



On the rates of convergence of fully discrete solutions of Hamilton-Jacobi equations

Roberto L. Gonzalez, Mabel M. Tidball

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UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél. (1) 39 63 55 11

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ON THE RATES OF CONVERGENCE OF FULLY DISCRETE SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

Roberto L.V. GONZALEZ
Mabel M. TIDBALL

Janvier 1991



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**ON THE RATES OF CONVERGENCE
OF FULLY DISCRETE SOLUTIONS
OF HAMILTON-JACOBI EQUATIONS**

**ESTIMATIONS DES VITESSES DE CONVERGENCE
DES SOLUTIONS APPROCHEES DES EQUATIONS
D'HAMILTON-JACOBI**

Roberto L. V. González[†] - Mabel M. Tidball^{†‡}

[†] Instituto de Matemática Beppo Levi
Facultad de Ciencias Exactas, Ingeniería y Agrimensura
Universidad Nacional de Rosario
Avda. Pellegrini 250 - 2000 Rosario - Argentine.

[‡] This paper is included in the activities developed in the frame of the
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Coordinators of the project: E. Rofman - R. L. V. González

ABSTRACT

In this paper we consider the Hamilton-Jacobi equation associated to an infinite horizon optimal deterministic control problem. We consider approximated solutions by discretization on time and spacial variables and we study the rate of convergence of the approximate solution to the optimal cost function of the original problem. We give several explicit estimates of the rate of convergence which depend on the regularity of the data and on the parameters of discretization. Finally we present some examples which show that these estimates are optimal.

RESUME

On considère ici l'équation de Hamilton-Jacobi associée à un problème de contrôle optimal avec horizon infini. On étudie les solutions approchées obtenues par discretisation dans les variables d'espace et de temps et la vitesse de convergence des solutions approchées vers la fonction de coût optimale. On donne des plusieurs estimations explicites de la vitesse de convergence qui dépendent de la régularité des données du problème et des paramètres de discretisation. Finalement on présente quelques exemples qui montrent que ces estimations sont optimales.

Key words and phrases. Hamilton-Jacobi equation, numerical approximations, error estimates, optimal estimates.

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REFERENCES

What is now proved, was once only imagined

William Blake

The Marriage of Heaven and Hell

INTRODUCTION

In this paper we study the Hamilton-Jacobi equation associated to an infinite horizon optimal deterministic control problem. We consider approximated solutions by discretization on time and spacial variables by means of finite elements techniques. We use the discretization procedures described in [3], [4], [5], [17].

Our aim is to obtain explicit estimates of the rate of convergence of the approximate solution to the optimal cost function of the original problem. Previous results in this direction have been obtained in [5], [11], [17]. Here, we have tried to get the optimal estimates depending on the regularity of the problem and on the parameters of discretization h, k (time and space discretization steps). In each case we have sought examples that show the optimality of the corresponding estimates. Also, we have studied the optimization of parameters h, k in order to obtain the best possible results available with a methodology of this type.

The principal results are the following:

- Theorem 3.1 gives an estimate of the error of the fully discrete solution as a function of parameters h, k .
- This estimate indicates that for the general case, when the optimal cost function is only Lipschitz continuous (for which heavy assumptions about the discount factor λ are necessary), the best result one can expect is a convergence of order $k^{1/2}$.
- Also, in the general case with a small discount coefficient λ , (where u is only Hölder continuous of order $\gamma < 1$) the best convergence rate one can expect to get is of order $k^{\gamma/2}$.
- We have obtained the optimal relation necessary to hold between the parameters h and k , in order to get the minimum error of discretization. The optimal value of h is different provided we deal with the general case or with the regular case. In the general case the optimal convergence rate is of order $k^{\gamma/2}$, (provided a step $h \leq k$ is used). In the regular case, where semiconcavity assumptions and the condition $\gamma > 1$ imply that the errors of the time discretization procedure are only of order h (not of order $h^{1/2}$, as it happens in the general case), the estimate and of course the numerical results can be improved. This occurs when an unusual large time step ($\bar{h} = k^\eta$) is employed, with $\eta = (2/(\gamma+1)) \wedge (2/3)$. The total error in this case is of order k^ξ , with $\xi = (\gamma/(\gamma+1)) \wedge (2/3)$.

1. GENERAL DESCRIPTION OF THE PROBLEM

1.1 The Optimal Cost Function u

Let u be the optimal cost function associated to an infinite horizon deterministic control problem in the following form:

$$u(x) = \inf_{a \in \mathcal{A}} J(x, a(\cdot)) \quad (1)$$

where

$$J(x, a(\cdot)) = \int_0^{\infty} f(y(t), a(t)) e^{-\lambda t} dt$$

Here \mathcal{A} denotes the set of all measurable functions of $[0, +\infty)$ to a given compact subset A of \mathbb{R}^n i.e. $\mathcal{A} = \{ a: [0, \infty) \rightarrow A, a(\cdot) \text{ measurable} \}$.

The state of the system is given by the following differential equation.

$$\dot{y}(t) = g(y(t), a(t)) \quad (2)$$

$$y(0) = x, x \in \Omega, \Omega \subset \mathbb{R}^n, \text{ open.}$$

The mapping $y: \mathbb{R}^n \times \mathcal{A} \rightarrow \mathbb{R}^n$ is called the *response* or the state corresponding to the control $a(\cdot)$. We suppose that $\forall t > 0, y(t) \in \Omega$, independently of the applied control. The constant λ is the *discount factor* and the function f is the *instantaneous cost*.

We assume f and g satisfies:

$$\begin{cases} \|g(x, a) - g(\bar{x}, a)\| \leq L_g \|x - \bar{x}\|, \\ \|g(x, a)\| \leq M_g \end{cases} \quad (3)$$

$$\begin{cases} |f(x, a) - f(\bar{x}, a)| \leq L_f \|x - \bar{x}\|, \\ |f(x, a)| \leq M_f \end{cases} \quad (4)$$

We will denote with H_1 the set of inequalities (3) and (4) and we will impose its validity along the rest of the paper.

1.2 Properties of u

It is well known (see [8], [22]) that under the assumptions above made, the value function u is the unique bounded uniformly continuous viscosity solution of the Hamilton-Jacobi-Bellman equation:

$$\min_{a \in A} \left(\frac{\partial u}{\partial x}(x) g(x, a) - \lambda u(x) + f(x, a) \right) = 0 \quad (5)$$

and that it satisfies (see [3], [4]):

$$|u(x)| \leq \frac{M_f}{\lambda}, \quad (6)$$

$$|u(x) - u(\hat{x})| \leq L_u \|x - \hat{x}\|^\gamma, \quad \forall x \in \Omega, \quad (7)$$

where

$$L_u = \begin{cases} \frac{L_f}{\lambda - L_g} & \gamma = 1 \quad \text{if } \lambda > L_g, \\ L_f^\gamma (2M_f)^{1-\gamma} \left(\frac{1}{\lambda} + \frac{1}{L_g - \lambda} \right), & \gamma = \frac{\lambda}{L_g} \quad \text{if } \lambda < L_g, \\ \left(\frac{2M_f}{L_f} \right)^{1-\gamma} \frac{1}{1-\gamma} e^{-\gamma}, & \gamma \in (0,1) \quad \text{if } \lambda = L_g. \end{cases} \quad (8)$$

i.e. u is Lipschitz continuous under assumption $\lambda > L_g$ and only Hölder continuous if $\lambda \leq L_g$ (see [4], [13], [14], [17]).

2. NUMERICAL SOLUTION

2.1 A Discrete Time Approximation

The first step to get a discrete problem amenable with numerical methods is to do a discretization in the time variable. This procedure also has good properties for the aim to obtain sub-optimal feedback policies (see [3]).

Let $h > 0$ and consider the Hamilton-Jacobi-Bellman equation discretized on time.

$$u^h(x) = \min_{a \in A} \left((1 - \lambda h) u^h(x + h g(x, a)) + h f(x, a) \right) \quad (9)$$

2.1.1 Properties of $u^h(x)$

It has been proved in [4] that (9) has a unique solution given by the unique fixed point of operator A^h , where:

$$\begin{aligned} A^h: C(\Omega) &\rightarrow C(\Omega) \\ (A^h w)(x) &= \min_{a \in A} ((1 - \lambda h) w(x + h g(x, a)) + h f(x, a)) \end{aligned} \quad (10)$$

In order to have a well defined operator A^h , we will suppose that:

$$x + h g(x, a) \in \Omega, \quad \forall x \in \Omega, \quad \forall a \in A, \quad \forall h \leq h_0 < \frac{1}{\lambda} \quad (11)$$

Moreover we define:

$$\begin{aligned} A_a^h: C(\Omega) &\rightarrow C(\Omega) \\ (A_a^h w)(x) &= ((1 - \lambda h) w(x + h g(x, a)) + h f(x, a)) \end{aligned} \quad (12)$$

trivially we have

$$(A^h w)(x) \leq (A_a^h w)(x) \quad \forall a \in A \quad (13)$$

For u^h , the following representation formula holds (see [4]):

$$u^h(x) = \inf_{a \in \mathcal{A}^h} J^h(x, a) \quad \forall x \in \Omega \quad (14)$$

where:

$$\mathcal{A}^h = \left\{ a(jh) / a(jh) \in A, j=1, \dots \right\}$$

and

$$J^h(x, a) = h \sum_{j=1}^{\infty} f(y^h(x, j), a(jh)) (1 - \lambda h)^{j-1} \quad (15)$$

In (15) the sequence $y^h(x, j)$ is determined by the recursion:

$$y^h(x, j+1) = y^h(x, j) + h g(y^h(x, j), a((j+1)h)), \quad j=0, \dots \quad (16)$$

with

$$y^h(x, 0) = x$$

u^h verifies the following properties (see [3])

$$|u^h(x)| \leq \frac{M_f}{\lambda}, \quad |u^h(x) - u^h(\hat{x})| \leq L_{u^h} \|x - \hat{x}\|^\gamma \quad (17)$$

$\forall h \in (0, \frac{1}{\lambda}), \forall x, \hat{x} \in \Omega$, where:

$$L_{u^h} = \begin{cases} \frac{L_f}{\lambda - L_g} & \gamma = 1 & \text{if } \lambda > L_g, \\ L_f^\gamma (2M_f)^{1-\gamma} \left(\frac{1}{\lambda} + \frac{1}{L_g - \lambda} \right) & \gamma = \frac{\lambda}{L_g} & \text{if } \lambda < L_g, \\ \left(\frac{2M_f}{L_f} \right)^{1-\gamma} \frac{1}{1-\gamma} e^{-\gamma}, & \gamma \in (0,1) & \text{if } \lambda = L_g. \end{cases} \quad (18)$$

2.1.2 Rate of convergence

The convergence of discrete time solution u^h to the continuous solution u is uniform; indeed, it is possible to establish explicit estimates of the rate of convergence.

Theorem 2.1: Under assumptions H_1 the following estimates holds:

$$|u(x) - u^h(x)| \leq Ch^{\gamma/2} \quad (19)$$

where, $\gamma = 1$ if $\lambda > L_g$, $\gamma = \frac{\lambda}{L_g}$ if $\lambda < L_g$, and γ is an arbitrary number in $(0,1)$ if $\lambda = L_g$.

Moreover, under the following additional hypotheses of semiconcavity of g, f :

$$\|g(x - \hat{x}) - 2g(x) + g(x + \hat{x})\| \leq C\|\hat{x}\|^2 \quad (20)$$

$$|f(x - \hat{x}) - 2f(x) + f(x + \hat{x})| \leq C\|\hat{x}\|^2 \quad (21)$$

the estimate

$$|u(x) - u^h(x)| \leq Ch^{\gamma/2 \wedge 1}$$

is valid for all γ .

Which implies in addition that if it is satisfied the condition

$$\lambda > 2L_g, \quad (22)$$

the following estimate, sharper than (19), holds

$$\left| u(x) - u^h(x) \right| \leq Ch \quad (23)$$

Remark 2.1: In (19), when $\gamma = L_g$ the value of C depends on the chosen value of γ .

Remark 2.2: The proof of these results will be omitted here. We will refer the reader to the proofs contained in [3], [20]. They are based in techniques of convex analysis [25] and classical methods usually employed in the treatment of viscosity solution [22].

Note: From here, and in order to obtain simplicity of notation and clarity of arguments, we will use letters C, M, K to denote arbitrary constants (which values depends on the context where they appear) which depend on the data of the problem (constants $\lambda, M_g, M_f, L_g, L_f$, etc.) but do not depend on the parameters of discretizations h, k , etc.

2.2 Fully Discrete Solution

The discretization above introduced remains a theoretical one. To obtain computational results it is also necessary to do a discretization of space variables and to give a numerical method to compute u^h . This is done here via a discretization in the state variables by means of finite elements techniques.

2.2.1 Description of the problem

a) Approximation of domain Ω

We will identify the discretization in space variable with the parameter k , which also indicate the size of the discretization.

Let Ω be an open set of \mathbb{R}^N and $\{S_j^k\}$ a family of regular triangulations of Ω , i.e. a finite sets of simplices (see [15],[16], [23]) that approaches Ω in the following sense:

$\Omega_k = \bigcup_j S_j^k$ is a polyhedron of \mathbb{R}^N such that the following properties are satisfied:

$$\bullet \max_j (\text{diám } S_j^k) = k \quad (24)$$

$$\bullet \exists h_0 > 0 \text{ such that } x + hg(x, a) \in \Omega_k \quad \forall x \in \Omega_k, \forall a \in A, \forall h < h_0 \quad (25)$$

- $\Omega_k \rightarrow \Omega$, when $k \rightarrow 0$, in the following sense:

$$\forall K \text{ compact contained in } \Omega, \exists \bar{k}(K) / K \subset \Omega_k \quad \forall k < \bar{k}(K) \quad (26)$$

- If d_i is the diameter of the simplex S_i^k , then: $\exists X_1 > 0$ such that for any simplex of Ω_k , there exists a sphere of radius

$$r \geq X_1 d_i \quad (27)$$

in the interior of the simplex.

