



# Some new families of finite elements for the Stokes equations

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## SOME NEW FAMILIES OF FINITE ELEMENTS FOR THE STOKES EQUATIONS

**Rolf STENBERG**

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# **SOME NEW FAMILIES OF FINITE ELEMENTS FOR THE STOKES EQUATIONS**

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*Abstract. We introduce a way of using the mixed finite element families of Raviart-Thomas-Nedelec [13,14] and Brezzi et al. [5,6,7] for constructing stable and optimally convergent discretizations for the Stokes problem.*

## **QUELQUES NOUVELLES FAMILLES D'ELEMENTS FINIS POUR LES EQUATIONS DE STOKES**

*Résumé. Nous introduisons une méthode utilisant les familles d'éléments finis mixtes de Raviart-Thomas-Nedelec [13,14] et Brezzi et al. [5,6,7] pour discrétiser le problème de Stokes. Nous montrons la stabilité et la convergence optimale de ces méthodes.*

## 1. INTRODUCTION

Recently there has been introduced a number of new mixed finite element spaces for the approximation of second order elliptic problems [5,6,7]. These new methods are of the same type as the methods in the classical Raviart-Thomas-Nedelec families [13,14], but they differ in that they are more accurate for the same computational cost.

One of the reasons for the increased interest in this field is the success of these methods in certain applications. For some problems in geophysics and semiconductor physics they have been shown to be more efficient than more conventional "displacement methods"; cf. [8,11,19] and the references therein.

In this note we will consider the application of these spaces for the approximation of the Stokes equations: Find the velocity  $\mathbf{u}$  and the pressure  $p$  such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where for simplicity  $\Omega \subset \mathbb{R}^N$ ,  $N = 2, 3$ , is assumed to be a bounded polygonal or polyhedral domain.

Let us assume that  $\mathbf{f} \in L^2(\Omega)^N$  and  $\mathbf{u}_0 \in H^{1/2}(\partial\Omega)^N$  with

$$\int_{\partial\Omega} \mathbf{u}_0 \cdot \mathbf{n} \, ds = 0$$

so that there is a unique solution to (1.1) (as usual the pressure is normalized to have a zero mean value over  $\Omega$ ).

When deriving optimal  $L^2$ -estimates for the velocity we will need the usual regularity assumption for the solution to (1.1) when  $\mathbf{u}_0 = 0$

$$\|\mathbf{u}\|_2 + \|p\|_1 \leq \|\mathbf{f}\|_0. \tag{1.2}$$

Below we will show that the spaces in [5,6,7,13,14] can be used as building blocks for producing "Taylor-Hood" type methods for the Stokes problem, i.e. methods with a continuous approximation for the pressure.

Let us here also mention that we in [16] used the same basic finite element spaces for the construction of a family for the discretization of the equations of linear elasticity. The estimates of [16] are uniformly valid with respect to the Poisson ratio, and hence a trivial notational change in the constitutive equation gives yet another family for discretizing Stokes problem.

## 2. THE FINITE ELEMENT FAMILIES

We will give the construction for the triangular and tetrahedral families of [5,7]. From the presentation it will be evident that the same construction can be done for the triangular and tetrahedral Raviart-Thomas-Nedelec elements [13,14], and also for all the rectangular and cubic elements of [5,6,7,13,14] except the lowest order methods with a piecewise constant approximation for the velocity. For the rectangular and cubic elements of [5,6,7] all estimates obtained are, however, not optimal.

In order to discretize (1.1) we write it as the system

$$\begin{aligned} \sigma - \nabla \mathbf{u} &= 0 & \text{in } \Omega \\ -\operatorname{div} \sigma + \nabla p &= \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_0 & \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where  $\sigma = \{\sigma_{ij}\}$ ,  $\sigma_{ij} = \partial_j u_i$ ,  $i, j = 1, \dots, N$ , and

$$(\operatorname{div} \sigma)_i = \sum_{j=1}^N \partial_j \sigma_{ij}, \quad i = 1, \dots, N.$$

(Since no confusion can arise we will use the same notation for the scalar and vector valued divergence operator.)

Let  $\mathcal{C}_h$  be the partitioning of  $\bar{\Omega}$  into closed triangles or tetrahedrons. The elements of  $\mathcal{C}_h$  are assumed to be regular in the usual sense (cf. [10]). The quasiuniformity of the mesh will not be assumed.

