



HAL
open science

Performance analysis of a transfer line with unreliable machines and finite buffers

Xiaolan Xie

► **To cite this version:**

Xiaolan Xie. Performance analysis of a transfer line with unreliable machines and finite buffers. [Research Report] RR-1089, INRIA. 1989, pp.24. inria-00075470

HAL Id: inria-00075470

<https://inria.hal.science/inria-00075470>

Submitted on 24 May 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INRIA

UNITÉ DE RECHERCHE
INRIA-LORRAINE

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
BP 105
78153 Le Chesnay Cedex
France
Tél. (1) 39 63 55 11

Rapports de Recherche

N° 1089

Programme 5
Automatique, Productique,
Traitement du Signal et des Données

PERFORMANCE ANALYSIS OF A TRANSFER LINE WITH UNRELIABLE MACHINES AND FINITE BUFFERS

Xiaolan XIE

Septembre 1989



**Performance analysis of a transfer line
with unreliable machines and finite buffers**

**Evaluation des performances
d'une ligne composée de machines
séparées par des stocks tampons de capacité limitée**

XIE Xiaolan

Projet SAGEP, INRIA-LORRAINE
Campus Scientifique, BP239
Bd des Aiguillettes, 54506 Vandoeuvre, France

Abstract. - This paper examines the performance analysis of a transfer line with unreliable machines and finite buffers. All machines have the same processing times.

We propose a new decomposition method which decomposes a line into a set of two-machine lines. A sufficient set of equations is established to find performance measures such as production rate and average buffer levels. A simple iterative algorithm is then proposed to solve these equations. We also prove that the set of decomposition equations has a unique solution and that the proposed algorithm converges to that unique solution. Experimental results show that the proposed algorithm leads to a good solution.

Keywords : Performance Evaluation, Transfer Lines, Unreliable Machines, Finite Buffers, Markov Chain Model, Decomposition, Convergence, Solution Uniqueness

Résumé. - Nous nous intéressons à l'évaluation des performances d'une ligne de fabrication composée de machines séparées par des stocks intermédiaires. Les machines peuvent tomber en panne. Les temps de passage sont identiques sur toutes les machines.

Nous proposons une méthode approchée qui décompose la ligne en plusieurs lignes composées de deux-machines. Un ensemble suffisant d'équations est établi pour calculer les performances de la ligne. Un algorithme itératif et simple est proposé pour résoudre ces équations. Nous montrons que l'ensemble des équations a une et une seule solution et que l'algorithme proposé converge vers cette solution unique. Des résultats numériques montrent que cette méthode est capable de fournir une bonne estimation des performances.

Mots clés : Evaluation des performances, ligne de fabrication, pannes, en-cours, chaîne de Markov, décomposition, convergence, unicité de solution

1. INTRODUCTION

A transfer line consists of a series of machines separated by buffers. A part arrives from outside the system to the first machine. After being processed by the first machine, it queues in the first buffer waiting to be processed by the second machine. It continues in this manner through all machines and leaves the system after being processed by the last machine. Each part spends a fixed amount of time on each machine called the processing time. In this paper, we assume that the processing times are the same for all machines. Such a line is called a homogeneous line in Dallery, David and Xie [1988b].

The performance of a transfer line is highly influenced by machines failures. When a machine breaks down, it cannot produce at all during the period of time required to repair it. When a machine is under repair, the number of parts in the downstream buffer decreases while the number of parts in the upstream buffer increases. If this condition persists, the downstream buffer becomes empty and the upstream buffer becomes full. Consequently, the downstream machines are starved and the upstream machines are blocked. These machines are said to be forced down.

The machine failures may be either operation dependent or time dependent. Operation dependent failures may occur only while a machine is processing a part, while time dependent failures can occur even when it is forced down. An operation dependent model is more appropriate than a time dependent one for modeling an automated transfer line (cf Buzacott and Hanifin [1978]). Only operation dependent failures are considered in this paper.

Significant research has been done on the modeling and analysis of transfer lines. A common assumption is that both the time between failures and the time to repair are exponentially distributed. Even in the case of a two-machine transfer line, no exact solution has yet been derived. As a result, two approximate models have been introduced : the discrete model and the continuous model.

The discrete model was first introduced by Buzacott [1967]. It is restricted to homogeneous lines. Without loss of generality, the processing time is considered to be one time unit. A basic assumption is that machines can break down or be repaired only at the beginning of a period. This model has been solved by Buzacott [1967], and Gershwin and Schick [1983] in the case of two-machine transfer lines.

The continuous model was first proposed by Zimmermann [1956]. It approximates the

flow of discrete parts in a transfer line by a continuous flow. In this model, the quantity of material in a buffer is a real number ranging from zero to the buffer capacity. When a machine is operational and is neither starved or blocked, it transfers material from the upstream buffer to the downstream buffer continuously at a constant rate. This model has been solved independently by Gershwin and Schick [1980] and Dubois and Forestier [1982] in the case of two-machine transfer lines.

Further work has been devoted to the analysis of longer transfer lines. Gershwin and Schick [1983] have established equations for solving a general transfer line of the discrete model. However, it seems impossible to solve it. An approximate method was proposed by Gershwin [1987] for the analysis of the discrete model of long lines. This method was based on the decomposition of a line into a set of two-machine lines. The performance parameters can be efficiently computed using the algorithm proposed by Dallery, David and Xie [1988a]. A similar decomposition technique has been proposed by Dallery, David and Xie [1988b] for the continuous model. The advantage of using the continuous model over the discrete model is that the equations involved in the decomposition method require fewer approximations.

This paper proposes a new decomposition method for performance analysis of a homogeneous line. A sufficient set of decomposition equations is established to solve the decomposition. An iterative algorithm is proposed to solve these equations. The proposed algorithm is simpler than the ones proposed earlier. Nevertheless, it provides comparable results.

Another major concern of this paper is to exhibit the uniqueness of the solution of the decomposition equations. The convergence of the proposed algorithm to this unique solution is also established. Convergence is critical for iterative procedures employed by most decomposition methods to obtain performance measures. It is hoped that the convergence proof presented in this paper will prove useful for other decomposition algorithms.

The paper is organized as follows. The continuous model and the decomposition algorithm are presented in section 2. In section 3, the convergence of the proposed algorithm to a solution of the decomposition equations is proved. In section 4, we show that the set of decomposition equations has a unique solution. Numerical examples are presented in section 5 and section 6 is a conclusion.