- Moreover there exists M , independent of the discretization such that:

$$\frac{k}{d_i} \leq M \quad \forall i. \quad (28)$$

b) Definition of the approximation space W_k

We consider the set W_k of functions $w: \Omega_k \rightarrow \mathbb{R}$, w continuous in Ω_k , $\frac{\partial w}{\partial x}$ constant in the interior of each simplex of Ω_k (i.e. w are linear finite elements)

Note: If denote with N the cardinality of set of vertices of Ω_k , being also $\{x_i, i = 1, \dots, N\}$ an appropriate order in the set of nodes (vertices of simplex) of Ω_k , it is obvious that any $w \in W_k$ is completely characterized by the values $w(x_i), \forall i = 1, \dots, N$.

c) Definition of operator A_k^h

We define operator A_k^h in the following way:

$$A_k^h: W_k \rightarrow W_k$$

$$(A_k^h w)(x_i) = \min_{a \in A} \left((1 - \lambda h) w(x_i + h g(x_i, a)) + h f(x_i, a) \right) \quad \forall i = 1, \dots, N \quad (29)$$

d) Fully discrete problem:

The final problem, amenable to be solved numerically is the following:

$$\boxed{\text{Problem } P_k: \text{ Find the fixed point of operator } A_k^h} \quad (30)$$

Note: It is clear that if A is an infinite set, problem P_k is not actually a fully discrete problem. In order to obtain a problem with those characteristics, a final step of discretization remains to be done: *the discretization of the set A by a finite set A_k^h* . This procedure introduces another error of discretization which size can be bounded by $d(A^k, A)/\lambda$. In this paper, we will not deal with this problem, which essentially only introduces a cumbersome notation; so, we will suppose from here that either A is finite or that the whole set A is considered in the definition of A_k^h .

e) An equivalent formulation

As every point x of Ω_k is a convex combination of vertices of the simplices to which it belongs, then $\forall a \in A \exists$ a matrix $\Lambda(a) \in \mathbb{R}^{N \times N}$ with entries $\alpha_j(x_i, a)$, such that the following properties hold:

$$\alpha_j(x_i, a) \geq 0 \quad \forall i, j \quad \sum_{j=1}^N \alpha_j(x_i, a) = 1, \quad \forall i \quad x_i + hg(x_i, a) = \sum_{j=1}^N \alpha_j(x_i, a)x_j \quad \forall i \quad (31)$$

Hence:

$$(A_k^h w)(x_i) = \min_{a \in A} \left((1 - \lambda h) \sum_{j=1}^N \alpha_j(x_i, a) w(x_j) + hf(x_i, a) \right) \quad (32)$$

We have the following relation between $w \in W_k$ and vectors $U \in \mathbb{R}^N$,

$$w(x_i) \Leftrightarrow u_i$$

and the corresponding fixed point problem has the equivalent form:

$$U = \min_{a \in A} \left((1 - \lambda h) \Lambda(a)U + hF(a) \right) \quad (33)$$

where $F(a) \in \mathbb{R}^N$ is defined by $F_i(a) = f(x_i, a)$ $i=1, \dots, N$, and $\Lambda(a)$ is the matrix with entries $\Lambda(a)_{ij} = \alpha_j(x_i, a)$

Remark 2.3: By virtue of A_k^h definition, we have, for any $w \in C(\Omega)$:

$$(A_k^h \tilde{w})(x_i) = \min_{a \in A} \left((1 - \lambda h) \sum_{j=1}^N \alpha_j(x_i, a) \tilde{w}(x_j) + h f(x_i, a) \right) = (A^h \tilde{w})(x_i) \quad (34)$$

with \tilde{w} the linear interpolation of w in each simplex of Ω_k , which coincides with w in any vertex of the triangulation Ω_k .

2.2.2 Existence and uniqueness of the fully discrete solution

Theorem 2.2: There exists a unique solution u_k^h of problem P_k .

Proof: We introduce in W_k the norm:

$$\|w\| = \max_i |w(x_i)|$$

It is clear that A_k^h is a contractive operator, i.e.

$$\|A_k^h(w) - A_k^h(\tilde{w})\| \leq (1 - \lambda h) \|w - \tilde{w}\|, \quad \forall w, \tilde{w} \in W_k \quad (35)$$

and then it has a unique fixed point. If we denote it with u_k^h , we have:

$$u_k^h = A_k^h(u_k^h) \quad (36)$$

□

Remark 2.4: Problem P_k is a nonlinear fixed point problem with a very special structure. It corresponds or is equivalent to an optimal stochastic control problem on a Markov Chain. (The reason of this relation is the existence of a discrete maximum principle (see [6], [7]) valid for the schemes of discretization used in the definition of A_k^h). So, it can be solved with any of the several methods of Picard or Howard type described in the current literature (see [2], [9], [10], [21], [26], [27]).

In this paper we will neither deal, with the computational procedure to solve ((30)) nor with any application but let us say that in the real application of the methodology theoretically studied in this paper (e.g. [1]) we have used the special techniques with acceleration procedures studied in [18], [19].

2.2.3 An equivalent problem

We will see that, in a broad sense, problem P_k is equivalent to problem \hat{P}_k where \hat{P}_k consists in to find the fixed point of \hat{A}_k^h , being

$$(\hat{A}_k^h w)(x_i) = \min_{a \in A} \left(\frac{1}{1 + \lambda h} w(x_i + h g(x_i, a)) + h f(x_i, a) \right) \quad \forall i=1, \dots, N \quad (37)$$

In a strict form, the following estimate holds:

$$\| u_k^h - \hat{u}_k^h \| \leq h M_f \quad (38)$$

In effect:

$$u_k^h(x) - \hat{u}_k^h(x) \leq (1 - \lambda h) u_k^h(x + h g(x, \bar{a})) - \frac{1}{1 + \lambda h} \hat{u}_k^h(x + h g(x, \bar{a})) \quad (39)$$

where \bar{a} realizes the minimum in (37). As

$$\| \hat{u}_k^h \| \leq \frac{1 + \lambda h}{\lambda} M_f$$

(39) becomes:

$$u_k^h(x) - \hat{u}_k^h(x) \leq (1 - \lambda h) \| u_k^h - \hat{u}_k^h \| + \left(\frac{1}{1 + \lambda h} - (1 - \lambda h) \right) \frac{1 + \lambda h}{\lambda} M_f \quad (40)$$

Analogously we can obtain the estimate

$$\hat{u}_k^h(x) - u_k^h(x) \leq (1 - \lambda h) \| u_k^h - \hat{u}_k^h \| + \left(\frac{1}{1 + \lambda h} - (1 - \lambda h) \right) \frac{1 + \lambda h}{\lambda} M_f \quad (41)$$

then by virtue of (40) and (41), we have:

$$\| u_k^h - \hat{u}_k^h \| \leq h M_f$$

Note: As it is obvious, no restrictions (of type (11)) on the value of h are necessary to hold in order to have a well defined problem \hat{P}_k .

3. CONVERGENCE OF FULLY DISCRETE PROBLEM

3.0 Presentation of the Central Result

In order to keep the employed ideas clear and to simplify the exposition of the arguments we will deal only with the case $\Omega = \mathbb{R}^n$. The general case can be treated without fundamental changes using the perturbation of domain techniques described in [17]. The main result is:

Theorem 3.1: *Assume (11), H_1 and the general properties of triangulation above described in §2.2.1 hold. Then, there is a constant M independent of h, k such that*

$$|u(x) - u_k^h(x)| \leq \begin{cases} M(\sqrt{h} + \frac{k}{\sqrt{h}}), & \text{if } L_g < \lambda \\ M(\sqrt{h} + \frac{k}{\sqrt{h}})^\gamma, & \text{with } \gamma = \frac{\lambda}{L_g} \text{ if } L_g > \lambda \\ M(\sqrt{h} + \frac{k}{\sqrt{h}})^\gamma, & \gamma \in (0, 1) \text{ if } L_g = \lambda \end{cases} \quad (42)$$

Also, if (20) and (21) are valid then the above estimations can be improved, i.e. it holds that:

$$|u(x) - u_k^h(x)| \leq \begin{cases} M(h + \frac{k}{\sqrt{h}}), & \text{if } 2L_g < \lambda \\ M(h^\gamma + \frac{k}{\sqrt{h}}), & \text{with } \gamma \in (0, 1) \text{ if } 2L_g = \lambda \\ M(h^{\gamma/2} + \frac{k}{\sqrt{h}}), & \text{with } \gamma = \frac{\lambda}{L_g} \text{ if } L_g < \lambda < 2L_g \\ M(h^{\gamma/2} + (\frac{k}{\sqrt{h}})^\gamma), & \text{with } \gamma \in (0, 1) \text{ if } L_g = \lambda \\ M(h^{\gamma/2} + (\frac{k}{\sqrt{h}})^\gamma), & \text{with } \gamma = \frac{\lambda}{L_g} \text{ if } L_g > \lambda \end{cases} \quad (43)$$

To achieve these results we need to prove some previous properties.

3.1 Introduction of Auxiliary Problems

3.1.1 Definition of u_T , optimal cost function for a problem with finite horizon

A key point of the proof of the explicit error estimates (42) and (43) is the introduction of the finite time optimization problem:

$$\inf_{a \in \mathcal{A}} \int_0^T f(y(t, a(t))) e^{-\lambda t} dt, \quad \text{where } y(\cdot) \text{ verifies (2)}$$

and the use of the associated optimal cost function u_T and its properties, with

$$u_T(x) = \inf_{a \in \mathcal{A}} \int_0^T f(y(t, a(t))) e^{-\lambda t} dt \quad (44)$$

3.1.2 Definition of u_n^h , discrete time optimal cost function for a problem with finite horizon

We define recursively for $0 \leq n \leq \mu = \frac{T}{h}$, the optimal cost function $u_n^h(x_i)$ associated to the optimization problem with finite horizon and step control functions corresponding to the discretization on time of problem (44). We proceed in the following way:

$$\begin{aligned} u_{n-1}^h &= A^h u_n^h \\ u_\mu^h &= 0 \end{aligned} \quad (45)$$

and we will denote $u_T^h = u_0^h$.

Note: In order to simplify the analysis of auxiliary problems and the proof of convergence properties, we will suppose, w.l.g. that $\frac{T}{h}$ is always an integer.

It is easy to see that:

$$u_T^h(x) = h \sum_{j=0}^{\mu-1} f(y^h(x, j), \bar{a}(jh)) (1 - \lambda h)^j, \quad \text{with } T = \mu h \quad (46)$$

where $\bar{a}(jh)$ is the control that realizes the minimum in (10), when A^h is applied to function u_j^h .

Remark 3.1: It must be noted that $\bar{a}(jh)$ depends on the initial position of the system x and it is defined recursively by the relation:

$$\begin{aligned}
 u_{n-1}^h(y^h(x, n-1)) &= (A^h u_n^h)(y^h(x, n-1)) = \\
 &= \min_{a \in A} \left((1 - \lambda h) u_n^h(y^h(x, n-1)) + h g(y^h(x, n-1), a) + h f(y^h(x, n-1), a) \right) = \\
 &= (1 - \lambda h) u_n^h(y^h(x, n-1)) + h g(y^h(x, n-1), \bar{a}((n-1)h)) + \\
 &\quad + h f(y^h(x, n-1), \bar{a}((n-1)h))
 \end{aligned} \tag{47}$$

where the sequence $y^h(x, j)$ is determined by the recursion (16).

3.1.3 Definition of $u_{k,n}^h$, fully discrete optimal cost function for a problem with finite horizon

We define recursively for $1 \leq n \leq \mu = \frac{T}{h}$ function $u_{k,n}^h$, fully discrete solution of the problem with finite horizon, in the following way:

$$\begin{aligned}
 u_{k,n-1}^h &= A_k^h u_{k,n}^h \\
 u_{k,\mu}^h &= 0
 \end{aligned} \tag{48}$$

and we will denote $u_{k,T}^h = u_{k,0}^h$.

3.2 Properties of Auxiliary Functions:

The procedure applied to prove the estimate of $|u(x) - u_k^h(x)|$, consists in the decomposition of it in the following four successive estimates:

$$\begin{aligned}
 |u(x) - u_k^h(x)| &\leq |u(x) - u_T(x)| + |u_T(x) - u_T^h(x)| + \\
 &\quad + |u_T^h(x) - u_{k,T}^h(x)| + |u_{k,T}^h(x) - u_k^h(x)|
 \end{aligned} \tag{49}$$

Estimates for the right terms are given by the properties established in the following lemmas:

Lemma 3.1: Under assumptions H_1 , we have:

$$|u(x) - u_T(x)| \leq \frac{M_f}{\lambda} e^{-\lambda T} \quad (50)$$

Lemma 3.2: Under assumptions H_1 and (11), we have:

$$|u_T(x) - u_T^h(x)| \leq \begin{cases} C\sqrt{h} & \text{if } L_g < \lambda \\ C\sqrt{h} e^{(L_g - \lambda)T} & \text{if } L_g > \lambda \\ C\sqrt{h} T & \text{if } L_g = \lambda \end{cases} \quad (51)$$

Also if (20) and (21) are valid, then the above estimations can be improved; i.e. it holds that:

$$|u_T(x) - u_T^h(x)| \leq \begin{cases} Ch & \text{if } 2L_g < \lambda \\ Ch e^{(2L_g - \lambda)T} & \text{if } 2L_g > \lambda \\ Ch T & \text{if } 2L_g = \lambda \end{cases} \quad (52)$$

Lemma 3.3: Let H_1 and (11) be satisfied. Then:

$$|u_{k,T}^h(x) - u_k^h(x)| \leq \frac{M_f}{\lambda} (1 - \lambda h)^\mu \leq \frac{M_f}{\lambda} e^{-\lambda T} \quad (53)$$

Lemma 3.4: Under assumptions H_1 , (11), (27) and (28) we have:

$$|u_T^h(x) - u_{k,T}^h(x)| \leq \begin{cases} M \frac{k}{\sqrt{h}} & \text{if } L_g < \lambda \\ M e^{(L_g - \lambda)T} \frac{k}{\sqrt{h}} & \text{if } L_g > \lambda \\ M T \frac{k}{\sqrt{h}} & \text{if } L_g = \lambda \end{cases} \quad (54)$$

3.3 Proof of Theorem 3.1

Proof: In order to simplify the proof of this theorem, we will refer the reader to the appendix to find there the proofs of Lemmas 3.1, 3.3 and 3.4; the proof of Lemma 3.2 is contained in [3], [20].