Our finite element approximation of (2.1) is now defined as: Find  $(\sigma_h, \mathbf{u}_h, p_h) \in H_h \times \mathbf{V}_h \times P_h$  such that

$$\begin{aligned} (\sigma_h, \tau) + (\operatorname{div} \tau, \mathbf{u}_h) &= \langle \mathbf{u}_0, \tau \cdot \mathbf{n} \rangle, \quad \tau \in H_h, \\ -(\operatorname{div} \sigma_h, \mathbf{v}) + (\nabla p_h, \mathbf{v}) &= (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h, \\ (\mathbf{u}_h, \nabla q) &= 0, \quad q \in P_h, \end{aligned} \quad (2.2)$$

where for the index  $k \geq 1$  the finite element spaces are defined through

$$H_h = \{ \tau \in H(\operatorname{div}; \Omega) \mid \tau|_K \in P_k(K)^{N \times N}, \quad K \in \mathcal{C}_h \}, \quad (2.3a)$$

$$\mathbf{V}_h = \{ \mathbf{v} \in L^2(\Omega)^N \mid \mathbf{v}|_K \in P_{k-1}(K)^N, \quad K \in \mathcal{C}_h \}, \quad (2.3b)$$

$$P_h = \{ p \in C(\Omega) \cap L_0^2(\Omega) \mid p|_K \in P_k(K), \quad K \in \mathcal{C}_h \}. \quad (2.3c)$$

As usual  $(\cdot, \cdot)$  denotes the inner product in  $L^2(\Omega)^N$  or  $L^2(\Omega)^{N \times N}$ , whereas  $\langle \cdot, \cdot \rangle$  stands for that in  $L^2(\partial\Omega)^N$ .  $L_0^2(\Omega)$  stands for the subspace of  $L^2(\Omega)$  consisting of functions with zero mean value over  $\Omega$ . We point out that for the functions in  $H_h$  the assumption  $\tau \in H(\operatorname{div}; \Omega)$ , where

$$H(\operatorname{div}; \Omega) = \{ \tau \in L^2(\Omega)^{N \times N} \mid \operatorname{div} \tau \in L^2(\Omega)^N \},$$

is equivalent to the continuity of the normal component of  $\tau$  along inter element boundaries. We also note that the spaces (2.3a), (2.3b) are  $N$  copies of the spaces of Brezzi et al. [5,7].

The finite element method (2.3) gives a poor approximation to the velocity, but it is possible to construct a considerably better approximation by various *Element-by-Element Postprocessing Techniques* [1,5,7,17]. Here we will consider a slight variation of a method introduced by us in [17]. We define

$$\mathbf{V}_h^* = \{ \mathbf{v} \in L^2(\Omega)^N \mid \mathbf{v}|_K \in P_{k+1}(K)^N, \quad K \in \mathcal{C}_h \}, \quad (2.4)$$

and calculate a new approximation  $\mathbf{u}_h^* \in \mathbf{V}_h^*$  to the velocity separately on each  $K \in \mathcal{C}_h$  by solving the following problem

$$\begin{aligned} (\nabla \mathbf{u}_h^*, \nabla \mathbf{v})_K &= (\sigma_h, \nabla \mathbf{v})_K, \quad \mathbf{v} \in (I - Q_h) \mathbf{V}_{h|K}^*, \\ Q_h \mathbf{u}_{h|K}^* &= \mathbf{u}_{h|K}, \end{aligned} \quad (2.5)$$

where  $Q_h : L^2(\Omega)^N \rightarrow \mathbf{V}_h^*$  denotes the  $L^2$ -projection and  $(\cdot, \cdot)_K$  stands for the inner product in  $L^2(K)^{N \times N}$ .

REMARK. We note that the postprocessing is performed separately for each component of  $\mathbf{u}$  and that the local stiffness matrices to invert on each element are the same for all components. ■

The analysis technique to be utilized in this paper is the one based on mesh dependent norms, first introduced in [2] for certain mixed approximations of the biharmonic equation and later adapted for mixed methods for second order problems [15]. The norms to be used are the following

$$\begin{aligned} \|\sigma\|_{0,h}^2 &= \|\sigma\|_0^2 + \sum_{T \in \Gamma_h} h_T \int_T |\sigma \cdot \mathbf{n}|^2 ds, \\ \text{for } \sigma &\in L^2(\Omega)^{N \times N} \text{ with } \sigma \cdot \mathbf{n} \in L^2(T)^N, \quad T \in \Gamma_h, \end{aligned} \quad (2.6)$$

$$\begin{aligned} \|\mathbf{u}\|_{1,h}^2 &= \sum_{K \in \mathcal{C}_h} \|\nabla \mathbf{u}\|_{0,K}^2 + \sum_{T \in \Gamma_h} h_T^{-1} \int_T |[\mathbf{u}]|^2 ds \\ \text{for } \mathbf{u} &\in L^2(\Omega)^N \text{ with } \mathbf{u}|_K \in H^1(K)^N, \quad K \in \mathcal{C}_h, \end{aligned} \quad (2.7)$$

