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ANALYSIS OF A MULTISTEP JACOBI MULTIGRID METHOD FOR SOLVING THE EULER EQUATIONS

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Analyse d'une méthode multigrille Jacobi-multipas pour résoudre les équations d'Euler

Résumé: Nous étudions la convergence de méthodes multigrilles avec itération de base multipas Jacobi pour résoudre les équations d'Euler stationnaires discrétisées par une approximation MUSCL. Ces méthodes étendent d'une certaine façon les approches à pas de temps local et caractéristique. Le principal outil utilisé est l'itération bigrille. Une étude analytique est réalisée en 1D. On propose une règle heuristique en 2D à partir de quelques expériences. Finalement, un algorithme de type W-cycle à faible complexité est décrit et comparé à l'itération bigrille.

Analysis of a multistep Jacobi multigrid method for solving the Euler equations

Abstract: We study the convergence of Jacobi multistep multigrid methods for solving the steady Euler equations with a MUSCL approximation; these methods involve some advantages of local and characteristic time stepping. The main tool is two-grid analysis; an analytic study is done for 1D case. Experiments are produced in 2D for proposing a heuristic rule. A low complexity W-cycle is compared to the 2-Grid iteration.

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1 INTRODUCTION

The Euler equations of compressible gas flow are expensive to solve numerically because they involve many variables, they are of hyperbolic type, and because many nodes are often necessary for capturing complex phenomena such as vortex flows. It is a striking fact that for many authors, the potential flow can be solved within five multi-grid cycles, while the Euler equivalent requires several hundreds of work units. In fact, many questions, that are not arising in the elliptic case or are now well solved, are still not answered for the so-called hyperbolic case. In particular, many people are interested in obtaining the solution of discrete systems relying on upwind schemes like MUSCL and other TVD ones; two difficulties arise:
Firstly, usual hyperbolic systems are severely non-isotropic since they involve trajectories; this fact is amplified by upwind schemes (whatever be the spatial order considered) since the internal numerical viscosity is non-isotropic (this fact is still more important with some two-dimensional or narrow schemes). This difficulty can lead to many incidents related in particular (but not only) to alignment pathologies. Secondly, second-order upwind schemes really lack of intrinseque dissipation; some explanation of the hyperbolic-MG convergence is related to error expulsion instead of damping, but this does not yet help a less dissipative scheme to work well. One issue is to use a first-order MG scheme inside a first-order/second-order defect correction iteration [1].
A third difficulty (not necessarily related to upwinding) is the frequent arising of a saturation phenomenon when too many levels are used. The limitation of the number of levels is an obstacle to mesh-independent convergence.
A fourth difficulty is a still more general one: when non-structured meshes are considered, one encounters difficulties to design a really implicit relaxation i.e. a relaxation with the help of non-diagonal direct methods; in this paper we restrict ourselves to explicit-like methods.
The preconditioning that we propose is block-diagonal; indeed, local time-stepping is a scalar-diagonal preconditioner that is less efficient at least in 1-D than both characteristic time-stepping [2] and Jacobi preconditioning that are block-diagonal; in this study, Jacobi is preferred in 2-D because it extends naturally to multi-dimensional systems.
Further, many ideas of the explicit approach can be extended to the Jacobi formulation; we introduce a four-stage Runge-Kutta-Jacobi iterative method.
Our main concern in this paper is to compare a 1-D study involving a sim-

plified advection model with its 2-D application to the Euler equations.
The 1-D analysis is of two-grid type because it takes into account transfers.
Extension to 2-D is complicated by the problem of anisotropy and mesh alignment; since we are working with non-structured meshes, we neglect the influence of a strong alignment.
The main parameter to optimize is the relaxation parameter; we present a set of experiments for studying the sensibility of the algorithm to this parameter and to compare the behaviour of a practical algorithm to analysis.

2 MODELS FOR HYPERBOLIC MULTIGRID

In order to compare analysis with practical calculation, we restrict ourselves to the Ideal 2-G scheme, in which "Ideal" non ambigously indicates that the coarse grid correction involves either a direct method or a fully converged iterative one.

2.1 Basic model

One very simple model of iteration for searching the steady solution of an hyperbolic problem is defined as follows: the advection equation is considered ($c > 0$) on the unit circle, ($0 = 2\pi$), $u_t + cu_x = 0$, and, an explicit upwind discretisation is choosen; we call it RK1:

$$u_j^{n+1} = u_j^n + c\frac{\Delta t}{\Delta x}(u_{j-1}^n - u_j^n) \tag{1}$$

this defines an iterative process $u^{n+1} = f(u^n)$ which converges to a constant solution, due to the numerical dissipation. From this one-grid example we can derive some 2-Grid schemes:

2.1.1 Finite volume 2-Grid

Usual 2-Grid: In order to understand the behaviour of a 2-Grid scheme it is interesting to write the correction built by such a scheme. Let us consider one cycle for the 1D-advection equation ($c = 1$), then define some terms:

- A coarse finite volume grid is generated by grouping $cell_{2p}$ and $cell_{2p+1}$
- The notations u_h or u define the value of u on the fine grid, and u_{2h} or $\bar{u}$ the value of u on the coarse grid.
- The basic iterative scheme:

$$u_j^{n+1} = u_j^n - \sigma(u_j^n - u_{j-1}^n)$$

where $\sigma = c\frac{\Delta t}{\Delta x}$ is the classical Courant number.

- The residual on the fine grid:

$$R_h(u_h) = R_h = \frac{(u_j - u_{j-1})}{\Delta x}$$

- The transfers from grid to grid:
  - The injection of the solution:

$$I_{h\to H}(U_h)_p = \frac{U_{h_{2p}} + U_{h_{2p+1}}}{2}$$

  - The restriction of the residuals:

$$\bar{I}_{h\to H}(R_h)_p = \frac{R_{h_{2p}} + R_{h_{2p+1}}}{2}$$

  - The interpolation of the correction:

$$\left.\begin{matrix} I_{H\to h}(Cor_H)_{2p} \\ I_{H\to h}(Cor_H)_{2p+1} \end{matrix}\right\} = Cor_{H_p}$$

It is now useful to write the source term used by the iterative scheme on the coarse grid (i.e.: we must solve the following problem: $u_t + cu_x = S$, S: source term, $S = 0$ on the fine grid):

$$\begin{aligned} P_{2h} &= R_{2h}(u_{2h}^{(0)}) - \frac{\sum R_h(u_h)}{n} = \\ \bar{P}_p &= \bar{R}(\bar{u}^{(0)}) - \frac{(R_{2p} + R_{2p+1})}{2} \\ &= \bar{R}(\bar{u}^{(0)}) - \frac{(u_{2p+1} - u_{2p-1})}{2\Delta x} \end{aligned}$$

where $\bar{u}^{(0)}$ is the restriction of the fine grid solution that initializes the processus on the coarse grid:

$$\bar{u}_p^{(0)} = \frac{(u_{2p} + u_{2p+1})}{2} \quad ; \quad \bar{u}_{p-1}^{(0)} = \frac{(u_{2p-2} + u_{2p-1})}{2}$$

Thus, the first step of the scheme on the coarse grid can be written as follows:

$$\bar{u}_p^{(1)} = \bar{u}_p^{(0)} - 2\Delta t[\frac{(\bar{u}_p^{(0)} - \bar{u}_{p-1}^{(0)})}{2\Delta x} - \frac{(\bar{u}_p^{(0)} - \bar{u}_{p-1}^{(0)})}{2\Delta x} + \frac{(u_{2p+1} - u_{2p-1})}{2\Delta x}]$$

The coarse grid correction is then defined by:

$$\bar{u}_p^{(1)} - \bar{u}_p^{(0)} = -\Delta t \frac{(u_{2p+1} - u_{2p-1})}{\Delta x}$$

If we now use the interpolation of the coarse grid correction on the fine grid, we can write the iterate solution after one cycle:

$$\begin{aligned} u_{2p}^{c+1} &= u_{2p}^{c} - \Delta t \frac{(u_{2p+1} - u_{2p-1})}{\Delta x} \\ u_{2p+1}^{c+1} &= u_{2p+1}^{c} - \Delta t \frac{(u_{2p+1} - u_{2p-1})}{\Delta x} \end{aligned}$$