By virtue of Lemmas 3.1 to 3.4, we have that the analysis of (49) is reduced to deal with the following three cases:

• **Case $L_g > \lambda$**

In this case (50), (51), (53) and (54) give:

$$\begin{aligned} \left| u(x) - u_k^h(x) \right| &\leq \left| u(x) - u_T(x) \right| + \left| u_T(x) - u_T^h(x) \right| + \\ &+ \left| u_T^h(x) - u_{k,T}^h(x) \right| + \left| u_{k,T}^h(x) - u_k^h(x) \right| \leq \\ &\leq \frac{M_f}{\lambda} e^{-\lambda T} + C\sqrt{h} e^{(L_g - \lambda)T} + M e^{(L_g - \lambda)T} \frac{k}{\sqrt{h}} + \frac{M_f}{\lambda} e^{-\lambda T} \end{aligned} \quad (55)$$

and we obtain:

$$\left| u(x) - u_k^h(x) \right| \leq M_1 \left(e^{-\lambda T} + e^{(L_g - \lambda)T} \left(\sqrt{h} + \frac{k}{\sqrt{h}} \right) \right) \quad (56)$$

where

$$M_1 = \max \left(2 \frac{M_f}{\lambda}, C, M \right)$$

Minimizing (56) in T , we obtain:

$$\left| u(x) - u_k^h(x) \right| \leq K \left(\sqrt{h} + \frac{k}{\sqrt{h}} \right)^\gamma, \quad (57)$$

where

$$K = M_1 \left(\left(\frac{L_g - \lambda}{\lambda} \right)^{\frac{\lambda}{L_g}} + \left(\frac{L_g - \lambda}{\lambda} \right)^{\frac{\lambda}{L_g} - 1} \right)$$

with

$$\gamma = \frac{\lambda}{L_g}$$

Note: Indeed, the minimum in (56) should be computed on the discrete set $\{nh/n=0,1,\dots\}$; if we do the

minimization procedure on the set $[0, \infty)$, we obtain an equivalent result up to an additional term of order h , that does not essentially modifies the estimate (57).

• Case $L_g < \lambda$

In this case, by virtue of (51), (55) becomes:

$$|u(x) - u_k(x)| \leq \frac{M_f}{\lambda} e^{-\lambda T} + C\sqrt{h} + M \frac{k}{\sqrt{h}} + \frac{M_f}{\lambda} e^{-\lambda T}$$

then, by taking the limite $T \rightarrow \infty$, we obtain

$$|u(x) - u_k^h(x)| \leq M_1 \left(\sqrt{h} + \frac{k}{\sqrt{h}} \right) \quad (58)$$

where $M_1 = \max(C, M)$

• Case $L_g = \lambda$

In this case, by virtue of (51) and (54), (55) becomes:

$$|u(x) - u_k^h(x)| \leq \frac{M_f}{\lambda} e^{-\lambda T} + C\sqrt{h} T + M \frac{k}{\sqrt{h}} T + \frac{M_f}{\lambda} e^{-\lambda T}$$

and we have

$$|u(x) - u_k^h(x)| \leq M_1 \left(e^{-\lambda T} + \left(\sqrt{h} + \frac{k}{\sqrt{h}} \right) T \right) \quad (59)$$

where $M_1 = \max\left(2 \frac{M_f}{\lambda}, C, M\right)$

For $\frac{1}{\lambda}(\sqrt{h} + \frac{k}{\sqrt{h}}) \leq 1$, the minimum of the rigth hand side of (59) is realized by

$$\bar{T} = -\frac{1}{\lambda} \ln \frac{1}{\lambda} \left(\sqrt{h} + \frac{k}{\sqrt{h}} \right)$$

so, replacing it in (59), we have:

$$|u(x) - u_k^h(x)| \leq M_1 \left(h + \frac{1}{\lambda} \left(h + \frac{k}{\sqrt{h}} \right) - \frac{1}{\lambda} \left(h + \frac{k}{\sqrt{h}} \right) \ln \frac{1}{\lambda} \left(h + \frac{k}{\sqrt{h}} \right) \right)$$

but, as the following property holds

$$-x \ln x \leq K x^\gamma, \gamma \in (0,1), K = \frac{1}{1-\gamma} e^{-1},$$

then we have:

$$\left| u(x) - u_k^h(x) \right| \leq C \left(\sqrt{h} + \frac{k}{\sqrt{h}} \right)^\gamma$$

where $C = \frac{M_1}{\lambda} \max(1, \frac{1}{1-\gamma} e^{-1})$

The estimate (43) can be proved in a similar way, taking in mind (52)

□

3.4 Discussion. Relation between h and k

Estimates (42), (43) give a general expression of the error estimates, depending on both parameters h, k. From them, two issues arise immediatly:

- the optimal relation between h and k.
- the optimality of the estimates.

The following paragraphs deal with both issues.

3.4.1 Optimality studies

Theorem 3.1 gives an estimate of the error of the fully discrete solution as a function of both parameters h, k. The parameter k is related to the size of the triangulation Ω_k and to the cardinality N of the mesh; obviously, it measures the quantity of information to be processed. Once this is realized, it arises the problem of parameter h optimization, i.e. to determine the value of h that brings the least total error considering that a mesh of fixed size k is given. This optimal value of h is found minimizing the expression (42) and (43); the answer is different provided we deal with the general case (where only (42) is valid) or the regular case, where the semiconcavity assumptions imply the sharper estimate (43).

a) General case

Here, only the estimate (42) holds. If we consider the minimization in h , we obtain that $\bar{h} = k$ realizes the minimum in (42). This result coincides with the usual case, because generally it is taken a time step h of the same order of k , i.e. there exists m_1, m_2 such that:

$$m_1 k \leq h \leq m_2 k$$

In this case the estimate (42) becomes

$$\left| u(x) - u_k^h(x) \right| \leq \begin{cases} M k^{1/2} & \text{if } L_g < \lambda \\ M k^{\gamma/2}, & \text{with } \gamma = \frac{\lambda}{L_g} \quad \text{if } L_g > \lambda \\ M k^{\gamma/2}, & \gamma \in (0,1) \quad \text{if } L_g = \lambda \end{cases} \quad (60)$$

Remark 3.2: Estimate (60) indicates which is the best result it can be expected for this methodology of approximation, because estimate (60) is optimal as it is clearly seen looking at the counterexample given in §4.

Remark 3.3: In [5], [11], [12], [29], several errors estimates have been discussed, in [12], M. Falcone has established the estimate:

$$\left| u(x) - u_k^h(x) \right| \leq C \left(h + \frac{k}{h} \right)$$

In the usual case $h \sim k$ this estimate is useless, because convergence cannot be derived from it. Indeed, the real phenomenon of convergence is completely described by the estimation (60) and the counterexample given in §4.

b) Regular case

Here, the estimate (43) holds. Now we consider the minimization in h , in the different cases we obtain:

• Case $\lambda > 2L_g$

We obtain that $\bar{h} = \frac{1}{2}k^{2/3}$ realizes the minimum; then the associated estimation becomes:

$$|u(x) - u_k^h(x)| \leq M k^{2/3}$$

• Case $L_g < \lambda < 2L_g$

We obtain that $\bar{h} = \frac{1}{\gamma} k^{2/(\gamma+1)}$ realizes the minimum, in this case the estimation is

$$|u(x) - u_k^h(x)| \leq M k^{\gamma/(\gamma+1)}$$

• Case $L_g > \lambda$

We obtain that $\bar{h} = k$ realizes the minimum, in this case the estimation is:

$$|u(x) - u_k^h(x)| \leq M k^{\gamma/2}$$

Making a summary we obtain:

$$|u(x) - u_k^h(x)| \leq \begin{cases} M k^{2/3} & \text{if } 2L_g < \lambda \\ M k^{\gamma/(\gamma+1)} & \text{if } L_g < \lambda < 2L_g \\ M k^{\gamma/2} & \text{if } L_g > \lambda \end{cases} \quad (61)$$

Remark 3.4: The estimates (61) indicate a continuous behavior of the maximum error corresponding to this type of discretizations. Also, there is a continuous variation of the point that realizes the minimum. This phenomenon is illustrated in Figure 1 and 2.

3.4.3 A problematic case $h \ll k$

When $h \ll k$ the estimate obtained in (42) although true, is useless and one might wonder if in this case convergence of the method is lost. Fortunately, this is not true and what is really happening with the convergence of solution u_k^h is shown by the equivalence given in the following lines. In a few words, we will prove that in this case h can be replaced by \hat{h} with $\hat{h} \sim k$, without any harmful effects because the following inequality is valid:

$$\| u_k^h - u_k^{\tilde{h}} \| \leq (h + \tilde{h} + |h - \tilde{h}|) M_f \quad (62)$$

To see this, we are going to use the following estimate:

$$\| \hat{u}_k^h - \hat{u}_k^{\tilde{h}} \| \leq M_f |h - \tilde{h}| \quad (63)$$

where \hat{u}_k^h is the unique solution of the problem \hat{P}_k .

Let b be as in Figure 3. By virtue of (28) there exists m_1 such that for any $h \leq m_1 k$, for any vertex x_i and for any control $a \in A$, the point $x_i + hg(x_i, a)$ belongs to a neighboring simplex S_j^k .

We will suppose in the rest of this section that the following inequality holds:

$$h \leq m_1 k \quad (64)$$

So

$$\begin{aligned} \hat{u}_k^h(x) &= \min_{a \in A} \left(\frac{1}{1 + \lambda h} \hat{u}_k^h(x + hg(x, a)) + hf(x, a) \right) = \\ &= \min_{a \in A} \left(\frac{1}{1 + \lambda h} \left(1 - \frac{\|g(x, a)\| h}{d} \right) \hat{u}_k^h(x) + \frac{h \|g(x, a)\|}{d} \hat{u}_k^h(b) \right) + hf(x, a) \end{aligned} \quad (65)$$

with $d = \|x - b\|$, then

$$\hat{u}_k^h(x) \leq \frac{\|g(x, \bar{a}_1)\|}{d \lambda + \|g(x, \bar{a}_1)\|} \hat{u}_k^h(b) + \frac{(1 + \lambda h) d}{\lambda d + \|g(x, \bar{a}_1)\|} f(x, \bar{a}_1) \quad (66)$$

where \bar{a}_1 realizes the minimum in (65), for $h = \tilde{h}$. Hence

$$\hat{u}_k^h(x) - \hat{u}_k^{\tilde{h}}(x) \leq \beta \left(\hat{u}_k^h(b) - \hat{u}_k^{\tilde{h}}(b) \right) + \left(\frac{(1 + \lambda h) d}{\lambda d + \|g(x, \bar{a}_1)\|} - \frac{(1 + \lambda \tilde{h}) d}{\lambda d + \|g(x, \bar{a}_1)\|} \right) f(x, \bar{a}_1) \quad (67)$$

with

$$\beta = \frac{\|g(x, \bar{a}_1)\|}{d \lambda + \|g(x, \bar{a}_1)\|}$$

Analogously, we can estimate $\hat{u}_k^{\tilde{h}}(x) - \hat{u}_k^h(x)$ with the same bound appearing in (67) and then, we have:

$$\left\| \hat{u}_k^h - \hat{u}_k^{\tilde{h}} \right\| \leq M_f |h - \tilde{h}|$$

By (38) and (63), we obtain:

$$\left\| u_k^h - u_k^{\tilde{h}} \right\| \leq \left\| u_k^h - \hat{u}_k^h \right\| + \left\| \hat{u}_k^h - \hat{u}_k^{\tilde{h}} \right\| + \left\| \hat{u}_k^{\tilde{h}} - u_k^{\tilde{h}} \right\| \leq M_f (h + \tilde{h} + |h - \tilde{h}|)$$

From this inequality arises the above mentioned equivalence (62).

3.4.4 Study of an asymptotic case

We may consider the problem \hat{P}_k and let the parameter h tend to zero. In that case we will arrive to the following fixed point problem

$$w_k(x) = \min_{a \in A} \left(\frac{\|g(x, a)\|}{\lambda d + \|g(x, a)\|} w_k(b) + \frac{d}{\lambda d + \|g(x, a)\|} f(x, a) \right) \quad (68)$$

where $d = \|b - x\|$, and b is the point described in the previous section and depicted in Figure 3.

This expression of the fully discrete problem was introduced in [17], where the convergence of the discrete solution w_k to the continuous solution u was proved for the case of Lipschitz continuous solution (for which the heavy assumption $\lambda > L_g$ is necessary). As a simple by-product of the results of this paper we can obtain, for the expression (68) of the fully discrete problem, the convergence of solutions w_k in the general case (where λ is arbitrary and u is only Hölder continuous).

By (65) we have

$$\hat{u}_k^h(x) \leq \frac{\|g(x, \bar{a})\|}{\lambda d + \|g(x, \bar{a})\|} \hat{u}_k^h(b) + \frac{(1 + \lambda h) d}{\lambda d + \|g(x, \bar{a})\|} f(x, \bar{a})$$

where \bar{a} realizes the minimum in (68).

Then, we have:

$$\hat{u}_k^h(x) - w_k(x) \leq \beta (\hat{u}_k^h(b) - w_k(b)) + \left(\frac{(1 + \lambda h) d}{\lambda d + \|g\|} - \frac{d}{\lambda d + \|g\|} \right) f(x, \bar{a})$$

where

$$\beta = \frac{\|g(x, \bar{a})\|}{\lambda d + \|g(x, \bar{a})\|}$$

Then

$$\hat{u}_k^h(x) - w_k(x) \leq \beta \|\hat{u}_k^h - w_k\| + \frac{\lambda h d}{\lambda d + \|g(x, \bar{a})\|} M_f$$

Analogously, we can obtain the estimate:

$$w_k(x) - \hat{u}_k^h(x) \leq \hat{\beta} \|\hat{u}_k^h - w_k\| + \frac{\lambda h d}{\lambda d + \|g(x, \hat{a})\|} M_f$$

where

$$\hat{\beta} = \frac{\|g(x, \hat{a})\|}{\lambda d + \|g(x, \hat{a})\|}$$

and \hat{a} realizes the minimum in (65).

By simple algebraic manipulations we arrive to the estimate:

$$\|\hat{u}_k^h - w_k\| \leq M_f h \quad (69)$$

Now if we let $h \rightarrow 0$, we obtain $\hat{u}_k^h \rightarrow w_k$.

