HAL
open science

Comparison of service disciplines in G/GI/m queues

Serguei Foss

To cite this version:

Serguei Foss. Comparison of service disciplines in G/GI/m queues. [Research Report] RR-1097, INRIA. 1989. inria-00075462

HAL Id: inria-00075462

https://inria.hal.science/inria-00075462

Submitted on 24 May 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

UNITÉ DE RECHERCHE
INRIA-SOPHIA ANTIPOLIS

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél. (1) 39 63 55 11

Rapports de Recherche

N° 1097

Programme 3
Réseaux et Systèmes Répartis

COMPARISON OF SERVICE DISCIPLINES IN G / GI / m QUEUES

Serguei FOSS

Octobre 1989

RR_1097

Comparison of Service Disciplines in G/GI/m Queues

Serguei FOSS*
Novosibirsk State University
Pirogova 2
630090 Novosibirsk
URSS

July 25, 1989

Abstract

We present a short summary of the author's results on the comparison of service disciplines in the G/GI/m queues which were published in Russian (see References).

The main result of the paper states that the so-called FCFS discipline is the best among all disciplines exhibiting some "independence of the future" property.

Keywords: Service discipline, independence of the future, stochastic ordering, partial ordering, waiting time, departure epoch.

1 Introduction, notations

Consider a queueing system with m servers numbered $1, 2, \ldots, m$. Customers come into the system with inter-arrival times $\{\tau_n\}$, i.e. n-th customer comes into the system at time $t_n = \tau_1 + \cdots + \tau_{n-1}$ (where $t_1 = 0$) and has the service time s_n. The k-th server ($k = 1, 2, \ldots, m$) can begin to serve customers at time $w_{1,k} \geq 0$.

We assume that the r.v.'s $\{s_n\}$ are i.i.d. and are independent of $\{\tau_n\}$, $W_1 = (w_{1,1}, \ldots, w_{1,m})$.

*This paper was completed during the visit of the author at INRIA-Sophia Antipolis in February and March of 1989.

Comparaison des Disciplines de Service dans les Files d'Attente G/GI/m

Juillet 1989

Serguei FOSS

Université d'état de Novossibirsk

Pirogova 2, 630090, Novossibirsk URSS

RESUME

Nous présentons un court résumé des résultats de l'auteur sur la comparaison des disciplines de service dans les files d'attente G/GI/m. Ces résultats sont parus en russe (cf Références).

Le résultat principal de cet article est que la discipline Premier Arrivé Premier Servi est la meilleure parmi les disciplines présentant la propriété d'*indépendance de l'avenir.*

Cet article a été rédigé pendant la visite de l'auteur à l'INRIA-Sophia Antipolis en Février et Mars 1989.

Definition 1 *A service discipline T is defined as a sequence $\{T_n, n \geq 1\}$ of $\{1, 2, \ldots, m\}$-valued r.v.'s. Here T_n is the number of the n-th customer's server, i.e., for $n = 1, 2, \ldots$, the n-th customer comes at time t_n and joins the queue of server T_n.*

EXAMPLES.

1. Cyclic discipline T^c. Here $T_n = (n-1) \pmod m + 1$ a.e.

2. Random discipline T^R. Here $T_n = k$ with probability $1/m$ for all $k = 1, 2, \ldots, m$ and r.v.'s $\{T_n\}$ are i.i.d. and are independent of $\{\tau_n, s_n\}$, W_1.