2. DECOMPOSITION ALGORITHM

Let us consider a transfer line consisting of a series of K machines (M_1, M_2, \dots, M_K) separated by $K-1$ buffers (B_1, B_2, \dots, B_{K-1}). A Part arrives from outside the system to machine M_1 . After being processed by machine M_1 , it queues in buffer B_1 . In this way, it is sequentially processed by machines M_1, M_2, \dots, M_K . It leaves the system after being processed by machine M_K .

The buffer capacities are respectively C_1, C_2, \dots, C_{K-1} . The processing times are deterministic. Moreover, we assume that the processing time is the same on all machines, and without loss of generality is considered to be one time unit. While a machine M_i is processing a part, it may fail. The time between failures is exponentially distributed with a mean of $1/\lambda_i$. The time necessary to repair a machine M_i is also exponentially distributed with a mean of $1/\mu_i$. Figure 1 illustrates a four-machine line.

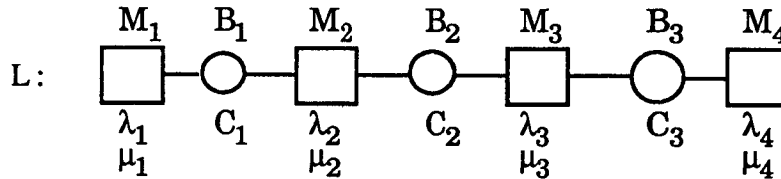


Figure 1: A four-machine transfer line

2.1 Continuous model

We approximate the flow of discrete parts in the transfer line by a continuous flow. Let $h_i(t) \in \mathbb{R}^+$ denote the quantity of material in buffer B_i at time t . Obviously, $h_i(t) \in [0, C_i]$. The behavior of the continuous model is described as follows :

1. Each machine is either operational or down. Let $\alpha_i(t) \in \{0,1\}$ indicate the state of the machine M_i at time t . $\alpha_i(t) = 1$ if M_i is **operational** and $\alpha_i(t) = 0$ if M_i is **down**.
2. A machine can be **starved** or **blocked**. Machine M_i is starved at time t if one of the upstream machines is down and all buffers in between this machine and machine M_i are empty, i.e.

$$\exists j < i \text{ such that } \alpha_j(t) = 0 \text{ and } h_k(t) = 0 \quad \forall k = j, \dots, i-1$$

Machine M_i is blocked at time t if one of the downstream machines is down and all buffers in between this machine and machine M_i are full, i.e.

$\exists j > i$ such that $\alpha_j(t) = 0$ and $h_k(t) = C_k \quad \forall k = i, \dots, j-1$

Let $s_i(t) \in \{0,1\}$ (resp. $b_i(t)$) indicate the starvation (resp. blocking) condition of machine M_i at time t . $s_i(t) = 0$ if M_i is starved and $s_i(t) = 1$ otherwise; $b_i(t) = 0$ if M_i is blocked and $b_i(t) = 1$ otherwise. A machine is **idle** if it is either starved or blocked.

3. A machine which is operational and not idle, is **busy**. When machine M_i is busy, it transfers material from its upstream buffer B_{i-1} to its downstream buffer B_i at a continuous rate of unity.

4. When a machine is busy, it may fail. The time between failures and the time to repair of machine M_i are exponentially distributed with mean $1/\lambda_i$ and $1/\mu_i$ respectively. That is :

$$\begin{aligned} \text{prob}[\alpha_i(t+dt)=0 \mid \alpha_i(t)=1, s_i(t)=1, b_i(t)=1] &= \lambda_i dt + o(dt) \\ \text{prob}[\alpha_i(t+dt)=1 \mid \alpha_i(t)=0] &= \mu_i dt + o(dt) \end{aligned}$$

5. It is assumed that there are always parts available at the input of the system and spaces available at the output of the system. In other words, machine M_1 is never starved, and machine M_K is never blocked.

2.2. Characteristics of the continuous model

In this subsection, we establish some basic relationships for steady-state performance parameters of the continuous model of the transfer line L . Let us first introduce the following quantities related to machine M_i :

$$I_i = \frac{\lambda_i}{\mu_i} \quad \text{and} \quad e_i = \frac{1}{1 + I_i} \quad (1)$$

where e_i is the isolated efficiency of machine M_i .

As only two parameters of λ_i , μ_i , I_i and e_i are independent, we choose I_i and μ_i as the elementary parameters of a machine, and the other two parameters can be obtained from equations (1).

We define the following performance parameters of the continuous model in steady state :

E_i : probability of machine M_i being busy (also called efficiency),
 ps_i : probability of machine M_i being starved,
 pb_i : probability of machine M_i being blocked.

Some relationships can be established between performance parameters. The first relation is related to the conservation of flow :

$$E_i = E_1 \quad \forall i = 2, 3, \dots, K \quad (2)$$

Let us consider the different performance parameters of machine M_i . They satisfy the following condition :

$$\begin{aligned}
 E_i &= \text{prob}[M_i \text{ is busy}] \\
 &= \text{prob}[\alpha_i(t)=1, s_i(t)=1, b_i(t)=1] \\
 &= \text{prob}[\alpha_i(t)=1 / s_i(t)=1, b_i(t)=1] \text{prob} [s_i(t)=1, b_i(t)=1].
 \end{aligned}$$

The quantity $\text{prob}[\alpha_i(t)=1 / s_i(t)=1, b_i(t)=1]$ is the isolated efficiency of machine M_i , i.e.

$$\text{prob}[\alpha_i(t)=1 / s_i(t)=1, b_i(t)=1] = e_i.$$

In the continuous model, the probability of a machine being starved and blocked simultaneously is 0. Thus, we have :

$$\begin{aligned}
 \text{prob}[s_i(t)=1, b_i(t)=1] &= 1 - \text{prob}[s_i(t)=0 \text{ or } b_i(t)=0] \\
 &= 1 - ps_i - pb_i.
 \end{aligned}$$

Combining the above relations, we obtain :

$$E_i = e_i (1 - ps_i - pb_i) \quad \forall i=1, 2, \dots, K \quad (3)$$

2.3. Decomposition

As mentioned in the introduction, the performance parameters of the continuous model can easily be computed in the case of a two-machine transfer line. However, it appears impossible to extend it to longer lines. We propose an approximate method which decomposes the K -machine line L into a set of $K-1$ two-machine lines $L(i)$; $i=1, \dots, K-1$. Each line $L(i)$ is composed of an upstream machine, $M_u(i)$, and a downstream machine, $M_d(i)$, separated by a buffer $B(i)$ (see figure 2).