Finally, if u is the viscosity solution of problem (5), then by (38), (42) and (69), we have:

$$\|u - w_k\| \leq \|u - u_k^h\| + \|u_k^h - \hat{u}_k^h\| + \|\hat{u}_k^h - w_k\| \leq (\sqrt{h} + \frac{k}{\sqrt{h}})^\gamma + 2h M_f$$

This last inequality implies the convergence of solutions w_k for the set-up (68) of the discrete problem. Also, an explicit estimate of the rate of convergence can be obtained, because by virtue of (64), if we take $h = m_1 k$ in the above expression, we get

$$\|u - w_k\| \leq C k^{\gamma/2} \quad (70)$$

4. OPTIMALITY OF ESTIMATIONS $k^{1/2}$, $k^{2/3}$ and $k^{\gamma/2}$

4.1 Optimality of Estimate $k^{1/2}$

Let us consider a problem with the following characteristics:

The domain where the system moves is the circular ring of extremal radii \bar{r}_1, \bar{r}_3 . Let (θ, r) be the polar coordinates of a general point x , i.e. $x=(\theta, r)$, $\theta \in [-\pi, \pi]$ and $\bar{r}_1 \leq r \leq \bar{r}_3$. The evolution of the system is described by the equations:

$$\dot{\theta} = 0$$

$$\dot{r} = \rho(r, a)$$

i.e. the system follows a radial movement; in consequence, we have for g written in polar coordinates:

$$g(\theta, r, a) = (0, \rho(r, a))'$$

For $\bar{r}_1 > 0$, $\bar{r}_2 = 2\bar{r}_1$, $\bar{r}_3 = 2\bar{r}_2$, $p < 1$, we define $A = \{a_1, a_2\}$ and:

$$\begin{cases} \rho(r, a_1) = 0 & \forall -\pi \leq \theta \leq \pi, \quad \bar{r}_1 \leq r \leq \bar{r}_3 \\ f(\theta, r, a_1) = \lambda(|\theta|+1)(|r-r_2|+1) & \forall -\pi \leq \theta \leq \pi, \quad \bar{r}_1 \leq r \leq \bar{r}_3, \end{cases}$$

$$\rho(r, a_2) = \begin{cases} 1 & \forall -\pi \leq \theta \leq \pi, \quad \bar{r}_1 \leq r \leq \bar{r}_2 \\ -\frac{1}{2\bar{r}_1}r + 2 & \forall -\pi \leq \theta \leq \pi, \quad \bar{r}_2 \leq r \leq \bar{r}_3 \end{cases}$$

$$f(\theta, r, a_2) = \begin{cases} \lambda(|\theta|+1) & \forall -\pi \leq \theta \leq \pi, \quad \bar{r}_1 \leq r \leq \bar{r}_2 \\ \lambda(|\theta|+1)(5(r-r_2)+1) & \forall -\pi \leq \theta \leq \pi, \quad \bar{r}_2 \leq r \leq \bar{r}_3 \end{cases} \quad (71)$$

It is easy to see that

$$u(0, \bar{r}_1) = 1$$

and that the optimal feedback control is a_2 for $\bar{r}_1 \leq r \leq \bar{r}_2$ and a_1 for $\bar{r}_2 \leq r \leq \bar{r}_3$.

We want to compute the discretized solution, so, we consider a triangulation of Ω of the type shown in

Figure 4, with Ω a circular ring of radii \bar{r}_1 and \bar{r}_3 . Let $p \in \mathcal{N}^+$, for the chosen triangulation the following relation holds:

$$n = p \quad (72)$$

$$\alpha = \frac{2\pi}{p}$$

$$k = 2 r_3 \sin\left(\frac{\pi}{p}\right) \quad (73)$$

$$\hat{h}_\nu = \frac{h - r_\nu(\cos(\alpha/2) - 1)}{\cos(\alpha/2)}$$

$$r_{\nu+1} = r_\nu + \hat{h}_\nu \quad \nu = 0, 1, \dots, n-1 \quad r_0 = \bar{r}_1$$

$$h \text{ such that } \sum_{q=0}^{n-1} \hat{h}_q = \bar{r}_2 - \bar{r}_1 \quad (74)$$

For $\bar{r}_1 \leq r \leq \bar{r}_2$, we will identify the nodes of the triangulation with a pair of indices ν, μ

$$0 \leq \nu \leq n \quad (75)$$

$$-\frac{p}{2} \leq \mu \leq \frac{p}{2} \quad (76)$$

Denoting with $x(\nu, \mu)$ the node associated to the pair (ν, μ) , where if $x(\nu, \mu) = (\theta, r)$ it is

$$\theta = \mu \alpha - \frac{\alpha}{2} (\nu - 2 \lfloor \frac{\nu}{2} \rfloor)$$

$$r = r_\nu$$

For simplicity of writing we will use in this section the notation $w = u_k^h$.

For the values of (ν, μ) given by (75) and (76), w can be computed recursively as we will do in the following lines. The key point of the development is the fact that:

$$x(\nu, \mu) + g(x(\nu, \mu), a_2) h = \frac{1}{2} (x(\nu+1, \mu) + x(\nu+1, \mu+1))$$

Let $x(0, 0) = (0, r_1)$, then by (29):

$$w(x(0,0)) = \left((1-\lambda h) w(x(0,0) + h g(x(0,0), a_2)) + h f(x(0,0), a_2) \right)$$

but

$$w(x(0,0) + h g(x(0,0), a_2)) = \frac{1}{2} w(x(1,0)) + \frac{1}{2} w(x(1,1))$$

then the above formula becomes:

$$w(x(0,0)) = (1-\lambda h) \left(\frac{1}{2} w(x(1,0)) + \frac{1}{2} w(x(1,1)) \right) + h f(x(0,0), a_2)$$

Duplicating this procedure and replacing $w(x(1,0))$ y $w(x(1,1))$ in formula (29), we obtain

$$\begin{aligned} w(x(0,0)) = & (1-\lambda h)^2 \left(\frac{1}{2} \right)^2 \left(w(x(2,-1)) + 2w(x(2,0)) + w(x(2,1)) \right) + \\ & + h f(x(0,0), a_2) + h \frac{1-\lambda h}{2} \left(f(x(1,0), a_2) + f(x(1,1), a_2) \right) \end{aligned} \quad (77)$$

considering that the minimum of (71) is realized at $\theta=0$ and has the value λ , we obtain:

$$w(x(0,0)) \geq (1-\lambda h)^2 \left(\frac{1}{2} \right)^2 \left(w(x(2,-1)) + 2w(x(2,0)) + w(x(2,1)) \right) + h \lambda (1 + (1-\lambda h))$$

After n steps

$$w(x(0,0)) \geq (1-\lambda h)^n \left(\frac{1}{2} \right)^n \sum_{j=0}^n \binom{n}{j} w(x(n, j - \lfloor \frac{n}{2} \rfloor)) + h \lambda \sum_{i=1}^n (1-\lambda h)^{i-1} \quad (78)$$

where

$$\|x(n, j - \lfloor \frac{n}{2} \rfloor)\| = r_2$$

but when $\|x\|=r_2$

$$w(x) = (1-\lambda h) w(x + h g(x, a_1)) + h f(x, a_1) = (1-\lambda h) w(x) + h f(x, a_1)$$

and then

$$w(x) = \frac{f(x, a_1)}{\lambda} = |\theta| + 1$$

by formula (78), we have:

$$w(x(0,0)) \geq (1-\lambda h)^n \left(\frac{1}{2} \right)^n \sum_{j=0}^n \binom{n}{j} \left(\left| j - \frac{1}{2} n \right| \alpha + 1 \right) + h \lambda \sum_{i=1}^n (1-\lambda h)^{i-1} \quad (79)$$

by Stirling's inequality, we have

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}}$$

we can estimate the individual terms $\binom{n}{j} \left(\frac{1}{2}\right)^n$ because we know that:

$$\binom{n}{j} \left(\frac{1}{2}\right)^n \geq \binom{n}{\frac{n}{2} + \sqrt{n}} \left(\frac{1}{2}\right)^n \quad \forall j / \frac{n}{2} - \sqrt{n} \leq j \leq \frac{n}{2} + \sqrt{n}$$

assuming \sqrt{n} integer, we have

$$\frac{n!}{(\frac{n}{2} + \sqrt{n})! (\frac{n}{2} - \sqrt{n})!} > \frac{1}{\sqrt{2\pi}} e^{\eta} \left(\frac{1}{2}\right)^{-n-1} \frac{1}{\sqrt{n}} \left(1 - \frac{4}{n}\right)^{-\frac{n}{2} - \frac{1}{2}} (\phi(n))^{-\sqrt{n}}$$

where

$$\eta = \frac{1}{12n+1} - \frac{1}{6n+12\sqrt{n}} - \frac{1}{6n-12\sqrt{n}}$$

and

$$\phi(n) = \frac{(\frac{n}{2} - 2\sqrt{n})}{(\frac{n}{2} + 2\sqrt{n})}$$

but:

$$e^{\eta} \rightarrow 1, \text{ when } n \rightarrow \infty$$

$$\left(1 - \frac{4}{n}\right)^{-\frac{n}{2} - \frac{1}{2}} \rightarrow e^2, \text{ when } n \rightarrow \infty$$

$$(\phi(n))^{-\sqrt{n}} \rightarrow e^{-4}, \text{ when } n \rightarrow \infty$$

then:

$$\binom{n}{j} \left(\frac{1}{2}\right)^n \geq \frac{C}{\sqrt{n}} \quad \forall j / \frac{n}{2} - \sqrt{n} \leq j \leq \frac{n}{2} + \sqrt{n},$$

for a large enough n .

By virtue of this last inequality we can estimate (79) in the following way, for a large enough n , considering also that:

$$h \lambda \sum_{i=1}^n (1 - \lambda h)^{i-1} \rightarrow 1 - e^{-\lambda T}$$

and, $T = \bar{r}_2 - \bar{r}_1 = n h$

$$(1 - \lambda h)^n \rightarrow e^{-\lambda T}, \text{ i.e. } (1 - \lambda h)^n - e^{-\lambda T} \geq -Ch$$

$$\begin{aligned}
w(x(0,0)) &\geq (1-\lambda h)^n \left(\frac{1}{2}\right)^n \sum_{j=0}^n \binom{n}{j} \left(\left|j - \frac{1}{2}n\right|\alpha + 1\right) + h\lambda \sum_{i=1}^n (1-\lambda h)^{i-1} \geq \\
&\geq e^{-\lambda T} \frac{C}{\sqrt{n}} \left(\sum_{j=\xi_1}^{\xi_2} \left|j - \frac{1}{2}n\right|\alpha\right) + e^{-\lambda T} \sum_{j=0}^n \binom{n}{j} \left(\frac{1}{2}\right)^n + 1 - e^{-\lambda T} - Ch
\end{aligned}$$

where

$$\xi_1 = \frac{n}{2} - \sqrt{n} \quad \text{and} \quad \xi_2 = \frac{n}{2} + \sqrt{n}, \text{ then:}$$

$$w(x(0,0)) \geq e^{-\lambda T} \frac{C}{\sqrt{n}} \alpha \left(\sum_{j=0}^{\sqrt{n}} j\right) + 1 - Ch \geq C \frac{1}{\sqrt{n}} + 1 - Ch \geq C_1 \sqrt{k} + 1$$

for a large enough n , by virtue of (74), (72) and (73). Hence

$$\max_{x \in \Omega} \left| u_k^h(x) - u(x) \right| \geq C_1 \sqrt{k}$$

and we see that the estimate (60) is optimal for this type of approximations.

4.2 Optimality of Estimate $k^{2/3}$

As it is shown in Figure 5, we consider the example above, but now each point x_i is related with the point $x_i + hg(x_i, a)$, where the new time step h is of order $k^{2/3}$, i.e., strictly

$$h = (\bar{r}_2 - \bar{r}_1)/p^{1/3}, \quad k = 2\bar{r}_3 \sin(\pi/(p+1)),$$

which implies

$$\lim_{p \rightarrow \infty} (h/k^{2/3}) = \bar{r}_1^{1/3} / (4\pi^{2/3})$$

Following the same procedure we obtain:

$$w(x(0,0)) = (1-\lambda h) \left(\frac{1}{2} w(x(p^{1/3}, 0)) + \frac{1}{2} w(x(p^{1/3}, 1)) \right) + h f(x(0,0), a_2)$$

Duplicating this procedure we obtain an equation similar to (78), and in parallel with the same estimate above, we have (we suppose $p^{1/3}$ integer):

$$w(x(0,0)) \geq (1-\lambda h)^m \left(\frac{1}{2}\right)^m \sum_{j=0}^m \binom{m}{j} w(x(m p^{1/3} j - [\frac{m}{2}])) + h \lambda \sum_{i=1}^m (1-\lambda h)^{i-1}$$

when $m = \bar{m} = p^{2/3}$, we are at $r = r_2$ and then we have:

$$w(x(0,0)) \geq C_1 \frac{1}{\sqrt{\bar{m}}} \bar{m} \alpha + 1 = C_2 k^{2/3} + 1$$

hence

$$\max_{x \in A} \left| u_k^h(x) - u(x) \right| > C_2 k^{2/3}$$

So, we see in this example that not only the procedure of approximation can be improved in the regular case (where (61) holds) but, that actually this improvement is the best one we can expect to obtain,

4.3 Optimality of Estimate $k^{\gamma/2}$

In this example, we consider that the domain Ω where the system moves is the circular ring of extremal radii \bar{r}_1, \bar{r}_3 . If (θ, r) are the polar coordinate of a general point x ,

$$x = (\theta, r), \bar{r}_1 = 1, \bar{r}_2 = 2, \bar{r}_3 = 3, -\pi \leq \theta \leq \pi$$

the evolution of the system is described by the equations:

$$\dot{\theta} = 2\pi$$

$$\dot{r} = \rho(r, a)$$

in consequence, we have for g written in polar coordinates:

$$g(\theta, r, a) = (2\pi, \rho(r, a))'$$

We define:

$$\left| \begin{array}{ll} \rho(r, a_1) = 0 & \forall -\pi \leq \theta \leq \pi, \quad 1 \leq r \leq 3 \\ f(\theta, r, a_1) = \lambda & \forall -\pi \leq \theta \leq \pi, \quad 1 \leq r \leq 3, \end{array} \right.$$

$$\rho(r, a_2) = \left| \begin{array}{ll} r-2 & \forall -\pi \leq \theta \leq \pi, \quad 1.2 \leq r \leq 2.8 \\ 4(1-r) & \forall -\pi \leq \theta \leq \pi, \quad 1 \leq r \leq 1.2 \\ 4(3-r) & \forall -\pi \leq \theta \leq \pi, \quad 2.8 \leq r \leq 3 \end{array} \right.$$

$$f(\theta, r, a_2) = \begin{cases} 0 & \forall -\pi \leq \theta \leq \pi, \quad 1.5 \leq r \leq 2.5 \\ \frac{1.5-r}{0.3} \lambda & \forall -\pi \leq \theta \leq \pi, \quad 1 \leq r \leq 1.5 \\ \frac{r-2.5}{0.3} \lambda & \forall -\pi \leq \theta \leq \pi, \quad 2.5 \leq r \leq 3 \end{cases}$$

i.e. the system follows a spiral movement (in fact, at $r=\bar{r}_2$ a circular movement along the unstable limite cycle $r = \bar{r}_2$).