3. FCFS discipline T^0. Here the sequence $\{T_n\}$ is defined by induction. Let $l = \min\{k : w_{1,k} = \min_{1 \leq j \leq m} w_{1,j}\}$. Then $T_1 = l$ and

$$w_{2,k} = \begin{cases} (w_{1,k} - \tau_1)^+ & \text{if } k \neq l \\ (w_{1,l} + s_1 - \tau_1)^+ & \text{if } k = l \end{cases}$$

Suppose that we constructed $T_1, T_2, \ldots, T_n$ and $W_1, W_2, \ldots, W_{n+1}$. Then

$$T_{n+1} = \min\{k : w_{n+1,k} = \min_{1 \leq j \leq m} w_{n+1,j}\} \tag{1.1}$$

and

$$w_{n+2,k} = \begin{cases} (w_{n+1,k} - \tau_{n+1})^+ & \text{if } k \neq T_{n+1} \\ (w_{n+1,k} + s_{n+1} - \tau_{n+1})^+ & \text{if } k = T_{n+1} \end{cases}$$

Remark 1. In [3] we introduced a more general class of service disciplines $\{T = \{T_n, \nu_n, \delta_n\}, n \geq 0\}$ where $\{\nu_n\}$ is the order of service (i.e., $\{\nu_n\}$ is a random permutation of the set $N = \{1, 2, \ldots\}$) and $\{\delta_n\}$ are the so-called "delays". For example, the discipline LCFS belongs to this class.

Definition 2 *The discipline $T = \{T_n\}$ exhibits the property of "independence of future" if for all $n = 1, 2, \ldots$ the sequences of r.v's $\{T_k, k \leq n\}$ and $\{s_k, k \geq n\}$ are independent.*

We shall consider independent of future (IF) disciplines only.

Remark 2. For sake of simplicity, we shall assume that for every discipline $T = \{T_n\}$ and for every n, the random variable T_n may be represented as the measurable non-random function $h_{n,T}$ of the "history", i.e.

$$T_n = h_{n,T}(W_1, \tau_1, \ldots, \tau_n, s_1, \ldots, s_{n-1}, T_1, \ldots, T_{n-1}). \tag{1.2}$$

For $k = 1, 2, \ldots, m; n = 1, 2, \ldots$ denote by $w_{n,k}$ the length of the time interval between t_n and the moment of the last departure epoch of customers

with numbers from the set $\{1, 2, \ldots, n-1\}$ from the k-th server. Denote also by γ_n the departure epoch of the n-th customer, $\Gamma_n = (\gamma_1, \ldots, \gamma_n)$. The vectors $W_n = (w_{n,1}, \ldots, w_{n,m})$ satisfy the well-known recurrent relations:

$$W_{n+1} = (W_n + e_{T_n} s_n - i\tau_n)^+ \tag{1.3}$$

where $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $i = (1, 1, \ldots, 1)$.

Borovkov ([5]) conjectured that the discipline T^0 is better than T^c in the following sense; for all $n = 1, 2, \ldots$

$$\max_{1 \le k \le m} w^c_{n,k} \ge \max_{1 \le k \le m} w^0_{n,k} \tag{1.4}$$

a.e. But Stoyan ([6]) gave a counterexample. And he stated the new conjecture that this inequality is valid in the stochastic sense, i.e. for all $x > 0, n = 1, 2, \ldots$

$$P\{\max_k w^c_{n,k} > x\} \ge P\{\max_k w^0_{n,k} > x\}. \tag{1.5}$$

Gittins ([7]) formulated two propositions for $GI/GI/m$ queues: if T is an arbitrary IF discipline then for all $x > 0, n = 1, 2, \ldots$

$$P\{\max_k w_{n,k} > x\} \ \ge \ P\{\max_k w^0_{n,k} > x\}, \tag{1.6}$$

$$P\{\sum_{i=1}^n (\gamma_i - t_i) > x\} \ \ge \ P\{\sum_{i=1}^n (\gamma^0_i - t_i) > x\}. \tag{1.7}$$

But he did not give an exact mathematical proof.

Vasicek ([8]) showed that for non-idling disciplines T (i.e., it is impossible to have an idle server and customers waiting for service in the system) and all convex functions g

$$\sum_{i=1}^n Mg(\gamma^0_i - t_i) \le \sum_{i=1}^n Mg(\gamma_i - t_i). \tag{1.8}$$

We shall generalize these three last inequalities.