We can notice first that it corresponds to a centered scheme for the even nodes, and an upwind scheme for the odd nodes. Then we also see that the fixed points (undamped frequencies) of the coarse grid correction are of the following type:

$$u_{2p+1} = const.$$

Ideal 2-Grid: Another way to study the behaviour of the 2-Grid scheme is to consider an Ideal 2-Grid cycle. In order to do that, we keep the same definitions as previously written except what concerns the convergence: an Ideal 2-Grid is defined by the complete convergence on the coarse grid during the cycle. If we now write the processus on the coarse grid we obtain ($\bar{u}_p^{(\infty)}$ is the converged solution on the coarse grid):

$$\bar{u}_p^{(\infty)} = \bar{u}_p^{(\infty)} - 2\Delta t \left[\frac{(\bar{u}_p^{(\infty)} - \bar{u}_{p-1}^{(\infty)})}{2\Delta x} - \frac{(\bar{u}_p^{(0)} - \bar{u}_{p-1}^{(0)})}{2\Delta x} + \frac{(u_{2p+1} - u_{2p-1})}{2\Delta x} \right]$$

Thus we have:

$$(\bar{u}_p^{(\infty)} - \bar{u}_{p-1}^{(\infty)}) - (\bar{u}_p^{(0)} - \bar{u}_{p-1}^{(0)}) = -(u_{2p+1} - u_{2p-1})$$

As written previously, the coarse grid correction is the difference between the converged coarse grid solution and the injection of the fine grid solution, so we define $\delta\bar{u}$ by:

$$\delta\bar{u} = \bar{u}^{(\infty)} - \bar{u}^{(0)}$$

Lemma 2.1 *$\delta\bar{u}$ is given by:*

$$\delta\bar{u} = \bar{v} + K \ , \ K = \frac{1}{p} \sum_p u_{2p+1} \in I\!R$$

Proof:
We can write:

$$\delta\bar{u}_p - \delta\bar{u}_{p-1} = -(u_{2p+1} - u_{2p-1})$$

In order to find the coarse grid correction we must solve the previous equation, so we put:

$$\bar{v}_p = -u_{2p+1} \Longleftrightarrow \bar{v}_{p-1} = -u_{2(p-1)+1} = -u_{(2p-1)}$$

We have now to solve the following equation:

$$\delta\bar{u}_p - \delta\bar{u}_{p-1} = \bar{v}_p - \bar{v}_{p-1}$$

The right-hand side of the equation corresponds to the derivative of a periodic function, thus if we introduce the operator $A : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ defined by:

$$A\delta\bar{u} = \delta\bar{u}_p - \delta\bar{u}_{p-1} = \bar{v}_p - \bar{v}_{p-1} = A\bar{v}$$

we can write:

- The Kernel of A has a dimension of 1 thus A can not be inverted.
- We apply Fredholm's alternative, we have the isomorphism:

  $$E/Ker(A) \mapsto Im(A)$$

  where A is the circulant matrix:

  $$A = Diag\ (-1\ 1\ 0)$$

  and there exists a solution:

  $$\delta\bar{u} = \bar{v} + K\ ,\ K \in I\!R$$

  K is determined by the initialization and the conservativity.

In order to calculate the value of K, let us consider the coarse grid processus that we integrate on [0 , 2 π]:

$$\int \bar{u}_p^{(n+1)} = \int \{\bar{u}_p^{(n)} - 2\Delta t[\frac{(\bar{u}_p^{(n)} - \bar{u}_{p-1}^{(n)})}{2\Delta x} - \frac{(\bar{u}_p^{(0)} - \bar{u}_{p-1}^{(0)})}{2\Delta x} + \frac{(u_{2p+1} - u_{2p-1})}{2\Delta x}]\}$$

Most of the above terms are equal to zero (derivation of a periodic function) except the last term $\frac{(u_{2p+1} - u_{2p-1})}{2\Delta x}$. A condition for this term to be equal to zero is that it must correspond to the derivation of a periodic function on the coarse grid, and that occurs if the number of nodes (cells) of the fine grid G_h is an even number. If this condition is verified we can write:

$$\int \bar{u}^{(n+1)} = \int \bar{u}^{(n)} = \int \bar{u}^{(0)} = \int u$$

And we deduce the value of K:

$$\int \delta\bar{u} = 0 \Longrightarrow \int K = \sum_p u_{2p+1} \Longrightarrow K = \frac{1}{p}\sum_p u_{2p+1}$$

End of the proof

Therefore we have a solution, and the coarse grid correction is:

$$\delta\bar{u}_p = -u_{2p+1} + K$$

The iterate solution after one cycle is given by:

$$\begin{aligned} u_{2p}^{c+1} &= u_{2p}^{c} - u_{2p+1}^{c} + K \\ u_{2p+1}^{c+1} &= u_{2p+1}^{c} - u_{2p+1}^{c} + K \;=\; K \end{aligned}$$

Conclusions:

1. The ideal coarse grid correction is uniform for the odd nodes.
2. We define the Ideal 2-Grid scheme by:
   1 cycle = 1 fine grid iteration + 1 ideal coarse grid correction

Lemma 2.2 *In the particular case where $\sigma = \frac{1}{2}$ the solution is obtained in two cycles.*

Proof:
After one cycle, the solution on the odd nodes is uniform: Then the iterative basic scheme is:

$$u_j^{n+1} = \frac{(u_j^n + u_{j-1}^n)}{2}$$

It means that after one fine grid iteration the solution is averaged, and two successive nodes have the same value. Thus the iterate solution, after the coarse grid correction, is:

$$\begin{array}{rcl} u_{2p}^{c+1} & = & u_{2p}^c - u_{2p+1}^c + K \; = \; K \\ u_{2p+1}^{c+1} & = & K \end{array}$$

And the fine grid solution is constant (end of the proof).

2.1.2 Local mode analysis

One way to analyze a scheme is the usual Fourier analysis, building its amplification factor $G(\theta)$. This is usually done for one grid and, for that reason, could limitate the field of investigations. It is usually said that the frequencies to be considered must be the high ones (in the interval $[\ \frac{\pi}{2}\ ,\ \pi\ ]$). Indeed, it seems that these frequencies can be excited by the transfer processus and thus must be damped by the smoother. Therefore, we define the local mode analysis by $\max_{\theta \in [\ \frac{\pi}{2}\ ,\ \pi\ ]} |\ G(\theta)\ |$ as a function of ω (Courant number or relaxation parameter in a Jacobi method). If we compare the results obtained with this analysis and with an experiment solving the 1D advection equation it is obvious that some informations have been lost. The behaviour of the analysis seems to tend to the result of the experiment but is much more pessimistic. Some reasons to explain that, can be that such an analysis does not take into account neither the transfers used in the multigrid processus nor the effect of the coarse grid correction.

2.1.3 2-Grid analysis

Local Mode Analysis can give some approximate information about the above 2-grid schemes. Since there are only two levels, a non-approximate Fourier analysis can be applied; we proceed as in Couaillier-Peyret [3] and Leclercq-Stoufflet [4]. This kind of Fourier analysis consists in giving two Fourier

modes for every node, in the iterative scheme. We consider differently the even nodes and the odd nodes for our transfers, thus we have the following kind of solution to be found:

$$\begin{cases} U^n_{h_{2p}} = \widehat{U}^n_h \; e^{i(2p)\theta_h} \\ V^n_{h_{2p+1}} = \widehat{V}^n_h \; e^{i(2p+1)\theta_h} \\ \widehat{U}^0_h = \widehat{V}^0_h \end{cases}$$

Then, setting in a first row the equation at the $2p$ nodes and in a second the equation at the $2p+1$ nodes, we obtain an amplification matrix:

$$\begin{pmatrix} \widehat{U}^{n+1}_h \\ \widehat{V}^{n+1}_h \end{pmatrix} = G \begin{pmatrix} \widehat{U}^n_h \\ \widehat{V}^n_h \end{pmatrix} = G^{n_h} \begin{pmatrix} \widehat{U}^0_h \\ \widehat{V}^0_h \end{pmatrix}$$

n_h is the number of iterations on the fine grid and G is a 2×2 matrix built as follows:

$$\begin{cases} G = \begin{pmatrix} g_e(\theta_h) & g_o(\theta_h) \\ g_o(\theta_h) & g_e(\theta_h) \end{pmatrix} \\ g_e(\theta_h) + g_o(\theta_h) = g(\theta_h) \end{cases}$$

where $g(\theta_h)$ is issued from the usual Fourier analysis of the equation. The amplification factor g can be written as follows:

$$g(\theta_h) = \sum_j \eta_j e^{ij\theta_h}$$

Then we can deduce g_e and g_o:

$$g_e(\theta_h) = \sum_q \eta_{2q} e^{i2q\theta_h} \; ; \; g_o(\theta_h) = \sum_q \eta_{2q+1} e^{i(2q+1)\theta_h}$$