It is obvious that for this system, $L_g = 1$; so, to consider the case $\gamma = \lambda / L_g < 1$, we will suppose $\lambda < 1$.

It is easy to see that

$$u(\theta, \bar{r}_2) = 0, \quad \forall \theta$$

and that the optimal control is:

$$a_2 \text{ for } 1.2 < r < 2.8 \text{ and } a_1 \text{ for } 1 \leq r \leq 1.2, \text{ or } 2.8 \leq r \leq 3$$

To compute the discretized solution, we consider the triangulation of Ω shown in Figure 6 (to depict more clearly the triangulation employed, in Figure 6 only a portion of Ω is shown). For $-n \leq i \leq n$, $0 \leq j \leq n$, we denote with $x(i,j)$ the node associated to the pair (i,j) . If $x(i,j) = (\theta, r)$, we have: (being $nh = 1$ and $\sqrt{n}/2$ integer)

$$\begin{aligned} \theta(i,j) &= j \frac{2\pi}{n} \\ r(i,j) &= 2 + ih \quad -\sqrt{n}/2 \leq i \leq \sqrt{n}/2, \text{ for } j \text{ even} \\ r(i,j) &= 2 + (i - \frac{1}{2})h \quad 0 < i \leq \sqrt{n}/2, \text{ for } j \text{ odd} \\ r(i,j) &= 2 + (i + \frac{1}{2})h \quad -\sqrt{n}/2 \leq i < 0, \text{ for } j \text{ odd} \\ r(i,j) &= 2 + (i - \frac{1}{2})h \quad \sqrt{n}/2 < i \leq n \quad \forall j \\ r(i,j) &= 2 + (i + \frac{1}{2})h \quad -n < i < -\sqrt{n}/2 \quad \forall j \end{aligned}$$

From here results

$$k = h \sqrt{1 + (6\pi)^2}$$

The proof of inequality:

$$\max_{x \in A} |u_k^h(x) - u(x)| > C_2 k^{7/2} \quad (80)$$

will be divided in two steps:

$$\text{a) } w(i,j) \geq M |ih|^\lambda, \text{ for all } i / n' \geq |i| > \frac{\sqrt{n}}{2}, n' = 0.8/h \quad (81)$$

$$\text{b) } w(0,j) \geq C w(\frac{\sqrt{n}}{2}, j) \quad (82)$$

a) As for $\sqrt{n}/2 < i \leq n'$ $0 \leq j \leq m$, $f(x(i,j)) \geq 0$, the discrete equation (36) implies:

$$w(x(i,j)) \geq (1-\lambda h) \left((\frac{1}{2} + r(i,j)-2) w(x(i+1,j+1)) + (\frac{1}{2} - r(i,j)+2) w(x(i,j+1)) \right) \quad (83)$$

with the obvious symmetry condition for j :

$$w(x(i,j)) = w(x(i,j')) \quad \forall j, j'$$

and the final condition

$$w(x(n',j)) = 1 \quad \forall j$$

If we define $\sigma(i,j) = \frac{1}{0.8^\lambda} (ih)^\lambda$, it follows that:

$$\sigma(i,j) = (1-\lambda h) \left((\frac{1}{2} + x(i,j)) \sigma(i+1,j+1) + (\frac{1}{2} - x(i,j)) \sigma(i-1,j+1) \right) + \phi(h) \quad (84)$$

where $\phi(h) \leq Ch^2$.

By (83) and (84), we have

$$w(i,j) - \sigma(i,j) \geq -Ch$$

from where it easily follows (80) for $i > 0$. Also, by reasons of symmetry, (80) is also valid for $i < 0$.

b) To prove (82), we restrict the study of (11) to $-\frac{\sqrt{n}}{2} \leq i \leq \frac{\sqrt{n}}{2}$, in this case the equation (11) verified by w takes the form:

$$w(x(i,j)) = (1-\lambda h) \left(\left(\frac{1}{2} + r(i,j)-2\right) w(x(i+1,j+1)) + \left(\frac{1}{2} - r(i,j)+2\right) w(x(i-1,j+1)) \right)$$

with the boundary condition $w(x(-\frac{\sqrt{n}}{2}+1,j))$, which has just been estimated in a).

The proof of (82) comprises three steps with the proofs of the properties described in (b1), (b2) and (b3).

$$b1) \quad w(x(i,j)) \leq w(x(i+1,j)) \quad i \geq 0$$

$$b2) \quad w(x(i,j)) \geq \hat{w}(x(i,j)) \quad -\frac{\sqrt{n}}{2} \leq i \leq \frac{\sqrt{n}}{2} \quad \forall j$$

where

$$\hat{w}(x(i,j)) = (1-\lambda h) \left(\frac{1}{2} \hat{w}(x(i+1,j+1)) + \frac{1}{2} \hat{w}(x(i-1,j+1)) \right)$$

with the boundary conditions :

$$\hat{w}(x(\sqrt{n}/2 + 1,j)) = w(x(\sqrt{n}/2 + 1,j)) \quad \forall j$$

$$\hat{w}(x(-\sqrt{n}/2 - 1,j)) = w(x(-\sqrt{n}/2 - 1,j)) \quad \forall j$$

Also, the following obvious symmetry condition for i holds:

$$w(x(-i,j)) = w(x(i,j))$$

and the symmetry condition for j :

$$w(x(i,j)) = w(x(i,j+2)) \quad (85)$$

where in (85) the addition is understood in the sense of $\text{mod}(n)$.

$$b3) \quad \hat{w}(x(i,j)) \geq \bar{w}(x(i,j)) \quad -\frac{\sqrt{n}}{2} \leq i \leq \frac{\sqrt{n}}{2} \quad \forall j$$

where:

$$\bar{w}(x(i,j)) = \left(1 - \frac{\lambda}{h} q^2 + \frac{\lambda}{h} (ih)^2 \right) \xi$$

with $q=(\sqrt{n}/2+1)h$, and $\xi = \hat{w}(x(\frac{\sqrt{n}}{2}+1, j))$, then (for enough large n):

$$\bar{w}(x(0, j)) \geq \frac{1}{2} \hat{w}(x(\frac{\sqrt{n}}{2}+1, j)),$$

finally by (b2) and (b3) we obtain (82), because:

$$w(0, j) \geq C \hat{w}(0, j) \geq C \bar{w}(0, j) \geq \frac{C}{2} \hat{w}(\frac{\sqrt{n}}{2}+1, j) = \frac{C}{2} w(\frac{\sqrt{n}}{2}+1, j)$$

The proof of (b1), (b2) and (b3) are straightforward and will be omitted here in order to simplify the presentation of the counterexample.

Finally, we arrive to the proof of (80), which is very easy taking into account (81) , (82) , also that $\gamma=\lambda$ (in this case $L_g=1$) and that

$$k = h \sqrt{1+(6\pi)^2}$$

By virtue of those mentioned inequalities we obtain:

$$w(0, j) \geq \frac{C}{2} w(\frac{\sqrt{n}}{2}+1, j) \geq M (\sqrt{n} h)^\lambda = M_1 k^{\gamma/2}$$

Inequality which prove the optimality of estimate $k^{\gamma/2}$.

CONCLUSIONS

This paper has analysed the problem of the error estimation in some usual discretization schemes of Hamilton-Jacobi equations corresponding to infinite horizon deterministic optimal control problems. We have given explicit error estimates and we have shown that they are optimal. The explicit estimates depend on the regularity of the data; in general, only an estimate of the type \sqrt{k} is attainable, but in the regular case, where semiconcavity assumptions hold, estimate of type $k^{\gamma/(\gamma+1)}$ can be obtained for values of $\gamma \in (1,2)$. It should be remarked that some of the optimality results, which indicate the convenience of using an unusual large time step h in relation with the space discretization step k (in fact $h \sim k^{2/3}$) give rise to a new methodology of discretization and emphasize the harmful effect of the space discretization.

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APPENDIX

Proof of technical results.

A.1 Proof of Lemma 3.1

Proof: Let $\{a_p(\cdot) / p=0,1,\dots\} \subset \mathcal{A}$ be a minimizing sequence, i.e.

$$\inf_{a \in \mathcal{A}} \int_0^T f(y(t,a(t))) e^{-\lambda t} dt = \lim_{p \rightarrow \infty} \int_0^T f(y(t,a_p(t))) e^{-\lambda t} dt$$

then:

$$\begin{aligned} u(x) - u_T(x) &= \inf_{a \in \mathcal{A}} \int_0^\infty f(y(t,a(t))) e^{-\lambda t} dt - \inf_{a \in \mathcal{A}} \int_0^T f(y(t,a(t))) e^{-\lambda t} dt \leq \\ &\leq \int_0^T f(y(t,a_p(t))) e^{-\lambda t} dt + \frac{M_f}{\lambda} e^{-\lambda T} - \lim_{p \rightarrow \infty} \int_0^T f(y(t,a_p(t))) e^{-\lambda t} dt \quad \forall p \end{aligned}$$

Then

$$u(x) - u_T(x) \leq \frac{M_f}{\lambda} e^{-\lambda T}$$

In a similar way, we can obtain:

$$u_T(x) - u(x) \leq \inf_{a \in \mathcal{A}} \int_0^T f(y(t,a(t))) e^{-\lambda t} dt - \lim_{p \rightarrow \infty} \int_0^T f(y(t,a_p(t))) e^{-\lambda t} dt + \frac{M_f}{\lambda} e^{-\lambda T} \leq \frac{M_f}{\lambda} e^{-\lambda T}$$

By virtue of these inequalities

$$|u(x) - u_T(x)| \leq \frac{M_f}{\lambda} e^{-\lambda T}$$

□

A.2 Proof of Lemma 3.3

Proof: A_k^h is a contractive operator, with fixed point u_k^h , i.e.

$$u_k^h = A_k^h u_k^h,$$

also, $u_{k,T}^h$ is defined recursively by:

$$u_{k,T}^h = (A_k^h)^\mu(w_T),$$

being $w_T = 0$. So, by (35) we have:

$$\left| u_{k,T}^h(x) - u_k^h(x) \right| \leq (1-\lambda h)^\mu \left| u_k^h \right|,$$

By (17), we have:

$$\left| u_{k,T}^h(x) - u_k^h(x) \right| \leq \frac{M_f}{\lambda} (1-\lambda h)^\mu \leq \frac{M_f}{\lambda} e^{-\lambda T} \quad (86)$$

□

A.3 Preliminaries for the proof of Lemma 3.4

To prove Lemma 3.4 we need some auxiliary results.

A.3.1 Regularity of u_T^h

Proposition 1: u_T^h is Lipschitz continuous and moreover

$$L_{u_T^h} \leq \begin{cases} L_f \frac{1}{L_g - \lambda} e^{(L_g - \lambda)T} & \text{if } L_g > \lambda \\ L_f \frac{1}{\lambda - L_g} & \text{if } L_g < \lambda \\ L_f T & \text{if } L_g = \lambda \end{cases}$$

Proof: It is clear that for $n = \mu$, $u_\mu^h = 0$ is Lipschitz continuous and $L_{u_\mu^h} = 0$

To complete the induction procedure, we consider the following inequality:

$$u_n^h(x) - u_n^h(y) \leq (1-\lambda h) u_{n+1}^h(x + h g(x, \tilde{a})) + h f(x, \tilde{a}) - (1-\lambda h) u_{n+1}^h(y + h g(y, \tilde{a})) + h f(y, \tilde{a})$$

where \tilde{a} realizes the minimum of (9) for $u_n^h(y)$, then

$$u_n^h(x) - u_n^h(y) \leq \left((1-\lambda h)(1 + L_g h) L_{u_{n+1}^h} + L_f h \right) \|x - y\|$$

Analogously for $u_n^h(y) - u_n^h(x)$, then we obtain:

$$L_{u_n^h} \leq (1-\lambda h)(1+L_g h) L_{u_{n+1}^h} + L_f h \quad (87)$$

Analysis of different cases

• $L_g = \lambda$

In this case we have that $(1-\lambda h)(1+L_g h) = 1-(\lambda h)^2 < 1$, then:

$$L_{u_n^h} \leq L_{u_{n+1}^h} + L_f h$$

which implies

$$L_{u_0^h} \leq L_{u_\mu^h} + \mu L_f h = \mu h L_f = L_f T$$

• $L_g < \lambda$

The formula (87) becomes

$$L_{u_n^h} \leq (1 - (\lambda - L_g)h) L_{u_{n+1}^h} + L_f h$$

then

$$L_{u_n^h} \leq L_f h \frac{1 - (1 - (\lambda - L_g)h)^n}{h(\lambda - L_g)} \leq L_f \frac{1}{\lambda - L_g}$$

• $L_g > \lambda$

The inequality (87) becomes

$$L_{u_n^h} \leq L_f h \frac{(1 + (L_g - \lambda)h)^n}{h(L_g - \lambda)}$$

so

$$L_{u_T^h} \leq L_f \frac{1}{L_g - \lambda} e^{(L_g - \lambda)T}$$

□

Remark A.3.1: From here we will denote with L_u the Lipschitz constant of u_T^h and we will discriminate the three cases between them when it were necessary.