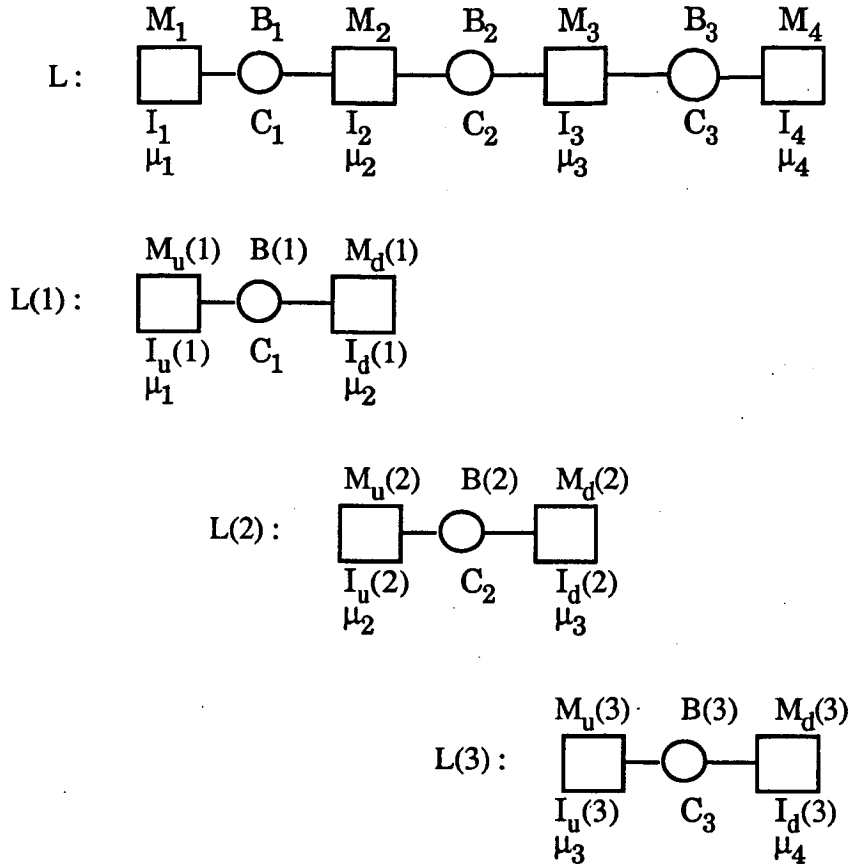


Figure 2 : Decomposition of the four-machine line into three two-machine lines

Machine $M_u(i)$ represents the portion of line L upstream of B_i and machine $M_d(i)$ represents the portion of line L downstream from B_i . The principle of the decomposition is that the behavior of the material flow in the buffer $B(i)$ closely matches that of the flow in buffer B_i of line L .

It seems natural to choose the processing times of both machines of line $L(i)$ to be one time unit. Furthermore, the capacity of buffer $B(i)$ is chosen to be equal to that of B_i , i.e. C_i . Unlike the decomposition techniques of Gershwin [1987] and Dallery, David and Xie [1988b], we choose the repair rates of machines $M_u(i)$ and $M_d(i-1)$ to be equal to the repair rate of machine M_i of line L , i.e. μ_i . The unknown parameters of the decomposition are $I_u(i)$ and $I_d(i)$ of each line $L(i)$.

The failure rate and the repair rate of the equivalent machine $M_u(i)$ (resp. $M_d(i)$) are $I_u(i) * \mu_i$ and μ_i (resp. $I_d(i) * \mu_{i+1}$ and μ_{i+1}). The isolated efficiency of these equivalent machines are given by :

$$e_u(i) = 1/(1+I_u(i)) \quad \text{and} \quad e_d(i) = 1/(1+I_d(i)). \quad (4)$$

For each line $L(i)$, we define the following performance parameters :

$E(i)$: probability of machine $M_d(i)$ being busy (efficiency of line $L(i)$),
 $p_s(i)$: probability of machine $M_d(i)$ being starved,
 $p_b(i)$: probability of machine $M_u(i)$ being blocked.

By similar arguments of equation (3), we can obtain :

$$E(i) = e_u(i) (1 - p_b(i)) \quad \forall i = 1, \dots, K-1 \quad (5)$$

$$E(i) = e_d(i) (1 - p_s(i)) \quad \forall i = 1, \dots, K-1 \quad (6)$$

We are now ready to establish equations for solving the decomposition. As stated earlier, the principle of the decomposition is to determine the parameters of each line $L(i)$ such that the behavior of material flow through buffer $B(i)$ closely matches that of the buffer B_i of line L . In particular, the efficiency of line $L(i)$ should be equal to that of machine M_{i+1} of line L ; the probability of machine $M_d(i)$ being starved should be equal to that of machine M_{i+1} of line L ; the probability of machine $M_u(i)$ being blocked should be equal to that of machine M_i of line L . That is :

$$E(i) = E_{i+1} \quad \forall i = 1, 2, \dots, K-1 \quad (7)$$

$$p_s(i) = p_{s_{i+1}} \quad \forall i = 1, 2, \dots, K-1 \quad (8)$$

$$p_b(i) = p_{b_i} \quad \forall i = 1, 2, \dots, K-1 \quad (9)$$

Combination of equations (2) and (7) gives :

$$E(i) = E(1) \quad \forall i = 2, \dots, K-1 \quad (10)$$

Combination of equations (3), (7), (8) and (9) gives :

$$E(i-1) = e_i(1 - p_s(i-1) - p_b(i)) \quad \forall i = 2, \dots, K \quad (11)$$

Using equations (5), (6), (10) and (11), it can be shown that :

$$I_d(i-1) + I_u(i) = 1/E(i-1) + I_i - 1 \quad \forall i = 2, \dots, K-1 \quad (12)$$

Finally, there are boundary conditions :

$$I_u(1) = I_1 \quad (13-1)$$

$$I_d(K-1) = I_K \quad (13-2)$$

The efficiencies $E(i)$ are non-linear functions of the $I_u(i)$ and $I_d(i)$. Expressions for $E(i)$ are given in Appendix A. Among equations (10), (12) and (13), there are $2(K-1)$ equations and $2(K-1)$ unknowns : $I_u(i)$ and $I_d(i)$.