For example, we consider the 1D advection equation with only one grid. The matrix is:

$$G = \begin{pmatrix} (1-\sigma) & \sigma e^{i\theta_h} \\ \sigma e^{i\theta_h} & (1-\sigma) \end{pmatrix} \quad \sigma = c\frac{\Delta t}{\Delta x}$$

There are two eigenvalues: $\lambda^{\pm} = (1-\sigma) \pm \sigma e^{i\theta_h}$ The definition of the amplification factor we will use is the one given by [4], i.e. "the spectral radius of the amplification matrix". For the two-grid analysis, the transfers and the calculus on the coarse grid are contained in a matrix that multiplies the previous one-grid matrix in order to give the amplification matrix of the scheme: let us consider the different steps of a cycle ($(\widehat{u}^a_h, \widehat{v}^a_h)$ is the iterate fine grid solution):

Initialization on the coarse grid: The coarse grid source term $\widehat{S}_{\tau_H}$ and coarse grid interpolation of the correction are given by:

$$\begin{cases} \widehat{S}_{\tau_H} = \widehat{R}_H^0 - \bar{I}_{h\to H}(\widehat{R}_h^a) = c\, z_H(\theta_H)\, \widehat{u}_h^0 - \bar{I}_{h\to H}(\widehat{R}_h^a) \\ \qquad\qquad and \\ \widehat{u}_h^0 = I_{h\to H}(\widehat{u}_h^a) \end{cases}$$

where $z_H(\theta_H)$ is issued from the 1D advection explicit flux on the coarse grid. Notice that:

$$\theta_H = -2\pi\, k\, \Delta x_H = 2\theta_H \quad k = 1, 3, ..., N_H - 1 \quad (k \ odd\ number)$$

Let us exhibit the residual restriction term: $\bar{I}_{h\to H}(\widehat{R}_h^a)$:

$$\bar{I}_{h\to H}(\widehat{R}_h^a) = \bar{I}_{h\to H}\left(c\, Z_h \begin{bmatrix} \widehat{u}_h^a \\ \widehat{v}_h^a \end{bmatrix}\right) = c\,[1, e^{-i\theta_h}]\, Z_h(\theta_h) \begin{bmatrix} \widehat{u}_h^a \\ \widehat{v}_h^a \end{bmatrix}$$

where $Z_h(\theta_h)$ is a 2×2 matrix given by:

$$\begin{cases} Z_h(\theta_h) = \begin{bmatrix} z_{h_e}(\theta_h) & z_{h_o}(\theta_h) \\ z_{h_o}(\theta_h) & z_{h_e}(\theta_h) \end{bmatrix} \\ z_{h_e}(\theta_h) + z_{h_o}(\theta_h) = z_h(\theta_h) \end{cases}$$

where $z_h(\theta_h)$ is issued from the 1D advection explicit flux on the fine grid. If we denote $z_h(\theta_h) = \sum \eta_j e^{ij\theta_h}$ then $z_{h_e}(\theta_h)$ and $z_{h_o}(\theta_h)$ are expressions of the following type:

$$z_{h_e}(\theta_h) = \sum_q \eta_{2q} e^{i2q\theta_h} \ ; \ z_{h_o}(\theta_h) = \sum_q \eta_{2q+1} e^{i(2q+1)\theta_h}$$

The coarse grid solution $\widehat{u}_H^*$: After n_H iterations of the scheme on the coarse grid we obtain the solution, with $\widehat{u}_H^0$ as first iterate:

$$\begin{aligned} \widehat{u}_H^* &= g_H(\theta_H)\widehat{u}_H^{n_H-1} + \frac{1 - g_H(\theta_H)}{c z_H(\theta_H)} \widehat{Sr}_H \\ &= (g_H(\theta_H))^{n_H} \widehat{u}_H^0 + \frac{1 - g_H(\theta_H)}{c z_H(\theta_H)} \widehat{Sr}_H \sum_{k=1}^{n_H-1} (g_H(\theta_H))^k \end{aligned}$$

The coarse grid correction $\widehat{Cor}_H$: Its definition is given as the difference of the calculated solution on the coarse grid and the injection of the fine grid solution:

$$\begin{aligned}\widehat{Cor}_H &= \widehat{u}_H^* - \widehat{u}_H^0 \\ &= \widehat{u}_H^* - I_{h\to H}(\widehat{u}_h^a) \\ &= -\frac{(1-g_H(\theta_H))^{n_H}}{cz_H(\theta_H)}\bar{I}_{h\to H}(\widehat{R}_h^a)\end{aligned}$$

We can notice that for an Ideal-2-Grid scheme the convergence is supposed to be complete on the coarse grid ($g_H(\theta_H)^{n_H} \approx 0$), thus:

$$\widehat{Cor}_H = \frac{-1}{cz_H(\theta_H)}\bar{I}_{h\to H}(\widehat{R}_h^a)$$

The interpolation of the corrections on the fine grid: After one multigrid cycle the fine grid solution is given by:

$$\begin{aligned}\begin{bmatrix}\widehat{u}_h^{n+1}\\ \widehat{v}_h^{n+1}\end{bmatrix} &= \begin{bmatrix}\widehat{u}_h^a\\ \widehat{v}_h^a\end{bmatrix} + I_{H\to h}(\widehat{Cor}_H) \\ &= \begin{bmatrix}\widehat{u}_h^a\\ \widehat{v}_h^a\end{bmatrix} + \begin{bmatrix}1\\ e^{i\theta_h}\end{bmatrix}\widehat{Cor}_H\end{aligned}$$

We obtain the amplification matrix as follows: let n_h be the number of iterations of the scheme on the fine grid,

$$MG = \begin{bmatrix}M_1 & M_2\\ M_3 & M_4\end{bmatrix}(G)^{n_h}$$

The terms of the matrix are given by:

$$\begin{cases} M_1 = 1 - \dfrac{(1-g_H(\theta_H))^{n_H}}{cz_H(\theta_H)}[z_{h_e}(\theta_h) + e^{-i\theta_h}z_{h_o}(\theta_h)] \\ M_2 = -\dfrac{(1-g_H(\theta_H))^{n_H}}{cz_H(\theta_H)}[z_{h_o}(\theta_h) + e^{-i\theta_h}z_{h_e}(\theta_h)] \\ M_3 = [M_1(\theta_h) - 1]e^{i\theta_h} \\ M_4 = 1 + M_2(\theta_h)e^{i\theta_h} \end{cases}$$

Application to 1-Grid In order to clarify this kind of analysis we use it on a classical 1D advection equation to compare it with what is already known. We build the amplification matrix and apply the definition kept for the amplification factor. We obtain then the two eigenvalues $\lambda^{\pm} = (1-\sigma) \pm \sigma e^{i\theta_h}$ where we recognize the usual amplification factor in λ_+. It occurs that in the interval $[\ 0\ ,\ \frac{\pi}{2}\ ]$, λ_+ is the greatest and we have exactly the same amplification factor, but in the interval $[\ \frac{\pi}{2}\ ,\ \pi\ ]$, λ_- is the greatest. Thus we deduce that this analysis can be pessimistic for two grids, since it is already for only one (Fig.1).

Application to 2-Grid Runge-Kutta 1 In this case we use an Ideal-2-Grid analysis (the solution is supposed to be converged on the coarse grid). Choosing $K = c\dfrac{\Delta t}{\Delta x}$ as a parameter, we obtain the convergence rate presented in Fig.2a. We can numerically measure these rates; we obtain (Fig.2b) - in this actual case - exactly the predicted curve. In order to compare this analysis with the one made previously, let us consider the particular case where $K = \dfrac{1}{2}$. This choice should produce convergence within only one cycle (direct solver). But we have seen that the convergence is obtained in two cycles. Actually, the analysis gives us the eigenvalues (eigenfunctions) of the system, thus it is possible that the matrix reduces to a triangular one A.