A.3.2 Definition of $u_{n\rho}^h$, regularization of u_n^h

By convolution with a smooth function $\beta(\cdot)$, we obtain a regular approximation of u_n^h

$$u_{n\rho}^h(x) = (u_n^h * \beta_\rho)(x) = \int_{B(\rho)} u_n^h(x-y) \beta_\rho(y) dy \quad (88)$$

where

$$\beta_1(\cdot) \in C^\infty(\mathbb{R}^\nu), \quad (89)$$

$$\beta_1(x) \geq 0 \quad \forall x, \text{ support of } \beta_1 \subset B_1 = \{ x \in \mathbb{R}^\nu / \|x\| \leq 1 \} \quad (90)$$

$$\int_{\mathbb{R}^\nu} \beta_1(x) dx = 1 \quad (91)$$

$$\beta_\rho(x) = \frac{1}{\rho^\nu} \beta_1\left(\frac{x}{\rho}\right) \geq 0 \quad \forall \rho \in \mathbb{R}^+ \quad (92)$$

A.3.3 Definition of $\tilde{u}_{n\rho}^h$, linear interpolation of $u_{n\rho}^h$

We define $\tilde{u}_{n\rho}^h$ the element of W_k such that:

$$\tilde{u}_{n\rho}^h(x_i) = u_{n\rho}^h(x_i), \quad \forall i = 1, N \quad (93)$$

Proposition 2: Let g, f be functions such that u_n^h is Lipschitz continuous, then:

$$\left| u_n^h(x) - u_{n,\rho}^h(x) \right| \leq L_u \rho \quad (94)$$

Proof: By definition of convolution we have:

$$\begin{aligned} \left| u_n^h(x) - u_{n,\rho}^h(x) \right| &= \left| u_n^h(x) - \int_{B(\rho)} u_n^h(x-y) \beta_\rho(y) dy \right| = \\ &= \int_{B(\rho)} \left| u_n^h(x) - u_n^h(x-y) \right| \beta_\rho(y) dy \end{aligned}$$

but, being u_n^h Lipschitz continuous, it holds

$$\left| u_n^h(x) - u_{n,\rho}^h(x) \right| \leq L_u \int_{B(\rho)} \|y\| \beta_\rho(y) dy \leq L_u \rho$$

□

Proposition 3: Functions u_n^h , $u_{n,\rho}^h$ and operator A_a^h verify inequalities (95) and (96)

$$u_n^h(x) \leq (A_a^h u_{n+1}^h)(x) \quad \forall a \in A \quad (95)$$

$$u_{n,\rho}^h(x) \leq (A_a^h u_{n+1}^h * \beta_\rho)(x) \quad \forall a \in A \quad (96)$$

also \forall Lipschitz continuous function w , we have:

$$\left| (A_a^h w) * \beta - A_a^h (w * \beta) \right| \leq (1 - \lambda h) L_w L_g \rho h + L_f \rho h \quad (97)$$

in addition

$$\left| (A_a^h u_{n,\rho}^h)(x) - (A_a^h \tilde{u}_{n,\rho}^h)(x) \right| \leq C(1 - \lambda h) L_u \frac{k^2}{\rho} \quad (98)$$

Proof: (95) and (96) are trivial, for (97) we have:

$$\begin{aligned} \left((A_a^h w) * \beta - A_a^h (w * \beta) \right)(x) &= \int_{B(\rho)} \left((1 - \lambda h) w(x - \eta + h g(x - \eta, a)) + h f(x - \eta, a) \right) \beta_\rho(\eta) d\eta - \\ &\quad - (1 - \lambda h) \int_{B(\rho)} w(x - \eta + h g(x, a)) \beta_\rho(\eta) d\eta - h f(x, a) \end{aligned}$$

then

$$\begin{aligned} \left| \left((A_a^h w) * \beta - A_a^h (w * \beta) \right)(x) \right| &\leq \int_{B(\rho)} (1 - \lambda h) |w(x - \eta + h g(x - \eta, a)) - w(x - \eta + h g(x, a))| \beta_\rho(\eta) d\eta + \\ &\quad + h \int_{B(\rho)} |f(x - \eta, a) - f(x, a)| \beta_\rho(\eta) d\eta \\ &\leq (1 - \lambda h) L_w L_g h \rho + L_f h \rho \end{aligned}$$

and (97) is proved.

Function $u_{n,\rho}^h$ has second derivatives bounded by

$$\|D^2 u_{n,\rho}^h\| \leq \tilde{C} \frac{L_u}{\rho} \quad (99)$$

where \tilde{C} is a constant depending only of $\beta_1(\cdot)$, because it is the regularization (by the function β_ρ) of the Lipschitz continuous function u_n^h and by virtue of (89)–(92). In consequence by virtue of condition (27) and (99) the difference between $u_{n,\rho}^h$ and its linear interpolation $\tilde{u}_{n,\rho}^h$ is bounded by $C L_u \frac{k^2}{\rho}$ (for a detailed proof of these properties see [17], [28]), and then we have:

$$\left| (A_a^h u_{n,\rho}^h)(x) - (A_a^h \tilde{u}_{n,\rho}^h)(x) \right| \leq (1-\lambda h) \left| (u_{n,\rho}^h - \tilde{u}_{n,\rho}^h)(x + g(x,a)h) \right| \leq C(1-\lambda h) L_u \frac{k^2}{\rho}$$

and (98) is proved. □

Proposition 4: *The following estimate holds*

$$\tilde{u}_{T\rho}^h(x) - u_{kT}^h(x) \leq \frac{1-\lambda h}{\lambda} L_u L_g \rho + \frac{L_f}{\lambda} \rho + \frac{1-\lambda h}{\lambda h} C_1 \frac{L_u k^2}{\rho} \quad (100)$$

Proof: We define:

$$E_n = \sup_{x \in \Omega_k} \left(\tilde{u}_{n\rho}^h(x) - u_{kn}^h(x) \right) = \max_{i=1,\dots,N} \left(\tilde{u}_{n\rho}^h(x_i) - u_{kn}^h(x_i) \right) \quad (101)$$

By properties (96) and (97) we have:

$$\begin{aligned} u_{n-1,\rho}^h(x) &= (A^h u_n^h * \beta_\rho)(x) \leq (A_a^h u_n^h * \beta_\rho)(x) \leq (A_a^h (u_n^h * \beta))(x) + \phi(x,h,\rho) = \\ &= (A_a^h u_{n,\rho}^h)(x) + \phi(x,h,\rho) \end{aligned} \quad (102)$$

with

$$|\phi(x,h,\rho)| \leq (1-\lambda h) L_u L_g h \rho + L_f h \rho$$

By (34) and (48), if $w \in W^k$, then:

$$(A_k^h w)(x) = \min_{a \in \mathcal{A}} \left((1-\lambda h) w(x+h g(x,a)) + h f(x,a) \right) = (A_a^h \tilde{w})(x) \leq (A_a^h \tilde{w})(x) \quad (103)$$

then, by (103), (102) and (98) we have:

$$\begin{aligned} u_{n-1,\rho}^h(x_i) &\leq (A_{\bar{a}}^h \bar{u}_{n,\rho}^h)(x_i) + \phi(x_i, h, \rho) + \psi(x_i, h, \rho, k) \leq \\ &\leq (A_{\bar{a}}^h \bar{u}_{n,\rho}^h)(x_i) + (1-\lambda h) L_w L_g h \rho + L_f h \rho + C(1-\lambda h) L_u \frac{k^2}{\rho} \end{aligned}$$

so

$$u_{n-1,\rho}^h(x_i) \leq (A_{\bar{a}}^h \bar{u}_{n,\rho}^h)(x_i) + \phi(x_i, h, \rho) + \psi(x_i, h, \rho, k) \quad \forall a$$

and

$$u_{k,n-1}^h(x_i) = (A_k^h u_{k,n}^h)(x_i) = (A_{\bar{a}}^h u_{k,n}^h)(x_i)$$

where \bar{a} realizes the minimum in (103) for $u_{k,n}^h$, then:

$$u_{n-1,\rho}^h(x_i) - u_{k,n-1}^h(x_i) \leq (A_{\bar{a}}^h \bar{u}_{n,\rho}^h)(x_i) + \phi(x_i, h, \rho) + \psi(x_i, h, \rho, k) - (A_{\bar{a}}^h u_{k,n}^h)(x_i)$$

By (93), we have

$$\begin{aligned} \bar{u}_{n-1,\rho}^h(x_i) - u_{k,n-1}^h(x_i) &\leq \\ &\leq (1-\lambda h) (\bar{u}_{n,\rho}^h(x_i) - u_{k,n}^h(x_i)) + (1-\lambda h) L_u L_g h \rho + L_f h \rho + C(1-\lambda h) L_u \frac{k^2}{\rho} \end{aligned}$$

so, by definition (101)

$$E_{n-1} \leq (1-\lambda h) E_n + (1-\lambda h) L_u L_g h \rho + L_f h \rho + C(1-\lambda h) L_u \frac{k^2}{\rho} \quad (104)$$

As $E_{n-1} \leq \beta E_n + b$ implies $E_0 \leq \beta^n E_n + \frac{b}{1-\beta}$, then replacing this estimate in (104) and taking into account that $E_\mu = 0$, we obtain (100).

□

Proposition 5: *The following estimate holds:*

$$u_{kT}^h(x) - \bar{u}_{T\rho}^h(x) \leq \frac{1-\lambda h}{\lambda} L_u L_g \rho + \frac{L_f}{\lambda} \rho + \frac{1-\lambda h}{\lambda h} C_1 \frac{L_u k^2}{\rho} + 2 L_u \rho \quad (105)$$

Proof: Let \hat{x}_j be an arbitrary vertex of Ω_k , then there exist $\bar{a}_1(\hat{x}_j)$, ..., $\bar{a}_\mu(\hat{x}_j)$ such that:

$$u_T^h(\hat{x}_j) = u_0^h(\hat{x}_j) = (A_{\bar{a}_1}^h u_1^h)(\hat{x}_j) = ((A_{\bar{a}_1}^h A_{\bar{a}_2}^h \dots A_{\bar{a}_\mu}^h) u_\mu^h)(\hat{x}_j) \quad (106)$$

where \bar{a}_n realizes the minimum that appears in the definition of operator A^h applied to function u_n^h and computed at the point defined recursively by

$$y^h(\hat{x}_j, n) = y^h(\hat{x}_j, n-1) + h g(y^h(\hat{x}_j, n-1), \bar{a}_n)$$

with initial condition

$$y^h(\hat{x}_j, 0) = \hat{x}_j$$

We define v_n^h in the following way

$$\begin{aligned} v_n^h &= A_{\bar{a}_{n+1}}^h v_{n+1}^h \quad \forall 0 \leq n \leq \mu-1 \\ v_\mu^h &= 0 \end{aligned}$$

Obviously, by (106), we have:

$$v_T^h(\hat{x}_j) = u_T^h(\hat{x}_j)$$

We define

$$\tilde{E}_n = \sup_{x \in \Omega_k} (u_{k,n}^h(x) - \tilde{v}_{n\rho}^h(x)) \quad (107)$$

Note: The construction of function v_n^h depends on \hat{x}_j , but for simplicity of notation we will write it without denoting explicitly this dependence.

It is easy to prove that v_n^h has the same regularity properties of u_n^h , i.e. it is Lipschitz continuous with the same Lipschitz constant and also it holds for it the estimates (97) and (98). Then, we have for any x_i vertex of Ω_k that:

$$v_{n+1,\rho}^h(x_i) = (A_{\bar{a}_{n+1}}^h \tilde{v}_{n,\rho}^h)(x_i) + \phi(x_i, h, \rho) + \psi(x_i, h, \rho, k)$$

As

$$u_{k,n+1}^h(x_i) = (A_{\bar{a}_{n+1}}^h u_{k,n}^h)(x_i) \leq (A_{\bar{a}_{n+1}}^h u_{k,n}^h)(x_i),$$

we have:

$$u_{k,n+1}^h(x_i) - v_{n+1,\rho}^h(x_i) \leq (A_{\bar{a}_{n+1}}^h u_{k,n}^h)(x_i) - (A_{\bar{a}_{n+1}}^h \tilde{v}_{n,\rho}^h)(x_i) + |\phi(x_i, h, \rho)| + |\psi(x_i, h, \rho, k)|$$

by the above inequality, (12) and (107), we have

$$\tilde{E}_{n+1} \leq (1-\lambda h) \tilde{E}_n + (1-\lambda h) L_u L_g h \rho + h L_f \rho + C_1 (1-\lambda h) \frac{L_u}{\rho} k^2$$

From here we obtain:

$$u_{kT}^h(x_i) - v_{T,\rho}^h(x_i) \leq \frac{1-\lambda h}{\lambda} L_u L_g \rho + \frac{L_f}{\lambda} \rho + \frac{1-\lambda h}{\lambda h} C_1 \frac{L_u k^2}{\rho} \quad (108)$$

As for $n=\mu$ $v_T^h(\hat{x}_j) = u_T^h(\hat{x}_j)$, we have:

$$\left| u_{T,\rho}^h(\hat{x}_j) - v_{T,\rho}^h(\hat{x}_j) \right| \leq \left| u_{T,\rho}^h(\hat{x}_j) - u_T^h(\hat{x}_j) \right| + \left| u_T^h(\hat{x}_j) - v_T^h(\hat{x}_j) \right| + \left| v_T^h(\hat{x}_j) - v_{T,\rho}^h(\hat{x}_j) \right| \leq 2 L_u \rho$$

the inequality (108) becomes:

$$u_{kT}^h(\hat{x}_j) - u_{T,\rho}^h(\hat{x}_j) \leq \frac{1-\lambda h}{\lambda} L_u L_g \rho + \frac{L_f}{\lambda} \rho + \frac{1-\lambda h}{\lambda h} C_1 \frac{L_u k^2}{\rho} + 2 L_u \rho$$

Finally, as \hat{x}_j is an arbitrary vertex of Ω_k (105) holds.