2 The main results

Let $\mathbb{R}$ denote the real line, $\mathbb{R}^n$ denote the n-dimentional space. To the vector $X = (x_1, x_2, \ldots, x_n)$ we associate the vector $R(X) = (x_{(1)}, \ldots, x_{(n)})$ defined as the permutation in increasing order of the coordinates of X, and the vector $(X)^+ = (x_1^+, \ldots, x_n^+)$.

For vectors $X, Y \in \mathbb{R}^n$ we introduce the following partial orderings:

Definition 3 *1.* $X \leq_1 Y$ *, if* $x_i \leq y_i$ *for all* i *;*

2. $X \leq_2 Y$ *, if* $R(X) \leq_1 R(Y)$ *;*

3. $X \leq_3 Y$ *, if* $\sum_{k=j}^{n} x_k \leq \sum_{k=j}^{n} y_k$ *for all* j *;*

4. $X \leq_4 Y$ *, if* $R(X) \leq_3 R(Y)$.

Properties. 1) If $X = R(X), Y = R(Y)$ and $X \leq_1 Y$ then $(X)^+ \leq_1 (Y)^+, X + ia \leq_1 Y + ia, X + e_k a \leq_1 Y + e_k a$ for all k, a. 2) If $X \leq_4 Y$ then $(X)^+ \leq_4 (Y)^+$ and $X + ia \leq_4 Y + ia$ for all a. 3) If $X = (x_1, \ldots, x_n)$ and $x_k \leq x_l$ then $X + e_k b \leq_4 X + e_l b$ for all $b \geq 0$.

Definition 4 *Let* ξ *and* η *be random variables. We shall write* $\xi \leq_{st} \eta$ *if* $P(\xi > a) \leq P(\eta > a)$ *for all* a.

Consider the following classes of functions F and $G^{(n)}, n = 1, 2, \ldots$:

1) F is the class of functions $f : \mathbb{R}^m \to \mathbb{R}$ such that if $X \leq_4 Y$ then $f(X) \leq f(Y)$ (the so-called class of Schur-convex functions);

2) $G^{(n)}$ is the class of functions $g : \mathbb{R}^n \to \mathbb{R}$ such that if $X \leq_2 Y$ then $g(X) \leq g(Y)$.

Note that the functions $\sum x_i, \max x_i$ belong to F, and the functions $\sum x_i$, $\max x_i$ and $\min x_i$ belong to $G^{(n)}$.

Theorem 2.1 *(see [1-3]). For every IF discipline* T *and every* $n = 1, 2, \ldots$

(i) $f(W_n^0) \leq_{st} f(W_n)$ *for all* $f \in F$*;*

(ii) $g(\Gamma_n^0) \leq_{st} g(\Gamma_n)$ *for all* $g \in G^{(n)}$.

Corollary 2.1 *For all* $n = 1, 2, \ldots$

$\max_k w_{n,k}^0 \leq_{st} \max_k w_{n,k}$ *and* $\sum_k w_{n,k}^0 \leq_{st} \sum_k w_{n,k}$.

Corollary 2.2 *Let* $A(t) = \{n : t_n \leq t\}$ *and* $\gamma(t) = \max_{n \in A(t)} \gamma_n - t$. *Then* $\gamma^0(t) \leq_{st} \gamma(t)$ *for all* $t \geq 0$.