and

$$\|\sigma, \mathbf{u}\|_h^2 = \|\sigma\|_{0,h}^2 + \|\mathbf{u}\|_{1,h}^2.$$

Here  $T$  stands for a side of an element,  $\mathbf{n}$  is the normal to  $T$  and  $\Gamma_h$  denotes the collection of the sides of the elements of  $\mathcal{C}_h$ . For element sides in the interior of  $\Omega$   $[\mathbf{u}]$  denotes the jump in  $\mathbf{u}$  whereas for sides on  $\partial\Omega$  it denotes the value of  $\mathbf{u}$ . We note that an integration by part on each  $K \in \mathcal{C}_h$  yields

$$(\operatorname{div} \sigma, \mathbf{u}) \leq \|\sigma\|_{0,h} \|\mathbf{u}\|_{1,h} \quad (2.8)$$

for each  $\sigma$  and  $\mathbf{u}$

ments (cf. [2]) gives the interpolation estimates

$$\inf_{\tau \in H_h} \|\sigma - \tau\|_{0,h} \leq Ch^r |\sigma|_r, \quad 1/2 < r \leq k+1, \quad (2.9)$$

and

$$\inf_{\mathbf{v} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_{1,h} \leq Ch^{s-1} |\mathbf{u}|_s, \quad 1 \leq s \leq k. \quad (2.10)$$

In order to see that the problem (2.2) can be analyzed with Brezzi's theory of saddle-point problems [3] we write it in the following way: Find  $(\sigma_h, \mathbf{u}_h, p_h) \in H_h \times \mathbf{V}_h \times P_h$  such that

$$\begin{aligned} a(\sigma_h, \mathbf{u}_h; \tau, \mathbf{v}) + b(\tau, \mathbf{v}; p_h) &= (\mathbf{f}, \mathbf{v}) + \langle \mathbf{u}_0, \tau \cdot \mathbf{n} \rangle, \quad (\tau, \mathbf{v}) \in H_h \times \mathbf{V}_h \\ b(\sigma_h, \mathbf{u}_h; q) &= 0, \quad q \in P_h, \end{aligned} \quad (2.11)$$

where

$$a(\sigma, \mathbf{u}; \tau, \mathbf{v}) = (\sigma, \tau) + (\operatorname{div} \tau, \mathbf{u}) - (\operatorname{div} \sigma, \mathbf{v})$$

and

$$b(\tau, \mathbf{v}; p) = (\mathbf{v}, \nabla p).$$

Hence, the analysis now consist of the verification of the two following lemmas.

**LEMMA 2.1.** *There is a positive constant  $C$  such that*

$$\sup_{\substack{0 \neq (\sigma, \mathbf{u}) \in \\ H_h \times \mathbf{V}_h}} \frac{a(\sigma, \mathbf{u}; \tau, \mathbf{v})}{\|(\sigma, \mathbf{u})\|_h} \geq C \|\tau, \mathbf{v}\|_h \quad (\tau, \mathbf{v}) \in H_h \times \mathbf{V}_h.$$

**PROOF:** This inequality states the stability of the discretization of the (vector) Laplace operator by the finite element spaces (2.3). This is a well known fact [5,7] even if it is usually not stated in this form. Let us therefore sketch the proof.

By applying the theory of Brezzi [3] we note that the assertion follows from the the following stability estimate

$$\sup_{\substack{\tau \in H_h \\ \|\tau\|_{0,h}=1}} (\operatorname{div} \tau, \mathbf{u}) \geq C \|\mathbf{u}\|_{1,h} \quad \mathbf{u} \in \mathbf{V}_h.$$



To prove this it is sufficient to take  $\tau$  in the Raviart-Thomas-Nedelec subspace  $\tilde{H}_h \subset H_h$ , defined through

$$\tilde{H}_h = \{ \sigma \in H(\text{div}; \Omega) \mid (\sigma_{i1}, \dots, \sigma_{iN})|_K \in P_{k-1}(K)^N \oplus \tilde{P}_{k-1}(K)\mathbf{x}, \quad K \in \mathcal{C}_h, \quad i = 1, \dots, N \},$$

where  $\tilde{P}_{k-1}(K)$  is the space of homogeneous polynomials of degree  $k-1$ . In [13,14] it is shown that for the degrees of freedom for  $\tau \in \tilde{H}_h$  one can choose

$$\begin{aligned} (i) \quad & \langle \tau \cdot \mathbf{n}, \mathbf{z} \rangle_T, \quad \mathbf{z} \in P_k(T)^N, \quad T \subset \partial K, \\ (ii) \quad & (\tau, \nabla \mathbf{w})_K, \quad \mathbf{w} \in P_{k-1}(K)^N, \end{aligned}$$

for each  $K \in \mathcal{C}_h$ .

