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Abstract. A class of functions with a finite number of return values is defined over combinatorial structures. We prove that the characterization of combinatorial structures by such a function is equivalent to a set of grammar productions, and we show how to derive these productions from the body of the function. In the field of automatic average case analysis, this result allows the use of functions, which makes the description of algorithms more natural and smaller.

Analyse de complexité moyenne de fonctions à nombre fini de valeurs

Résumé. On définit une classe de fonctions opérant sur des structures combinatoires, telle que chaque fonction a un ensemble image fini. On démontre que la caractérisation de structures combinatoires par de telles fonctions est équivalente à la donnée de productions d’une grammaire, et on indique comment obtenir ces productions à partir de la définition de la fonction. Dans le cadre de l’analyse en moyenne automatique d’algorithmes, ce résultat autorise l’utilisation de fonctions, ce qui rend la description des algorithmes plus aisée et concise.

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Analysis of functions with a finite number of return values

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Abstract

A class of functions with a finite number of return values is defined over combinatorial structures. We prove that the characterization of combinatorial structures by such a function is equivalent to a set of grammar productions, and we show how to derive these productions from the body of the function. In the field of automatic average case analysis, this result allows the use of functions, which makes the description of algorithms more natural and smaller.

Keywords: combinatorial structures, automatic average case analysis, generating functions.

1 Introduction

Since Wegbreit’s paper [11], different authors tried to formalize the analysis of algorithms: Ramshaw in his PhD Thesis [9], Cohen and Hickey [2, 7]. Recent works are more concerned in the design of systems that really perform an automatic analysis [8, 13, 5]. In the particular field of average case analysis, generating functions have proven to be a very useful tool [6]. Briefly, programs translate into equations for generating functions (Algebraic Analysis [12]) that contain all the required informations for average case analysis; in particular, some well-known analytic theorems give an asymptotic expansion of the average cost directly from the generating functions (Analytic Analysis [10]).

In this paper, we extend the Algebraic Analysis process to a wider class of programs than the one described in [12]. More precisely, we allow the use of (some kind of) functions, and we show that the resulting programs still translate into generating functions. This result is interesting, because it enables to write smaller (and usually more comprehensive) programs to describe the cost of an algorithm, for example in the Lambda-Upsilon-Omega system. Before we reveal the aim of this paper, let us just give a brief description of this system.

Lambda-Upsilon-Omega (or simply ΛΠΩ) is an assistant system, that analyzes in the average case some well-defined classes of algorithms [4, 5]. To describe an algorithm, one writes a program

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in a special purpose language called ADL (Algorithm Description Language). This program contains declarations for data types, procedures and complexity measures. For example, the set $\mathcal{N}$ of arithmetic expressions constructed with the integers 0, 1, the binary operators $+, \times$ could be described in the following manner:

\[
type N = \text{zero} \mid \text{one} \mid \text{plus} \; N \; \text{N} \mid \text{times} \; N \; N;
\]

\[
\begin{align*}
\text{zero}, & \quad \text{one} = \text{atom}(0); \\
\text{plus}, & \quad \text{times} = \text{atom}(1);
\end{align*}
\]

Here, the keyword \texttt{type} begins the data type declarations, the first line says that an element of $\mathcal{N}$ is either \texttt{zero} or \texttt{one}, or the Cartesian product of \texttt{plus} and two elements of $\mathcal{N}$, and so on. The second line defines \texttt{zero} and \texttt{one} as \texttt{atoms} (terminals in usual grammars) of size 0. The atoms \texttt{plus} and \texttt{times} are of size 1. The size is additive, thus the size of an arithmetic expression as defined by the above lines equals the number of \texttt{plus} and \texttt{times} it contains. For example, the sequence

\[
\texttt{plus times zero times one times plus one times plus one one}
\]

stands for the expression $\left(0 \times \left(1 \times \left((1 + 1) \times ((1 + 1) \times 1))\right)\right) + 1$, whose size is 7.