The multistage Runge-Kutta schemes A. Jameson proposed in [9] to use the multistage schemes to solve the Euler equations. The reason is not the order of accuracy obtained (4 in the classical Runge-Kutta), but their properties of damping and stabilities. This kind of Runge-Kutta p-stage schemes can be written as follows:

$$\begin{aligned} u^{(0)} &= u^n \\ u^{(1)} &= u^{(0)} - \alpha_1 \Delta t D u^{(0)} \\ u^{(2)} &= u^{(0)} - \alpha_2 \Delta t D u^{(1)} \qquad (\alpha_p = 1) \\ &\cdots \\ u^{(p)} &= u^{(0)} - \alpha_p \Delta t D u^{(p-1)} \\ u^{n+1} &= u^{(p)} \end{aligned}$$

$Du^{(p)}$ represents the fluxes calculated with the value of $u^{(p)}$. It has been shown [8] that the Runge-Kutta 4 scheme can present good properties as a

smoother of the high frequencies.

2.2 Imposing an optimal CFL number

2.2.1 Characteristic preconditioning and Jacobi iteration

Clearly the efficiency of the above Ideal-2-Grid schemes is very sensitive to the choice of ω, that is a scaling for the wave speed. When two waves are considered:

$$U_t + AU_x = 0 \quad , \quad U(x,t) \in I\!R$$

a 2x2 scaling matrix must be introduced in order to impose e.g. $CFL = 0.5$ for each wave; this is the idea of **"characteristic time stepping"** (see Van Leer *et al.* [2]) that is written in short as follows

$$U_j^{n+1} = U_j^n + T \begin{pmatrix} \Delta t_1 & 0 \\ 0 & \Delta t_2 \end{pmatrix} T^{-1} \frac{A}{\Delta x}(U_{j-1}^n - U_j^n)$$

with

$$A = T \begin{pmatrix} \lambda 1 & 0 \\ 0 & \lambda 2 \end{pmatrix} T^{-1} \quad , \quad \frac{\lambda_1 \Delta t_1}{\Delta x} = \frac{\lambda_2 \Delta t_2}{\Delta x} = \omega$$

The idea uses a diagonalization of A at node j and is not easily extended to multidimensional systems. It is interesting to note that this iteration scheme can also be understood as a **Jacobi** process:

$$U_j^{n+1} = U_j^n + \omega A^{-1} A(U_{j-1}^n - U_j^n)$$

in which the block-diagonal of the whole system is thus used for preconditioning. While characteristic-time-stepping requires, in the case of variable coefficients or non-linearity, to choose a local value of T, Jacobi iteration is applied in a more straightforward manner both in this simplified case and in 2D non-linear systems: unfortunately, in the 2D case, the different wave speed are not uniformized (this would require a non diagonal preconditionner). In the sequel, we shall consider only **Jacobi-like schemes** and the design parameter will be the damping parameter ω.

2.2.2 Multistage Jacobi

The idea of preconditioning the system to solve leads us to another kind of schemes deduced from the Runge-Kutta ones. Thus we will use several four-stage Jacobi iterations defined as follows [10]:

$$
\begin{aligned}
&X^{(0)} = X_{iter} \\
&For\ l = 1\ to\ 4: \quad X^{(l)} = X^{(0)} + \omega\ \alpha_l\ D^{-1}(B - MX^{(l-1)}) \\
&X_{iter+1} = X^{(4)}
\end{aligned}
$$

Where $MX = B$ is the system to be solved and D the block-diagonal of M. Several choices of the α_l coefficients are possible:

(a)	RK4-S	Standard	1/4	1/3	1/2	1
(b)	RK4-L	From Lallemand [8]	0.11	0.2766	0.5	1.
(c)	RK4-VL	From van Leer et al [2]	0.0833	0.2069	0.4265	1.

Referring to RK1 for the one-stage Jacobi, we observe a strong relation between wideness of the set of CFL that produce damping (Fig. 2, 6a, 6b, 6c) and the corresponding maximal efficiency measures on the NACA practical test (Fig. 4, 7.1a, 7.1b, 7.1c). Van Leer's coefficients are the best among the above choices; some attempt to locally vary them did not bring any noticeable improvement.

2.3 A 2D Euler experiment

2.3.1 Numerical method

Because non linear effects may have consequences on the convergence of a MG process, we shall first restrict to the study of the MG solution of a linear system that is a linearization of the dicretized Euler equations. The computer program that we use was initially written by M.H. Lallemand and is described in [5]. In short, a Finite Volume upwind scheme is applied on a maybe non structured finite element triangular mesh; an unfactored linearized implicit time stepping is applied, relying on the Steger-Warming flux difference splitting:

$$\Phi(U, V) = A^+(U)U + A^-(V)V$$

and linearization is the usual one:

$$\Phi(U^n, V^n, \delta U^{n+1}, \delta V^{n+1}) = A(U^n)\delta U^{n+1} + B(V^n)\delta V^{n+1}$$

So that the implicit part is "first order accurate". We refer to [6] for a further description. The solution of the linearized system is obtained by applying a MG process: The MG scheme is a finite volume one (after the above explanations) and coarse grid cells are group of fine grid cells; some experiments showed that this "unstructured" point of view does not loose convergence speed with respect to structured ones [7].

2.3.2 A first 2D experiment

The flow around a NACA0012 airfoil (Fig.3, Mach=0.85, α=0 degrees) is computed and converged up to a 10^{-4} residual; then the linearized system is picked up and we measure the MG performance. The MG scheme is an Ideal-2-Grid one, with a sawtooth V-cycle, and Jacobi as an iteration on the fine level (block 4x4); we verified that the results presented are mesh size independent. While a rough application of the 1D analysis would have lead to choosing $\omega = \frac{1}{2}$, we see that $\omega_{opt} = 1.01$, while $\omega_{max} = 1.13$.

2.4 A 2-D scalar study

Our purpose now is to evaluate the different sources of discrepancy between the 1-D analysis and the 2-D practical example. We consider a 2-D equation: $au_x + bu_y = 0$ *and periodic b.c.* In order to get a straightforward extension of the 1-D model, we apply the usual Donor Cell upwind scheme on a regular mesh of squares and the Jacobi iteration.

2.4.1 The alignment phenomenon

This occurs when the coarse grid correction has no effect on the iterative solution. That means the scheme behaves just like a 1-Grid one. Let us consider the following example: the 2-D advection equation and a 2-Grid cycle.

$$\begin{cases} u_t + cu_x + \varepsilon u_y = 0 \\ +\textit{periodic boundary conditions} \end{cases}$$

To solve this equation we will suppose:

$$\begin{cases} c = 1 \\ \varepsilon = 0 \end{cases}$$

The upwind scheme used is then:

$$u_{j,k}^{n+1} = u_{j,k}^{n} + \sigma(u_{j-1,k}^{n} - u_{j,k}^{n}) \ , \ \sigma = \frac{\Delta t}{\Delta x}$$

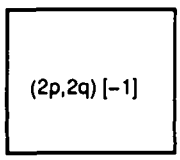
Just as previously we can define the injection on the coarse grid of the fine grid solution by:

$$\bar{u}_{p,q}^{(0)} = \frac{1}{4}\sum(u_{2p\pm1,2q\pm1})$$
$$\bar{u}_{p-1,q}^{(0)} = \frac{1}{4}\sum(u_{2p-2\pm1,2q\pm1})$$

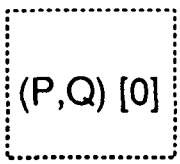
We now write the process on the coarse grid and we determine the coarse grid correction:

$$\begin{aligned} \bar{u}_{p,q}^{(n+1)} &= \bar{u}_{p,q}^{(n)} + \sigma(\bar{u}_{p-1,q}^{(n)} - \bar{u}_{p,q}^{(n)}) \\ &\quad +[-\sigma(\bar{u}_{p-1,q}^{(0)} - \bar{u}_{p,q}^{(0)}) + \frac{\sigma}{2}\{(u_{2p-1,2q} - u_{2p+1,2q}) + (u_{2p-1,2q+1} - u_{2p+1,2q+1})\}] \end{aligned}$$