Now an integration by parts on each  $K \in \mathcal{C}_h$  yields

$$(\text{div } \tau, \mathbf{u}) = \sum_{K \in \mathcal{C}_h} -(\tau, \nabla \mathbf{u})_K + \sum_{T \in \Gamma_h} \langle \tau \cdot \mathbf{n}, [\mathbf{u}] \rangle_T.$$

Hence, since  $([\mathbf{u}])|_T \in P_k(T)^N$ ,  $T \in \Gamma_h$  and  $\mathbf{u} \in P_{k-1}(K)^N$  for  $K \in \mathcal{C}_h$ , we can choose  $\tau \in \tilde{H}_h$  such that

$$\begin{aligned} (i) \quad & \tau \cdot \mathbf{n}|_T = h_T^{-1/2}([\mathbf{u}])|_T, \quad T \in \Gamma_h, \\ (ii) \quad & (\tau, \nabla \mathbf{u})_K = \|\nabla \mathbf{u}\|_{0,K}^2, \quad K \in \mathcal{C}_h. \end{aligned}$$

This gives

$$(\text{div } \tau, \mathbf{u}) = \|\mathbf{u}\|_{1,h}^2.$$

By scaling (cf. [2,14]) one proves that

$$\|\tau\|_{0,h} \leq C \|\mathbf{u}\|_{1,h},$$

which verifies the assertion.  $\blacksquare$

LEMMA 2.2. *There is a positive constant  $C$  such that*

$$\sup_{\substack{0 \neq (\sigma, \mathbf{u}) \in \\ H_h \times \mathbf{V}_h}} \frac{b(\sigma, \mathbf{u}; q)}{\|(\sigma, \mathbf{u})\|_h} \geq C \|q\|_0 \quad q \in P_h.$$

PROOF: Since for  $q \in P_h$  it holds  $\nabla q \in \mathbf{V}_h$ , we can choose  $\mathbf{u}$  through

$$\mathbf{u}|_K = h_K^2 \nabla q|_K, \quad K \in \mathcal{C}_h.$$

With  $\sigma = 0$  this gives

$$b(\sigma, \mathbf{u}; q) = \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2. \quad (2.12)$$

Scaling arguments yield

$$\|\mathbf{u}\|_{1,h}^2 \leq C \sum_{K \in \mathcal{C}_h} h_K^{-2} \|\mathbf{u}\|_{0,K}^2 = C \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2. \quad (2.13)$$

By combining (2.12) and (2.13) we get

$$\sup_{\substack{\sigma \neq (\sigma, \mathbf{u}) \in \\ \mathbf{H}_h \times \mathbf{V}_h}} \frac{b(\sigma, \mathbf{u}; q)}{\|\sigma, \mathbf{u}\|_h} \geq C \left( \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla q\|_{0,K}^2 \right)^{1/2}.$$

The assertion now follows from an argument by Verfürth [18, Proposition 3.3]. We remark that by using the above locally weighted norm for the pressure, the quasiuniformity assumption (i.e.  $h_K \geq Ch \ \forall K \in \mathcal{C}_h$ ) of [18] can be avoided. ■

We now get the following error estimates.

**THEOREM 2.1.** *For the solution  $\mathbf{u}_h^* \in \mathbf{V}_h^*$  and  $p_h \in P_h$  to (2.5) and (2.2) we have*

$$\|\mathbf{u} - \mathbf{u}_h^*\|_{1,h} + \|p - p_h\|_0 \leq Ch^{k+1}(|\mathbf{u}|_{k+2} + |p|_{k+1}).$$

Moreover, if (1.2) is valid we have

$$\|\mathbf{u} - \mathbf{u}_h^*\|_0 \leq Ch^{k+2}(|\mathbf{u}|_{k+2} + |p|_{k+1}) \quad \text{for } k \geq 2, \quad (2.14)$$

$$\|\mathbf{u} - \mathbf{u}_h^*\|_0 \leq Ch^2(|\mathbf{u}|_3 + |p|_2) \quad \text{for } k = 1.$$

If in addition  $\mathbf{f} \in \mathbf{V}_h$ , then (2.14) is also valid for  $k = 1$ .

PROOF: Let us split the proof into four steps.