To describe the cost of an algorithm, one writes some procedures that recursively go through the data structures. For example, one could count the number of $\times$ symbols that are just followed by a 0 in an arithmetic expression by the following ADL procedure

\[
\text{procedure } P \left( n : \text{N} \right);
\]

\[
\begin{align*}
&\text{begin} \\
&\quad \text{case } n \; \text{of} \\
&\quad \quad \text{zero} : \text{found_zero}; \\
&\quad \quad \text{one} : \text{found_zero}; \\
&\quad \quad \text{plus}(i,j) : \text{begin } P(i); \text{P}(j) \text{ end}; \\
&\quad \quad \text{times}(\text{zero},j) : \text{begin } \text{found_one}; \text{P}(j) \text{ end}; \\
&\quad \quad \text{times}(i,j) : \text{begin } P(i); \text{P}(j) \text{ end}; \\
&\quad \text{end} \\
&\text{end}; \\
&\text{measure } \text{found_zero} : 0; \text{ found_one} : 1;
\end{align*}
\]

In this procedure, the \texttt{case ... end} instruction distinguishes between different kinds of expressions, where the pattern \texttt{plus}(i,j) filters all sums, \texttt{times}(\text{zero},j) filters all products whose first component is 0, and \texttt{times}(i,j) filters all remaining products.

Using algebraic analysis rules [12], the system translates the program into generating function\footnote{If $\mathcal{A}$ is a set of combinatorial structures, the (counting) generating function of $\mathcal{A}$ is $A(z) = \sum_{a \in \mathcal{A}} z^{|a|}$, where $| \cdot |$ denotes the size function. If $P$ is a procedure that takes inputs in $\mathcal{A}$, the generating function (of cost) associated to $P$ is $\tau P(z) = \sum_{a \in \mathcal{A}} \tau P(a) z^{|a|}$, where $\tau P(a)$ is the cost of the evaluation of $P$ on $a$. Thus the average cost of $P$ over inputs of size $n$ is simply $\tau P_n/A_n$ where $f_n$ denotes the coefficient of $z^n$ in the Taylor expansion of $f$ around $z = 0$.}1 equations. These equations are then solved by a computer algebra system and we get:

\[
N(z) = \frac{1 - \sqrt{1 - 16z}}{4z}, \quad \tau P(z) = \frac{1 - \sqrt{1 - 16z}}{4\sqrt{1 - 16z}}.
\]

Using singularity analysis [10], the Analytic Analyzer extracts from these generating functions an asymptotic expansion of the average cost, here $n/8 + O(\sqrt{n})$ for expressions of size $n$. The interested reader will find a more complete introduction to the ADL system in [5].

The capabilities of the ADL language, which is extensively described in [12, section 1.6], are powerful enough to describe rather complex problems, like Banach’s famous matchbox
problem [4, Report 16]. But in some cases, the problem description becomes very intricate in
ADL, although its formulation in an usual computer language is easy. Suppose for example
that, among all arithmetic expressions of size $n$ in $N$, we want to know what proportion
have an even value. One (tedious) solution would be to write an ADL grammar for even and
odd expressions; an odd expression is for example either 1, or the sum of an even expression
and an odd one, or the product of two odd expressions. An easier (and more natural) way
would be to write a boolean function $\text{is\_even}$, and then use this function in other parts of
the program, for example if $\text{is\_even}(n)$ then count1 else count0. But the ADL language
as described in [12] does not allow any function.

We provide in this paper an extension of the ADL language with some kind of functions
that compute properties over data structures. These functions are useful to describe algo-
rithms in a natural manner. In this extension, the Algebraic Analysis is still automatic: the
equations for generating functions are computable by a machine. This is a consequence of the
Reduction Theorem (Section 3) that proves in addition that the class of equations produced
is not modified by this extension. For example, we shall see how to obtain automatically the
following result:

The probability $P_n$ that an expression of size $n$ in $N$ has an even value is

$$P_0 = \frac{1}{2}, \quad P_1 = \frac{5}{8}, \quad P_2 = \frac{21}{32}, \quad P_3 = \frac{429}{640}, \quad \ldots, \quad P_n = \frac{1}{\sqrt{2}} + O(n^{-1/2}).$$

The paper is organized as follows: in Section 2, we define a class of functions with a finite
number of return values over combinatorial structures like Cartesian products, lists and sets
(Definition 2); in Section 3, we show that every such function translates into characteristic
data type specifications (Theorem 1 or Reduction Theorem); in Section 4, we apply the
results of Section 3 to the complexity analysis of programs with functions (Theorem 2) and
we provide two examples.