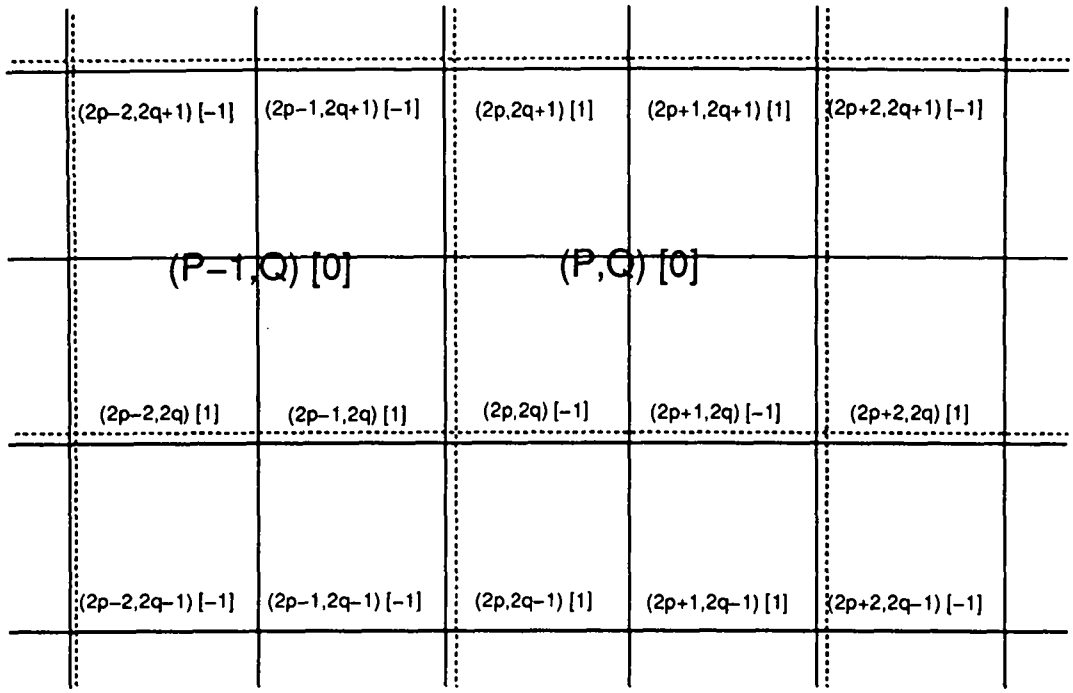
In order to understand this phenomenon we take the following practical example:



: The value of the function is -1 in the fine grid cell (2p,2q)



: The value of the function is 0 in the coarse grid cell (P,Q)



Lemma 2.3 *(i) in 1D: (1,1,-1,-1) is damped by the coarse grid correction because it is captured by the correction (the convergence occurs in two cycles)*

(ii) in 2D the coarse grid correction has no effect on the iterate solution because $u_{2p+1,j} = -u_{2p-1,j}$ *and thus the source term is equal to zero so that:*

$$\bar{u}^{(1)} = \bar{u}^{(0)}$$
$$\bar{u}^{(2)} = \bar{u}^{(1)}$$
$$\ldots$$

Therefore the fine grid keeps the property that $u_{2p+1,j} = -u_{2p-1,j}$ *and the convergence is identical to a 1-Grid convergence.*

2.4.2 Experiments and an heuristic rule

Let us return to $au_x + bu_y = 0$; with an upwind scheme; the results that we present in Figs.5.1a,5.1b,5.1c are obtained by experiments (with verification of mesh independency). These curves can be interpreted as follows: The case

$a = b$ looks like the 1-D case, (with fastest convergence at $\omega = .5$, but with a reduction factor of .6 instead of 0, and maximum stable ω about .7 ; due to 2-D stability condition). In the case $a = 0$, $b = 1$, mesh and equations are strongly aligned, this anihilates the efficiency of intermediary values of ω; the best factor is .8 for $\omega = .2$ or .8, but this measure was not mesh independent. Case $a = 5\ b$ is somewhat intermediary; to understand it, we suggest to compare it with a 1-D experiment in which the system $au_x = 0$; $bv_x = 0$ solved with only an explicit (non-Jacobi) iteration; the resulting curve presents a first segment related to the slow wave with a slope five times smaller than with one equation, and then the fast wave part; this is sketched in Fig.5.2. From the comparison with Fig.4, we have the following comments: No effect resembling the strongly aligned case appears. The dissymmetry in Fig.4 is similar to the dissymmetry in Fig.5.1b. The two-wave-speed model of Fig.5.2 is a rather accurate model of what is measured in Fig.4. We derive the following prospective rule:

1-D optimisation rule: *one important feature of the scheme is its ability in damping simultaneously waves with disparate speeds; therefore we need a good damping for a large interval of CFL numbers.*

3 PRACTICAL APPLICATIONS

3.1 Illustration of the rule on several schemes

We present now some 2-D linearized-Euler experiments for illustrating the above rule. We compare the 2-grid efficiency of several Jacobi iterations: - the usual "one-stage" Jacobi iteration

3.2 From two-grid study to efficient MG

Although (rather) fastly converging, the two-grid scheme is not an efficient one since a complete solution of the coarse grid system is assumed; we consider now a MG scheme with five levels and a W^*-cycle defined as follows: a W^*-cycle is a W-cycle for which smoothing is suppressed when returning at the finest level and on intermediate levels when coming from a coarser one going to a finer one. We think that such a cycle is one (among others) natural extension of the two-grid scheme; in particular it involves only one fine-grid smoothing per cycle; the complexity in 2-D is bounded by 2; the motivation in choosing this scheme is to try to obtain a convergence rather close to the 2-grid cycle, and in particular to avoid saturation effects possibly arising when many levels are used (in [5], 7 levels are used successfully).

3.3 Results

The convergence of the W^*-cycle has been measured to be only slightly mesh dependent (see [5]). We present in Fig.7.2c the reduction factor versus damping parameter for the RK4-VL iteration; this is to be compared to Fig.5.1c (2-grid). The best reduction is obtained for $\omega = 3.3$, a figure very close to the 2-grid one; the corresponding factor is only .6 instead of .5 (for the 2-grid). It is interesting to note that 4 W^*-cycles with RK1 iteration (about same expense) give a reduction of .659, and one W^*-cycle with 4 RK1 iterations as smoother gives a reduction of .76; thus RK4-VL is the efficient option. At last, we evaluate the behaviour of the nonlinear algorithm with implicit linearized time stepping; one W^*-cycle is applied for (partly) solving the linearized system at every time step. The convergence is still slower, with a best reduction factor equal to .7; again the optimal parameter is not very

different, around 3.25 (Fig.7.2d).
As a conclusion of this section, the 2-grid model gives an optimistic but rather good prediction of the performance of the W^*-cycle, with a good guess of the optimal parameter.

4 CONCLUSION

In this report we studied a class of explicit-like multi-stage multi-grid schemes for solving the steady Euler equations; in this class, we suggest to include the RK-Jacobi iteration that presents a natural block-diagonal preconditioner.

This class of schemes are studied as explicit ones although time-consistency is not used for analysing the MG convergence; also, in order to easily take into account transfers and aliasing phenomena that are not measured in the usual local mode analysis, we concentrate the study on 2-grid analysis.

A two-mode Fourier formulation proves to be a predictive tool for the 1-D case.

For the 2-D case, we tried to build a realistic rule, not taking into account alignment problems, since they are not generally encountered on unstructured meshes and would lead to a too much pessimistic view of the convergence.

The proposed rule were verified on a set of RK-Jacobi iterations, and the discrepancy between each stage from models to practical calculation have been quantified. In practise, linear convergence factor with 5 levels is as good as .6 with a cheap W-cycle and a four-stage iteration as smoother.

For obtaining faster MG schemes, we think that we must extend the family of schemes under comparison and optimization. In particular, adding some extra linear algebra, that is, using non-diagonal preconditionners, should be investigated.

5 ACKNOWLEDGEMENT

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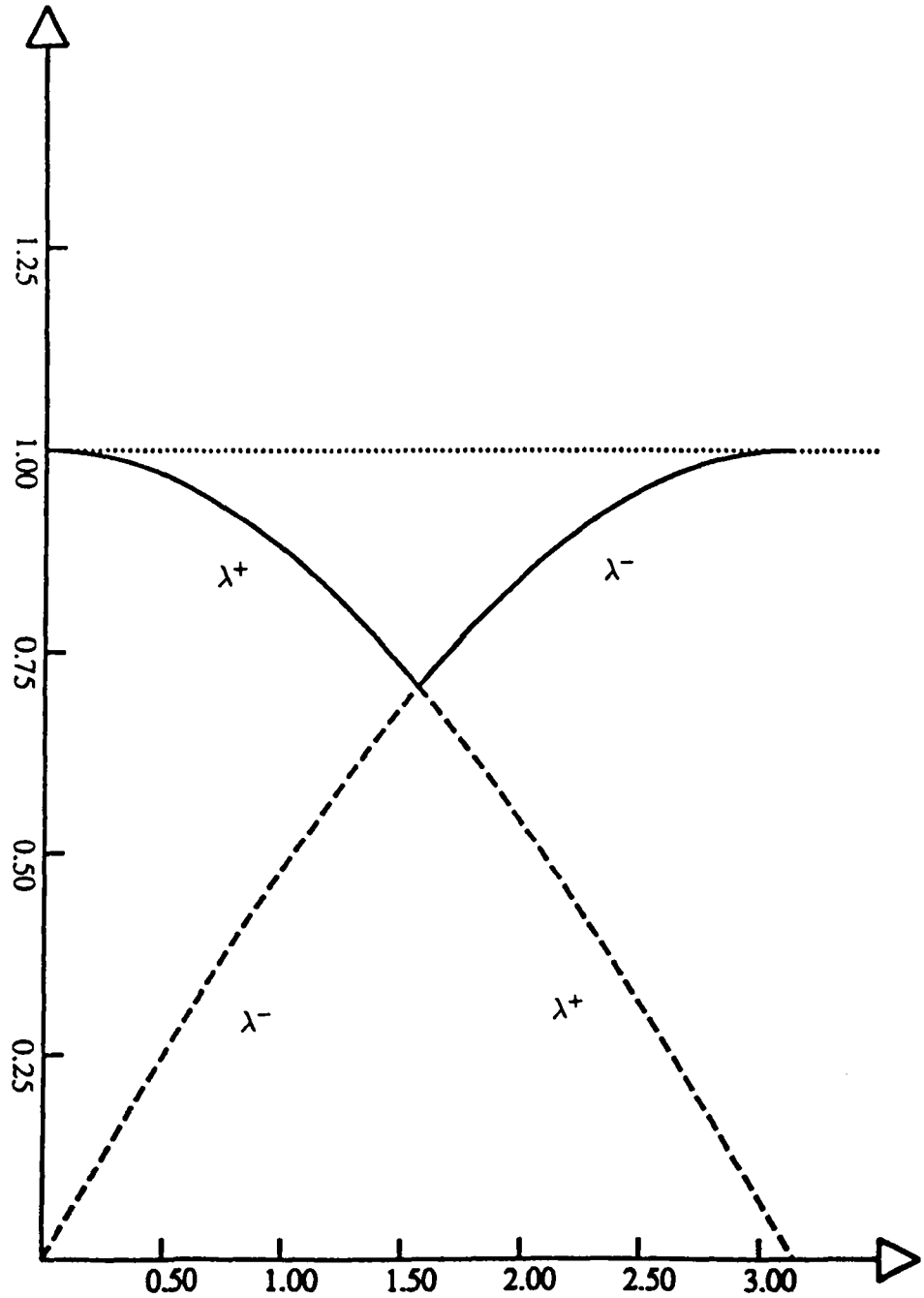
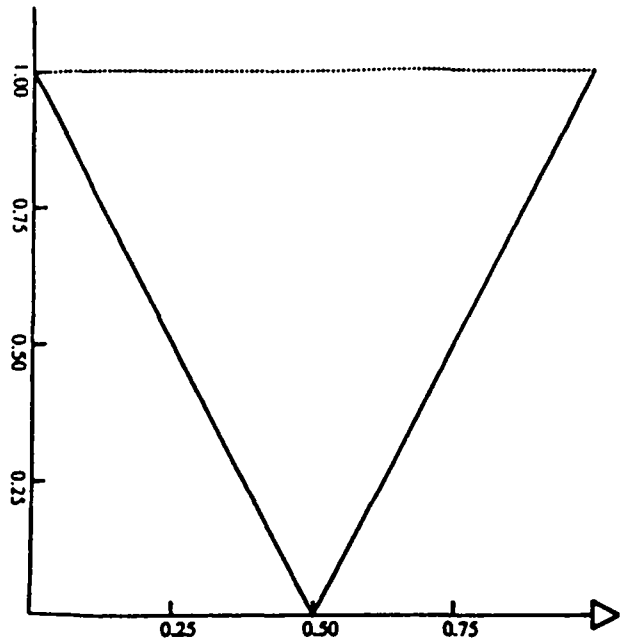
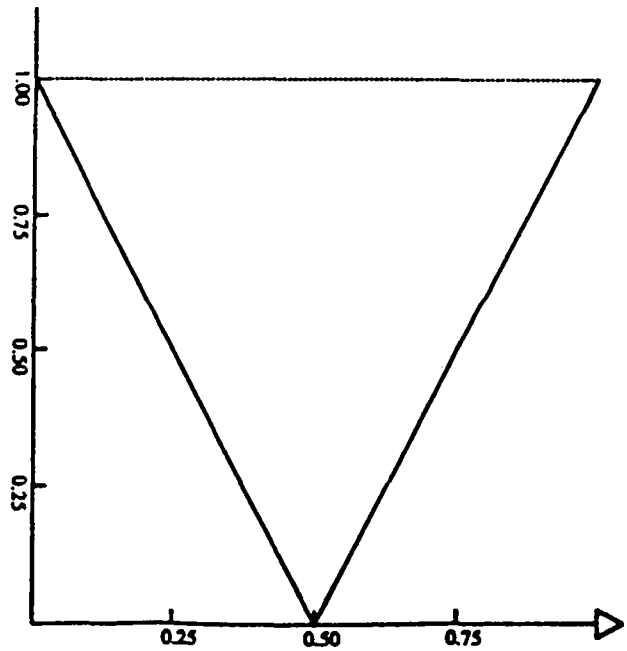