Step 1. Let us prove the estimate

$$\|\sigma - \sigma_h\|_{0,h} + \|\mathbf{u}_h - Q_h \mathbf{u}\|_{1,h} + \|p - p_h\|_0 \leq Ch^{k+1}(|\sigma|_{k+1} + |p|_{k+1}). \quad (2.15)$$

Lemmas 2.1 and 2.2 implies the existence of a triple  $(\tau, \mathbf{v}, q) \in H_h \times \mathbf{V}_h \times P_h$  such that

$$\|\tau\|_{0,h} + \|\mathbf{v}\|_{1,h} + \|q\| \leq C \quad (2.16)$$

and

$$\begin{aligned} & \|\sigma_h - \tilde{\sigma}\|_{0,h} + \|\mathbf{u}_h - Q_h \mathbf{u}\|_{1,h} + \|p_h - \tilde{p}\| \\ & \leq a(\sigma_h - \tilde{\sigma}, \mathbf{u}_h - Q_h \mathbf{u}; \tau, \mathbf{v}) + b(\tau, \mathbf{v}; p_h - \tilde{p}) + b(\sigma_h - \tilde{\sigma}, \mathbf{u}_h - Q_h \mathbf{u}; q), \end{aligned} \quad (2.17)$$

where  $\tilde{\sigma}$  and  $\tilde{p}$  are the interpolants to  $\sigma$  and  $p$ , respectively. Using (2.1) and (2.2) we get

$$\begin{aligned} & a(\sigma_h - \tilde{\sigma}, \mathbf{u}_h - Q_h \mathbf{u}; \tau, \mathbf{v}) + b(\tau, \mathbf{v}; p_h - \tilde{p}) + b(\sigma_h - \tilde{\sigma}, \mathbf{u}_h - Q_h \mathbf{u}; q) \\ & = a(\sigma - \tilde{\sigma}, \mathbf{u} - Q_h \mathbf{u}; \tau, \mathbf{v}) + b(\tau, \mathbf{v}; p - \tilde{p}) + b(\sigma - \tilde{\sigma}, \mathbf{u} - Q_h \mathbf{u}; q) \\ & = (\sigma - \tilde{\sigma}, \tau) + (\mathbf{u} - Q_h \mathbf{u}, \operatorname{div} \tau) - (\mathbf{v}, \operatorname{div} (\sigma - \tilde{\sigma})) \\ & \quad + (\mathbf{v}, \nabla(p - \tilde{p})) + (\mathbf{u} - Q_h \mathbf{u}, \nabla q). \end{aligned} \quad (2.18)$$

Let us now estimate the terms on the right hand side of (2.18). Since  $\nabla q \in \mathbf{V}_h$  and  $\operatorname{div} \tau \in \mathbf{V}_h$  the definition of  $Q_h$  yields

$$(\mathbf{u} - Q_h \mathbf{u}, \operatorname{div} \tau) = 0 \quad (2.19)$$

and

$$(\mathbf{u} - Q_h \mathbf{u}, \nabla q) = 0. \quad (2.20)$$

Further, standard interpolation theory gives

$$(\sigma - \tilde{\sigma}, \tau) \leq \|\sigma - \tilde{\sigma}\|_0 \|\tau\|_0 \leq Ch^{k+1} |\sigma|_{k+1}, \quad (2.21)$$

and using (2.8) and (2.9) we get

$$(\mathbf{v}, \operatorname{div} (\sigma - \tilde{\sigma})) \leq C \|\sigma - \tilde{\sigma}\|_{0,h} \|\mathbf{v}\|_{1,h} \leq Ch^{k+1} |\sigma|_{k+1}. \quad (2.22)$$

Finally, the last term is estimated as follows

$$\begin{aligned}
(\mathbf{v}, \nabla(p - \tilde{p})) &= - \sum_{K \in \mathcal{C}_h} (\operatorname{div} \mathbf{v}, p - \tilde{p}) + \sum_{T \in \Gamma_h} \int_T ([\mathbf{v} \cdot \mathbf{n}]) (p - \tilde{p}) \, ds \\
&\leq \left( \sum_{K \in \mathcal{C}_h} \|\operatorname{div} \mathbf{v}\|_{0,K}^2 + \sum_{T \in \Gamma_h} h_T^{-1} \int_T |[\mathbf{v} \cdot \mathbf{n}]|^2 \, ds \right)^{1/2} \cdot \left( \|p - \tilde{p}\|_0^2 + \sum_{T \in \Gamma_h} h_T \int_T |p - \tilde{p}|^2 \, ds \right)^{1/2} \\
&\leq C \|\mathbf{v}\|_{1,h} (\|p - \tilde{p}\|_0^2 + \sum_{T \in \Gamma_h} h_T \int_T |p - \tilde{p}|^2 \, ds)^{1/2} \\
&\leq Ch^{k+1} |p|_{k+1},
\end{aligned} \tag{2.23}$$

where we in the last step used an interpolation estimate easily obtained by scaling; cf. [2, Lemma 3].