Corollary 2.3 *For all* $n = 1, 2, \ldots$

$$\sum_{i=1}^{n} c_i^0 \leq_{st} \sum_{i=1}^{n} c_i, \tag{2.1}$$

where a) $c_i = \gamma_i$*; b)* $c_i = \gamma_i - t_i$*; c)* $c_i = \gamma_i - s_i$*; d)* $c_i = \gamma_i - s_i - t_i$*. Here* γ_i *is the time when the i-th customer's service starts;* $\gamma_i - t_i$ *is the response time of the i-th customer;* $\gamma_i - s_i - t_i$ *is the actual waiting time of the i-th customer.*

Corollary 2.4 *Let T be any IF discipline. Suppose that the real waiting times have the stationary distributions W (under discipline T) and W^0 (under discipline T^0). For example, this is the case if $W_1 = 0$ and $\{\tau_n\}$ is a stationary metrically transitive sequence and $Es_1 < mE\tau_1 \leq \infty$. Suppose, moreover, that $EW^0 < \infty$ and $EW < \infty$. Then $EW^0 < EW$.*

Corollary 2.5 *Let q_n be the queue length at time t_n and $q(t)$ be the queue length at time t. Then $q_n^0 \leq_{st} q_n$ and $q^0(t) \leq_{st} q(t)$ for all $n = 1, 2, \ldots; t > 0$.*

Theorem 2.2 *(see [4]). For any IF discipline T and every $n = 1, 2, \ldots$*

$$\min_{1\leq i\leq n} \max_{1\leq k\leq m} w_{i,k}^0 \leq_{st} \min_{1\leq i\leq n} \max_{1\leq k\leq m} w_{i,k}. \tag{2.2}$$

Corollary 2.6 *Let $\nu = \min\{n > 0 : \max_{1\leq k\leq m} w_{n,k} < a\}$, where $a \geq 0$. Then $\nu^0 \leq_{st} \nu$ for every IF discipline. Therefore, one can obtain ergodicity and stability theorems and estimates of the rate of convergence (for $G/GI/m$ queues) with the help of the method of renovating events (see [4,9]).*

Remark 3. Theorem 2.2 may be generalized in a natural way.

Remark 4. The results of Theorems 2.1-2.2 may be formulated in the case when $\{(s_n, \tau_n)\}$ is the sequence of i.i.d. random vectors (but s_n and τ_n may be dependent, see [3]).

Remark 5. In [3] we formulated and proved the similar results for tandem queues.

3 Proofs

Lemma 3.1 *(see [1,3]). Let $W_1 = (w_{1,1}, w_{1,2})$ be a random vector and $w_{1,2} \geq w_{1,1}$ a.e.; τ_1 and τ_2 be arbitrary r.v.'s ; s_1, s_2 be i.i.d. non-negative r.v.'s , which are independent of W_1, τ_1, τ_2. Then there exist r.v.'s s_1' and s_2' such that*

(i) $s_1' =_{st} s_2' =_{st} s_1$; $s_1 + s_2 = s_1' + s_2'$ a.e.;
(ii) s_1', s_2' are i.i.d.r.v.'s;
(iii) s_1', s_2' are independent of W_1, τ_1, τ_2 ;

$$(iv) W_3' \equiv ((W_1 + e_1 s_1' - i\tau_1)^+ + e_2 s_2' - i\tau_2)^+ \leq_2 \tag{3.1}$$

$$W_3 \equiv ((W_1 + e_2 s_1 - i\tau_1)^+ + e_1 s_2 - i\tau_2)^+ \tag{3.2}$$

a.e.

Proof. Let $A = \{w_{1,2} - \tau_1 < 0\}$ and $s_1' = s_1 I(A) + s_2 I(\Omega - A)$; $s_2' = s_2 I(A) + s_1 I(\Omega - A)$, where $I(.)$ is the indicator function. Then $s_1' + s_2' = s_1 + s_2$ a.e. and

(i) $P(s_1' \in B) = P(s_1 \in B, A) + P(s_2 \in B, \Omega - A) =$
$P(s_1 \in B)P(A) + P(s_1 \in B)P(\Omega - A) = P(s_1 \in B)$;

(ii-iii) $P(s_1' \in B_1, s_2' \in B_2, \tau_1 \in C_1, \tau_2 \in C_2, W_1 \in C_3) = P(\ldots, A) + P(\ldots, \Omega - A)$ and so on ;