□

A.4 Proof of Lemma 3.4

Proof: Obviously

$$\left| u_T^h(x) - u_{k,T}^h(x) \right| \leq \left| u_T^h(x) - u_{T,\rho}^h(x) \right| + \left| u_{T,\rho}^h(x) - \bar{u}_{T,\rho}^h(x) \right| + \left| \bar{u}_{T,\rho}^h(x) - u_{k,T}^h(x) \right| \quad (109)$$

so, applying proposition 2, (100) y (105) we can estimate the right hand side of (109) and we have:

$$\left| u_T^h(x) - u_{k,T}^h(x) \right| \leq L_u \rho + C \frac{L_u k^2}{\rho} + \frac{1}{\lambda} L_u L_g \rho + \frac{L_f}{\lambda} \rho + \frac{1}{\lambda h} C_1 \frac{L_u k^2}{\rho} + 2 L_u \rho \quad (110)$$

Analysis of different cases

• $L_g > \lambda$

In this case, L_u takes the form:

$$L_u = L_f \frac{1}{L_g - \lambda} e^{(L_g - \lambda)T}$$

Inequality (110) becomes

$$\left| u_T^h(x) - u_{k,T}^h(x) \right| \leq M e^{(L_g - \lambda)T} \left(\rho + \frac{k^2}{h\rho} \right) \quad (111)$$

where

$$M = \max \left(3 + \frac{L_g}{\lambda} + L_g - \lambda, \frac{C}{\lambda} + \frac{C_1}{\lambda} \right) \frac{L_f}{L_g - \lambda}$$

Minimizing in ρ the expression (111) $\left(\rho + \frac{k^2}{h\rho} \right)$ we have the minimum is realized by $\rho = \frac{k}{\sqrt{h}}$, so:

$$\left| u_T^h(x) - u_{k,T}^h(x) \right| \leq 2 M e^{(L_g - \lambda)T} \frac{k}{\sqrt{h}}$$

• $L_g < \lambda$

in this case $L_u = \frac{L_f}{\lambda - L_g}$ and the inequality (110), becomes:

$$\left| u(x) - u_k^h(x) \right| \leq M \left(\rho + \frac{k^2}{h\rho} \right)$$

where

$$M = \max \left(3 + \frac{L_g}{\lambda} + \lambda - L_g, \frac{C}{\lambda} + \frac{C_1}{\lambda} \right) \frac{L_f}{\lambda - L_g}$$

and in the same way that it has been done above, we have:

$$\left| u_T^h(x) - u_{k,T}^h(x) \right| \leq 2 M \frac{k}{\sqrt{h}}$$

• $L_g = \lambda$

In this case $L_u = L_f T$ and inequality (110) becomes

$$\left| u(x) - u_k^h(x) \right| \leq M T \left(\rho + \frac{k^2}{h\rho} \right)$$