By collecting (2.17)-(2.23), using the triangle inequality and (2.9) we get (2.15) which contains the asserted estimate for the pressure.

*Step 2.* Let us derive an estimate for  $\|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_0$ .

We first note that  $Q_h \mathbf{u}_h^* = \mathbf{u}_h$ .

Suppose that (1.2) is valid so that the solution  $(\mathbf{z}, \gamma, q) \in H_0^1(\Omega)^N \times L^2(\Omega)^{N \times N} \times L_0^2(\Omega)$  to

$$\begin{aligned}
\gamma - \nabla \mathbf{z} &= 0 && \text{in } \Omega, \\
-\operatorname{div} \gamma + \nabla q &= \mathbf{u}_h - Q_h \mathbf{u} && \text{in } \Omega, \\
\operatorname{div} \mathbf{z} &= 0 && \text{in } \Omega, \\
\mathbf{z} &= 0 && \text{on } \partial\Omega,
\end{aligned} \tag{2.24}$$

satisfies

$$\|\mathbf{z}\|_2 + \|\gamma\|_1 + \|q\|_1 \leq \|\mathbf{u}_h - Q_h \mathbf{u}\|_0. \tag{2.25}$$

Let  $\tilde{\gamma}$  and  $\tilde{q}$  be the interpolants to  $\gamma$  and  $q$ . Using the relations

$$(\operatorname{div} \tilde{\gamma}, \mathbf{u}) = (\operatorname{div} \tilde{\gamma}, Q_h \mathbf{u}) \quad \text{and} \quad (\nabla \tilde{q}, \mathbf{u}) = (\nabla \tilde{q}, Q_h \mathbf{u})$$

standard arguments gives

$$\begin{aligned}
& \| \mathbf{u}_h - Q_h \mathbf{u} \|_0^2 \\
&= (\sigma - \sigma_h, \tilde{\gamma} - \gamma) + (\operatorname{div} (\sigma - \sigma_h), \mathbf{z} - Q_h \mathbf{z}) - (\nabla(p - p_h), \mathbf{z} - Q_h \mathbf{z}) \\
&+ (\operatorname{div} \gamma, Q_h \mathbf{u} - \mathbf{u}_h) - (\operatorname{div} \tilde{\gamma}, \mathbf{u} - \mathbf{u}_h) - (\nabla q, Q_h \mathbf{u} - \mathbf{u}_h) + (\nabla \tilde{q}, \mathbf{u} - \mathbf{u}_h) \\
&= (\sigma - \sigma_h, \tilde{\gamma} - \gamma) + (\operatorname{div} (\sigma - \sigma_h), \mathbf{z} - Q_h \mathbf{z}) - (\nabla(p - p_h), \mathbf{z} - Q_h \mathbf{z}) \\
&+ (\operatorname{div} (\gamma - \tilde{\gamma}), Q_h \mathbf{u} - \mathbf{u}_h) - (\nabla(q - \tilde{q}), Q_h \mathbf{u} - \mathbf{u}_h) \\
&\leq C(\|\sigma - \sigma_h\|_{0,h} + \|\mathbf{u}_h - Q_h \mathbf{u}\|_{1,h} + \|p - p_h\|_0 + (\sum_{T \in \Gamma_h} h_T \int_T |p - p_h|^2 ds)^{1/2}) \\
&\cdot (\|\gamma - \tilde{\gamma}\|_{0,h} + \|\mathbf{z} - Q_h \mathbf{z}\|_{1,h} + \|q - \tilde{q}\|_0 + (\sum_{T \in \Gamma_h} h_T \int_T |q - \tilde{q}|^2 ds)^{1/2}) \\
&\leq Ch^{k+1}(|\sigma|_{k+1} + |p|_{k+1}) \\
&\cdot (\|\gamma - \tilde{\gamma}\|_{0,h} + \|\mathbf{z} - Q_h \mathbf{z}\|_{1,h} + \|q - \tilde{q}\|_0 + (\sum_{T \in \Gamma_h} h_T \int_T |q - \tilde{q}|^2 ds)^{1/2}).
\end{aligned} \tag{2.26}$$

For  $k \geq 2$ , (2.9), (2.10) and the interpolation estimate of [2, Lemma 3] give

$$\begin{aligned}
& (\|\gamma - \tilde{\gamma}\|_{0,h} + \|\mathbf{z} - Q_h \mathbf{z}\|_{1,h} + \|q - \tilde{q}\|_0 + (\sum_{T \in \Gamma_h} h_T \int_T |q - \tilde{q}|^2 ds)^{1/2}) \\
&\leq Ch(|\gamma|_1 + |q|_1 + |\mathbf{z}|_2).
\end{aligned} \tag{2.27}$$

