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***Products of Irreducible Random Matrices
in the (Max,+) Algebra - Part I***

Jean MAIRESSE

N° 1939

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PROGRAMME 1

Architectures parallèles,
bases de données,
réseaux et systèmes distribués



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Products of Irreducible Random Matrices in the $(\text{Max}, +)$ Algebra - Part I

Jean MAIRESSE*

Programme 1 — Architectures parallèles, bases de données, réseaux
et systèmes distribués
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Abstract: The study of networks with synchronization, and more particularly of Stochastic Event Graphs has raised an interest for products of random matrices in the $(\text{Max}, +)$ algebra. We consider a general model of type “ $x(n+1) = A(n)x(n)$ ” where $x(n+1)$ and $x(n)$ are \mathbb{R}^J -valued vectors and $A(n)$ is an irreducible random matrix of size $J \times J$. The exogeneous sequence $\{A(n), n \in \mathbb{N}\}$ is i.i.d or more generally stationary and ergodic. The model is completely solved in a sense, as we give necessary and sufficient conditions for the system to couple in finite time to a unique stationary regime. Moreover, we characterize the systems which converge to a unique regime with ε -coupling only. They appear to be limit cases of the former ones.

Key-words: Discrete event systems, stochastic event graphs, stationary processes, stochastic recursive sequences, coupling, ε -coupling, $(\text{Max}, +)$ algebra, spectral theory, product of random matrices.

(Résumé : tsvp)

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Produits de Matrices Aléatoires

Irréductibles dans l'Algèbre (Max,+) - Partie I

Résumé : L'étude des réseaux avec synchronisation et plus particulièrement celle des Graphes d'Événements Stochastiques a suscité un intérêt pour les produits de matrices aléatoires dans l'algèbre $(Max, +)$. On s'intéresse à un modèle général du type " $x(n+1) = A(n)x(n)$ " où $x(n+1)$ et $x(n)$ sont des vecteurs de \mathbb{R}^J et $A(n)$ est une matrice aléatoire irréductible de taille $J \times J$. La suite exogène $\{A(n), n \in \mathbb{N}\}$ est i.i.d. ou plus généralement stationnaire et ergodique. Le modèle est entièrement résolu dans la mesure où l'on obtient des conditions nécessaires et suffisantes pour que de tels systèmes couplent en temps fini avec un unique régime stationnaire. D'autre part on caractérise les systèmes pour lesquels la convergence vers un régime unique n'a lieu qu'avec ε -couplage. Ces systèmes apparaissent comme des cas limites des précédents.

Les conditions sont basées sur la structure de matrices déterministes choisies dans le support de $A(0)$. Le résultat principal est que le système couple en temps fini avec un régime stationnaire unique si et seulement si le support de $A(0)$ contient un nombre fini de matrices, $\{A_1, \dots, A_k\}$, telles que le produit $A_1 \cdots A_k$ ait un unique régime périodique.

1 Introduction

A large class of computer or communication networks accept a description as Stochastic Discrete Events Systems and can be modeled by Stochastic Petri Networks (SPN). More precisely, a class of SPN, Event Graphs (EG), appears to be very efficient in describing models with synchronization, blocking and/or fork-join properties. We can mention Job-Shop models and asymmetric exclusion models as examples. One of the present approaches is to study those Event Graphs by means of a linear dynamic equation in a non-standard algebra, the $(\text{Max}, +)$ algebra. More precisely, we look at dates associated with transitions of an EG, for example we consider the dates of the beginning of the firing of each transition. We are interested by two kinds of results, which are respectively called first and second order results. The first order deals with ratios of dates and accounts for the existence and the computation of flow or throughput rates. The second order is concerned by differences of dates, and is related to problems on sojourn or idle times, queue length or frequency of occupation. In fact we are going to exhibit conditions under which all models which fit into our framework

The non-autonomous case has been exhaustively treated by Baccelli ([3], [4]) (for both first and second order aspects). First order problems of existence and uniqueness in autonomous (closed) systems have also been solved (same references). This paper deals with the open question of second order problems for autonomous systems. We are going to exhibit necessary and sufficient conditions under which all models which fit into our framework have a unique stationary distribution for quantities such as queue length, sojourn or idle times. In fact the model we are studying includes much more general systems than EG. For example some colored Petri Nets, or task graphs with random precedence (see [5]) fits also into our framework. In [12], Kaspi and Mandelbaum propose sufficient conditions for some closed networks to be ergodic. Those networks are covered by the present study and we obtain hence necessary and sufficient conditions of ergodicity. We can also deal with models of queues with general blocking (Kanban blocking, . . .) as those proposed in [9].

The paper is organized as follows. Sections **2,3** and **4** are presenting the tools we are using in the paper, they can be skipped by people knowing the subject. Section **2** is devoted to the $(\text{Max}, +)$ algebra, section **3** to the spectral theory in this algebra and section **4** to Borovkov's theory of renovating events. Section **5** presents the main results. In **5.1** we define precisely our model, in **5.2** we state some preliminary results. In **5.3** and **5.4** we give sufficient conditions for the stability of discrete and general models respectively. Paragraph **5.5** is devoted to converses of results from the two previous subsections. In section **6**, we weaken the assumptions under which some of our results apply and we make a short comparison with the non-autonomous case. At last, for commodity reasons, we have put the proofs of two theorems in Appendix.

2 (Max,+) linear systems

Definition 2.1 ((Max,+) algebra) We consider the algebra $(\mathbb{R}^*, \oplus, \otimes)$, where $\mathbb{R}^* = \mathbb{R} \cup \{-\infty\}$. The law \oplus is “Max” and \otimes is the usual addition. We set $\varepsilon = -\infty$ and $e = 0$. The element ε is neutral for the operation \oplus and absorbing for \otimes . The element e is neutral for \otimes . The law \oplus is idempotent, i.e. $a \oplus a = a$. $(\mathbb{R}^*, \oplus, \otimes)$ is an idempotent semiring, called a dioid. It is moreover a commutative dioid. We shall write it \mathbb{R}_{Max} .

In the rest of the paper, the notations “+,” will stand for the usual addition and multiplication. Nevertheless, we will write ab for $a \otimes b$ whenever there is no possible confusion.

We define the product spaces $\mathbb{R}_{Max}^J, \mathbb{R}_{Max}^{J \times J}$. We define the product of a vector by a scalar: $a \in \mathbb{R}_{Max}^J, u \in \mathbb{R}_{Max}^J, (a \otimes u)_i = a \otimes u_i$.

Matrix product is defined in the following way. Let $A, B \in \mathbb{R}_{Max}^{J \times J}$,

$$(A \otimes B)_{ij} = \text{Max}_k (A_{ik} + B_{kj}) = \bigoplus_k A_{ik} \otimes B_{kj}.$$

Matrix-vector product is defined in a similar way.

Let us consider a probability space (Ω, \mathcal{F}, P) . We are interested in systems of the type:

$$\begin{aligned} x(n+1) &= A(n) \otimes x(n), \quad n \in \mathbb{N} \\ x(0) &= x_0, \end{aligned}$$

where $x(n)$ and $A(n)$ ($\forall n$) are finite, respectively \mathbb{R}^J and $\mathbb{R}^{J \times J}$ -valued, random variables. $\forall \omega \in \Omega, \forall n \in \mathbb{N} \quad x(n, \omega) \in \mathbb{R}_{Max}^J$ and $A(n, \omega) \in \mathbb{R}_{Max}^{J \times J}$. We are sometimes going to use the notation $x(n, x_0)$ to emphasize the value of the initial condition. We will consider models where the sequences $\{A(n), n \in \mathbb{N}\}$ are respectively i.i.d or stationary and ergodic. Let us recall some definitions adapted to the \mathbb{R}_{Max} algebra.

Definition 2.2 The communication graph of a square matrix A is a directed graph having a number of nodes equal to the size of A . This graph contains an arc from i to j iff $A_{ji} \neq \varepsilon$. The valuation of this arc is A_{ji} .

Definition 2.3 A square matrix A is irreducible if: $\forall i, j \quad \exists m \geq 0 \quad | \quad (A^m)_{ij} > \varepsilon$ (or equivalently if its communication graph is strictly connected).

Definition 2.4 A square matrix A is aperiodic if: $\exists N \quad \forall n \geq N \quad A^n > \varepsilon$.

Definition 2.5 A stochastic matrix $\{A(\omega), \omega \in \Omega\}$ has a fixed structure if $P(A_{ij} = \varepsilon) = 1$ or $P(A_{ij} = \varepsilon) = 0, \forall i, j$.

We will consider only models where $A(n)$ is $P - a.s$ irreducible.

There are no further assumptions on the matrices, in particular they are not supposed to have a fixed structure, to be positive or aperiodic. Our only assumption is not too restrictive as the problems we are going to treat are already solved when matrices $A(n)$ have a fixed structure and are reducibles (see [3], [4]). On the other hand, when matrices $A(n)$ are reducible and have no fixed structure, the problem is still open and we discuss it a little further in section 6.

Definition 2.6 ($\mathbb{P}\mathbb{R}_{Max}^J$) *The projective space $\mathbb{P}\mathbb{R}_{Max}^J$ is defined as the quotient of \mathbb{R}_{Max}^J by the parallelism relation:*

$$u, v \in \mathbb{R}^J \quad u \simeq v \iff \exists a \in \mathbb{R}_{Max} \setminus \{\varepsilon\} \text{ such that } u = a \otimes v.$$

Let π be the canonical projection of \mathbb{R}_{Max}^J into $\mathbb{P}\mathbb{R}_{Max}^J$.

For example $(e, -1)'$ and $(1, e)'$ are in the same parallelism class, i.e. are two representatives of the same vector of $\mathbb{P}\mathbb{R}_{Max}^J$. We define in the same way $\mathbb{P}\mathbb{R}^J$, $\mathbb{P}\mathbb{R}_{Max}^J$, $\mathbb{P}\mathbb{R}_{Max}^{J \times J}$. We define a distance on $\mathbb{P}\mathbb{R}^J$ which we are going to call the projective distance.

Definition 2.7 *Let $x, y \in \mathbb{P}\mathbb{R}^J$ and $u, v \in \mathbb{R}^J$ be two representatives of x and y respectively ($\pi(u) = x$, $\pi(v) = y$). We define:*

$$d(x, y) = d(u, v) = \bigoplus_i (u_i - v_i) \otimes \bigoplus_i (v_i - u_i).$$

It is easy to check that $d(x, y)$ does not depend on the representatives u and v . It is also easy to check that it is a distance in $\mathbb{P}\mathbb{R}^J$. It is nothing else than the \mathcal{L}_∞ norm on the projective space $\mathbb{P}\mathbb{R}^J$. We write either $d(x, y)$ or $d(u, v)$ with a little abuse of notation. It is also worth mentioning that it is the \mathbb{R}_{Max} analogue of a distance used in classical Perron-Frobenius theory, which is called the Hilbert's projective metric and is defined by " $\delta(u, v) = \ln(\inf\{\mu/\lambda \mid \lambda u \leq v \leq \mu u\})$ ". We have the following very important property:

Proposition 2.1 *Let $A \in \mathbb{R}_{Max}^{J \times J}$ be an irreducible matrix. Let u, v be two vectors of \mathbb{R}^J . We have:*

$$d(Au, Av) \leq d(u, v).$$

There is no simple criterion to get a strict inequality. This monotonicity is characteristic of the \mathbb{R}_{Max} algebra and has to be interpreted as a synchronization property.

Definition 2.8 We consider $A \in \mathbb{R}_{Max}^{J \times J}$. We set

$$D(A) = \sup_{u,v \in \mathbb{R}^J} d(Au, Av).$$

We call $D(A)$ the projective diameter of A .

A matrix A can be viewed as a linear (in an \mathbb{R}_{Max} sense) operator of $\mathbb{P}\mathbb{R}^J$ into $\mathbb{P}\mathbb{R}^J$. It is bounded if the (decreasing) sequence $D(A^k)$ has a finite limit.

3 Deterministic spectral theory

We will now recall some results of deterministic spectral theory in the \mathbb{R}_{Max} algebra. Those results can be found in [4].

We want to find non trivial solutions of the equation:

$$A \otimes x = \lambda \otimes x,$$

where $A \in \mathbb{R}^{J \times J}$ is an irreducible matrix, x is a column vector (the ‘‘eigenvector’’) and λ is a real constant (the ‘‘eigenvalue’’). We define also periodic solutions of the eigenvalue problem. A periodic solution of period d is a set of vectors $\{x_1, \dots, x_d\}$ of \mathbb{R}^J verifying $Ax_i = \lambda x_{i+1}$, $i = 1, \dots, d-1$ and $Ax_d = \lambda x_1$.

Definition 3.1 For each path $\zeta = \{t_1, t_2, \dots, t_j, t_{j+1} = t_1\}$, we define the average weight by:

$$p(\zeta) = \frac{a_{t_1 t_j} \otimes \dots \otimes a_{t_3 t_2} \otimes a_{t_2 t_1}}{j},$$

(here the division is the conventional one).

Theorem 3.1 There is a unique (non ε) eigenvalue, λ . It satisfies the relation

$$\lambda = \max_{\zeta} p(\zeta),$$

where ζ covers all the circuits of (the communication graph of) A . We call also λ the **Lyapunov exponent** of A .

There might be several eigenvectors. A linear combination (in \mathbb{R}_{Max}) of eigenvectors is an eigenvector. An eigenvector has all its coordinates different from ε (due to the irreducibility assumption).

Definition 3.2 For a matrix A , we define:

Critical circuit A circuit ζ of A is said to be critical if its average weight is maximal, i.e. if $p(\zeta) = \lambda$.

Critical graph *It consists of the nodes and arcs of A belonging to a critical circuit.*

Cyclicity *The cyclicity of a strictly connected graph (i.e. of an irreducible matrix) is the g.c.d. of the lengths of all the circuits. The cyclicity of a connected graph is the l.c.m. of the cyclicities of its strictly connected subgraphs (s.c.s.).*

We normalize a matrix by dividing (in \mathbb{R}_{Max} i.e. by subtracting in the conventional algebra) all its entries by its eigenvalue. A normalized matrix has e as eigenvalue. For a normalized matrix A of size J , we define:

$$A^+ = A \oplus A^2 \oplus \cdots \oplus A^J .$$

We check that $A^+ \oplus A^{J+1} = A^+$.

Theorem 3.2 *Let A be a normalized matrix. Every eigenvector of A writes as a linear combination of columns of A^+ . More accurately, we have:*

- A_i^+ , i belonging to the critical graph, is an eigenvector.
- A_i^+ and A_j^+ are independent iff i and j belong to two different s.c.s. of the critical graph.

A corollary of this result will be of particular use for us:

An irreducible matrix has a unique eigenvector (up to a multiplicative (\otimes) constant) iff its critical graph has a unique s.c.s.

In \mathbb{R}^{Max} , every irreducible matrix is cyclic in the sense of the following theorem.

Theorem 3.3 *For an irreducible matrix A of size J and whose eigenvalue is λ , there exists integers d and M such that:*

$$\forall m \geq M, \quad A^{m+d} = \lambda^{\otimes d} \otimes A^m ,$$

furthermore the smallest d verifying the property is equal to the cyclicity of the critical graph of A . We will call it the cyclicity of A .

A cyclicity greater than one will provide periodic solutions of period greater than one for the eigenvalue problem.

An irreducible matrix has a unique eigenvector and no periodic solutions of period greater than one for the eigenvalue problem, iff its critical graph has a unique s.c.s. and its cyclicity is one. Such a matrix will be called a **scs1-cyc1** matrix.

4 Borovkov Renovating Events Theory

Borovkov's theory deals with the problem of regeneration in so-called "stochastic recursive sequences". For a complete treatment, the reader is referred to [7] or [8].

Definition 4.1 We call stochastic recursive sequence (SRS), a sequence $\{x(k)\}$ of \mathbb{R}^J -valued random variables defined by

$$x(k+1) = f(x(k), a(k)), \quad k \geq 0$$

$$x(0) = x_0,$$

where $\{a(k)\}$ is an exogeneous sequence of \mathbb{R}^N -valued random variables, stationary by the translation shift θ . f is a measurable function from $\mathbb{R}^J \times \mathbb{R}^N$ into \mathbb{R}^J . The vector x_0 is the initial condition. In order to stress the value of the initial condition, we will sometimes write the SRS: $\{x(k, x_0)\}$.

We will talk of an i.i.d. SRS when the sequence $a(k)$ is i.i.d. (an i.i.d. SRS is a Markov Chain and the converse is true!).

Definition 4.2 We consider $\{x(k)\}$, a SRS. We write \mathcal{F}_n the σ -algebra $\mathcal{F}_n = \sigma\{a(k), k \in \{-\infty, \dots, n-1\}\}$. The sequence of events $\{\mathcal{A}(k) \in \mathcal{F}_{k+m}, k \in \mathbb{N}\}$ is said to be a renovating sequence of length m and of associated function $\phi: \mathbb{R}^{m \times N} \rightarrow \mathbb{R}^J$ if:

$$\exists k_0 \forall k \geq k_0, x(k+m) = \phi(a(k), a(k+1), \dots, a(k+m-1)) \text{ on } \mathcal{A}(k).$$

A sequence $\{\mathcal{A}(k), k \in \mathbb{N}\}$ of renovating events of same length and associated function is said to be stationary if $\mathcal{A}(k) = \mathcal{A}(0) \circ \theta^k = \theta^{-k} \mathcal{A}(0)$.

We will need the following notions of convergence:

Definition 4.3 We say that there is coupling convergence in finite time (or even only coupling) of sequence X_k to the stationary sequence $Y \circ \theta^k$ if

$$P(X_{n+k} = Y \circ \theta^{n+k}, \forall k \geq 0) \xrightarrow{n \rightarrow +\infty} 1.$$

It is easy to show that this notion of coupling convergence implies total variation convergence ($X_n \rightarrow Y$ in total variation if $\sup_A |P(X_n \in A) - P(Y \in A)| \xrightarrow{n \rightarrow +\infty} 0$).

Definition 4.4 We say that a SRS $\{X_n\}$ strongly coupling converges in finite time (or strongly converges) to the stationary sequence $\{Y \circ \theta^n\}$ if:

$$\nu = \min \left\{ n \geq 0 \mid X_{k+n} \circ \theta^{-(k+n)} = Y, \forall k \geq 0 \right\} \text{ is a.s finite,}$$

(the sequence $\{X_n \circ \theta^{-n}\}$ corresponds to the famous Loynes scheme).

Remark Strong coupling implies coupling but the converse is not true.

Theorem 4.1 (Borovkov's renovating events) *Let (Ω, \mathcal{F}, P) be a probability space. Let $\{a(k)\}$ be a sequence of θ -stationary random variables, \mathbb{R}^N -valued. We consider a SRS $\{x(k)\}$ defined by:*

$$\begin{aligned} x(k+1) &= f(x(k), a(k)), \quad k \geq 0 \\ x(0) &= x_0. \end{aligned}$$

If the random process $\{x(k), k \in \mathbb{N}\}$ admits a stationary sequence of renovating events $\{\mathcal{A}(k)\}$ such that $P(\mathcal{A}(0)) > 0$, then there exists a finite random variable Z such that:

$$Z \circ \theta = f(Z, a(0)),$$

and $x(n)$ strongly coupling converges in finite time to $Z \circ \theta^n$. (For an i.i.d. SRS (= Markov Chain), the conditions of this theorem are equivalent to those ensuring Harris ergodicity! [8]).

In the preceding results, we have considered SRS defined with a unique initial condition, x_0 . In the rest of the paper, we will be interested by having results uniformly over the initial conditions. We will then use the following generalization of Borovkov's theorem.

Theorem 4.2 *We consider an increasing sequence of events $\{V_k\}$ such that $\bigcup_{k=1}^{+\infty} V_k = \mathbb{R}^J$. We suppose that for all k , there exists a stationary sequence of events $\{\mathcal{A}_n^k\}$ verifying $P(\mathcal{A}_0^k) > 0$ and which is renovating for the SRS $\{x(k, x_0)\}$, $\forall x_0 \in V_k$. Then $\{x(k, x_0)\}$ converges with strong coupling to a unique stationary regime, $\forall x_0 \in \mathbb{R}^J$.*

Theorem 4.3 (converse of theorems 4.1 and 4.2) *Conditions of theorem 4.1 (resp. 4.2) are necessary and sufficient for strong coupling convergence (resp. uniformly over \mathbb{R}^J).*

There is a strong connection between Borovkov's theory in an i.i.d. setting and regenerative processes. For an explanation of this, the reader is referred to [10]. In fact, for an i.i.d. SRS, the existence of renovating events of length 1 implies that the SRS is regenerative in the classical sense (regeneration cycles are i.i.d.). The existence of renovating events of length > 1 implies that the SRS is regenerative in the sense of Asmussen [1] (regeneration cycles can be dependent given that any cycle is independent of previous regeneration times).

5 Presentation of the results

5.1 Introduction

The model is the one introduced in section 2: " $x(n+1) = A(n)x(n)$ ". We are interested in two kinds of asymptotic results.

- First order results, on ratios:

$$\lim_k \frac{\|x(k)\|}{k}, \quad \lim_k \frac{x_i(k)}{k}.$$

- Second order results, on differences:

$$\lim_k x_i(k+1) - x_i(k), \quad \forall i,$$

$$\lim_k x_j(k) - x_i(k), \quad \forall i \neq j.$$

First order's theorems have been treated by Baccelli ([3] or [4]) using Kingman's sub-additive ergodic theorem. Let us state one of the main results. For $x \in \mathbb{R}^J$, we use the notation $\|x\| = \bigoplus_{i=1}^J x_i$.

Proposition 5.1 *Let $\{A(n)\}$ be a stationary and ergodic sequence of (P -a.s.) irreducible matrices. There exists a constant $\lambda \in \mathbb{R}$ such that, for all initial condition x_0 ,*

$$\lim_k \frac{\|x(k, x_0)\|}{k} = \lim_k \frac{x_i(k, x_0)}{k} = \lambda, \quad P - a.s.$$

*The constant λ is called the **Lyapunov exponent** of the stochastic matrix $A(0)$ (cf theorem 3.1).*

The second order's problems have been solved in the non-autonomous case (when matrices $A(n)$ have a fixed and reducible structure) (same references) and the results are recalled in section 6.

The rest of the paper deals with second order's problems in the autonomous ($A(n)$ irreducible) case. The limit, if it exists, is expected to be a random variable. We are interested in determining whether there is one or several limit laws, and also in the kind of convergence occurring. It appears that the notion of coupling convergence in finite time is central in our model. In fact, we show that some of the quantities we are interested in are SRS (def 4.1).

We write π the canonical projection $\mathbb{R}_{Max}^J \xrightarrow{\pi} \mathbb{P}\mathbb{R}_{Max}^J$. It is clear that the recursive equation $x(n+1) = A(n)x(n)$ defines a SRS. It is then easy to show that $\pi(x(n))$ is also a SRS. Indeed, let's consider $x(n)$ and $x'(n)$ such that $\pi(x(n)) = \pi(x'(n))$. We define $x(n+1) = A(n)x(n)$ and $x'(n+1) = A(n)x'(n)$. It is straightforward that $\pi(x(n+1)) = \pi(x'(n+1))$. We can write with a little abuse of notation that $\{\pi(x(n)), n \in \mathbb{N}\}$ verify the recursive equation " $\pi(x(n+1)) = A(n)\pi(x(n))$ " (identifying $A(n)$ and its canonical projection in $\mathbb{P}\mathbb{R}_{Max}^{J \times J}$).

Proposition 5.2 For $i \in \{1, \dots, J\}$, we define $z_i(n) = x_i(n) - x_i(n-1)$. The vector $(z_i(n), \pi(x(n)))$ is a SRS.

Proof This proposition was proved in [15].

$$\begin{aligned} z_i(n) &= \bigoplus_k (A_{ik}(n-1)x_k(n-1)) - x_i(n-1) \\ &= A_{ii}(n-1) \oplus \bigoplus_k A_{ik}(n-1) \otimes (x_k(n-1) - x_i(n-1)), \end{aligned}$$

and $(x'_k - x'_i)$ is independent of x' in the parallelism class of $x(n-1)$. ■

More precisely, we have shown that: $\forall i, z_i(n) = F_i(A(n-1), \pi(x(n-1)))$, where F_i is an absolutely continuous function. The sequence $\{A(n)\}$ being stationary by hypothesis, it implies the following corollary.

Corollary 5.1 A sufficient condition for $(z_1(n), \dots, z_J(n))'$ to couple in finite time (resp. to converge almost surely) to a unique stationary sequence, uniformly over initial conditions in $\mathbb{P}\mathbb{R}^J$, is that $\pi(x(n))$ has the same property.

Remark This sufficient condition is not necessary... as it is shown by examples arising from the deterministic spectral theory. Let us consider

$$A = \begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix}.$$

We have $A^+ = A$, so $u_1 = (1, e)'$ and $u_2 = (e, 1)'$ are eigenvectors of A (theorem 3.2). As a consequence $\{u_\lambda = \lambda \otimes u_1 \oplus (1-\lambda) \otimes u_2, \lambda \in [0, 1]\}$ is the set of eigenvectors of A . There is a continuum of stationary regimes for $\pi(x(n))$. It is easy to check that for an initial condition $u_\lambda, \lambda \in [0, 1]$, we have:

$$x_1(n, u_\lambda) - x_2(n, u_\lambda) = 2 \times \lambda - 1.$$

But on the other hand, we have a unique stationary regime for $z_i(n)$. As a direct consequence of the equality $A^2 = A$, we have $z_1(n) = z_2(n) = 1, \forall n \geq 2$.

We can also build stochastic counter-examples based on the model of dimension 2 to appear in example at the end of section 5.2.

Remark The results on $\pi(x(n))$ depend only on the sequence $\{\pi(A(n))\}$ where π is the canonical projection in $\mathbb{P}\mathbb{R}_{Max}^{J \times J}$. Hence, all results on $\pi(x(n))$ would still be true under the weaker assumption that only the sequence $\{\pi(A(n))\}$ is stationary and ergodic. But then, the previous corollary would not be true anymore.

In the rest of the paper we will say that there is a unique stationary regime for our model when the SRS $\pi(x(n))$ (and $z_i(n), \forall i$) converges to a unique stationary distribution, uniformly over $x_0 \in \mathbb{R}^J$. We will also say that the model is **stable**. This convergence will occur in different senses, weak convergence, total variation convergence, etc...

5.2 Preliminary results

The first results we are going to show will enable us to understand why convergence occurs in finite time in some of our models, suggesting that Borovkov's theory is the right way to deal with them.

5.2.1 Finite models

We begin by considering finite models. Let A_i , $i \in \{1, \dots, I\}$ be a finite collection of irreducible matrices of size $J \times J$. We suppose there exists a discrete probability law $\{p_i, p_i > 0\}$ such that $A(n, \omega) = A_i$ with probability p_i . The sequence $\{A(n)\}$ is stationary and ergodic. Let $d(.,.)$ denote the projective distance.

Proposition 5.3 *Let $x_0, y_0 \in \mathbb{R}^J$, be two initial conditions. We have:*

$$d(A(n, \omega)A(n-1, \omega) \cdots A(0, \omega)x_0, A(n, \omega)A(n-1, \omega) \cdots A(0, \omega)y_0) \xrightarrow{n \rightarrow +\infty} c(\omega).$$

There is a unique stationary regime iff $c = 0$, $P - a.s.$ When $c = 0$, $P - a.s.$, this limit is attained in finite time.

Proof In a deterministic setting (i.e. $P(A(n, \omega) = A) = 1$), there is coupling in finite time with a periodic regime, as a direct consequence of theorem 3.3. We have $c = d(A^M x_0, A^M y_0)$ and c is attained in less than M steps, where M is such that $A^{m+d} = \lambda^{\otimes d} A^m$, $\forall m \geq M$. In the more general setting of our proposition, the proof is given in Appendix. ■

Corollary 5.2 *We consider a finite model. If there is a unique stationary regime, then convergence to this stationary regime occurs with coupling in finite time.*

Proof The uniqueness of stationary regimes implies that $c = 0$, $P - a.s.$ Proposition 5.3 assures that convergence to 0 happens in finite time and there is coupling because we consider only SRS. ■

5.2.2 Discrete models

We now consider a discrete model " $x(n+1) = A(n)x(n)$ ". We assume there exists $\{A_i, i \in I \text{ or } i \in \mathbb{N}\}$, a finite or countable collection of irreducible matrices of size $J \times J$ such that $A(n, \omega) = A_i$ with probability $p_i > 0$.

In the **i.i.d.** setting if there exist a periodic stationary regime, there is not one but d stationary regimes in the sense defined previously, where d is the period. It would then be interesting to determine if it is possible to get a unique periodic regime, uniformly over

initial conditions. In fact, it is not possible under an i.i.d assumption. We are going to show an even better result which is that there is uniqueness or an infinity of stationary regimes.

Proposition 5.4 *We consider a model with an i.i.d. sequence $\{A(n)\}$. If there is a finite number of stationary or periodic regimes (i.e. if there is a finite number of extremal stationary probability measures), then there is a unique stationary regime uniformly over initial conditions (possibly random) in \mathbb{R}^J .*

Proof The demonstration, rather long, is given in Appendix. ■

Corollary 5.3 *There can not be a unique periodic regime (period $d > 1$) uniformly over initial conditions in \mathbb{R}^J .*

Remark A good illustration of this theorem is given by the \mathbb{R}_{Max} deterministic systems. We consider a matrix

$$A = \begin{pmatrix} e & a \\ a & e \end{pmatrix}.$$

For $a > e$, there is a continuum of periodic regimes of period 2 for $\pi(x(n), x_0)$, when the initial condition x_0 vary. Those periodic regimes are $(\pi(b, e), \pi(e, b))$, $e \leq b \leq a$. For $a = e$, there is a unique eigenvector, (e, e) , and no periodic regimes of period greater than 1. For $a < e$, there is a continuum of eigenvectors, $\pi(e, b)$, $-a \leq b \leq a$. In all cases, there is either one or an infinity of stationary regimes.

5.2.3 General models

We now come back to a general model “ $x(n+1) = A(n)x(n)$ ” as presented in section 2. We still have that there is a unique stationary regime only if $c = 0$ P -a.s. We would like to show the converse property. Let us introduce a new definition.

Definition 5.1 (ε -coupling) *We consider a metric space (E, d) . We consider two sequences $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ defined on E . We say that there is ε -coupling of those two sequences if for each $\varepsilon > 0$, one can find versions of $\{X_n\}$ and $\{Y_n\}$ defined on a common probability space and an a.s. finite random time N such that*

$$n \geq N \implies d(X_n, Y_n) \leq \varepsilon.$$

The following proposition is shown in Asmussen [2].

Proposition 5.5 *We consider a sequence $\{X_n\}_{n \in \mathbb{N}}$ and a stationary sequence $\{Y \circ \theta^n\}_{n \in \mathbb{N}}$ defined on E . Let μ be the invariant distribution of Y . If there is ε -coupling of the two sequences, then $\{X_n\}$ converges weakly to μ .*

We are now ready to prove the following proposition.

Proposition 5.6 *We consider a general model “ $x(n+1) = A(n)x(n)$ ”. If we have the property $\forall x_0, y_0 \in \mathbb{R}^J$, $d(x(n, x_0), x(n, y_0)) \xrightarrow{n} 0$, P -a.s. then there is a unique stationary regime $\{Z \circ \theta^n\}$, i.e. a unique r.v. Z verifying $Z \circ \theta = A(0)Z$. Convergence to the stationary sequence $Z \circ \theta^n$ occurs with ε -coupling. It implies weak convergence of $x(n, x_0)$, $\forall x_0 \in \mathbb{R}^J$ to the stationary distribution of Z .*

Proof We define $\{Z \circ \theta_n\}$ a stationary sequence such that Z is a finite r.v. verifying $Z \circ \theta = A(0)Z$. The existence of such a sequence comes from a backward scheme of Loynes kind. It is detailed in Appendix, in the proof of proposition 5.3. $d(\cdot, \cdot)$ denotes the projective distance, we have:

$$\forall x_0 \in \mathbb{R}^J, d(x(n, x_0), Z \circ \theta_n) \xrightarrow{n \rightarrow +\infty} 0, P - a.s. \quad (1)$$

We conclude that there is ε -coupling of $\{x(n, x_0)\}$ and $\{Z \circ \theta_n\}$. We can apply proposition 5.5. There is weak convergence of $\{x(n, x_0)\}$ to the distribution of Z and relation 1 establishes the a.s. convergence of $\{x(n, x_0)\}$ to $Z \circ \theta_n$. ■

Remark In fact we do not need to introduce ε -coupling to prove the previous proposition. The backward scheme (see in Appendix) gives us the following result.

$$\forall x_0 \in \mathbb{R}^J, A(-1)A(-2) \cdots A(-n)x_0 \xrightarrow{n \rightarrow +\infty} Z(\omega), P - a.s.$$

We consider a function $f : \mathbb{R}^J \rightarrow \mathbb{R}$ continuous and bounded. We have

$$\begin{aligned} E(f(x(n, x_0))) &= E(f(A(n-1) \cdots A(0)x_0)) \\ &= E(f(A(-1) \cdots A(-n)x_0)) \xrightarrow{n} E(f(Z)), P - a.s., \end{aligned}$$

using Lebesgue’s dominated convergence theorem (f is bounded). It proves weak convergence. We conclude in the same manner. The reason why we have introduced ε -coupling is to show the continuity with what happens in a finite model. In one case there is coupling in finite time and total variation convergence, and in the other case, only ε -coupling and weak convergence. Furthermore, it enables us to prove proposition 5.6 without using backward induction, only with the assumption that there exists a solution to $Z \circ \theta = A(0)Z$.

For the moment, we have proven some results uniformly over initial conditions. There is another problem worth considering: what happens for a fixed deterministic initial condition x_0 . But for that specific problem, we can only state negative results. More precisely, for an i.i.d model $x(n+1) = A(n)x(n)$ and $x_0 \in \mathbb{R}^J$, $x(n, x_0)$ can have transient regimes and/or several classes of recurrence. Let us illustrate this.

Example

The matrices $A(n), n \in \mathbb{N}$ are **i.i.d.**, and $A(k) = A$ or B with $P(A(0) = A) = p > 0$ and $P(A(0) = B) = 1 - p > 0$.

$$A = \begin{pmatrix} 4 & e & e \\ e & 4 & e \\ e & e & e \end{pmatrix}, \quad B = \begin{pmatrix} 1 & e & 4 \\ e & 1 & 4 \\ -4 & -4 & 1 \end{pmatrix}.$$

We consider the initial condition $x_0 = (\eta, e, e)'$, where $e < \eta \ll 1$. In this example, there are two absorbing states. We verify that $\pi(Ax_0) = \pi(4, 4 - \eta, e)'$ and $\pi(ABx_0) = \pi(AB^kx_0) = \pi(4, 4, e)'$. Both vectors $(4, 4 - \eta, e)'$ and $(4, 4, e)'$ are eigenvectors of both matrices A and B . We conclude that with probability p , we have $\pi(x(n, x_0)) = \pi(4, 4 - \eta, e)'$ and with probability $1 - p$, we have $\pi(x(n, x_0)) = \pi(4, 4, e)'$.

Example Let us consider an example in dimension 2. The sequence $\{A(n)\}$ is **i.i.d.**

$$A(n) = \begin{pmatrix} U(n) & e \\ e & U(n) \end{pmatrix},$$

where the random variables $U(n)$ are uniform over $[1, 2]$. For an initial condition $x_0 = (a, b)'$ with $a \geq b + 2$, we have $\pi(x(n, x_0)) = \pi((\inf_{k \leq n} U(k), 0)')$. We have a transient MC.

Example We can mix the previous examples and we get one where we have for a fixed initial condition, a transient MC with probability p and a recurrent MC with probability $1 - p$. More precisely, we consider the matrix A as in the first example. With probability p , $A(0) = A$ and with probability $1 - p$, $A(0)$ has the following law:

$$A(0) = \begin{pmatrix} U & e & 4 \\ e & U & 4 \\ -4 & -4 & U \end{pmatrix},$$

where the random variable U is uniform over $[0.5, 1]$. We take for initial condition $(\eta, e, e)'$.

Example In a stationary and ergodic framing, there are a lot of examples of the same kind. We can for example look at pseudo periodic systems of the kind described in the forthcoming example of theorem 5.2.

5.3 Stability for discrete models

Let $\{A_i, i \in I \text{ or } i \in \mathbb{N}\}$, be a finite or countable collection of irreducible matrices of size $J \times J$. We suppose there exists a discrete probability law $\{p_i\}$ such that $A(n, \omega) = A_i$ with probability $p_i > 0$.

Theorem 5.1 *Matrices $A(k)$ are i.i.d. A product of the kind $A_{u_K} \otimes \cdots \otimes A_{u_1}$ with $u_1, \dots, u_K \in I$ (or \mathbb{N}) is called a pattern of length K . If there exists a pattern of finite length whose critical graph has a unique s.c.s. and whose cyclicity is 1 (scs1-cycl matrix), then the model has a unique stationary regime, which is attained by strong coupling in finite time.*

Proof Let $C = A_{u_K} \otimes \cdots \otimes A_{u_1}$ be a scs1-cycl pattern. We have, using the cyclicity 1 assumption (theorem 3.3):

$$\exists M \mid \forall m \geq M, C^m = \lambda C^{m-1},$$

where λ is the Lyapunov exponent of C . We conclude that for all initial condition y , $C^{M-1}y$ is an eigenvector of C . By the assumption on the critical graph of C , there is a unique eigenvector up to a constant, denoted y_0 . We have $C^{M-1}y = \mu(y) \otimes y_0$, $\mu(y) \in \mathbb{R}$. We define

$$\mathcal{B}_i = \{\omega \mid A(i + KM - 1, \omega) \otimes \cdots \otimes A(i + 1, \omega)A(i, \omega) = C^M\}.$$

From the i.i.d. assumption, it follows that $P(\mathcal{B}_i) > 0$. On \mathcal{B}_i , we have $\forall z$, initial condition:

$$\begin{aligned} x(i + MK) &= CC^{M-1}x(i) \\ &= \lambda C^{M-1}x(i) \\ &= \lambda \mu(x(i))y_0 \\ \implies \pi(x(i + MK)) &= \pi(y_0), \end{aligned}$$

We check that the sequence \mathcal{B}_i is compatible with the shift i.e. $\mathcal{B}_i = \mathcal{B}_0 \circ \theta^i$. We conclude that \mathcal{B}_i is a stationary renovating event sequence for the SRS $\pi(x(n))$. We apply Borovkov Renovating Events Theorem (version 4.2) and the uniqueness of the stationary regimes follows. ■

Remark This theorem is in particular true in the important case where one of the matrices A_i is a scs1-cycl matrix.

We are now going to give a version of this theorem in a stationary and ergodic framework.

Theorem 5.2 *The sequence of matrices $A(k)$ is stationary and ergodic. We suppose there exists a finite pattern $C = A_{u_K} \otimes \cdots \otimes A_{u_1}$ which is a scs1-cycl matrix. We suppose also that $\mathcal{B} = \{\omega \mid A(MK - 1)A(MK - 2) \otimes \cdots \otimes A(1)A(0) = C^M\}$ is of strictly positive probability, where M is the smallest integer verifying: $\forall m \geq M, C^m = \lambda \otimes C^{m-1}$ (λ is the Lyapunov exponent of C). Then there is a unique stationary regime attained by strong coupling in finite time..*

Proof It looks exactly like the one of theorem 5.1. On $\mathcal{B}_n = \mathcal{B} \circ \theta^n$, we have $\forall x$, the initial condition

$$\begin{aligned} \pi(x(n + MK)) &= CC^{M-1}\pi(x) \\ &= \lambda C^{M-1}\pi(x) \\ &= C^{M-1}\pi(x) = \pi(y_0), \end{aligned}$$

where y_0 is the unique eigenvector of C . We check that the sequence \mathcal{B}_n is compatible with the shift and we apply Borovkov's theorem . ■

Remark The conditions of this theorem are of course weaker than the i.i.d. assumption of theorem 5.1. They are nevertheless stronger than the simple ergodicity of θ . Condition $P(\mathcal{B}) > 0$ can not be released, as shown by the following counter-example.

Let $\Omega = \{\omega_1, \omega_2\}$ be the probability space, $P = \{\frac{1}{2}, \frac{1}{2}\}$ the probability law, and θ the ergodic shift defined by: $\theta(\omega_1) = \omega_2$ and $\theta(\omega_2) = \omega_1$. We consider

$$A = \begin{pmatrix} 1 - \eta & e \\ e & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & e \\ e & 1 - \eta \end{pmatrix}, \quad e < \eta \ll 1.$$

$$\{A(n, \omega_1)\} = A, B, A, B, \dots \quad \{A(n, \omega_2)\} = B, A, B, A, \dots$$

Both matrices A and B constitute a pattern of length 1 which is a scs1-cycl matrix, but there is a continuum of possible limits which are periodic. In fact, consider $x_0 = (a, b)'$ with $-1 + \eta < a - b < 1 - \eta$. Then the limiting regime of $\pi(x(n))$ has a state space which is either $\{\pi((a, b)'), \pi((a + \eta, b)')\}$ (with probability $\frac{1}{2}$), or $\{\pi((a, b)'), \pi((a, b + \eta)')\}$ (with probability $\frac{1}{2}$).

When η becomes arbitrary small, we can build more complex counter-examples to theorem 5.2 by limiting the number of times we can get the same matrix A or B in a row. Such models, where there exist patterns C which are scs1-cycl but where none of them fulfill the assumption $P(\mathcal{B}) > 0$, will be called pseudo-periodic.

If the dependence between matrices is markovian, a necessary and sufficient condition to get $P(\mathcal{B}) > 0$ is that $p(A_{u_K}, A_{u_1}) > 0$ and $p(A_{u_i}, A_{u_{i+1}}) > 0, \forall i = 1, \dots, K - 1$, where $p(.,.)$ is the markovian transition kernel.

5.4 Stability of general models

In this section, we consider a general model where the entries of our matrices have a support which can be discrete, absolutely continuous with respect to Lebesgue's measure or a mixing of those two cases.

We will need the following definition, extending the notion of pattern we have been using for finite models.

Definition 5.2 (pattern) *Let A be a random matrix. We say that \tilde{A} is a pattern of A if \tilde{A} is a deterministic matrix which has the following properties:*

- $\forall i, j, \tilde{A}_{ij} \in \text{support of } A_{ij}$
- $P(A^\eta = \tilde{A}) > 0, \forall \eta > 0$ (where A^η is defined in the following).

Another way to put it is to say that \tilde{A} , deterministic vector of size $J \times J$, belongs to the support of A , closed subset of $\mathbb{R}^{J \times J}$ (this definition includes the cases where \tilde{A} is only accumulation point (discrete case) or boundary point (continuous case) of A).

In the finite case, this definition simplifies to: 1. $\forall i, j, \tilde{A}_{ij} \in \text{support of } A_{ij}$, 2. $P(A(0) = \tilde{A}) > 0$. The general idea consists in using theorems 5.1 and 5.2 after having discretized matrices $A(k)$. This discretization goes in the following way. Suppose there exist $a_{ij}, i, j = 1, \dots, J$ belonging to the support of $A_{ij}, i, j = 1, \dots, J$. We define A_0 by $(A_0)_{ij} = a_{ij}$. Generally, we have: $P(A(0, \omega) = A_0) = 0$. We consider $\eta > 0$ fixed. We want to build matrices $A^\eta(k, \omega)$ such that $\|A(k, \omega) - A^\eta(k, \omega)\|_\infty \leq \eta$. For i, j fixed, we define:

$$A_{ij}^\eta(0, \omega) = (a_{ij} + k\eta) \text{ if } A_{ij}(0, \omega) \in]a_{ij} + (2k - 1)\frac{\eta}{2}, a_{ij} + (2k + 1)\frac{\eta}{2}], k \in \mathbb{Z}.$$

We check that in all cases we have: $P(A_{ij}^\eta(0, \omega) = a_{ij}) > 0$. If the random variables A_{ij} are independent, we also have $P(A^\eta(0, \omega) = A_0) > 0, \forall \eta > 0$, which says that A_0 is a pattern of A . When they are not independent, we will have to state explicitly that A_0 is a pattern of A . We check also that we have $\|A(k, \omega) - A^\eta(k, \omega)\|_\infty \leq \eta$ with probability 1.

Remark Using theorem 7.8 p.50 of Billingsley [6], we can prove that $A^\eta(0)$ converges weakly to $A(0)$ when $\eta \rightarrow 0$. This enables us to understand the limit properties we are going to obtain in the next theorems.

Theorem 5.3 *The matrices $A(n)$ are i.i.d. We suppose there exists a scs1-cycl matrix $C = C_{K-1} \otimes \dots \otimes C_0$ where C_i is a pattern of $A(0), \forall i \in \{0, \dots, K-1\}$. Then the SRS $\{\pi(x(n))\}$ has a unique stationary regime $\{Z \circ \theta^n\}$. The convergence occurs with ε -coupling, which implies weak convergence of the SRS to its unique stationary distribution.*

We will prove directly a more general result, in a stationary and ergodic framing.

Theorem 5.4 *The matrices $A(n)$ are stationary and ergodic. We suppose there exist a matrix $C = C_{K-1} \otimes \dots \otimes C_0$ which is a pattern of $A(K-1) \otimes \dots \otimes A(0)$. We suppose that this matrix C is scs1-cycl. Let M be such that: $\forall m \geq M, C^m = \lambda \otimes C^{m-1}$. We suppose that C^M is a pattern of $A(MK-1) \otimes \dots \otimes A(0)$. Then the SRS $\{\pi(x(n))\}$ has a unique stationary regime. The convergence occurs with ε -coupling, which implies weak convergence of the SRS to its unique stationary distribution.*

Proof In the whole proof, $d(\cdot, \cdot)$ represents the distance defined in 2.7. Let M be such that: $\forall m \geq M, C^m = \lambda \otimes C^{m-1}$. We define a decreasing sequence $\eta_i > 0$ such that: $4MK\eta_i \leq \frac{1}{i}$. The K matrices C_0, \dots, C_{K-1} define K possible ways to discretize matrices $A(n)$. For $k \in \{0, \dots, K-1\}$, we define the intervals: $](C_k)_{ij} + (2p-1)\frac{\eta_i}{2}, (C_k)_{ij} + (2p+1)\frac{\eta_i}{2}]$, $p \in \mathbb{Z}$. By realizing all possible intersection between those intervals, we define a new countable set of disjoint intervals, whose union is \mathbb{R}^J . The discretization of $A(n)$ will be done with respect to this new set. The following lemma is true (and the proof, easy, is left to the reader).

Lemma 5.1 *Let u and v be two vectors of initial conditions. We have:*

$$| \|A(0)u, A(0)v\|_\infty - \|A^\eta(0)u, A^\eta(0)v\|_\infty | \leq 4\eta.$$

We define events \mathcal{B}_j^i in the following way:

$$\mathcal{B}_0^i = \left\{ \omega \mid A^{\eta_i}(KM-1, \omega) \otimes \dots \otimes A^{\eta_i}(0, \omega) = C^M \right\},$$

$$\mathcal{B}_k^i = \mathcal{B}_0^i \circ \theta^k.$$

We fix i . The event $\left\{ \omega \mid \sum_{n=0}^{\infty} 1_{\mathcal{B}_n^i} = +\infty \right\}$ is left invariant by the shift θ . Using the ergodicity hypothesis, we deduce that it is of probability 0 or 1.

Furthermore, we have:

$$\frac{\sum_{n=0}^k 1_{\mathcal{B}_n^i}}{k} \xrightarrow{k \rightarrow +\infty} P(\mathcal{B}_1^i) \quad P\text{-a.s.},$$

by Birkhoff's ergodic theorem. Because of the assumption that C^M is a pattern, we have $P(\mathcal{B}_0^i) > 0$.

We conclude that $P\left(\sum_{n=0}^{\infty} 1_{\mathcal{B}_n^i} = +\infty\right) = 1$.

But we have also:

$$\left\{ \omega \mid \sum_{n=0}^{\infty} 1_{\mathcal{B}_n^i} = +\infty \right\} \subset \left\{ \omega \mid \lim_{n \rightarrow +\infty} d(A(n) \cdots A(0)u, A(n) \cdots A(0)v) \leq \frac{1}{i} \right\}.$$

We prove it by remarking that on \mathcal{B}_n^i , we have:

$$\begin{aligned} d \left(A^{\eta_i}(KM+n-1) \otimes \dots \otimes A^{\eta_i}(n)A(n-1) \otimes \dots \otimes A(0)u, \right. \\ \left. A^{\eta_i}(KM+n-1) \otimes \dots \otimes A^{\eta_i}(n)A(n-1) \otimes \dots \otimes A(0)v \right) = 0. \end{aligned}$$

Here, we strongly use the fact that coupling convergence for the discretized model $\{A^{\eta_i}\}$ occurs uniformly over the initial conditions. According to lemma 5.1, we have:

$$d(A(KM+n-1) \otimes \dots \otimes A(0)u, A(KM+n-1) \otimes \dots \otimes A(0)v) \leq 4KM\eta_i \leq \frac{1}{i},$$

and we finish by using the monotonicity of $d(.,.)$ (see proposition 2.1).