## 2 Definitions

By specifying their grammar productions, we now define a class $\Omega$ of data structures (Sec-
tion 2.1) and a class $\Pi$ of functions on these structures (Section 2.2).

### 2.1 The class $\Omega$

We consider the class $\Omega$ of grammars with productions among the three following classical
kinds, where upper-case letters ($A, B, C, \ldots$) stand for non-terminals, and lower-case letters
($a, b, c, \ldots$) for terminals:

$$A \rightarrow a,$$

$$A \rightarrow B \mid C,$$

$$A \rightarrow B \times C,$$
and other productions that produce respectively lists, sets and multisets, that is sets with repetitions (the +-form of each constructor does not produce the empty structure):

\[
A \rightarrow \text{sequence}(B), \ A \rightarrow \text{sequence}^+(B), \\
A \rightarrow \text{set}(B), \ A \rightarrow \text{set}^+(B), \\
A \rightarrow \text{multiset}(B), \ A \rightarrow \text{multiset}^+(B).
\]

**Definition 1** We call \(\Omega\) the class of grammars whose productions use union (\(\{}\)), product (\(\times\)), and the constructors sequence, sequence\(^+\), set, set\(^+\), multiset, multiset\(^+\).

Every such grammar derives a set of objects (even if these objects are no longer words, because of the set productions, we keep the word “language” for a set of objects). For example, the grammar

\[
A \rightarrow a, \ B \rightarrow b, \ C \rightarrow A \mid B, \ S \rightarrow \text{multiset}(C),
\]

belongs to \(\Omega\), and the non-terminal \(S\) derives all multisets with letters \(a\) and \(b\), including the empty multiset \(\{}\).

### 2.2 The class \(\Pi\) of functions on \(\Omega\)

We now define a class, that we call \(\Pi\), of programs on \(\Omega\). A program is a set of definitions of functions on \(\Omega\). These functions are constructed by means of *programming schemes*. We distinguish between two kinds of programming schemes: *general* schemes do not look at the inner object structure, whereas *descending* schemes depend on the object structure (product, set, sequence, ...).

#### 2.2.1 General schemes

Our notation for a function definition is the following:

\[
\langle \text{function name} \rangle (\langle \text{argument name} \rangle : \langle \text{argument type} \rangle) := \langle \text{function body} \rangle.
\]

In the following lines, the names \(f, g, h\) denote functions of the class \(\Pi\) that is currently being defined. The first general scheme is the *return* scheme:

**Scheme 1 (return)** \(f(a : A) := \langle \text{value} \rangle\),

that defines a constant function over inputs of type \(A\). To combine the results of different functions on the same input, we use the *combine* scheme:

**Scheme 2 (combine)** \(f(a : A) := g(a) \diamond h(a)\)

where \(\diamond\) stands for a binary operator between finite sets. This operator could be one of the logical operators *and*, *or* in the case where \(f, g, h\) return boolean values, that is true or false.
2.2.2 Descending schemes

Combining the general schemes defined above, we are only able to construct constant functions, because we have no branching scheme. The following schemes that depend on the data structure enable us to define more interesting functions.

**Scheme 3 (descend in union)**

\[ A \rightarrow B \mid C \quad f(a : A) := \text{if } a \in B \text{ then } g(a) \text{ else } h(a) \]

This scheme enables us to return a different result according to the type of the function argument. If one compares the union constructor (\(\mid\)) to the record with variant declaration of the PASCAL language, then the descend in union scheme becomes very similar to a test of the record selector:

```plaintext
type A = record case sel : boolean of
  true : (b : B);
  false : (c : C);
end;

if a.sel = true then ... else ...
if a \in B then ... else ...
```

When a data structure is the product of several sub-objects, the following scheme enables one to define functions depending on one component only:

**Scheme 4 (descend in product)** \(A \rightarrow B \times C\)

\[ f((b, c) : A) := g(b). \]

The last schemes to be introduced test whether a given property is true for all components (forall) or at least for one (forone) in a list, set or multiset, denoted below by the generic constructor \(\Phi\). Therefore, these schemes are available only for boolean functions.