Recalling that  $Q_h \mathbf{u}_h^* = \mathbf{u}_h$ , (2.25)-(2.27) give

$$\|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_0 \leq Ch^{k+2}(|\sigma|_{k+1} + |p|_{k+1}) \quad \text{for } k \geq 2. \tag{2.28}$$

For the case  $k = 1$  and  $\mathbf{f} \in \mathbf{V}_h$  we have

$$\begin{aligned}
& (\operatorname{div} (\sigma - \sigma_h) + \nabla(p - p_h), \mathbf{z} - Q_h \mathbf{z}) \\
&= (\mathbf{f} - \operatorname{div} \sigma_h + \nabla p_h, \mathbf{z} - Q_h \mathbf{z}) = 0,
\end{aligned}$$

and (2.28) is still valid.

If  $\mathbf{f} \notin \mathbf{V}_h$  we only get

$$\|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_0 \leq Ch^2(|\sigma|_2 + |p|_2) \quad \text{for } k = 1. \tag{2.29}$$

*Step 3.* Next we estimate  $\|(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_0$  and  $\|(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_{1,h}$ .

First, since  $(I - Q_h|_K)\mathbf{v}|_K = 0$  for  $\mathbf{v} \in P_0(K)^N$ , we have

$$\|(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_{0,K} \leq Ch_K |(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})|_{1,K}. \quad (2.30)$$

Next we write

$$\begin{aligned} |(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})|_{1,K}^2 &= (\nabla(I - Q_h)(\mathbf{u}_h^* - \mathbf{u}), \nabla(I - Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K \\ &= (\nabla(\mathbf{u}_h^* - \mathbf{u}), \nabla(I - Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K - (\nabla Q_h(\mathbf{u}_h^* - \mathbf{u}), \nabla(I - Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K. \end{aligned} \quad (2.31)$$

Now, (2.1) and (2.5) gives

$$\begin{aligned} (\nabla(\mathbf{u}_h^* - \mathbf{u}), \nabla(I - Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K &= (\sigma_h - \sigma, \nabla(I - Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K \\ &\leq \|\sigma - \sigma_h\|_{0,K} |(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})|_{1,K}. \end{aligned} \quad (2.32)$$

Further, a local inverse inequality yields

$$\begin{aligned} &(\nabla Q_h(\mathbf{u}_h^* - \mathbf{u}), \nabla(I - Q_h)(\mathbf{u}_h^* - \mathbf{u}))_K \\ &\leq |Q_h(\mathbf{u}_h^* - \mathbf{u})|_{1,K} |(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})|_{1,K} \\ &\leq Ch_K^{-1} \|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_{0,K} |(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})|_{1,K}. \end{aligned} \quad (2.33)$$

Combining (2.30)-(2.33) and summing over all  $K \in \mathcal{C}_h$  gives

$$\|(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_0 \leq C(h\|\sigma - \sigma_h\|_0 + \|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_0). \quad (2.34)$$

Also, from (2.31),(2.32),(2.33) and the inequality (proven by scaling)

$$h_K^{-1} \int_{\partial K} |(I - Q_h)\mathbf{v}|^2 ds \leq C |\nabla(I - Q_h)\mathbf{v}|_{0,K}^2, \quad \mathbf{v} \in \mathbf{V}_{h|K}^*,$$

we get

$$\|(I - Q_h)(\mathbf{u}_h^* - \mathbf{u})\|_{1,h} \leq C(\|\sigma - \sigma_h\|_0 + \|Q_h(\mathbf{u}_h^* - \mathbf{u})\|_{1,h}). \quad (2.35)$$

*Step 4.* By combining (2.28),(2.29),(2.34) and (2.15) we obtain the asserted estimates for  $\|\mathbf{u} - \mathbf{u}_h^*\|_0$ , and that for  $\|\mathbf{u} - \mathbf{u}_h^*\|_{1,h}$  follows from (2.15) and (2.35). ■

### 3. The implementation by hybridization

Let us close the paper by briefly considering the solution of the discrete system (2.2). The method advocated in [1,5,6,7] is the Fraijs de Veubeke hybridization technique. In that the condition  $\operatorname{div} \sigma \in L^2(\Omega)^N$  for  $\sigma \in H_h$ , which is equivalent to the continuity of  $\sigma \cdot \mathbf{n}$  along interelement boundaries, is enforced by introducing Lagrange multipliers. More precisely, define

$$\mathbf{M}_h = \{ \mathbf{m} \mid \mathbf{m}|_T \in P_k(T)^N, \quad T \in \Gamma_h, \quad \mathbf{m}|_T = 0 \text{ for } T \subset \partial\Omega \},$$