(iv) On the event A

$$w_{3,1}' = ((w_{1,1} + s_1 - \tau_1)^+ - \tau_2)^+ \leq ((w_{1,2} + s_1 - \tau_1)^+ - \tau_2)^+ = w_{3,2}; \quad (3.3)$$

$$w_{3,2}' = (s_2 - \tau_2)^+ = w_{3,1}. \quad (3.4)$$

On the event $\Omega - A$

$$w_{3,1}' = ((w_{1,1} + s_2 - \tau_1)^+ - \tau_2)^+ \leq ((w_{1,1} - \tau_1)^+ + s_2 - \tau_2)^+ = w_{3,1}; \quad (3.5)$$

$$w_{3,2}' = (w_{1,2} - \tau_1 + s_1 - \tau_2)^+ = w_{3,2}. \quad (3.6)$$

Remark 6. Note that the vectors $W_2' = (W_1 + e_1 s_1' - i\tau_1)^+$ and $W_2 = (W_1 + e_2 s_1 - i\tau_1)^+$ may be non-comparable.

Remark 7. The same result is valid if we suppose that $X = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and replace x_1 by some x_l , x_2 by some x_r , e_1 by e_l, e_2 by e_r.

Remark 8. Let $f \in F$ and $W_3'' = ((W_1 + e_l s_1 - i\tau_1)^+ + e_r s_2 - i\tau_2)^+$. Then $W_3'' =_{st} W_3'$ and, therefore, $f(W_3'') \leq_{st} f(W_3)$ for all $f \in F$.

Remark 9. If k and l are random variables independent of s_1, s_2 then the result of Lemma 1 is unchanged.

Remark 10. If we denote $\gamma_1 = w_{1,2} + s_1, \gamma_2 = \max(w_{1,1}, \tau_1) + s_2, \gamma_1' = w_{1,1} + s_1', \gamma_2' = \max(w_{1,2}, \tau_1) + s_2'$, then $\Gamma_2' \leq_2 \Gamma_2$.

Definition 5 *Consider three disciplines $T^{(1)} = \{T_n^{(1)}\}, T^{(2)} = \{T_n^{(2)}\}$ and $T^{(3)} = \{T_n^{(3)}\}$. We shall write $T_k^{(1)} = T_k^{(2)}$ if (see Remark 2) $h_{k,T^{(1)}} = h_{k,T^{(2)}}$. We shall write $T_{(k,l)}^{(1)} = T_{(k,l)}^{(2)}$ if $T_i^{(1)} = T_i^{(2)}$ for all $k \leq i \leq l$. We shall write $T_{(k,r)}^{(1)} = T_{(k,l-1)}^{(2)} \bigcup T_{(l,r)}^{(3)}$ if $T_{(k,l-1)}^{(1)} = T_{(k,l-1)}^{(2)}$ and $T_{(l,r)}^{(1)} = T_{(l,r)}^{(3)}$. For fixed $n = 1, 2, \ldots$ we shall write $T_{(1,n)}^{(1)} \geq T_{(1,n)}^{(2)}$ if $f(W_{n+1}^{(1)}) \geq_{st} f(W_{n+1}^{(2)})$ and $g(\Gamma_n^{(1)}) \geq_{st} g(\Gamma_n^{(2)})$ for all $f \in F, g \in G^{(n)}$.*

Lemma 3.2 *(see [1-3]). Let $k = 1, \ldots, n-1$ be an arbitrary number and assume that the discipline $T^{(1)}$ has the form*

$T^{(1)}_{(1,n)} = T^{(1)}_{(1,k)} \bigcup T^0_{(k+1,n)}$.