2.4. Computation algorithm

Instead of solving the previous set of equations, we will transform it into an equivalent set of equations. First, equation (12) can be equivalently written as :

$$I_u(i) = 1/E(i-1) + I_i - I_d(i-1) - 1 \quad \forall i = 2, \dots, K-1 \quad (14)$$

Now, using equation (10) and by shifting i to $i+1$, equation (12) can be written as :

$$I_d(i) = 1/E(i+1) + I_{i+1} - I_u(i+1) - 1 \quad \forall i = 1, \dots, K-2 \quad (15)$$

This new set of equations : (14) and (15), is equivalent to the previous set of equations (10) and (12). With this new formulation, it now appears that the unknown parameter $I_u(i)$ of the upstream machine of line $L(i)$ can be obtained from the parameters of line $L(i-1)$ by means of equation (14). Similarly, the unknown parameter $I_d(i)$ of the downstream machine of line $L(i)$ can be obtained from the parameters of line $L(i+1)$ by means of equation (15). This new formulation leads to the following algorithm which iteratively calculates the unknown parameters.

Algorithm :

1. Initialize :

$$I_d(i) = I_{i+1} \quad \forall i = 1, \dots, K-1;$$

$$I_u(1) = I_1;$$

$t = 0$ where t is the iteration number index;

2. Repeat

Step 1. $t := t+1$;

Step 2. For $i = 2$ to $K-1$ do

2.1. Calculate $E(i-1)$ using updated values of $I_u(i-1)$ and $I_d(i-1)$;

2.2. Calculate $I_u(i)$ using (14);

Step 3. For $i=K-2$ to 1 do

2.1. Calculate $E(i+1)$ using updated values of $I_u(i+1)$ and $I_d(i+1)$;

2.2. Calculate $I_d(i)$ using (15);

until convergence of the unknown parameters.

3. CONVERGENCE

In this section, we prove the convergence of the proposed algorithm. Remark that the algorithm iteratively modifies the parameters of the two-machine lines. In order to study the variation of the efficiencies of these two-machine lines, we first exhibit some basic relations between the efficiency and the parameters of a two-machine line. Subsection 3.2 gives an equivalent version of the proposed algorithm which is more suitable for the convergence proof, and subsection 3.3 shows the convergence.

3.1. Properties of a two-machine line

Consider a transfer line composed of two machines separated by a buffer. Let C be the capacity of the buffer, and let λ_i and μ_i be the failure rate and repair rate of machine M_i respectively.

Let $g(I_1, \mu_1, I_2, \mu_2, C)$ be the efficiency of the line where the definition of I_1 and I_2 is given by equation (1). Its value is given in appendix A. Using this notation, the efficiencies $E(i)$ introduced in Section 2.3. can be expressed as follows :

$$E(i) = g(I_u(i), \mu_i, I_d(i), \mu_{i+1}, C_i).$$

We introduce a quantity $y(I_1, \mu_1, I_2, \mu_2, C)$ defined as follows :

$$y(I_1, \mu_1, I_2, \mu_2, C) := 1 / g(I_1, \mu_1, I_2, \mu_2, C).$$

The quantities $g(I_1, \mu_1, I_2, \mu_2, C)$ and $y(I_1, \mu_1, I_2, \mu_2, C)$ have the following properties :

Property 1 : $g(I_1, \mu_1, I_2, \mu_2, C) \leq \min\{e_1, e_2\}$.

Property 2 : $0 < \nabla_{I_1} y \leq 1$ and $0 < \nabla_{I_2} y \leq 1$.

Property 3 : For any $\Delta > 0$,

$$0 < y(I_1 + \Delta, \mu_1, I_2, \mu_2, C) - y(I_1, \mu_1, I_2, \mu_2, C) \leq \Delta,$$

$$0 < y(I_1, \mu_1, I_2 + \Delta, \mu_2, C) - y(I_1, \mu_1, I_2, \mu_2, C) \leq \Delta.$$

Property 4 : $-1 < \nabla_{I_1} g < 0$ and $-1 < \nabla_{I_2} g < 0$.

Proofs of these properties are given in appendix B.

3.2 An equivalent version of the computation algorithm

In this subsection, we give an equivalent version of the algorithm proposed in section 2.4. This new version is more suitable for the convergence proof. For each iteration t of the proposed algorithm, we define :

$I_u^t(k)$: the parameter $I_u(k)$ of line $L(k)$ obtained at step 2 when $i = k$,
 $I_d^t(k)$: the parameter $I_d(k)$ of line $L(k)$ obtained at step 3 when $i = k$,
 $E_f^t(k)$: the efficiency of line $L(k)$ with $I_u(k) = I_u^t(k)$ and $I_d(k) = I_d^{t-1}(k)$,
 $E_b^t(k)$: the efficiency of line $L(k)$ with $I_u(k) = I_u^t(k)$ and $I_d(k) = I_d^{t-1}(k)$.

By using the notation introduced in the section 3.1,

$E_f^t(k) := g(I_u^t(k), \mu_k, I_d^{t-1}(k), \mu_{k+1}, C_k),$
 $E_b^t(k) := g(I_u^t(k), \mu_k, I_d^t(k), \mu_{k+1}, C_k).$

At step 2, the equation (14) implies :

$$I_u^t(i) = 1/E_f^t(i-1) + I_i - I_d^{t-1}(i-1) - 1 \quad \forall i = 2, \dots, K-1 \quad (16)$$

At step 3, the equation (15) implies :

$$I_d^t(i) = 1/E_b^t(i+1) + I_{i+1} - I_u^t(i+1) - 1 \quad \forall i = 1, \dots, K-2 \quad (17)$$

The boundary conditions imply :

$I_u^t(1) = I_1,$
 $I_d^t(K-1) = I_K,$
 $E_f^{t+1}(1) = E_b^t(1),$
 $E_f^t(K-1) = E_b^t(K-1).$

Now, the algorithm proposed in section 2.4 can equivalently be written as follows :

Algorithm :

1. **Initialize** : $t = 0$ and $I_d^0(i) = I_{i+1} \quad \forall i = 1, \dots, K-1;$
 2. **Repeat**
 - Step 1. $t := t+1;$
 - Step 2. $I_u^t(1) = I_1$, and for $i = 2, 3, \dots, K-1$, calculate $I_u^t(i)$ using (16);
 - Step 3. $I_d^t(K-1) = I_K$, and for $i = K-2, K-3, \dots, 1$, calculate $I_d^t(i)$ using (17);
- until** convergence of the unknown parameters $I_u^t(i)$ and $I_d^t(i)$.

Before giving the convergence proof, we derive two equations which are quite useful in the following. From equations (16) and (17), we have

$$\begin{aligned} I_u^t(i) &= 1/E_f^t(i-1) + I_i - I_d^{t-1}(i-1) - 1, \\ I_d^{t-1}(i-1) &= 1/E_b^{t-1}(i) + I_i - I_u^{t-1}(i) - 1. \end{aligned}$$

Combining the two equations, we obtain

$$I_u^t(i) - I_u^{t-1}(i) = \frac{1}{E_f^t(i-1)} - \frac{1}{E_b^{t-1}(i)} \quad \forall t \geq 2, \forall i = 2, \dots, K-1 \quad (18)$$

Similarly, it can be shown that :

$$I_d^t(i) - I_d^{t-1}(i) = \frac{1}{E_b^t(i+1)} - \frac{1}{E_f^t(i)} \quad \forall t \geq 1, \forall i = 1, \dots, K-2 \quad (19)$$

3.3. Convergence of the proposed algorithm

The central result of this subsection is the following theorem which exhibits the convergence of the algorithm proposed in this paper.

Theorem 1 : $\forall k \in \{1, 2, \dots, K-1\}$, the quantities $I_u^t(k)$, $I_d^t(k)$, $E_f^t(k)$ and $E_b^t(k)$ converge to finite and positive values as t tends to infinity and

$$\lim_{t \rightarrow \infty} E_f^t(k) = \lim_{t \rightarrow \infty} E_b^t(k) = E$$

From this result and equations (16) and (17), we can easily derive the following corollary which shows that the proposed algorithm converges to a solution of the set of decomposition equations.

Corollary : The set of equations (10), (12) and (13) has at least one solution and $\{I_u^t(i), I_d^t(i) \forall i=1, \dots, K-1\}$ converges to one of its solutions.

The proof of Theorem 1 is based on the following three lemmas.

Lemma 1 : $I_u^t(k) \geq I_k$ and $I_d^t(k) \geq I_{k+1} \quad \forall t \geq 1$ and $\forall k \in \{1, 2, \dots, K-1\}$.

Lemma 2 :

a) $E_f^1(i-1) \geq E_f^1(i) \quad \forall i = 2, \dots, K-1;$

$$\begin{aligned}
\text{b) } E_f^t(i-1) &\geq E_f^t(i) \geq E_b^{t-1}(i) && \forall t \geq 2 \text{ and } \forall i = 2, \dots, K-1; \\
\text{c) } E_b^t(i+1) &\leq E_b^t(i) \leq E_f^t(i) && \forall i = 1, \dots, K-2.
\end{aligned}$$

Lemma 3 : $I_u^t(k) \leq 1/E_f^1(K-1) - 1$ and $I_d^t(k) \leq 1/E_f^1(K-1) - 1$, for $t \geq 1$ and $\forall k \in \{1, 2, \dots, K-1\}$.

Lemma 1 shows that the unknown parameters of the two-machine lines $I_u^t(k)$ and $I_d^t(k)$ obtained from equations (16) and (17) are always positive. Lemmas 1 and 3 show that the quantities $I_u^t(k)$ and $I_d^t(k)$ are bounded. Lemma 2 shows the dynamical behavior of the efficiencies of the set of two-machine lines in the proposed algorithm.

Lemma 1 can easily be derived from equation (16) and (17) by using Property 1 and proofs of lemmas 2 and 3 are given in appendix C. Now we apply these lemmas to prove Theorem 1.

Proof of Theorem 1 : From lemma 2 and equation (18), we have

$$I_u^t(i) - I_u^{t-1}(i) = \frac{1}{E_f^t(i-1)} - \frac{1}{E_b^{t-1}(i)} \leq 0 \quad \forall t \geq 2, \forall i = 2, \dots, K-1$$

which means that $I_u^t(k)$ decreases as t increases. Since $I_u^t(k)$ is bounded and positive, it converges to a finite positive value as t tends to infinity.

From lemma 2 and equation (19), we have

$$I_d^t(i) - I_d^{t-1}(i) = \frac{1}{E_b^t(i+1)} - \frac{1}{E_f^t(i)} \geq 0 \quad \forall t \geq 1, \forall i = 1, \dots, K-2$$

which means that $I_d^t(k)$ increases as t increases. Since $I_d^t(k)$ is bounded and positive, it converges to a finite positive value as t tends to infinity.

To prove the convergence of the quantities $E_f^t(k)$ and $E_b^t(k)$, we first examine the convergence of the quantities $E_f^t(1)$ and $E_b^t(1)$. From lemma 2, we have

$$E_f^{t+1}(1) = E_b^t(1) \geq E_f^{t+2}(1) = E_b^{t+1}(1) \quad \forall t \geq 1$$

which implies that the quantities $E_f^t(1)$ and $E_b^t(1)$ converge to the same value, i.e.

$$\lim_{t \rightarrow \infty} E_f^t(1) = \lim_{t \rightarrow \infty} E_b^t(1) = E.$$

We are now ready to examine the convergence of the quantities $E_f^t(2)$ and $E_b^t(2)$. From equation (18) with $i = 2$, the convergence of the quantity $I_u^t(2)$ leads to :

$$\lim_{t \rightarrow \infty} \left(\frac{1}{E_f^t(1)} - \frac{1}{E_b^{t-1}(2)} \right) = 0$$

which implies that the quantity $E_b^t(2)$ also converges and

$$\lim_{t \rightarrow \infty} E_b^t(2) = \lim_{t \rightarrow \infty} E_f^t(1) = E.$$

From lemma 2, we have

$$E_f^t(1) \geq E_f^t(2) \geq E_b^{t-1}(2)$$

which means that the quantity $E_f^t(2)$ also converges to E , i.e.

$$\lim_{t \rightarrow \infty} E_f^t(2) = E.$$

In the same manner, we can show that $\forall k \in \{3, \dots, K-1\}$, the quantities $E_f^t(k)$ and $E_b^t(k)$ also converge to the quantity E as t tends to infinity. *Q.E.D.*

4. UNIQUENESS OF THE SOLUTION

The previous section shows that there exists at least one solution to the equations (10), (12) and (13). In this section, we show that the solution is unique.

First, the equations (10), (12) and (13) can equivalently be rewritten as follows :

$$g(I_u(i), \mu_i, I_d(i), \mu_{i+1}, C_i) = E \quad \forall i = 1, \dots, K-1 \quad (20)$$

$$I_d(i-1) + I_u(i) = 1/E + I_i - 1 \quad \forall i = 2, \dots, K-1 \quad (21)$$

$$I_u(1) = I_1 \text{ and } I_d(K-1) = I_K \quad (22)$$

Theorem 2 : There exists a unique solution which satisfies the equations (20), (21) and (22).

Proof : The proof is given in two steps. First, we show that there exists a unique E . Then, we show that the parameters $I_u(i)$ and $I_d(i)$ are also unique.

1. Uniqueness of the line efficiency E

Suppose that the set of equations (20), (21) and (22) has two solutions $\{E1, I1_u(i), I1_d(i)\}$ and $\{E2, I2_u(i), I2_d(i)\}$ such that

$$E1 \geq E2. \quad (23)$$

Since $E1 = g(I1_u(1), \mu_1, I1_d(1), \mu_2, C_1)$ and $E2 = g(I2_u(1), \mu_1, I2_d(1), \mu_2, C_1)$, Property 3 implies :

$$I2_d(1) - I1_d(1) \geq 1/E2 - 1/E1. \quad (24)$$

Combining this relation and equation (21) with $i = 1$, we obtain

$$I2_u(2) - I1_u(2) \leq 0 \quad (25)$$

Since $E1 = g(I1_u(2), \mu_2, I1_d(2), \mu_3, C_2)$ and $E1 = g(I2_u(2), \mu_2, I2_d(2), \mu_3, C_2)$, considering relation (25) and properties of $g(\cdot)$ we can show that :

$$I2_d(2) - I1_d(2) \geq 0. \quad (26)$$

From relations (21) with $i = 2$, (25) and (26),

$$\begin{aligned} 1/E2 - 1/E1 &= y(I2_u(2), \mu_2, I2_d(2), \mu_3, C_2) - y(I1_u(2), \mu_2, I1_d(2), \mu_3, C_2) \\ &\leq y(I1_u(2), \mu_2, I2_d(2), \mu_3, C_2) - y(I1_u(2), \mu_2, I1_d(2), \mu_3, C_2) \\ &\leq I2_d(2) - I1_d(2) \end{aligned}$$

Similarly, we can show that

$$I2_d(i) - I1_d(i) \geq 1/E2 - 1/E1 \quad \text{for } i = 2, \dots, K-2, \quad (27)$$

$$I2_u(i) - I1_u(i) \leq 0 \quad \text{for } i = 2, \dots, K-1, \quad (28)$$

From relations (20) and (28) with $i = K-1$,

$$\begin{aligned} 1/E2 - 1/E1 &= y(I2_u(K-1), \mu_{K-1}, I2_d(K-1), \mu_K, C_{K-1}) - y(I1_u(K-1), \mu_2, I1_d(K-1), \mu_K, C_{K-1}) \\ &\leq I2_u(K-1) - I1_u(K-1) \leq 0 \end{aligned} \quad (29)$$

Combination of relations (23) and (29) gives :

$$E1 = E2$$

which implies the uniqueness of E.

2. Uniqueness of equivalent machine parameters

Since $E = g(I_1, \mu_1, I_d(1), \mu_2, C_1)$ and the function $g(\cdot)$ is strictly decreasing in $I_d(1)$, there exists a unique $I_d(1)$ which satisfies equation (20) when given E .

Given E and $I_d(1)$, there exists a unique $I_u(2)$ which satisfies equation (21).

In the same manner, we show that given the quantity E , there exists a unique set of $I_u(i)$ and $I_d(i)$ which satisfies equations (20) and (21). *Q.E.D.*

5. NUMERICAL RESULTS

Table 1 shows the simulation results of four numerical experiments. The efficiency obtained by the proposed algorithm is always quite close to that obtained by simulation.

	$\lambda_1 = \lambda_2 = \lambda_3 = 0.03$		$\mu_1 = \mu_2 = \mu_3 = 0.1$	
	(C1, C2)	(20, 20)	(100, 2)	(2, 100)
3-machine	simulation	0.6696	0.6382	0.6353
	approximation	0.6637, - 0.9%	0.6381, - 0.0%	0.6381, 0.4%
10-machine	$\lambda_1 = \dots = \lambda_{10} = 0.1 \quad \mu_1 = \dots = \mu_{10} = 0.1 \quad C1 = \dots = C9 = 5$			
	simulation : 0.2477		approximation : 0.2422, - 2.2%	

Table 1 : Efficiencies of four transfer lines

6. CONCLUSION

This paper presents a decomposition method for the performance analysis of transfer lines. A sufficient set of equations is established to solve these equations. An iterative algorithm is proposed to solve these equations. We show that the set of decomposition equations has a unique solution, and the proposed algorithm converges always to this unique solution.

Further work is necessary to test how good the approximation is. Also, it is hoped that the convergence proof presented in this paper will prove useful for the convergence analysis of other decomposition algorithms.

