{(u-y)u(u-t)} = \frac{1}{(u-y)} \frac{1}{y(y-t)} + \frac{1}{(u-t)} \frac{1}{t(t-y)} + \frac{1}{uty} \quad (3.5)$$

in the last expression of (3.4), and note by a straightforward application of Cauchy's Formula that

$$G^*(y, t) = 1 + \frac{tF^*(y) - yF^*(t)}{y-t} + F^*(0). \quad (3.6)$$

The identity (2.14) and the fact that  $Z(0) = 0$  now imply that

$$G^*(y, t) = 1 + \frac{tZ(y) - yZ(t)}{y-t} \quad (3.7)$$

and (2.31) is now obtained by some elementary algebra after using this last expression in (2.32).  $\square$

Observe that  $N^*(z, t)$  also admits an integral representation in the form

$$N^*(z, t) = \frac{1}{1-Z(t)} \left[ 1 + \frac{\xi(z)t}{2i\pi} \int_C \frac{Z(u)}{(u-\xi(z))u(u-t)} du \right], \quad (3.8)$$

which follows from (2.32) and (3.4).

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