Then there exists an IF discipline $T^{(2)}$ such that

$$T^{(2)}_{(1,k)} = T^{(1)}_{(1,k-1)} \bigcup T^0_k \tag{3.7}$$

and

$$T^{(2)}_{(1,n)} \leq T^{(1)}_{(1,n)}. \tag{3.8}$$

Proof. Let $r = T^{(1)}_k$ and $l = \min\{i : w_{k,i} = \min_j w_{k,j}\}$. Note that $T^{(1)}_{k+1} = l$ on the event $\{l \neq k\}$, i.e. $T^{(1)}_{k+1}$ is independent of s_k under the condition $\{l \neq k\}$. Let

$A = \{w_{k,r} - \tau_k < 0\} \bigcup \{r = l\}$.

Define the sequence $\{s'_i, i \geq 1\}$ as follows:

1) $s'_i = s_i$ a.e. if $i \neq k, i \neq k+1$;

2) $s'_k = s_k I(A) + s_{k+1} I(\Omega - A); s'_{k+1} = s_k I(\Omega - A) + s_{k+1} I(A)$.

Construct the discipline $T^{(2)}$ defined by

1) $T^{(2)}_i = T^{(1)}_i, i = 1, \ldots, k-1$;

2) $T^{(2)}_k = l$ a.e.;

3)

$$T^{(2)}_{k+1} = \begin{cases} T^{(1)}_{k+1} & \text{if } r = l \\ r & \text{otherwise} \end{cases}$$

4) If $w_{k,r} \geq \tau_k$ or $r = l$ then $T^{(2)}_i = T^{(1)}_i$ a.e. for $i = k+2, \ldots, n, \ldots$. If $w_{k,r} < \tau_k$ and $r \neq l$ then $T^{(2)}_i = r$ if $T^{(1)}_i = l$; $T^{(2)}_i = l$ if $T^{(1)}_i = r$; $T^{(2)}_i = T^{(1)}_i$ in another cases (for $i = k+2, \ldots$). As s'_k and s'_{k+1} are independent of k, l then $T^{(2)}$ is IF discipline.

Under Lemma 3.1 we can see that $W^{(1)}_{k+1} \geq_2 W^{(2)}_{k+1}$ a.e. Moreover, the inequality $W^{(1)}_i \geq_2 W^{(2)}_i$ is valid for all $i \geq k+1$.Then if we construct $T^{(2)}$ on the r.v.'s $\{s_n\}$ (see Remark 8) we shall obtain $T^{(2)}_{(1,n)} \leq T^{(1)}_{(1,n)}$.

PROOF OF TH.2.1. We shall use a "backward" induction and prove that $T_{(1,n)} \geq T_{(1,k)} \bigcup T^0_{(k+1,n)}$ for all $k \geq 0$.

Let $k = n-1$. Then this inequality is trivial.

Let $k_0 \leq n-1$ and $T_{(1,n)} \geq T_{(1,k_0)} \bigcup T^0_{(k_0+1,n)} \equiv T^{(1)}_{(1,n)}$ for all IF disciplines T. Under Lemma 2.2, there exists a discipline $T^{(2)}$ such that $T^{(2)}_{(1,k_0)} = T^{(1)}_{(1,k_0-1)} \bigcup T^0_{k_0}$ and $T^{(1)}_{(1,n)} \geq T^{(2)}_{(1,n)}$. By induction assumption,

$T^{(2)}_{(1,n)} \geq T^{(2)}_{(1,k_0)} \bigcup T^0_{(k_0+1,n)} = T^{(1)}_{(1,k_0-1)} \bigcup T^0_{(k_0,n)}$.

Note that this partial ordering has the distributive property. Therefore the proof is completed. COMMENTS on the COROLLARIES. Corollary 3 follows from Theorem 2.1 under the condition (i) of Lemma 3.1. Corollary 4 follows from Corollary 3 because $1/n \sum_{i=1}^{n}(\gamma_i - s_i - t_i) \to EW$ a.e.