REFERENCES

- [1] J. A. Buzacott (1967), "Automatic Transfer Lines with Buffer Stocks," *International Journal of Production Research*, Vol. 6, 1967.
- [2] J. A. Buzacott and L. E. Hanifin (1978), "Models of Automatic Transfer Lines with Inventory Banks : a Review and Comparison," *AIIE Transactions*, Vol. 10, No. 2, 1978.
- [3] Y. Dallery, R. David and X. Xie (1988a), "An Efficient Algorithm for Analysis of Transfer Lines with Unreliable Machines and Finite Buffers," *IEE Transactions*, Vol. 20, No. 3, September 1988.
- [4] Y. Dallery, R. David and X. Xie (1988b), "Approximate Analysis of Transfer Lines with Unreliable Machines and Finite Buffers," to appear in *IEEE Transactions on Automatic Control*.
- [5] D. Dubois and J.P. Forestier (1982), "Productivité et en-cours moyens d'un ensemble de deux machines séparées par une zone de stockage," *RAIRO Automatique*, Vol. 16, No. 2, 1982.
- [6] S. B. Gershwin (1987), "An Efficient Decomposition Method for the Approximate Evaluation of Tandem Queues with Finite Storage Space and Blocking," *Operations Research*, Vol. 35, N° 2, March-April 1987.
- [7] S. B. Gershwin and I. C. Schick (1980), "Continuous Model of an Unreliable Two-Stage Material Flow System with a Finite Interstage Buffer," MIT Report, LIDS-R-1039, 1980.
- [8] S. B. Gershwin and I. C. Schick (1983), "Modeling and Analysis of Three-Stage Transfer Lines with Unreliable Machines and Finite Buffers," *Operations Research*, Vol. 31, No. 2, March-April 1983.
- [9] B. Zimmern (1956), "Etude de la propagation des arrêts aléatoires dans les chaînes de production," *Revue de Statistique Appliquées*, Vol. 4, pp. 85-104, 1956.

APPENDIX A

Performance measures of the continuous model of a two-machine homogeneous line

In this appendix, the formulas for calculating the performance parameters of the continuous model of a homogeneous two-machine line are given. These results are taken from Dubois and Forestier [1982]. Let λ_1 and μ_1 (λ_2 and μ_2) be the failure rate and the repair rate of the upstream (downstream) machine. Let C be the capacity of the buffer. Let I_1 and I_2 be defined as follows :

$$I_1 = \frac{\lambda_1}{\mu_1} \quad \text{and} \quad I_2 = \frac{\lambda_2}{\mu_2}$$

We distinguish two cases :

1) $I_1 \neq I_2$

The efficiency of the line is given by :

$$g(I_1, \mu_1, I_2, \mu_2, C) = \frac{1}{1 + \frac{I_2^2 e^{aC} - I_1^2}{I_2 e^{aC} - I_1}}$$

where

$$a = \mu_1 \mu_2 (I_2 - I_1) \left(\frac{1}{I_1 \mu_1 + I_2 \mu_2} + \frac{1}{\mu_1 + \mu_2} \right)$$

The average buffer level is given by :

$$Q(I_1, \mu_1, I_2, \mu_2, C) = \frac{\frac{I_1 I_2}{I_2 - I_1} \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} (1 - e^{aC}) + I_2 (1 + I_2) C e^{aC}}{I_2 (1 + I_2) e^{aC} - I_1 (1 + I_1)}$$

2) $I_1 = I_2$

The efficiency of the line is given by :

$$g(I, \mu_1, I, \mu_2, C) = \frac{(1 + I)C + \frac{\mu_1 + \mu_2}{\mu_1 \mu_2}}{(1 + I)^2 C + (1 + 2I) \frac{\mu_1 + \mu_2}{\mu_1 \mu_2}}$$

The average buffer level is given by :

$$Q(I, \mu_1, I, \mu_2, C) = \frac{(1+I)\frac{C}{2} + \frac{I\mu_1 + (1+I)\mu_2}{\mu_1\mu_2}}{(1+I)^2C + (1+2I)\frac{\mu_1 + \mu_2}{\mu_1\mu_2}} C$$

APPENDIX B

Properties of a two-machine line

Property 1 can easily be derived from equation (3) of subsection 2.2. Property 3 can easily be derived from Property 2 by using Taylor's expansion. Property 4 can be derived from Property 2 by considering that $g(I_1, \mu_1, I_2, \mu_2, C) < 1$.

Proof of Property 2 : The proof of $0 \leq \nabla_{I_1} y \leq 1$ only is given. The proof of $0 \leq \nabla_{I_2} y \leq 1$ is the similar. We distinguish two cases : $I_1 \neq I_2$ and $I_1 = I_2$.

Case $I_1 \neq I_2$

First, $y(I_1, \mu_1, I_2, \mu_2, C)$ and its partial derivative can be computed as follows :

$$y(I_1, \mu_1, I_2, \mu_2, C) = 1 + \frac{I_2^2 e^{aC} - I_1^2}{I_2 e^{aC} - I_1},$$

and

$$\frac{\partial y}{\partial I_1} = \frac{I_1 I_2 C e^{aC} \frac{\partial a}{\partial I_1} (I_1 - I_2) + I_2^2 e^{aC} + I_1^2 - 2I_1 I_2 e^{aC}}{(I_2 e^{aC} - I_1)^2}$$

where

$$\frac{\partial a}{\partial I_1} = - \left[\frac{\mu_1 \mu_2}{\mu_1 + \mu_2} + I_2 \frac{\mu_1 \mu_2 (\mu_1 + \mu_2)}{(I_1 \mu_1 + I_2 \mu_2)^2} \right]$$

1. Proof of $\nabla_{I_1} y \leq 1$

Since

$$\frac{\partial y}{\partial I_1} = 1 + \frac{I_2 e^{aC}}{(I_2 e^{aC} - I_1)^2} \left(I_1 C \frac{\partial a}{\partial I_1} (I_1 - I_2) + I_2 - I_2 e^{aC} \right),$$

it is equivalent to prove that

$$z(C) = I_1 C \frac{\partial a}{\partial I_1} (I_1 - I_2) + I_2 - I_2 e^{aC} \leq 0 \quad \forall C \geq 0.$$

The function $z(C)$ satisfies the following conditions :

$$z(0) = 0 \quad (B1)$$

$$\dot{z}(0) = - (I_1 - I_2)^2 \left(\frac{\mu_1 \mu_2}{\mu_1 + \mu_2} + I_2 \frac{\mu_1 \mu_2^2}{(I_1 \mu_1 + I_2 \mu_2)^2} \right) < 0 \quad (B2)$$

$$\frac{d^2 z(C)}{dC^2} = -a^2 I_2 e^{aC} < 0 \quad (B3)$$

Relations (B2) and (B3) imply that

$$dz(C)/dC < 0 \quad \forall C \geq 0 \quad (B4)$$

From (B1) and (B4), we obtain

$$z(C) \leq 0 \quad \forall C \geq 0$$

which implies that $\nabla_{I_1} y \leq 1$.