Figure 1: AMPLIFICATION FACTOR VERSUS THE FREQUENCIES FOR THE 1-D ADVECTION EQUATION FOR A 1-GRID METHOD



a. 2-GRID 1-D ADVECTION ANALYSIS



b. 2-GRID 1-D ADVECTION MEASUREMENT

Figure 2: CONVERGENCE RATE VERSUS DAMPING PARAMETER

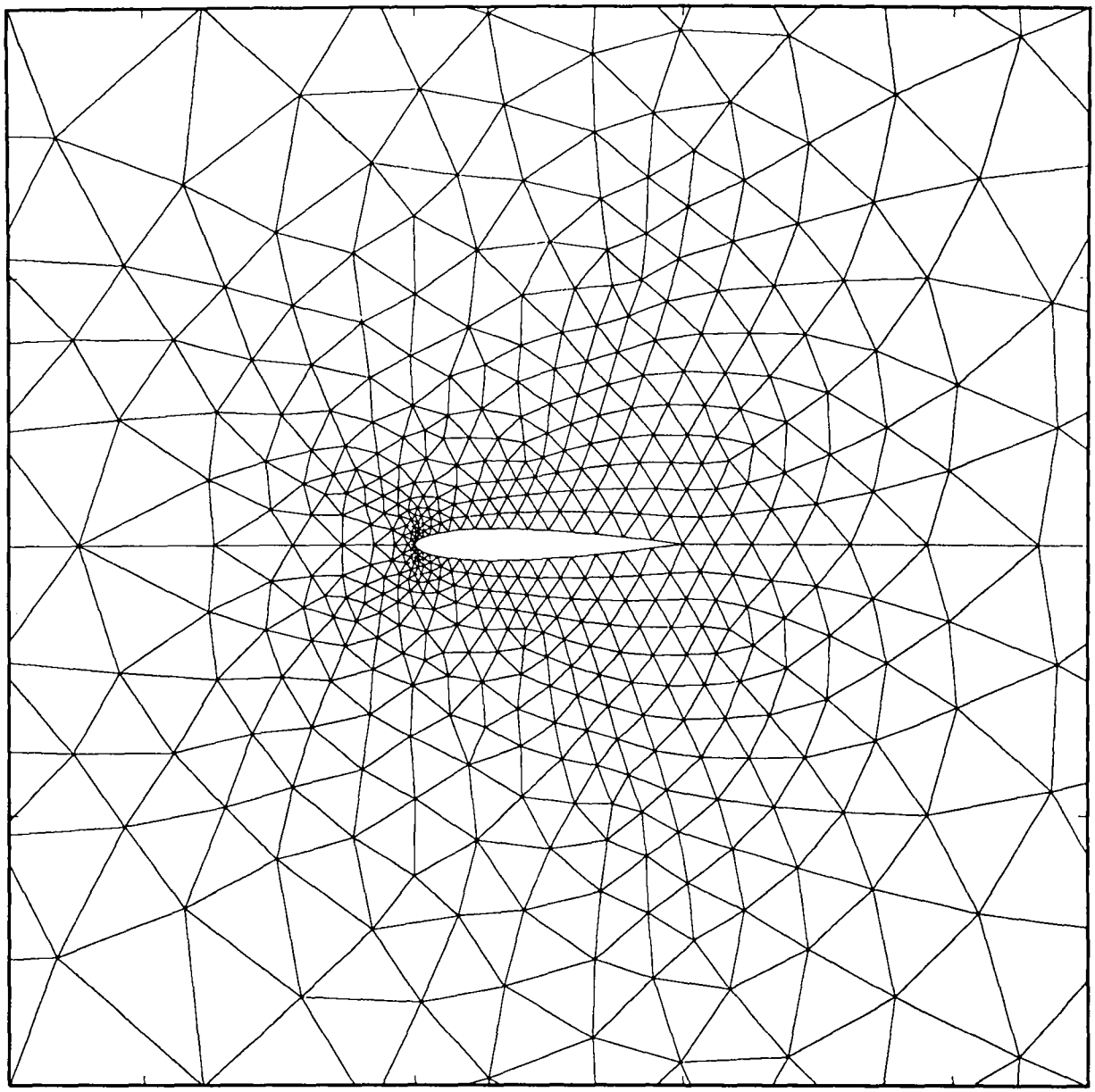


Figure 3: MESH AROUND A NACA0012 AIRFOIL

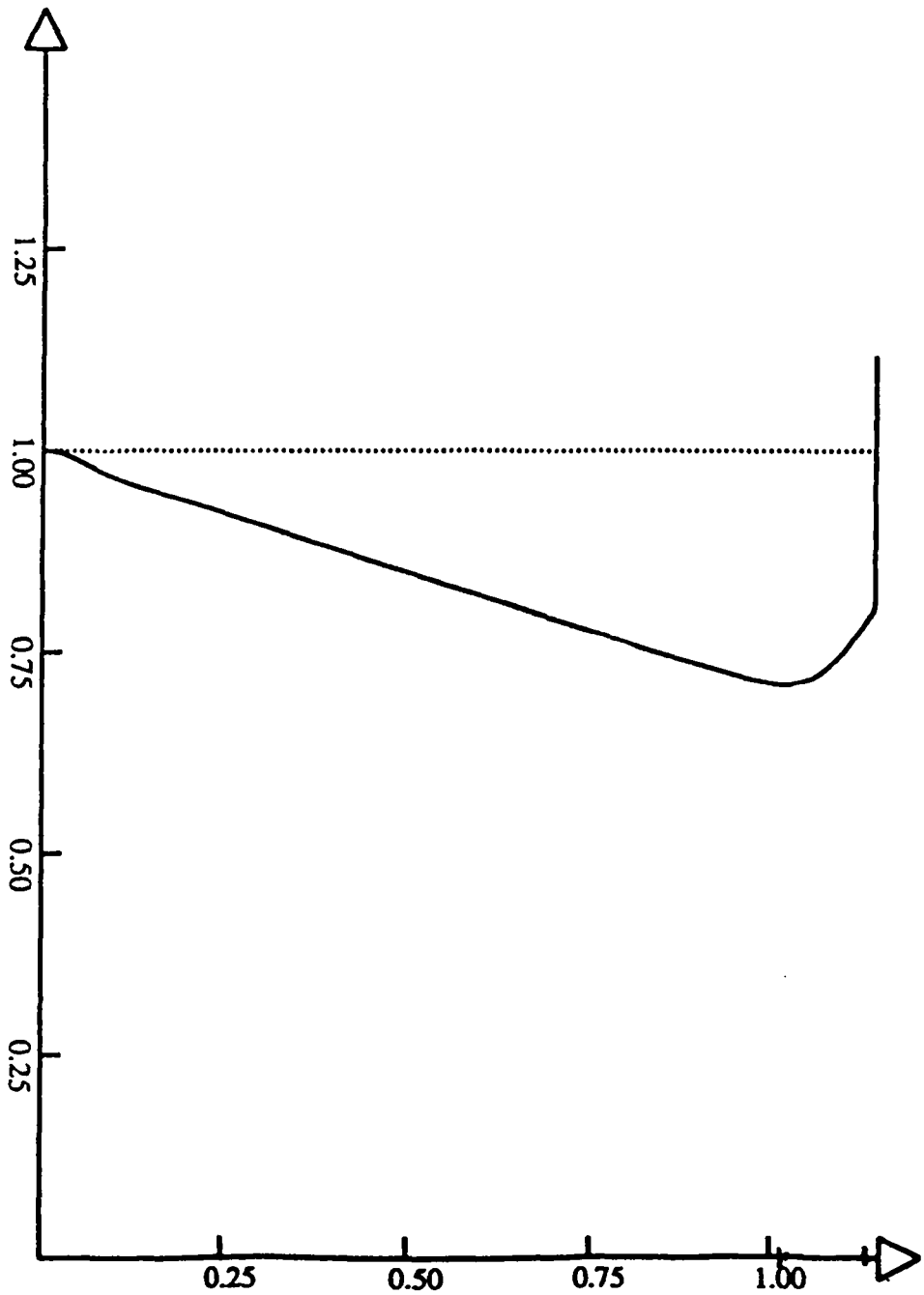
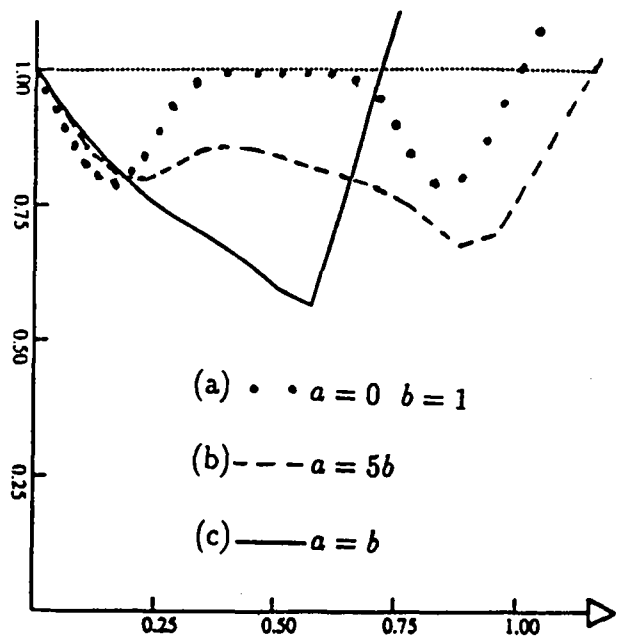
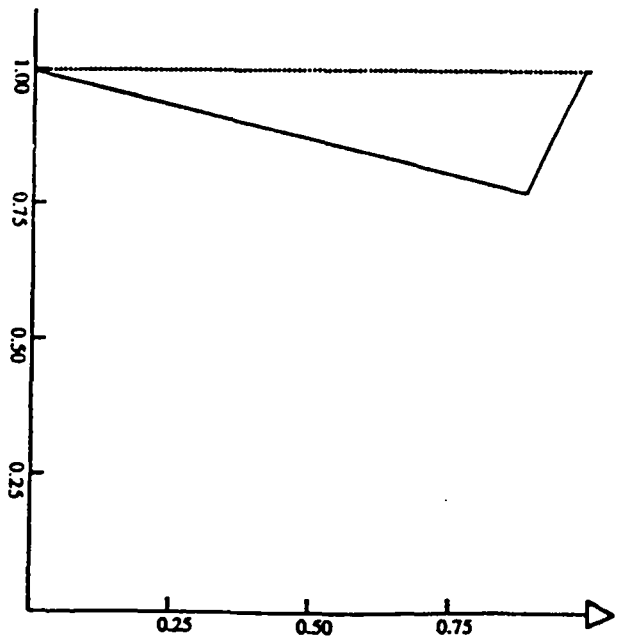


Figure 4: CONVERGENCE RATE VERSUS DAMPING PARAMETER FOR THE JACOBI SOLUTION OF LINEARIZED 2-D EULER SYSTEM (FIRST-ORDER UPWIND DISCRETIZATION; V*-CYCLE)



1. 2-GRID CONVERGENCE FOR THE 2-D SCALAR ADVECTION



2. A 1-D MODEL WITH TWO WAVE SPEEDS; $a = 5b$

Figure 5: A 2-D SCALAR STUDY

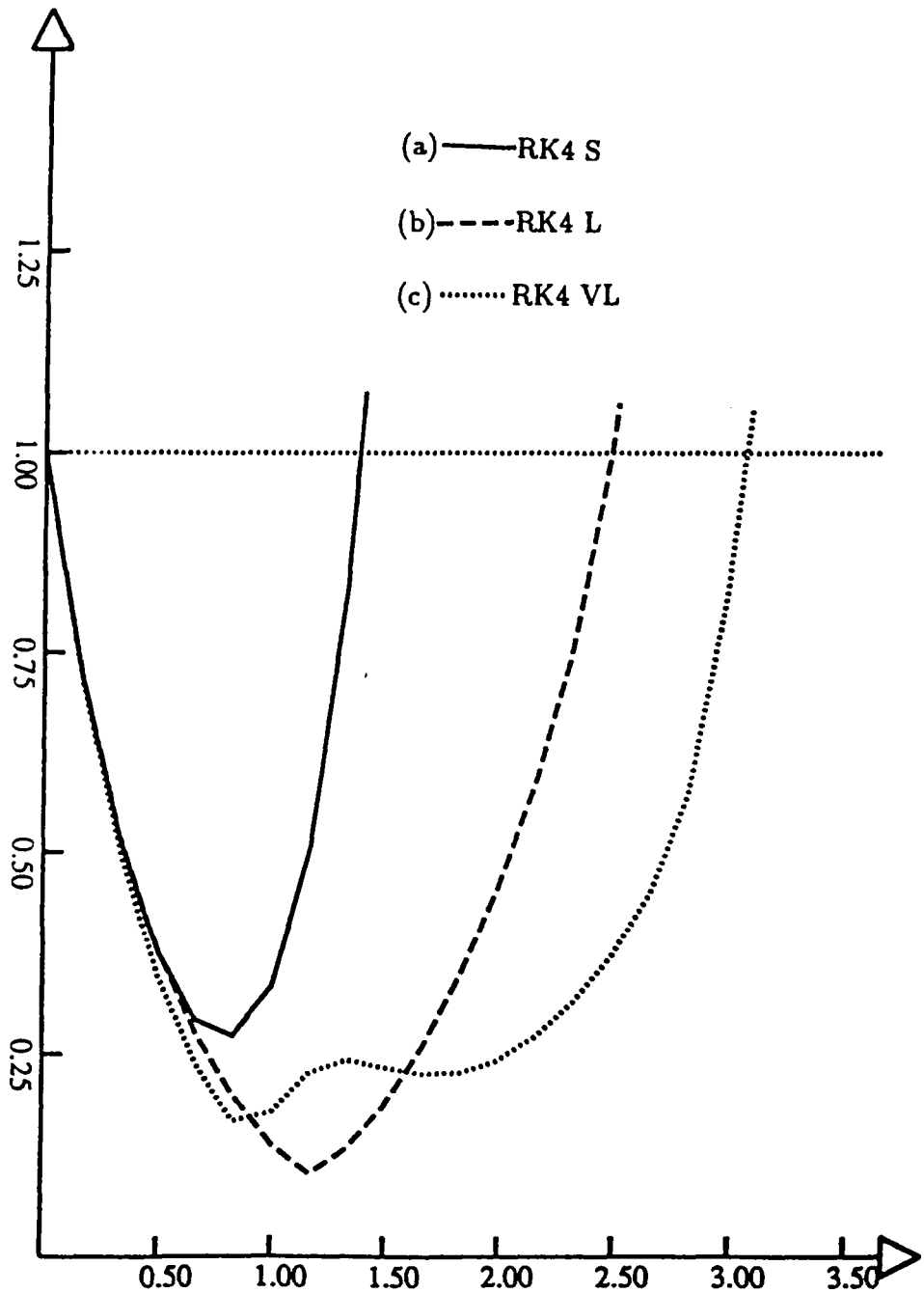
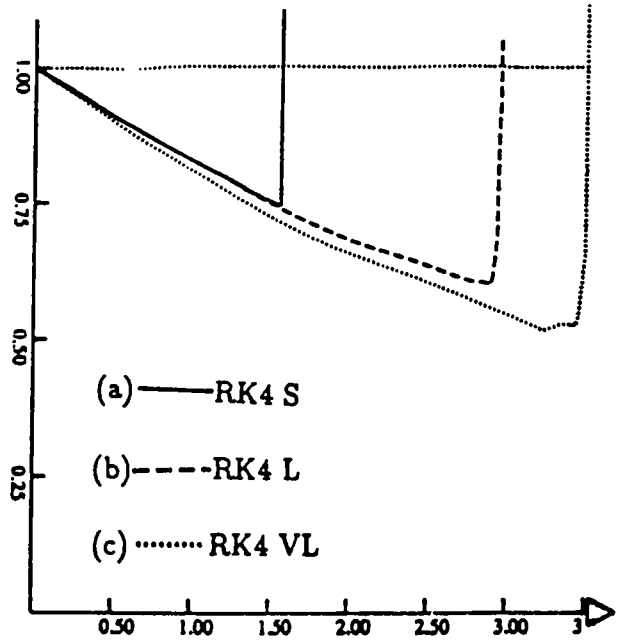
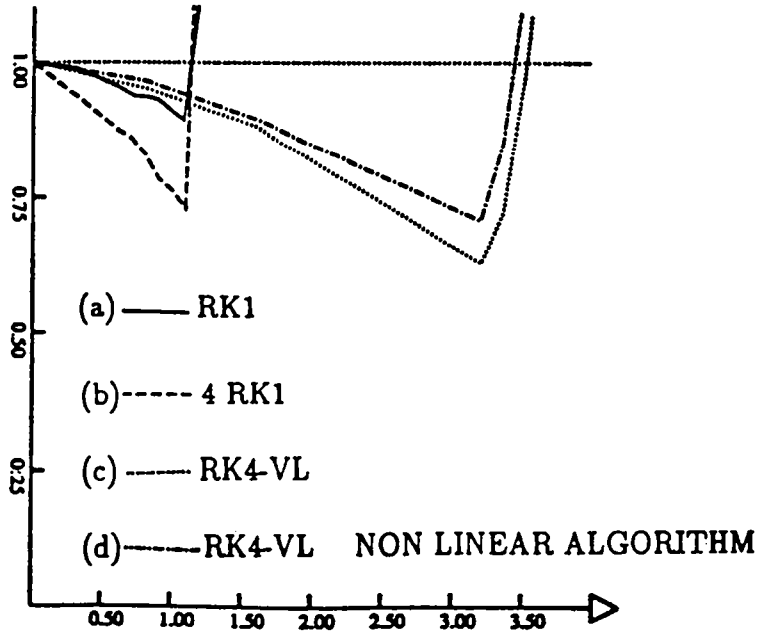


Figure 6: RK4 STUDY 1-D ANALYSIS



1. 2-D PRACTICAL TEST V*-CYCLE



2. 2-D PRACTICAL TEST W*-CYCLE

Figure 7: APPLICATION OF THE ANALYSIS

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