and let  $\hat{H}_h$  be as in (2.3a) with the condition  $\operatorname{div} \sigma \in L^2(\Omega)^N$  dropped. The modified discretization now reads: Find  $(\sigma_h, \mathbf{u}_h, \mathbf{m}_h, p_h) \in \hat{H}_h \times \mathbf{V}_h \times \mathbf{M}_h \times P_h$  such that

$$\begin{aligned} (\sigma_h, \tau) + \sum_{K \in \mathcal{C}_h} \{ (\operatorname{div} \tau, \mathbf{u}_h)_K - \langle \tau \cdot \mathbf{n}_K, \mathbf{m}_h \rangle_{\partial K} \} &= \langle \tau \cdot \mathbf{n}, \mathbf{u}_0 \rangle, \quad \tau \in \hat{H}_h, \\ - \sum_{K \in \mathcal{C}_h} (\operatorname{div} \sigma_h, \mathbf{v})_K + (\nabla p_h, \mathbf{u}) &= (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h, \\ (\mathbf{u}_h, \nabla q) &= 0, \quad q \in P_h, \\ \sum_{K \in \mathcal{C}_h} \langle \sigma_h \cdot \mathbf{n}_K, \mathbf{l} \rangle_{\partial K} &= 0, \quad \mathbf{l} \in \mathbf{M}_h. \end{aligned} \tag{3.1}$$

Above  $\mathbf{n}_K$  stands for the unit outward normal to  $\partial K$ .

Clearly, the three first components of (2.2) and (3.1) coincide. The physical meaning of the new unknown  $\mathbf{m}_h$  is that of an approximation to the velocity  $\mathbf{u}$  along the inter element boundaries.

For  $\mathbf{m}_h$  we have the following error estimate which is needed if any of the postprocessing methods of [1,5,7] is used.

LEMMA 4.1. *For  $T \in \Gamma_h$  let  $I_T : L^2(T)^N \rightarrow \mathbf{M}_{h|T}$  be the  $L^2$ -projection. Then we have*

$$\|\mathbf{m}_h - I_T \mathbf{u}\|_{0,T} \leq C \{ h_K^{1/2} \|\sigma - \sigma_h\|_{0,K} + h_K^{-1/2} \|Q_h \mathbf{u} - \mathbf{u}_h\|_{0,K} \}.$$

for  $T \subset \partial K$ ,  $K \in \mathcal{C}_h$ . ■

For the proof of this result we refer to [7, Lemma 4.1].

The algebraic equations generated by (3.1) are of the form

$$\mathcal{A}\Sigma + \mathcal{B}U - \mathcal{C}M = 0,$$

$$-\mathcal{B}^T\Sigma + \mathcal{D}P = F,$$

$$\mathcal{C}^T\Sigma = 0,$$

$$\mathcal{D}^TU = 0,$$

where  $\Sigma$ ,  $U$ ,  $M$  and  $P$  are the vectors for the degrees of freedom for  $\sigma_h$ ,  $\mathbf{u}_h$ ,  $\mathbf{m}_h$  and  $p_h$ , respectively.

Now the matrix  $\mathcal{A}$  is positively definite and block diagonal. Hence,  $\Sigma$  can be eliminated separately on each element which gives the system

$$\hat{\mathcal{A}}\hat{U} + \hat{\mathcal{D}}P = \hat{F},$$

$$\hat{\mathcal{D}}^T\hat{U} = 0,$$

where  $\hat{U}^T = (U, M)^T$ ,  $\hat{\mathcal{D}}^T = (\mathcal{D}^T, 0)$ ,  $\hat{F}^T = (F, 0)^T$  and

$$\hat{\mathcal{A}} = \begin{pmatrix} \mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}^T & -\mathcal{B}^T\mathcal{A}^{-1}\mathcal{C} \\ -\mathcal{C}^T\mathcal{A}^{-1}\mathcal{B} & \mathcal{C}^T\mathcal{A}^{-1}\mathcal{C} \end{pmatrix},$$

which is positively definite. Hence the system is of the same form as for more conventional Taylor-Hood type discretizations of Stokes problem. Thus any of the methods for solving linear systems of this kind (cf. [12]) can be utilized.

Further,  $\mathcal{B}^T\mathcal{A}^{-1}\mathcal{B}^T$  is also positively definite and block diagonal, and hence  $M$  can be eliminated on each element separately. This results in a system of the form

$$\mathcal{E}M + \mathcal{G}P = G,$$

$$\mathcal{G}^TM - \mathcal{H}P = H,$$

where both  $\mathcal{E}$  and  $\mathcal{H}$  are symmetric and positively definite. Now the system has the same form as that of some recent stabilized mixed methods [4,9].



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