2. Proof of $\nabla_{I_1} y > 0$

Since

$$\frac{\partial y}{\partial I_1} = \frac{e^{aC}}{(I_2 e^{aC} - I_1)^2} \left(I_1 I_2 C \frac{\partial a}{\partial I_1} (I_1 - I_2) + I_2^2 + I_1^2 e^{-aC} - 2I_1 I_2 \right),$$

it is equivalent to show that

$$w(C) = I_1 I_2 C \frac{\partial a}{\partial I_1} (I_1 - I_2) + I_2^2 + I_1^2 e^{-aC} - 2I_1 I_2 \geq 0 \quad \forall C \geq 0.$$

The function $w(C)$ satisfies the following conditions :

$$w(0) = I_2^2 + I_1^2 - 2I_1 I_2 = (I_1 - I_2)^2 > 0 \quad (B5)$$

$$\dot{w}(0) = I_1 (I_2 - I_1)^2 \left[\frac{\mu_1 \mu_2}{\mu_1 + \mu_2} + \frac{\mu_1 \mu_2}{(I_1 \mu_1 + I_2 \mu_2)^2} (I_2 \mu_2 + \mu_1 (I_2 + I_1)) \right] > 0 \quad (B6)$$

$$\frac{d^2 w(C)}{dC^2} = a^2 I_1 e^{aC} > 0 \quad (B7)$$

Relations (B6) and (B7) imply that

$$dw(C)/dC > 0 \quad \forall C \geq 0 \quad (B8)$$

From (B5) and (B8), we obtain

$$w(C) > 0 \quad \forall C \geq 0$$

which implies that $\nabla_{I_1} y > 0$.

Case $I_1 = I_2 = I$

The partial derivative of $y(I_1, \mu_1, I_2, \mu_2, C)$ on I_1 exists and

$$\frac{\partial y}{\partial I_1} = \frac{1 + Ia_1 C + I^2 a_2 C + I^2 a_1^2 C^2 / 2}{(1 + Ia_1 C)^2}$$

where

$$a_1 = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \left(1 + \frac{1}{I}\right) \quad \text{and} \quad a_2 = \frac{\mu_1^2 \mu_2}{(I\mu_1 + I\mu_2)^2}$$

It is obvious that

$$\frac{\partial y}{\partial I_1} > 0.$$

$\nabla_{I_1} y$ can be rewritten as follows :

$$\frac{\partial y}{\partial I_1} = 1 - \frac{Ia_1 - I^2 a_2}{(1 + Ia_1 C)^2} C - \frac{I^2 a_1^2 C^2 / 2}{(1 + Ia_1 C)^2}$$

Since

$$Ia_1 - I^2 a_2 = \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2} (\mu_2 + I(\mu_1 + \mu_2)) > 0,$$

we have

$$\frac{\partial y}{\partial I_1} \leq 1 \quad \forall C \geq 0.$$

Q.E.D.

APPENDIX C
Proof of lemmas 2 and 3

Proof of lemma 2 : We show this lemma iteration by iteration.

1. Step 2 of the first iteration (t=1)

From equation (16), we have

$$I_u^1(2) = \frac{1}{E_f^1(1)} + I_2 - I_d^0(1) - 1 = \frac{1}{E_f^1(1)} - 1.$$

From Property 1, we obtain

$$\begin{aligned} E_f^1(2) &= g(I_u^1(2), \mu_2 I_d^0(2), \mu_3, C_2) \\ &\leq \frac{1}{1 + I_u^1(2)} \\ &= E_f^1(1). \end{aligned}$$

Similarly, we can show that

$$E_f^1(i-1) \geq E_f^1(i) \quad \forall i = 2, \dots, K-1.$$

2. Step 3 of the first iteration (t=1)

According to the boundary conditions,

$$E_b^1(K-1) = E_f^1(K-1).$$

From equation (19), we have

$$I_d^1(K-2) - I_d^0(K-2) = \frac{1}{E_b^1(K-1)} - \frac{1}{E_f^1(K-2)} \geq 0.$$

Since

$$\frac{1}{E_b^1(K-2)} - \frac{1}{E_f^1(K-2)} = y(I_u^1(K-2), \mu_{K-2}, I_d^1(K-2), \mu_{K-1}, C_{K-2}) - y(I_u^1(K-2), \mu_{K-2}, I_d^0(K-2), \mu_{K-1}, C_{K-2}),$$

Property 3 leads to :

$$0 \leq \frac{1}{E_b^1(K-2)} - \frac{1}{E_f^1(K-2)} \leq I_d^1(K-2) - I_d^0(K-2)$$

which implies :

$$E_b^t(K-1) \leq E_b^t(K-2) \leq E_f^t(K-2).$$

Similarly, we can show that relation 3(c) is true for $i \leq K-3$ and $t = 1$.

3. Step 2 of iteration 2

According to the boundary conditions,

$$E_f^2(1) = E_b^1(1)$$

From equation (18), we have

$$I_u^2(2) - I_u^1(2) = \frac{1}{E_f^2(1)} - \frac{1}{E_b^1(2)} \leq 0. \quad (C1)$$

Since

$$\frac{1}{E_f^2(2)} - \frac{1}{E_b^1(2)} = y(I_u^2(2), \mu_2, I_d^1(2), \mu_3, C_2) - y(I_u^1(2), \mu_2, I_d^1(2), \mu_3, C_2)$$

Property 3 implies :

$$0 \geq \frac{1}{E_f^2(2)} - \frac{1}{E_b^1(2)} \geq I_u^2(2) - I_u^1(2) \quad (C2)$$

Combining relations (C1) and (C2), we obtain

$$E_f^2(1) \geq E_f^2(2) \geq E_b^1(2).$$

Similarly, we can show that relation 3(b) is true for $i = 3, \dots, K-1$ with $t=2$.

4. Step 3 of iteration 2

The proof is similar to that of iteration 1.

In the same manner, we can show that relations 3(b) and 3(c) are also true for $t \geq 3$.

Q.E.D.

Proof of lemma 3 : We only examine the first relation. The proof of the second one is the same. From lemma 2,

$$E_f^t(k-1) \geq E_f^t(K-1) \geq E_f^1(K-1).$$

Using this relation and lemma 1, equation (16) implies :

$$\begin{aligned} I_u^t(k) &= 1/E_f^t(k-1) + I_k - I_d^{t-1}(k-1) - 1 \\ &\leq 1/E_f^1(K-1) - 1. \end{aligned}$$

Q.E.D.