**Scheme 5 (descend in list, set, multiset [forall])**

\[ A \rightarrow \Phi(B) \quad f(a : A) := \text{forall } b \text{ in } a \text{ do } g(b) \]

**Scheme 6 (descend in list, set, multiset [forone])**

\[ A \rightarrow \Phi(B) \quad f(a : A) := \text{forone } b \text{ in } a \text{ do } g(b) \]

The meaning of the forall scheme is: if all components \(b\) are such that \(g(b) = \text{true}\), or if there is no component, then \(f\) returns true, otherwise it returns false. In the case of the forone scheme, if \(g(b) = \text{true}\) for at least one component \(b\), then \(f\) returns true, otherwise it returns false; if there is no component, then \(f\) returns false.\(^2\)

**Definition 2** A boolean program is a self-contained set of function specifications on \(\Omega\) defined with the schemes 1–6 (return, combine, descend in union, product, list, set, multiset); we denote by II the class of boolean programs.

As every return value is associated to a constant function (by the return scheme), the number of possible return values of a boolean program is less than the number of functions that it contains. Therefore, functions in programs of II have a finite number of return values.

\(^2\)The forall and the forone schemes may be considered as two particular instances of a general iteration scheme of the form \(d \diamond g(b_1) \diamond g(b_2) \cdots \diamond g(b_k)\), where \(d\) is a default value. With \(d = \text{true}\) and \(\diamond = \text{and}\), we obtain the forall scheme; with \(d = \text{false}\) and \(\diamond = \text{or}\), we get the forone scheme. With \(\diamond = \text{xor}\), we would get a scheme that depends on the parity of the number of \(b\) for which \(g(b)\) is true.
3 Computing characteristic data type specifications

Consider a function $f$ with input in a set $A$ of objects corresponding to a non terminal $A$. If $v$ is a possible value for the function $f$, let $A_{f=v}$ be the subset of objects of $A$ for which $f$ returns $v$.

**Definition 3** A characteristic data type specification of $f$ is a set of grammar specifications of the subsets $A_{f=v}$ for all possible return values $v$ for $f$.

In this section, we prove that programs of $\Pi$ have characteristic data type specifications in the class $\Omega$, and how to compute them from function definitions and initial data type specifications.

**Theorem 1 (Reduction Theorem)** Every program of $\Pi$ translates into characteristic data type specifications for its functions. Moreover, these specifications are in the class $\Omega$.

Let us now prove Theorem 1: The proof divides into two steps: first, every function of $\Pi$ translates into some specification in a class $\Omega_\cap$ (Lemma 1); secondly, these specifications of $\Omega_\cap$ are reduced to specifications of $\Omega$ (Lemma 2).

Let $\Omega_\cap$ be the class of specifications made from the constructors of $\Omega$ with in addition the intersection constructor $\cap$, such that $A \cap B$ derives the data structures derived by $A$ and by $B$.

**Lemma 1** Every program of $\Pi$ translates into data type specifications in $\Omega_\cap$.

**Proof:** The lemma results from the following rules that compute from a function its characteristic data type specification (for the sake of simplicity, the letter $v$ denotes a generic value of the function $f$, and the letter $w$ denotes another value, different from $v$):