PROOF OF TH.2.2. For sake of simplicity, we shall establish the proof in the case $m = 2, n = 3, W_1 = (w_{1,1}, w_{1,2}) = const, w_{1,2} \geq w_{1,1}, \tau_1 = const, \tau_2 = const$ and $T_1 = 2$ a.e. Firstly, observe that the inequality $T_{(1,2)} \geq T_1 \bigcup T_2^0 \equiv T'_{(1,2)}$ is trivial. Then notice that $T'_1 = 2, T'_2 = 1$ a.e.. Consider the vectors

$$W'_2 = ((w_{1,1} - \tau_1)^+, (w_{1,2} + s_1 - \tau_1)^+); \tag{3.9}$$

$$W'_3 = ((w_{1,1} - \tau_1)^+ + s_2 - \tau_2)^+, (w_{1,2} + s_1 - \tau_1 - \tau_2)^+). \tag{3.10}$$

If $w_{1,1} - \tau_1 \geq a$ then

$$P(\min_{j=2,3} \max_{i=1,2} w'_{j,i} \geq a) = P(\max_{i=1,2} w'_{3,i} \geq a) \geq P(\max_{i=1,2} w^0_{3,i} \geq a) \tag{3.11}$$

under Th.2.1.

If $w_{1,1} - \tau_1 < a$ then

$$P(\min \max w'_{j,i} \geq a) = P(w_{1,2} + s_1 - \tau_1 \geq a, \max_{i=1,2} w'_{3,i} \geq a) = \tag{3.12}$$

$$P(\max_{i=1,2} w'_{3,i} \geq a) - P((w_{1,1} - \tau_1)^+ + s_2 \geq a + \tau_2, w_{1,2} + s_1 - \tau_1 < a) \geq \tag{3.13}$$

$$P(\max_{i=1,2} w^0_{3,i} \geq a) - P((w_{1,1} - \tau_1)^+ \geq a + \tau_2 - s_2, w_{1,2} - \tau_1 < a - s_1) \geq \tag{3.14}$$

$$P(\max_{i=1,2} w^0_{3,i} \geq a) - P((w_{1,2} - \tau_1)^+ \geq a + \tau_2 - s_2, w_{1,1} - \tau_1 < a - s_1) = \tag{3.15}$$

$$P(\min_{j=2,3} \max_{i=1,2} w^0_{j,i} \geq a) \tag{3.16}$$

Remark 11. We are sure that the reader can reconstruct the full proof from this ideas and from the proof of Th.2.1.

References

[1] S.G.Foss, On Approximation of Multi-Server Queues, *Sibirsk.Mat.J.* **21**, 132-140, 1980 (in Russian).

[2] S.G.Foss, Comparison of Multi-Server Queues, *Sibirsk.Mat.J.* **22**, 190-197, 1981 (in Russian).

[3] S.G.Foss, Extremal Problems in Queueing Theory,Ph.D.Thesis,Institute of Mathematics, Novosibirsk, 1-109,1982 (in Russian).

[4] S.G.Foss, On the Certain Method of Estimation of the Rate of Convergence in the Ergodicity and Stability Theorems for Multi-Server Queueing Systems, *Predelnije Teoremi Teoriji Verojatnostej, Trudi Instituta Matematiki, 5, Nauka, Novosibirsk*, 126-137, 1985 (in Russian).

[5] A.A.Borovkov, *Stochastic Processes in Queueing Theory*, John Wiley & Sons, 1976.

[6] D.Stoyan, A Critical Remark on a System Approximation in Queueing Theory, *Math.Operationsf.Statist.* **7**, 953-956, 1976.

[7] J.C.Gittins, A Comparison of Service Disciplines for GI/G/m Queues, *Math.Operationsf.Statist.* **9**, 255-260, 1978.

[8] D.A.Vasicek, An Inequality for the Variance of Waiting Time under a General Queueing Discipline, *Oper.Reseach* **25**, 879-884, 1977.

[9] A.A.Borovkov, *Asymptotic Methods in Queueing Theory*, John Wiley & Sons,1984.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique