



**HAL**  
open science

# Numerical analysis for the heat flux in a mixed elliptic problem to obtain a discrete steady-state two-phase Stefan problem

Domingo Alberto Tarzia

► **To cite this version:**

Domingo Alberto Tarzia. Numerical analysis for the heat flux in a mixed elliptic problem to obtain a discrete steady-state two-phase Stefan problem. [Research Report] RR-1593, INRIA. 1992. inria-00074967

**HAL Id: inria-00074967**

**<https://inria.hal.science/inria-00074967>**

Submitted on 24 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# INRIA

UNITÉ DE RECHERCHE  
INRIA-ROCQUENCOURT

Institut National  
de Recherche  
en Informatique  
et en Automatique

Domaine de Voluceau  
Rocquencourt  
B.P.105  
78153 Le Chesnay Cedex  
France  
Tél.: (1) 39 63 55 11

## Rapports de Recherche

1992



ème

anniversaire

N° 1593

*Programme 5*

*Traitement du Signal,  
Automatique et Productique*

### NUMERICAL ANALYSIS FOR THE HEAT FLUX IN A MIXED ELLIPTIC PROBLEM TO OBTAIN A DISCRETE STEADY-STATE TWO-PHASE STEFAN PROBLEM

Domingo Alberto TARZIA

Février 1992



\* R R . 1 5 9 3 \*

# NUMERICAL ANALYSIS FOR THE HEAT FLUX IN A MIXED ELLIPTIC PROBLEM TO OBTAIN A DISCRETE STEADY-STATE TWO-PHASE STEFAN PROBLEM

## ANALYSE NUMERIQUE DU FLUX DE CHALEUR DANS UN PROBLEME ELLIPTIQUE MIXTE POUR OBTENIR UN CAS STATIONNAIRE DISCRET DU PROBLEME DE STEFAN A DEUX PHASES

Domingo Alberto TARZIA (\*)

### Abstract

We consider a material  $\Omega \subset \mathbb{R}^n$  which occupies a convex polygonal bounded domain, with regular boundary  $\Gamma = \Gamma_1 \cup \Gamma_2$  with  $\text{meas}(\Gamma_1) = |\Gamma_1| > 0$  and  $|\Gamma_2| > 0$ . We assume, without loss of generality, that the melting temperature is  $0^\circ\text{C}$ . We apply a temperature  $b = \text{Const.} > 0$  on  $\Gamma_1$  and a heat flux  $q = \text{Const.} > 0$  on  $\Gamma_2$ . We consider a steady-state heat conduction problem in  $\Omega$ .

We consider a regular triangulation of the domain  $\Omega$  with Lagrange triangles of type 1. We study sufficient (and/or necessary) conditions for the heat flux  $q$  on  $\Gamma_2$  to obtain a change of phase (steady-state two-phase discretized Stefan problem) into the corresponding discretized domain, that is a discrete temperature of non-constant sign in  $\Omega$ .

### Résumé

On considère un matériel  $\Omega \subset \mathbb{R}^n$ , un domaine polygonal borné et convexe avec une frontière régulière  $\Gamma = \Gamma_1 \cup \Gamma_2$  avec  $\text{mes}(\Gamma_1) = |\Gamma_1| > 0$  et  $|\Gamma_2| > 0$ . On suppose, sans perte de généralité, que la température de changement de phase est  $0^\circ\text{C}$ . On considère une température  $b = \text{Const.} > 0$  sur  $\Gamma_1$  et un flux de chaleur  $q = \text{Const.} > 0$  sur  $\Gamma_2$ . On considère le cas stationnaire du problème de conduction de la chaleur dans  $\Omega$ .

On considère une triangulation régulier de  $\Omega$  avec des triangles de Lagrange de type 1. On étudie des conditions suffisantes (et/ou nécessaires) pour le flux de chaleur  $q$  sur  $\Gamma_2$  pour obtenir un changement de phase (cas stationnaire discret d'un problème de Stefan à deux phases) dans le domaine discrète correspondante, c'est-à-dire une température discrète de signe non-constante dans  $\Omega$ .

Key words : Steady-state Stefan problem, finite element method, mixed elliptic problem, numerical analysis, variational inequalities, error bounds.

Mots Clés : Problème de Stefan stationnaire, méthode d'éléments finis, problème elliptique mixte, analyse numérique, inéquations variationnelles, estimation de l'erreur.

AMS Subject Classification : 35R35, 35J85, 65N15, 65N30.

(\*) Departamento de Matemática, FCE, Universidad Austral, Moreno 1056, (2000) Rosario, Argentina and PROMAR (CONICET-UNR), Instituto de Matemática "Beppo Levi", Av. Pellegrini 250, (2000) Rosario, Argentina.

## I. - INTRODUCTION.

We consider a material  $\Omega \subset \mathbb{R}^n$  which occupies a convex polygonal bounded domain, with a regular boundary  $\Gamma = \Gamma_1 \cup \Gamma_2$  with  $\text{meas}(\Gamma_1) = |\Gamma_1| > 0$  and  $|\Gamma_2| > 0$ . We assume, without loss of generality, that the melting temperature is  $0^\circ\text{C}$ . We apply a temperature  $b = \text{Const.} > 0$  on  $\Gamma_1$  and a heat flux  $q = \text{Const.} > 0$  on  $\Gamma_2$ . We consider a steady-state heat conduction problem in  $\Omega$ . Following [7], we study the temperature  $\theta = \theta(x)$  for  $x \in \Omega$ . The set  $\Omega$  can be written by

$$(1) \quad \Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L} .$$

where

$$(2) \quad \begin{aligned} \Omega_1 &= \left\{ x \in \Omega / \theta(x) < 0 \right\} \text{ (liquid phase) ,} \\ \Omega_2 &= \left\{ x \in \Omega / \theta(x) > 0 \right\} \text{ (solid phase) ,} \\ \mathcal{L} &= \left\{ x \in \Omega / \theta(x) = 0 \right\} \text{ (free boundary) ,} \end{aligned}$$

are the liquid phase, the solid phase and the free boundary which separates them respectively. The temperature  $\theta$  can be represented in  $\Omega$  in the following way

$$(3) \quad \begin{aligned} \theta_1(x) &< 0, & x \in \Omega_1, \\ \theta(x) &= 0, & x \in \mathcal{L}, \\ \theta_2(x) &> 0, & x \in \Omega_2, \end{aligned}$$

and satisfies the following conditions :

$$(4) \quad \begin{aligned} \Delta \theta_i &= 0 & \text{in } \Omega_i \text{ ( } i = 1, 2 \text{) ,} \\ \theta_1 = \theta_2 &= 0, \quad k_1 \frac{\partial \theta_1}{\partial n} = k_2 \frac{\partial \theta_2}{\partial n} & \text{on } \mathcal{L} , \\ \theta_2 |_{\Gamma_1} &= b > 0, \\ -k_2 \frac{\partial \theta_2}{\partial n} |_{\Gamma_2} &= q & \text{if } \theta |_{\Gamma_2} > 0 , \\ -k_1 \frac{\partial \theta_1}{\partial n} |_{\Gamma_2} &= q & \text{if } \theta |_{\Gamma_2} < 0 , \end{aligned}$$

where  $k_i > 0$  is the thermal conductivity in  $\Omega_i$  ( $i=1$  : solid phase,  $i=2$  : liquid phase). If we introduce the new unknown function [2, 7]

$$(5) \quad u = k_2 \theta^+ - k_1 \theta^- \quad \left( \theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \right) \quad \text{in } \Omega ,$$

where  $\theta^+$  and  $\theta^-$  represent the positive part and the negative part of the function  $\theta$  respectively, then problem (4) is transformed into the following mixed elliptic problem

$$(6) \quad \begin{aligned} \Delta u &= 0 \quad \text{dans } \Omega , \\ u |_{\Gamma_1} &= B , \\ -\frac{\partial u}{\partial n} |_{\Gamma_2} &= q , \end{aligned}$$

whose variational formulation is given by

$$(7) \quad \begin{aligned} a(u, v-u) &= L(v-u) \quad , \quad \forall v \in K , \\ u &\in K , \end{aligned}$$

where

$$(8) \quad \begin{aligned} V &= H^1(\Omega) \quad , \quad B = k_2 b > 0 \quad \text{on } \Gamma_1 , \\ K &= \left\{ v \in V / v |_{\Gamma_1} = B \right\} , \quad V_0 = \left\{ v \in V / v |_{\Gamma_1} = 0 \right\} , \\ a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad , \quad L(v) = L_q(v) = - \int_{\Gamma_2} q v \, d\gamma . \end{aligned}$$

Moreover, the solution of (7) is characterized by the following minimum problem [4]

$$(9) \quad \begin{aligned} J(u) &\leq J(v) \quad , \quad \forall v \in K , \\ u &\in K , \end{aligned}$$

where

$$(10) \quad J(v) = J_q(v) = \frac{1}{2} a(v, v) - L_q(v) = \frac{1}{2} a(v, v) + \int_{\Gamma_2} q v \, d\gamma .$$

We can define the real fonction  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  in the following way

$$(11) \quad f(q) = J_q(u(q)) = \frac{1}{2} a(u(q), u(q)) + q \int_{\Gamma_2} u_q \, d\gamma ,$$

where  $u = u(q)$  is the unique solution of the variational equality (7) for each heat flux  $q > 0$  (for a given  $B > 0$ ).

For the continuous problem (6) or (7), a sufficient condition to have a steady-state two-phase Stefan problem (i.e. the solution  $u(q)$  of (7) is a function of non-constant sign in  $\Omega$ ) was obtained in [8, 9].

**Theorem 1** .— (i) Function  $f$  is derivable. Moreover,  $f'$  is a continuous and strictly decreasing function, and it is given by the following expression

$$(12) \quad f'(q) = \int_{\Gamma_2} u(q) \, d\gamma .$$

(ii) There exists a geometrical constant  $C = C(\Omega, \Gamma_1, \Gamma_2) > 0$  such that

$$(13) \quad a(u(q), u(q)) = C q^2, \quad f(q) = -\frac{C}{2} q^2 + B |\Gamma_2| q, \quad \forall q > 0 .$$

Moreover, the constant  $C > 0$  is given by

$$(14) \quad C = a(u_3, u_3) = \int_{\Gamma_2} u_3 \, d\gamma > 0 ,$$

where  $u_3 \in V_0$  ( $u(q) = B - q u_3$  in  $\Omega$ ) is the unique solution of the following mixed elliptic problem

$$(15) \quad \begin{aligned} \Delta u_3 &= 0 \text{ in } \Omega, \\ u_3|_{\Gamma_1} &= 0, \quad \frac{\partial u_3}{\partial n}|_{\Gamma_2} = 1, \end{aligned}$$

whose variational formulation is given by

$$(16) \quad \begin{aligned} a(u_3, v) &= \int_{\Gamma_2} v \, d\gamma, \quad \forall v \in V_0, \\ u_3 &\in V_0 . \end{aligned}$$

(iii) If

$$(17) \quad q > q_0(B)$$

then (6) or (7) represents a steady-state two-phase Stefan problem (i.e. the solution  $u(q)$  of problem (7) is a function of non-constant sign in  $\Omega$ ), where  $q_0 = q_0(B)$  is given by

$$(18) \quad q_0(B) = \frac{B |\Gamma_2|}{C}, \quad \forall B > 0 .$$

(iv) If the function  $u(q)$  is constant over  $\Gamma_2$ , then the sufficient condition (given by (17)) is also necessary.

Proof .— See [8, 9].

Now, we consider  $\tau_h$ , a regular triangulation of the polygonal domain  $\Omega$  with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class  $C^0$ , where  $h > 0$  is a parameter which goes to zero. We can take  $h$  equals to the longest side of the triangles  $T \in \tau_h$  and we can approximate  $V_0$  by [1] :

$$(19) \quad V_h = \left\{ v_h \in C^0(\bar{\Omega}) / v_h|_T \in P_1(T), \forall T \in \tau_h, v_h|_{\Gamma_1} = 0 \right\},$$

where  $P_1$  is the set of the polynomials of degree less or equals than 1. Let  $\pi_h$  be the corresponding linear interpolation operator. Then, we can consider that there exists a constant  $C_0 > 0$  (independent of the parameter  $h$ ) such that

$$(20) \quad \|v - \pi_h v\|_V \leq C_0 h^{r-1} \|v\|_{r,\Omega}, \quad \forall v \in H^r(\Omega), \text{ with } 1 < r \leq 2.$$

We consider the following finite dimensional approximate variational problem, corresponding to the continuous variational problem (7), given by :

$$(21) \quad \begin{aligned} a(u_h, v_h) &= L(v_h), \quad \forall v_h \in V_h, \\ u_h &\in K_h = B + V_h, \end{aligned}$$

and we can obtain the following results.

Lema 2 .— We have

$$(22) \quad \lim_{h \rightarrow 0^+} \|u_h - u\|_V = 0,$$

where  $u$  is the unique solution of the variational equality (7).

Proof .— Owing to  $\text{meas}(\Gamma_1) > 0$ , we have that the bilinear form  $a$  is coercitivity over  $V_0$ , that is [4] :

$$(23) \quad \exists \alpha > 0 / a(v, v) = \|v\|_{V_0}^2 \geq \alpha \|v\|_V^2, \quad \forall v \in V_0,$$

and therefore  $\|\cdot\|_{V_0}$  and  $\|\cdot\|_V$  are two equivalent norms in  $V_0$ . We follow a similar method

developed in [1].

Corollary 3 .- If we define

$$(24) \quad \theta_h = \frac{1}{k_2} u_h^+ - \frac{1}{k_1} u_h^- \in V, \quad \theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \in V$$

then we have

$$(25) \quad \lim_{h \rightarrow 0^+} \|\theta_h - \theta\|_H = 0,$$

where  $H = L^2(\Omega)$ .

Proof .- If we consider the scalar product in  $H$ , defined by

$$(26) \quad (u, v) = \int_{\Omega} u v \, dx,$$

then, we deduce

$$(27) \quad \begin{aligned} \|u_h - u\|_H^2 &= \|u_h^+ - u^+\|_H^2 + \|u_h^- - u^-\|_H^2 + 2(u_h^+, u^-) + \\ &+ 2(u_h^+, u^+) \geq \|u_h^+ - u^+\|_H^2 + \|u_h^- - u^-\|_H^2, \end{aligned}$$

that is

$$(28) \quad \text{Max} (\|u_h^+ - u^+\|, \|u_h^- - u^-\|) \leq \|u_h - u\|_H.$$

From (24) we obtain :

$$(29) \quad \begin{aligned} \|\theta_h - \theta\|_H &\leq \frac{1}{k_2} \|u_h^+ - u^+\|_H + \frac{1}{k_1} \|u_h^- - u^-\|_H \leq \\ &\leq \left( \frac{1}{k_1} + \frac{1}{k_2} \right) \|u_h - u\|_H, \end{aligned}$$

i.e. (25).

The goal of this paper is to consider the numerical analysis of the inequality (17). We study sufficient (and/or necessary) conditions for the constant heat flux  $q$  on  $\Gamma_2$  to obtain a change of phase (steady-state two-phase discretized Stefan problem) into the corresponding discretized domain, that is a discrete temperature of non-constant sign in  $\Omega$ . We obtain that :

i) there exists a constant  $C_h > 0$  (which depends only of the geometry of the domain  $\Omega$  for each  $h > 0$  and it is characterized by a variational problem) such that if  $q > q_{0h}(B) = B|\Gamma_2|/C_h$  then the steady-state discretized problem presents two phases.



ii) we have the estimations  $C_h < C$  and  $q_0(B) < q_{0h}(B)$  where  $C$  and  $q_0(B)$  have been obtained for the continuous problem by (14) and (18) respectively [9].

iii) we deduce an error bounds for  $C - C_h$  and  $q_{0h}(B) - q_0(B)$  as a function of the parameter  $h$ .

In other words, we obtain for the mixed elliptic discretized problem, defined by  $u_h$ , analogous conditions to the ones obtained for the corresponding continuous problem [9], defined by  $u$ .

**II. - INEQUALITY FOR THE HEAT FLUX IN THE DISCRETIZED PROBLEM.**

For each  $q > 0$  we consider the functions  $u(q) \in K$  and  $u_h(q) \in K_h$ , as the unique solution of the variational equalities (7) (continuous problem) and (21) (discrete problem) respectively. We define the real function  $f_h : \mathbb{R}^+ \rightarrow \mathbb{R}$ , for each  $h > 0$ , in the following way

$$(30) \quad f_h(q) = J_q(u_h(q)) = \frac{1}{2} a(u_h(q), u_h(q)) + q \int_{\Gamma_2} u_h(q) \, d\gamma, \quad q > 0.$$

Therefore, we obtain the following properties :

Theorem 4 .- (i) If  $u_i = u_h(q_i)$  is the solution of (21) for  $q_i > 0$  ( $i = 1, 2$ ), then we have the following equalities :

$$(31) \quad a(u_2 - u_1, u_2 - u_1) = (q_1 - q_2) \int_{\Gamma_2} (u_2 - u_1) \, d\gamma,$$

$$(32) \quad a(u_2, u_2) - a(u_1, u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2) \int_{\Gamma_2} (u_1 - u_2) \, d\gamma.$$

(ii) For all real numbers  $q > 0$  and  $\Delta$  such that  $(q + \Delta) > 0$ , we obtain the following estimations :

$$(33) \quad \left\| \frac{1}{\Delta} [u_h(q) - u_h(q + \Delta)] \right\|_V \leq D_1 = \frac{|\gamma_0|}{\alpha} |\Gamma_2|^{1/2},$$

$$(34) \quad \left\| \frac{1}{\Delta} [u_h(q) - u_h(q + \Delta)] \right\|_{L^2(\Gamma_2)} \leq D_2 = D_1 |\gamma_0|,$$

where  $\gamma_0$  is the linear and continuous trace operator, defined over  $V$ . Moreover, the function

$$(35) \quad q > 0 \rightarrow \int_{\Gamma_2} u_h(q) \, d\gamma \in \mathbb{R},$$

is a continuous and strictly decreasing function.

(iii) Function  $f_h = f_h(q)$  is derivable. Moreover,  $f_h'$  is a continuous and strictly decreasing function and given by the following expression

$$(36) \quad f_h'(q) = \int_{\Gamma_2} u_h(q) \, d\gamma ,$$

Proof .— (i) If we take  $v = u_2 - u_1 \in V_h$  in the variational equality corresponding to  $u_1$  and  $v = u_1 - u_2 \in V_h$  in the one corresponding to  $u_2$ , and we add up and subtract both equalities, then we obtain (31) and (32) respectively.

(ii) Taking into account (23), the Cauchy-Schwarz inequality and the continuity of the operator  $\gamma_0$  we deduce (33). From (33) and the continuity of  $\gamma_0$  we have (34). Therefore we have (35) because

$$(37) \quad \left| \int_{\Gamma_2} [u_h(q) - u_h(q + \Delta)] \, d\gamma \right| \leq D_2 |\Gamma_2|^{1/2} \Delta .$$

Moreover, the monotony property is a consequence of (31).

(iii) From (30) and elementary computations, we deduce

$$(38) \quad \frac{1}{\Delta} [f_h(q + \Delta) - f_h(q)] = \frac{1}{2} \int_{\Gamma_2} [u_h(q) + u_h(q + \Delta)] \, d\gamma ,$$

that is (36), by using (35).

Lema 5 .— We have the following expressions :

$$(39) \quad a(u_h(q), u_h(q)) = q B |\Gamma_2| - q \int_{\Gamma_2} u_h(q) \, d\gamma ,$$

$$(40) \quad f_h(q) = q B |\Gamma_2| - \frac{1}{2} a(u_h(q), u_h(q)) ,$$

$$(41) \quad f_h(q) = \frac{q B |\Gamma_2|}{2} + \frac{q}{2} \int_{\Gamma_2} [u_h(q) \, d\gamma ,$$

$$(42) \quad q f_h''(q) = f_h'(q) - B |\Gamma_2| ,$$

$$(43) \quad \frac{d}{dq} [a(u_h(q), u_h(q))] = \frac{2}{q} a(u_h(q), u_h(q)) = 2 B |\Gamma_2| - 2 f_h'(q) .$$

Proof .— Taking  $v_h = B - u_h(q) \in V_h$  in (21) we obtain (31). We deduce (40) and (41) by using (30) and (39). From (36), (39)–(41) we obtain (42) and (43) by derivating with respect to variable  $q$ .

**Theorem 6.** – (i) There exists a constant  $C_h > 0$  such that

$$(44) \quad f_h(q) = q B |\Gamma_2| - \frac{1}{2} C_h q^2, \quad \forall q > 0,$$

$$(45) \quad a(u_h(q), u_h(q)) = C_h q^2, \quad \forall q > 0.$$

(ii) If  $q > q_{0h}(B)$ , then problem (21) represents a discretized steady-state two-phase Stefan problem (i.e.  $u_h(q)$  is a function of non-constant sign in  $\Omega$ ), where

$$(46) \quad q_{0h}(B) = \frac{B |\Gamma_2|}{C_h}.$$

(iii) The constant  $C_h > 0$  can be computed by the expression

$$(47) \quad C_h = \frac{1}{q} \int_{\Gamma_2} [B - u_h(q)] d\gamma,$$

for any  $q > 0$ . Moreover,  $C_h$  is given by

$$(48) \quad C_h = a(u_{3h}, u_{3h}) = \int_{\Gamma_2} u_{3h} d\gamma,$$

where  $u_{3h}$  is the unique solution of the variational equality

$$(49) \quad \begin{aligned} a(u_{3h}, v_h) &= \int_{\Gamma_2} v_h d\gamma, \quad \forall v_h \in V_h, \\ u_{3h} &\in V_h. \end{aligned}$$

**Proof.** – (i) Function  $f_h = f_h(q)$  satisfies  $f_h'''(q) = 0, \forall q > 0$  because (42). Owing to

$$(50) \quad u_h(0^+) = B, \quad f_h(0^+) = B |\Gamma_2| q, \quad f_h'(0^+) = B |\Gamma_2|,$$

there exists some constant  $C_h > 0$  such that (44) and (45) are holds.

(ii) It follows taking into account

$$(51) \quad f_h'(q_{0h}(B)) = 0,$$

and the monotony property of function  $f_h'$ .

(iii) From (39) and (45) we obtain (47) for any  $q > 0$ . Moreover, if we define  $w_h = w_h(q) \in V_h$  by the expression

$$(52) \quad w_h(q) = \frac{B - u_h(q)}{q},$$

then, we have :

$$(53) \quad a(w_h(q), v_h) = \frac{1}{q} a(B - u_h(q), v_h) = -\frac{1}{q} a(u_h(q), v_h) = \int_{\Gamma_2} v_h \, d\gamma, \quad \forall v_h \in V_h,$$

that is  $w_h(q) = u_{3h}$  by uniqueness of the solution of the variational equality (49). then, we deduce (48).

Theorem 7 .- (i) We have the following equality :

$$(54) \quad a(u(q), u_h(q)) = C_h q^2, \quad \forall q > 0.$$

(ii) We have the following inequalities :

$$(55) \quad (a) C_h < C, \quad (b) q_0(B) < q_{0h}(B).$$

Proof .- (i) If we take  $v = u_h(q) \in K_h = B + V_h \subset B + V_0 = K$  in the variational equality (7), and we take into account the expressions (13) and (44), then we obtain (54).

(ii) On the other hand, from (23) and (54) we have

$$(56) \quad \alpha \|u(q) - u_h(q)\|_V^2 \leq a(u(q) - u_h(q), u(q) - u_h(q)) = (C - C_h) q^2,$$

that is (55a). Moreover, (55b) follows from (18), (46) and (55).

Now, we shall use the interpolation result (20) for the function  $u_3 \in H^r(\Omega)$ , as a hypothesis of regularity of the continuous problem (7) (in general,  $1 < r < \frac{3}{2}$  [3, 5, 6]). In [8], we present three examples with explicit solution. In these cases, we have  $u(q), u_3 \in C^\infty(\Omega)$ .

Theorem 8 .- We have the following relations and estimations :

$$(57) \quad a(u(q) - u_h(q), v_h) = 0, \quad \forall v_h \in V_h,$$

$$(58) \quad (C - C_h) q^2 = a(u(q) - u_h(q), u(q) - u_h(q)) \leq \\ \leq \inf_{v_h \in V_h} a(u(q) - v_h, u(q) - v_h),$$

$$(59) \quad 0 < C - C_h \leq C_0^2 h^{2(r-1)} |u_3|_{r,\Omega}^2,$$

$$(60) \quad 0 < q_{0_h}(B) - q_0(B) \leq \frac{C_0^2 h^{2(r-1)}}{C} |u_3|_{r,\Omega}^2 q_{0_h}(B).$$

Proof .— If we take  $v = v_h \in V_h \subset V_0$  in the variational equality (7) and we subtract it with the variational equality (21), we obtain (57). By using (54), (56) and (57) we deduce

$$(61) \quad \begin{aligned} a(u(q) - u_h(q), u(q) - u_h(q)) &= a(u(q) - u_h(q), u(q)) - \\ a(u(q) - u_h(q), u_h(q)) &= a(u(q) - u_h(q), u(q)) - a(u(q) - u_h(q), v_h) = \\ &= a(u(q) - u_h(q), u(q) - v_h) \leq [a(u(q) - u_h(q), u(q) - u_h(q))]^{\frac{1}{2}} \cdot \\ &\cdot [a(u(q) - v_h, u(q) - v_h)]^{\frac{1}{2}}, \quad \forall v_h \in V_h, \end{aligned}$$

because  $a(.,.)$  is a scalar product in  $V_0$ , then we obtain (58).

By using (58), the facts that

$$(62) \quad \Pi_h(u(q)) \in B + V_h, \quad u(q) - \Pi_h(u(q)) \in V_h$$

and the interpolation result (20), we deduce (59). The relation (60) is obtained by using the definition of  $q_{0_h}(B)$  and  $q_0(B)$ , and (59).

Remark 1 .— If we only have  $u(q) \in V$  (i.e.  $u_3 \in V$ ) we can obtain

$$(63) \quad 0 < C - C_h \leq \frac{1}{q^2} \|u(q) - \Pi_h(u(q))\|_V^2 = \|u_3 - \Pi_h(u_3)\|_V^2,$$

where the second term converges to zero when  $h \rightarrow 0^+$  [1], but we cannot give an order of convergence.

Remark 2 .— If the constant heat flux on  $\Gamma_2$  verifies the inequality  $q > q_{0_h}(B)$ , then both discrete and continuous problem represent a steady-state two-phase Stefan problem, that is, their temperatures are of non-constant sign in  $\Omega$ .

Remark 3 .— When the function  $u_h(q)$  is constant on  $\Gamma_2$  (as a function of  $x \in \Gamma_2$ ), then the sufficient condition, given by the Theorem 6—part (ii), is also a necessary condition to have a

two-phase discrete problem, because

$$(64) \quad \int_{\Gamma_2} u_h(q) \, d\gamma < 0 \Leftrightarrow u_h(q) < 0 \text{ on } \Gamma_2 .$$

Theorem 9 .- If we consider  $h, B > 0$ , and  $0 < \epsilon_0 < 1$  ( $\epsilon_0$  is a parameter to be chosen arbitrarily), then we have the following estimations :

$$(65) \quad q_0(B) < q_{0_h}(B) \leq \frac{q_0(B)}{\epsilon_0} \quad \text{and} \quad C_h \geq C \epsilon_0 , \quad \forall h \leq h_r(\epsilon_0),$$

$$(66) \quad 0 < q_{0_h}(B) - q_0(B) \leq \frac{C_0^2 |u_3|_{r,\Omega}^2}{C \epsilon_0} q_0(B) h^{2(r-1)} , \quad \forall h \leq h_r(\epsilon_0),$$

where

$$(67) \quad h_r(\epsilon_0) = \left( \frac{C(1-\epsilon_0)}{C_0^2 |u_3|_{r,\Omega}^2} \right)^{\frac{1}{2(r-1)}} .$$

Proof .- From (60) we deduce

$$(68) \quad A(h) q_{0_h}(B) \leq q_0(B) ,$$

where

$$(69) \quad A(h) = 1 - \frac{C_0^2 |u_3|_{r,\Omega}^2}{C} h^{2(r-1)} < 1 .$$

If we consider, for each parameter  $0 < \epsilon_0 < 1$ , the following equivalence :

$$(70) \quad 0 < \epsilon_0 < A(h) < 1 \Leftrightarrow 0 < h < h_r(\epsilon_0) ,$$

we can deduce the inequalities (65) and (66).

Corollary 10 .- If  $B > 0$ , then we have the following limit

$$(71) \quad \lim_{h \rightarrow 0^+} q_{0_h}(B) = q_0(B) .$$

Remark 4 .- If  $r = 2$  then the convergence in Corollary 10 is of the order of  $h$ .

Remark 5 .— Every thing we proved in this paper is still valid if the boundary  $\Gamma$  of the bounded domain  $\Omega$  is represented by the union of the portions ( $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ ) such that they have the following characteristics :

- (i)  $\Gamma_1$  and  $\Gamma_2$  have the same conditions as the ones previously described in (4).
- (ii)  $\Gamma_3$  is a wall impermeable to heat, i.e. we have  $\frac{\partial \theta}{\partial n} |_{\Gamma_3} = 0$  in (4) and therefore  $\frac{\partial u}{\partial n} |_{\Gamma_3} = 0$  in (6).

Moreover, the first example considered in [8] verifies this condition.

Acknowledgments .— This paper has been sponsored by the Project "Análisis Numérico de Ecuaciones e Inecuaciones Variacionales" from CONICET, Rosario—Argentina.

## REFERENCES

- [1] P.G. CIARLET, "The finite element method for elliptic problems", North-Holland, Amsterdam (1978).
- [2] G. DUVAUT, "Problèmes à frontière libre en théorie des milieux continus", Rapport de Recherche N° 185, LABORIA—IRIA, Rocquencourt (1976).
- [3] P. GRISVARD, "Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain", in Numerical Solution of Partial Differential Equations III, SYNSPADE 1975, B. Hubbard (Ed.), Academic Press, New York (1976), 207—274.
- [4] D. KINDERLEHRER — G. STAMPACCHIA, "An introduction to variational inequalities and their applications", Academic Press, New York (1980).
- [5] M.K.V. MURTHY — G. STAMPACCHIA, "A variational inequality with mixed boundary conditions", Israel J. Math., 13(1972), 188—224.
- [6] E. SHAMIR, "Regularization of mixed second-order elliptic problems", Israel J. Math., 6 (1968), 150—168.
- [7] D.A. TARZIA, "Sur le problème de Stefan à deux phases", C. R. Acad. Sc. Paris, 288A(1979), 941—944; See also "Aplicación de métodos variacionales en el caso estacionario del problema de Stefan a dos fases", Math. Notae, 27(1979/80), 145—156 .
- [8] D.A. TARZIA, "The two-phase Stefan problem and some related conduction problems", Reuniões em Matemática Aplicada e Computação Científica, Vol. 5, SBMAC, Rio de Janeiro (1987).
- [9] D.A. TARZIA, "An inequality for the constant heat flux to obtain a steady-state two-phase Stefan problem", Engineering Analysis, 5(1988), 177—181.

**ISSN 0249 - 6399**