**Rule 1 (return)**

$$f(a : A) := v$$

$$A_{f=v} \rightarrow A, \quad A_{f=w} \rightarrow \emptyset$$

**Rule 2 (combine)**

$$f(a : A) := g(a) \circ h(a)$$

$$A_{f=v} \rightarrow \bigcup_{v_1, v_2} A_{g=v_1} \cap A_{h=v_2}$$

**Rule 3 (descent in union)**

$$A \rightarrow B | C \quad f(a : A) := \text{if } a \in B \text{ then } g(a) \text{ else } h(a)$$

$$A_{f=v} \rightarrow B_{g=v} | C_{h=v}$$

**Rule 4 (descent in product)**

$$A \rightarrow B \times C \quad f((b, c) : A) := g(b)$$

$$A_{f=v} \rightarrow B_{g=v} \times C$$
Rule 5 (forall on a sequence)

\[
A \rightarrow \text{sequence}(B) \quad f(a : A) := \text{forall } b \text{ in } a \text{ do } g(b)
\]

\[
\begin{align*}
A_f = \text{true} & \rightarrow \text{sequence}(B_g = \text{true}) \\
A_f = \text{false} & \rightarrow \text{sequence}(B_g = \text{true}) \times B_g = \text{false} \times \text{sequence}(B)
\end{align*}
\]

Rule 6 (forall on a set)

\[
A \rightarrow \text{set}(B) \quad f(a : A) := \text{forall } b \text{ in } a \text{ do } g(b)
\]

\[
\begin{align*}
A_f = \text{true} & \rightarrow \text{set}(B_g = \text{true}) \\
A_f = \text{false} & \rightarrow \text{set}(B_g = \text{true}) \times \text{set}^+(B_g = \text{false})
\end{align*}
\]

The remaining rules are very similar: for the +-form of the set constructor, just replace set by set in the production of \(A_f = \text{true}\); for the multiset constructor, just replace set by multiset in the rules; and for the forone scheme, just exchange true and false in rules 5 and 6.

Once put in Chomsky normal form, all rules give productions of \(\Omega\), except rule 2 that produces an intersection, therefore the whole specification belongs to \(\Omega_\cap\).

**Lemma 2** Each data type specification of \(\Omega_\cap\) produced by the preceding rules reduces to an equivalent (producing the same data structures) specification of \(\Omega\).

**Proof:** [sketch] This lemma is proven by the existence of an algorithm, called Reduction, that transforms specifications of \(\Omega_\cap\) into specifications of \(\Omega\). This algorithm is detailed in [12] in the case of a similar class of functions (schemes 5 and 6 were not allowed).

The main ideas of this algorithm are the following: first, for each new intersection \(U_1 \cap U_2\) to be computed, the non-terminals \(U_1\) and \(U_2\) have necessarily a common ancestor-type, that is a non-terminal \(U\) of the initial specification that derives at least all data structures derived by \(U_1\) and \(U_2\). The existence of this ancestor-type is proven by induction from rule 2. Secondly, the productions that define \(U_1\) and \(U_2\) are necessarily of the same type (terminal, union, product, set, ...). Thus it is possible to write a production for \(U_1 \cap U_2\) in terms of intersection of non-terminals appearing in the right hand side of the productions of \(U_1\) and \(U_2\). Then one replaces everywhere \(U_1 \cap U_2\) by a new non-terminal, and one proceeds. The algorithm halts because the number of intersections to be computed is bounded. One then obtains a specification without any intersection. ■

### 4 Application to complexity analysis

In this section, we apply our results to the complexity analysis of computer programs. On the one hand, Theorem 1 implies that all programs containing functions of the class II could be written without functions, thus functions are not useful. On the other hand, the same theorem implies that every program with functions of II could be rewritten automatically into a program without any function, for which several methods of complexity analysis are known (see for example [5]). Thus the rules of Section 3 are useful because functions help to write more understandable and usually much smaller programs.

In a program description, we use functions by means of the conditional scheme:
Scheme 7 (conditional)

\[ P(a : A) := \text{if } f(a) = v \text{ then } Q(a) \text{ else } R(a), \]

where \( f \) is a function of \( \Pi \), and \( P, Q, R \) are procedures. When the value of \( f \) on \( a \) is \( v \), \( P(a) \) calls \( Q(a) \), otherwise \( R(a) \).

**Theorem 2 (Automatic analysis)** The Algebraic Analysis of ADL programs with functions in the class \( \Pi \) and the conditional scheme is possible automatically (i.e., the equations satisfied by generating functions are automatically computable), and the corresponding average case costs are computable in polynomial time.