where

$$M = \max \left(4, \frac{C}{\lambda} + \frac{C_1}{\lambda} \right) L_f$$

then

$$\left| u_T^h(x) - u_{k,T}^h(x) \right| \leq 2 M T \frac{k}{\sqrt{h}}$$

□

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FIGURE 1

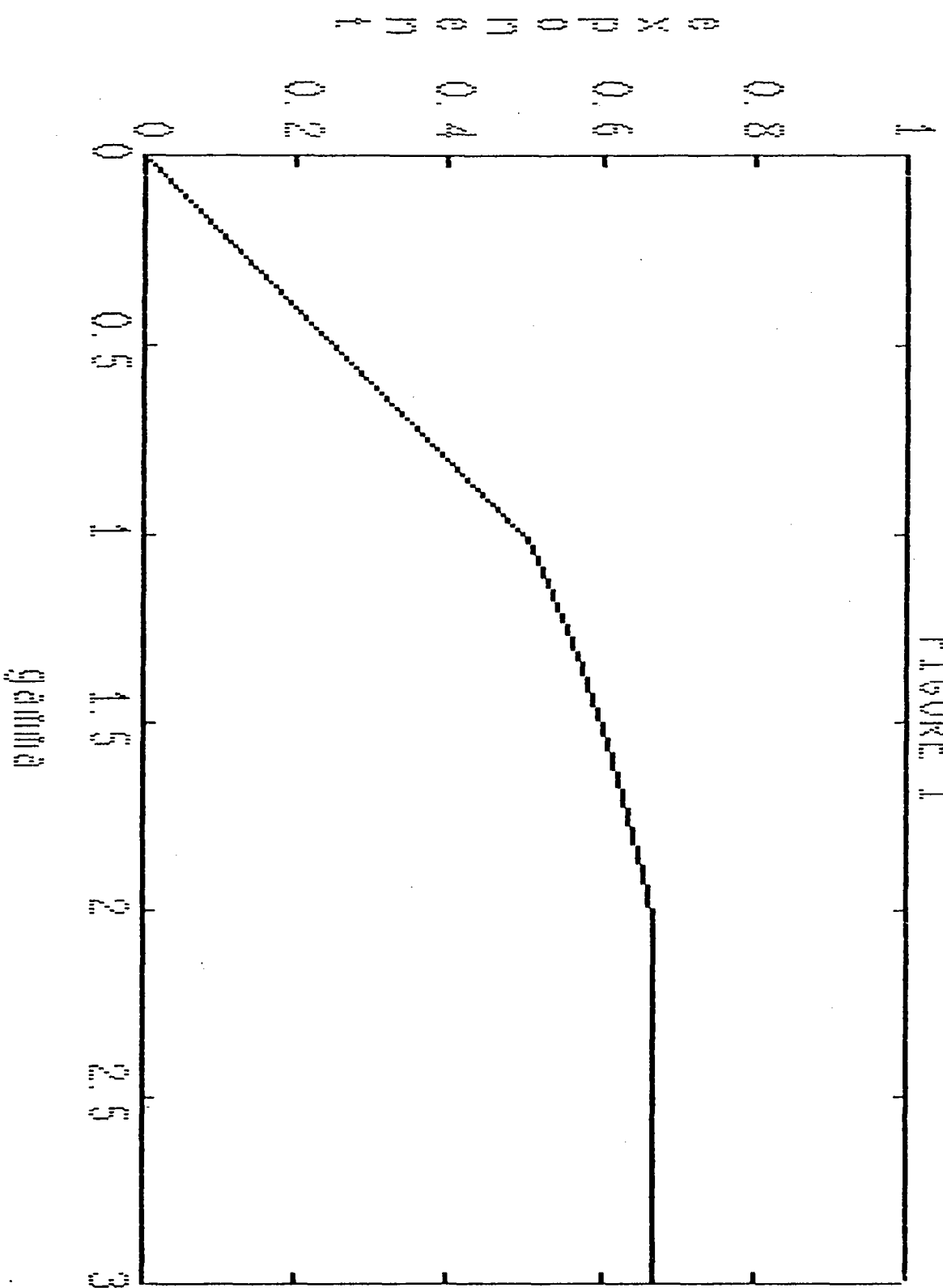


FIGURE 2

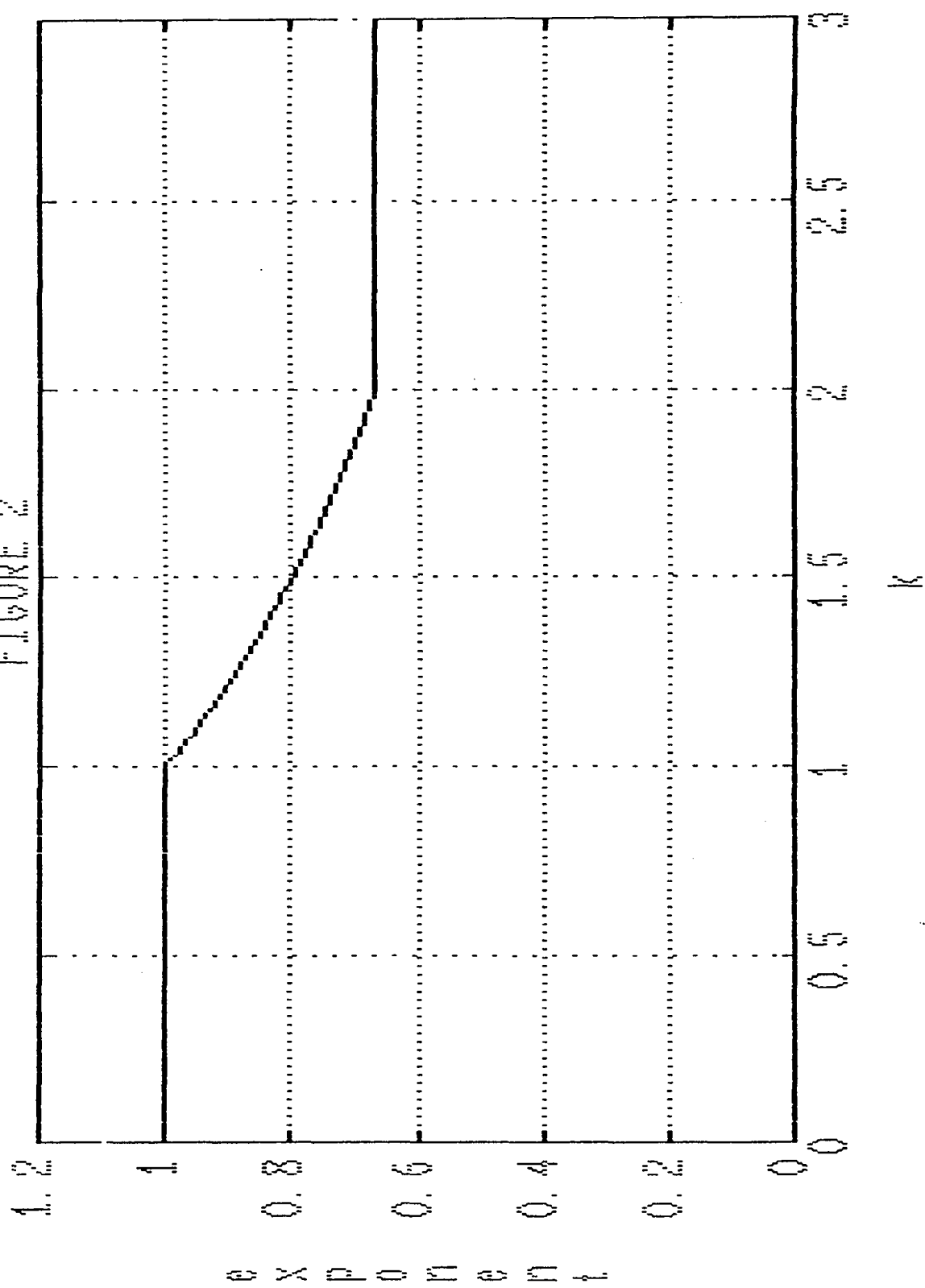


FIGURE 2

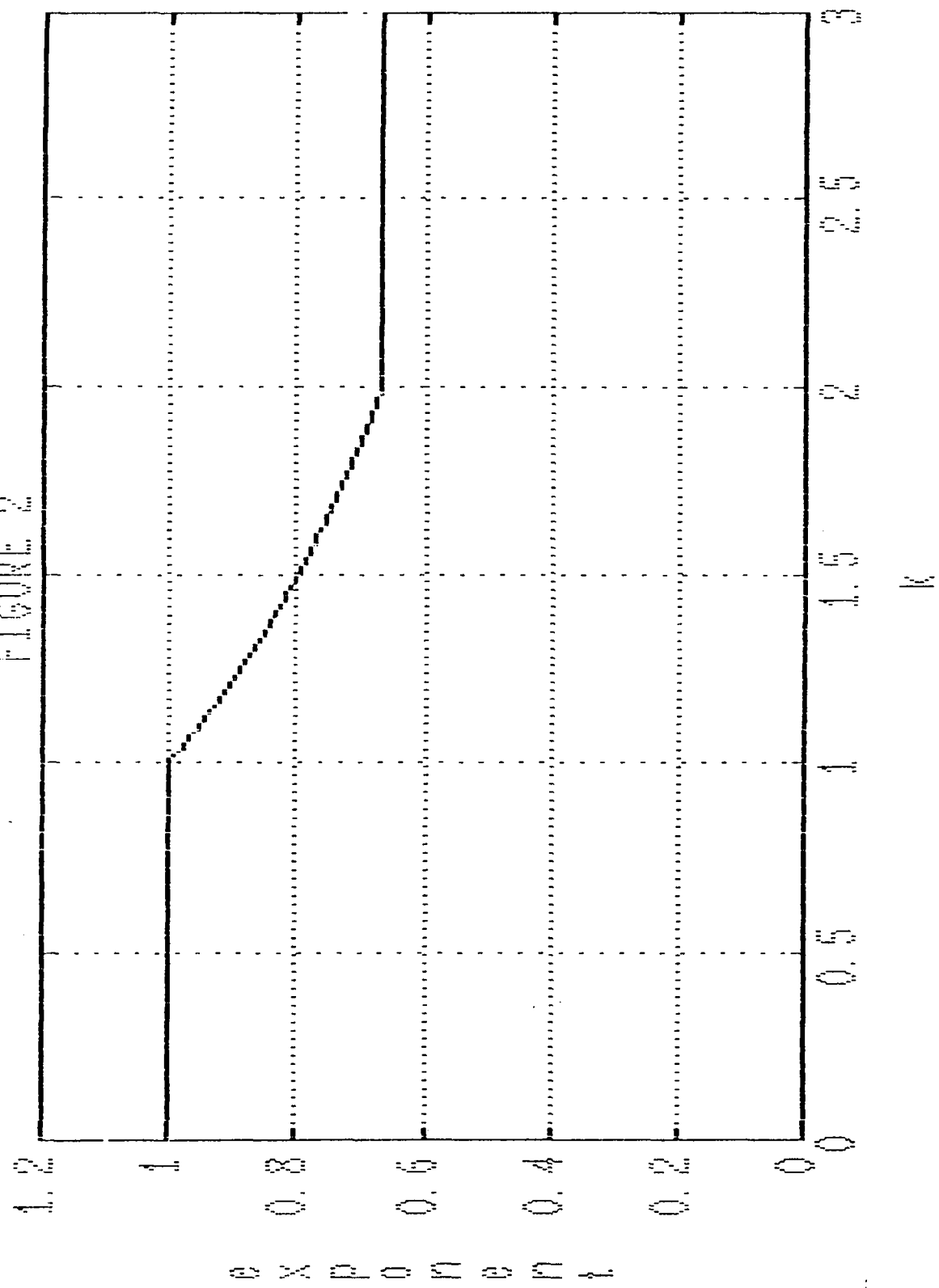


FIGURE 3

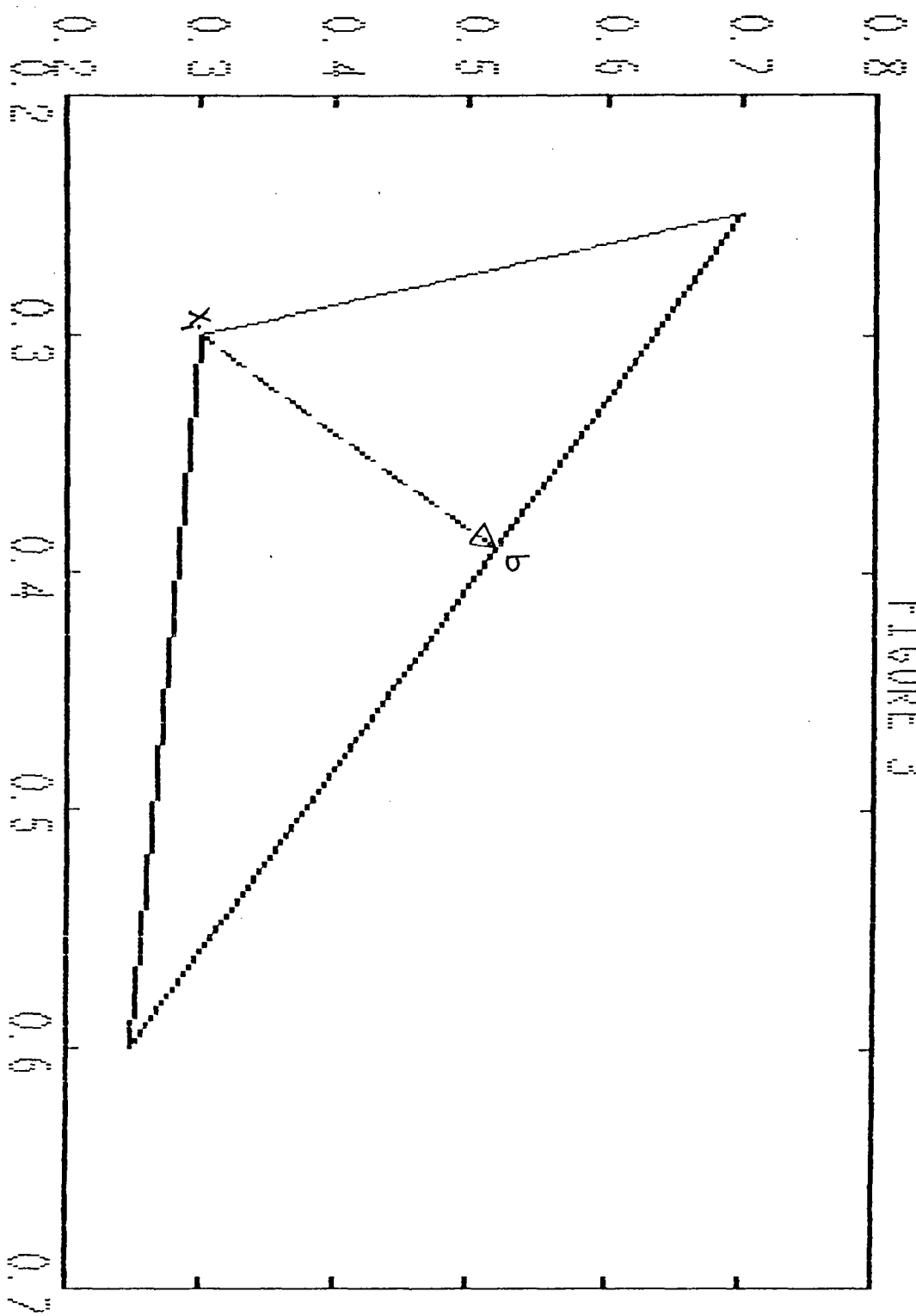


FIGURE 4

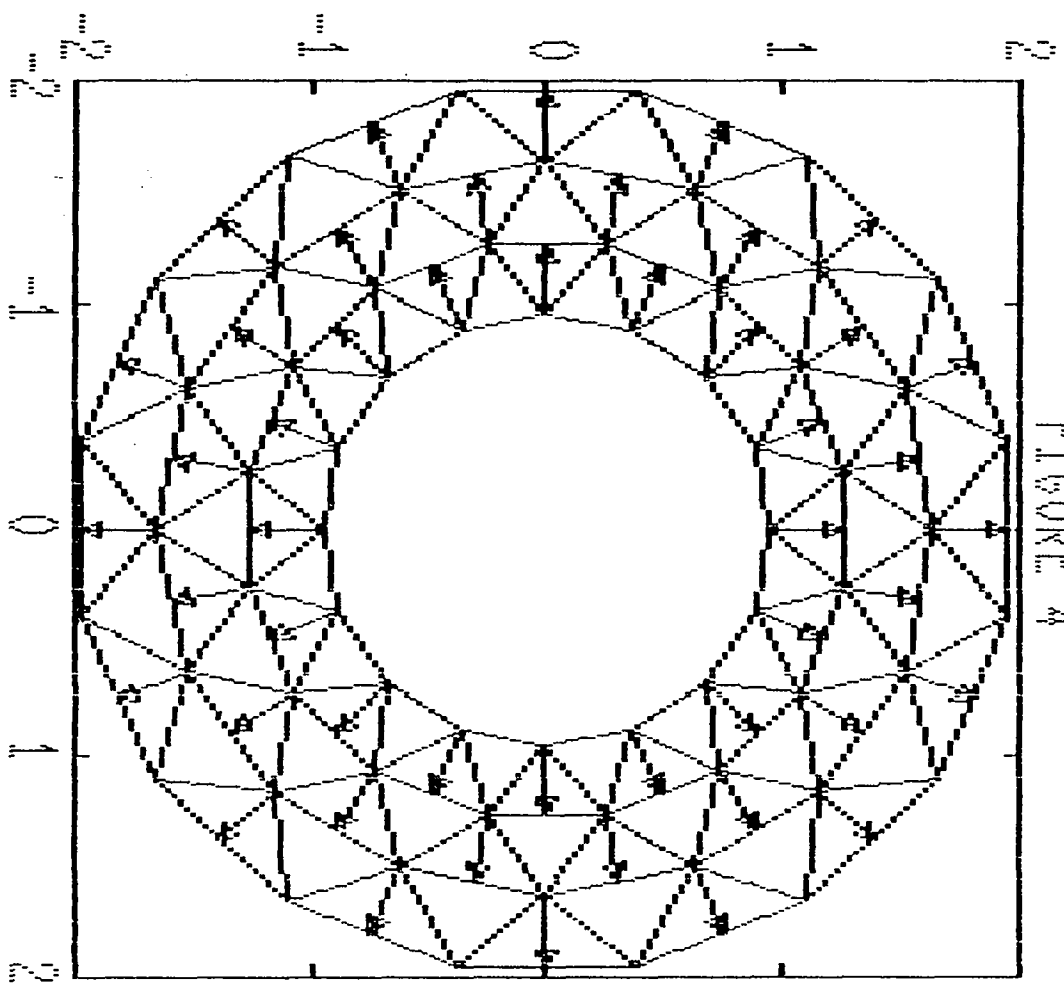
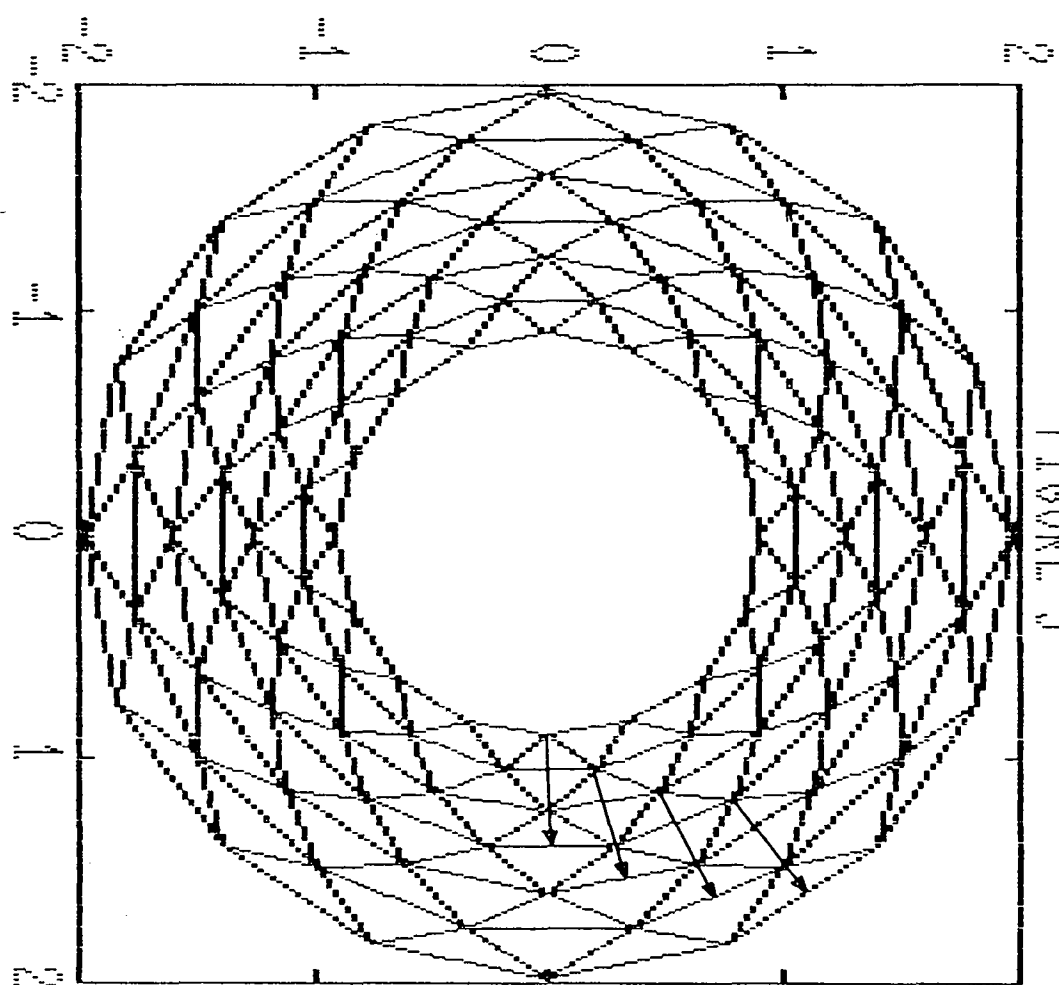
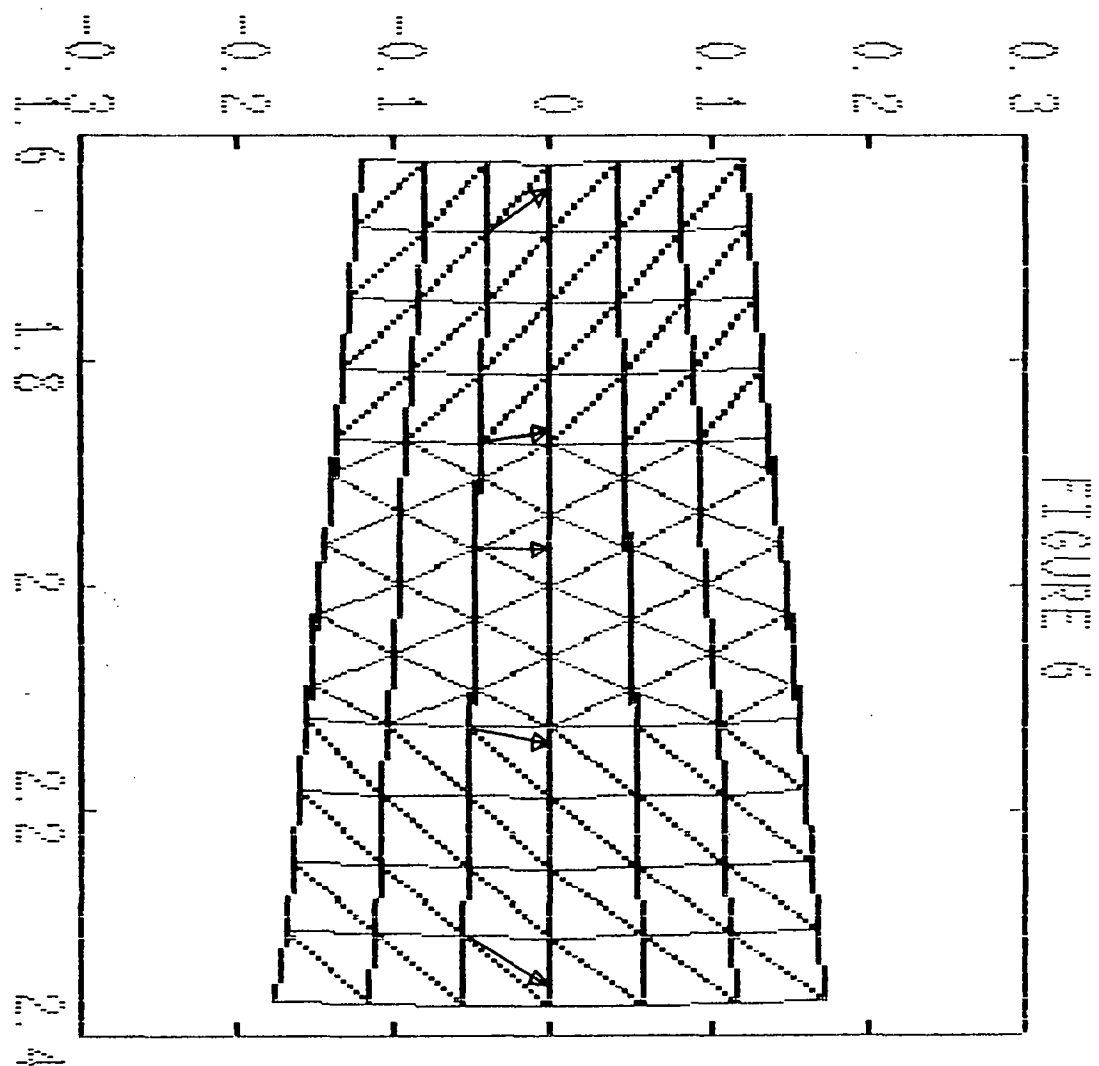


FIGURE 5





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