This last theorem is in fact a corollary of Theorem 1: the first assertion is a direct consequence of the reduction property and of the algebraic analysis of programs without functions on \( \Omega \) [12, theorem 4]; the second assertion comes from theorems 5 and 6 of [12].

We said in the introduction that the description of algorithms with functions is more natural and easier. As the following result proves it, the use of functions may also produce much smaller programs:

**Theorem 3** Using functions decreases exponentially the program size in some cases.

**Proof:** Consider a program that takes as input an integer in unary notation

\[ \text{type integer = one \mid one integer;} \]

and has cost 1 if this integer is divisible by some number \( p \), and 0 otherwise. Without functions, we write \( p \) mutually recursive procedures \( Q_0, Q_1, \ldots, Q_{p-1} \), where \( Q_k(\text{one}) \) has cost 1 if \( k = p - 1 \), 0 otherwise, and \( Q_k((\text{one}, j)) \) calls \( Q_{(k+1) \mod p}(j) \). This program, with \( Q_0 \) as main procedure, is a solution. Each procedure has a constant length, and there are \( p \) procedures, thus the program length is \( \Omega(p) \). In the same manner, we could obtain a set of boolean functions of total length \( O(p) \) that recognizes multiples of \( p \).

Suppose now that \( p \) is the product of the \( k \) first prime numbers \( p_1, \ldots, p_k \). Using functions, we would write a set of functions for each prime, where one function, say \( M_j \), recognizes multiples of \( p_j \), and a main function whose body would simply be

\[ M_1(i) \text{ and } M_2(i) \text{ and } \ldots \text{ and } M_k(i). \]

This program would have a length of \( O(p_1 + \cdots + p_k + k) = O(kp_k) \), which is exponentially smaller than \( \Omega(p_1 \ldots p_k) \) without functions.

Rules 1 to 6 have already been included in an experimental version of the \( \Pi \Omega \) system. Thus in the following examples, the complexity analysis is done automatically by a computer (namely a Sun 3/60).
4.1 Parity of arithmetic expressions

The purpose of this example is to show how characteristic data types specifications are automatically deduced from function definitions, following the rules of Lemma 1 and the algorithm Reduction of Lemma 2.

We consider the set $\mathcal{N}$ of arithmetic expressions defined in the introduction, and we are interested in the probability of an expression of size $n$ (with $n$ symbols $+$ or $\times$) to have an even value. For example, there are eight expressions of size 1,

$$0 + 0, \ 0 + 1, \ 1 + 0, \ 1 + 1, \ 0 \times 0, \ 0 \times 1, \ 1 \times 0, \ 1 \times 1,$$

and five of them have an even value. Thus the probability for $n = 1$ is $P_1 = 5/8$. Now let us write a boolean function that takes as input an arithmetic expression, and decides recursively whether it is even or not:

```plaintext
function is_even (n : N) : boolean;
begin
  case n of
    zero : true;
    one : false;
    plus(i,j) : if is_even(i) then is_even(j)
                else if is_even(j) then false else true;
    times(i,j) : if is_even(i) then true else is_even(j)
  end
end;
```

Using the conditional scheme, we then write the following procedure that calls a dummy instruction count of cost 1 for each even expression. The average cost of this procedure is thus exactly the probability wanted.

```plaintext
procedure proba_is_even (n : N);
begin
  if is_even(n) then count
end;

measure count : 1;
```

When we analyze this program with the $\mathcal{LY}\Omega$ system, we get the following answer:

```
% luo V1.4
Lambda-Upsilon-Omega V1.4 (Caml+Maple)

#printlevel:=2; analyze "parity"; (* we only type this *)

Introducing the new type N_is_even
Introducing the new type N_not_is_even

Counting generating functions:
N_is_even(z)=zero(z)+plus(z)*N_is_even(z)**2+plus(z)*N_not_is_even(z)**2
            +times(z)*N_is_even(z)*N(z)+times(z)*N_not_is_even(z)*N_is_even(z)
N_not_is_even(z)=one(z)+2*plus(z)*N_is_even(z)*N_not_is_even(z)+times(z)
            *N_not_is_even(z)**2
```

9
These last two equations, that are in fact the algebraic translation of grammar productions generated by the system for even and odd expressions, enable us to find the following generating function of cost for the procedure \textit{is\_even}:

\[ \tau P(z) = \frac{2 - \sqrt{2} \sqrt{1 + \sqrt{1 - 16z}}}{4z}, \]

and the asymptotic expansion \( P_n = 1/\sqrt{2} + O(1/\sqrt{n}) \) follows by means of well-known methods of singularity analysis presented in [5, 3] and implemented in a computer algebra system by B. Salvy [10].

### 4.2 Partitions with odd summands

This other example illustrates the use of the \texttt{forall} and \texttt{forone} schemes for functions over lists, sets or multisets. The following Adl program that belongs to the class II determines whether an integer partition contains only odd summands or not:

```adl
type partition = multiset(integer);
    integer = one | product(one,integer);
    one = atom(1);

function is_odd (i : integer) : boolean;
begin
    case i of
        one : true;
        (one,j) : if is_odd(j) then false else true;
    end;
end;

function has_only_odd_summands (p : partition) : boolean;
begin
    forall i in p do is_odd(i)
end;
```

The automatic complexity analysis of this program by the \texttt{Ap\Omega} system produces the following results:

- Introducing the new type \texttt{integer\_is\_odd}
- Introducing the new type \texttt{integer\_not\_is\_odd}

- Introducing the new type \texttt{partition\_has\_only\_odd\_summands}
- Introducing the new type \texttt{partition\_not\_has\_only\_odd\_summands}

- Counting generating functions:
  \( \text{partition\_not\_has\_only\_odd\_summands}(z) = \text{MP}(\text{integer\_is\_odd}(z)) \times (\text{MP}(\text{integer\_not\_is\_odd}(z)) - 1) \)
  \( \text{partition\_has\_only\_odd\_summands}(z) = \text{MP}(\text{integer\_is\_odd}(z)) \)
  \( \text{integer\_is\_odd}(z) = \text{one}(z) + \text{one}(z) \times \text{integer\_not\_is\_odd}(z) \)
  \( \text{integer\_not\_is\_odd}(z) = \text{one}(z) \times \text{integer\_is\_odd}(z) \)
\[
\tau_{\text{proba has only odd summands}}(z) = \frac{\text{MP}(z)}{2} - 1 + z
\]

where \text{MP} is the operator associated to the \textit{multiset} constructor: \text{MP}(\sum a_n z^n) = \prod \frac{1}{(1-a_n z)^{a_n}}.

5 Conclusion

In this paper, we have presented a model of combinatorial specifications with functions (Section 2) that is completely reducible to a simpler model (Theorem 1), which in turn is well-suited for automatic complexity analysis [5]. Hence we have not enlarged the class of complexity results that a computer is able to produce, but we provide new tools to make the program description easier and smaller, as shown by the examples of Section 4 and Theorem 3.

Our result is very similar to the well-known closure property of context-free languages under the action of finite automata [1]: the class \( \Omega \) introduced here describes context-free languages extended with set constructors (\textit{set, multiset}), and functions of \( \Pi \) behave like automata without memory. More precisely, the functions of \( \Pi \) are tree-automata over the derivation trees of data structures. Because of this analogy, we say that functions of \( \Pi \) compute \textit{regular} properties over the combinatorial data structures of \( \Omega \).

In some sense, \( \Pi \) is the largest class that preserves the closure property of Theorem 1, with respect to the class \( \Omega \). Namely, if we allow an integer counter per function, then we will be able to write a function \( f \) that recognizes words of the form \( a^n b^m c^l \), a function \( g \) that recognizes words of the form \( a^n b^m c^n \), whence \( f \) and \( g \) will recognize the well-known language \( \{a^n b^m c^n\} \), which does not belong to the class of languages produced by a specification of \( \Omega \).

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References