We have proved that $\forall i$ fixed, $\lim_n d(A(n) \cdots A(0)u, A(n) \cdots A(0)v) \leq \frac{1}{i}$ $P - a.s.$ By letting i go to $+\infty$, we obtain that $d(A(n) \cdots A(0)u, A(n) \cdots A(0)v) \xrightarrow{n \rightarrow +\infty} 0$ $P - a.s.$ We conclude by using proposition 5.6. \blacksquare

Remark Theorems 5.1 to 5.4 do not require any aperiodicity (def 2.4) assumption on matrices $A(n)$. Anyway, the pattern C whose existence is critic in all those theorems is irreducible and aperiodic. The condition “scs1-cyc1” implies aperiodicity.

5.5 A converse theorem

We are going to prove converses of theorems 5.1, 5.2, 5.3 and 5.4. We will consider successively finite, discrete and general models of type: “ $x(n+1) = A(n)x(n)$ ” where the matrices are of size $J \times J$ and $P - a.s$ irreducible. We will, moreover, always suppose that there exist, with positive probability, a pattern whose projective diameter is finite, i.e.:

$$\exists n \mid P \{D(A(n) \cdots A(0)) < +\infty\} > 0.$$

This condition is very weak, we comment further on it in section 6.

5.5.1 Finite models

We consider a finite model: “ $x(n+1) = A(n)x(n)$ ”. Let $\{A_i, i \in \mathcal{I} = \{1, \dots, I\}\}$, be a finite collection of irreducible matrices.

Theorem 5.5 *The sequence of matrices $\{A(n)\}$ is i.i.d. or stationary and ergodic. A necessary and sufficient condition for the model to have a unique stationary regime is that there exists a matrix $C = A_{u_K} \cdots A_{u_2} A_{u_1}$ with $u_i \in \mathcal{I}$, such that:*

1. C is a scs1-cyc1 matrix.
2. $\exists M$ such that C^M is a pattern of $A(MK-1)A(MK-2) \cdots A(0)$, where M verifies $\forall m \geq M, C^{m+1} = \lambda \otimes C^m$.

Remark If the sequence $\{A(n)\}$ is i.i.d., then the second condition is always verified. And a sufficient condition for having a pattern of finite projective diameter is that there exists a matrix $A_i, i \in \mathcal{I}$ aperiodic.

Proof We have already proved the sufficient part (theorems 5.1 and 5.2). The necessity of those conditions is proved in Appendix using a backward scheme. \blacksquare

5.5.2 Discrete models

We consider a discrete model: “ $x(n + 1) = A(n)x(n)$ ”. Let $\{A_i, i \in \mathbb{N}\}$ be a countable collection of irreducible matrices of size $J \times J$.

We do not have anymore that stability implies coupling in finite time.

A good example of this is to consider the model illustrating the conjecture of subsection

5.5.3. For our purpose, random variables $\{U(n)\}$ are i.i.d. and take values in $\{\frac{1}{k}, k \in \mathbb{Z} \setminus \{0\}\}$ with $\forall k \neq 0, P(U(0) = \frac{1}{k}) > 0$. The analysis remains the same. There is a unique stationary regime but only ε -coupling.

We can still, anyway, show the next result.

Theorem 5.6 *The sequence of matrices $\{A(n)\}$ is i.i.d. or stationary and ergodic. A necessary and sufficient condition for the model to couple in finite time to a unique stationary regime is that there exists a matrix $C = A_{u_K} \cdots A_{u_2} A_{u_1}$ with $u_i \in \mathbb{N}$, such that:*

1. C is a scs1-cyc1 matrix.
2. $\exists M$ such that C^M is a pattern of $A(MK - 1)A(MK - 2) \cdots A(0)$, where M verifies $\forall m \geq M, C^{m+1} = \lambda \otimes C^m$.

Proof We have already proved the sufficient part (theorems 5.1 and 5.2). Let us prove that those conditions are necessary. We suppose that our model couples in finite time to a unique stationary regime, uniformly over initial conditions in \mathbb{R}^J . Let us prove a lemma first.

Lemma 5.2 *If there is a unique stationary regime, convergence of the SRS $\pi(x(n))$ occurs with strong coupling in finite time, uniformly over initial conditions.*

Proof Let $\{Z \circ \theta^n\}$ be the unique stationary regime with which the SRS $\pi(x(n))$ couples. We consider the event:

$$\mathcal{Y}_n = \left\{ \omega \mid \pi(x(n, x_0)) \circ \theta^{-n} \omega = Z\omega, \forall x_0 \in \mathbb{R}^J \right\}.$$

The assumption of coupling in finite time uniformly over \mathbb{R}^J writes:

$$P(\mathcal{Y}_n) \xrightarrow{n \rightarrow +\infty} 1.$$

Here we implicitly use the assumption that the projective image of $A(-1) \cdots A(-n)$ is asymptotically bounded (see the proof of proposition 5.3). Let us consider $\omega \in \mathcal{Y}_n$ and k an integer > 0 , we have:

$$\begin{aligned} \pi(x(n + k, x_0)) \circ \theta^{-n-k} \omega &= \pi \left(x(n, x(k, x_0) \circ \theta^{-k}) \right) \circ \theta^{-n} \omega \\ &= Z\omega \quad (\text{as } \omega \in \mathcal{Y}_n). \end{aligned}$$

Here we use in a critical way the fact that coupling occurs uniformly over initial conditions. We have proved that events \mathcal{Y}_n were increasing, so we have:

$$\mathcal{Y}_n = \left\{ \omega \mid \pi(x(n+k, x_0)) \circ \theta^{-(n+k)} \omega = Z\omega, \quad \forall k \geq 0, \quad \forall x_0 \in \mathbb{R}^J \right\},$$

and

$$P(\mathcal{Y}_n) \xrightarrow{n \rightarrow +\infty} 1.$$

This is exactly the definition of strong coupling (definition 4.4). ■

We can now use the converse theorem 4.3. There exists a stationary sequence of events $\{\mathcal{A} \circ \theta^n\}$ which is renovating for the SRS $\{\pi(x(n, x_0))\}$, $\forall x_0 \in \Pi$ (or equivalently $\forall x_0 \in \mathbb{R}^J$). Let m be the common length and Φ the common function of those renovating events. We have, on \mathcal{A} :

$$\pi(x(m)) = \Phi(A(m-1), \dots, A(0)), \quad \forall x_0 \in \mathbb{R}^J.$$

But we also have:

$$x(m) = A(m-1) \otimes \dots \otimes A(0)x_0, \quad \forall x_0 \in \mathbb{R}^J.$$

We conclude that, on \mathcal{A} , $\pi(A(m-1, \omega) \otimes \dots \otimes A(0, \omega)x_0)$ is independent of x_0 which implies that $A(m-1, \omega) \otimes \dots \otimes A(0, \omega)$ is a scs1-cycl matrix. But products of the form $A(m-1) \otimes \dots \otimes A(0)$ can only take a countable number of values with strictly positive probability. Then there exists a matrix $C = A_{u_m} \dots A_{u_1}$, $u_i \in \mathcal{I}$ which is a scs1-cycl matrix and verify $P(\mathcal{A} \cap \{A(m-1, \omega) \dots A(0, \omega) = C\}) > 0$. It means exactly that there exists a pattern of finite length wich is a scs1-cycl matrix, and it ends the proof. ■

5.5.3 General models

Let's come back to a general model of type " $x(n+1) = A(n)x(n)$ ".

We have an exact analog of the previous theorem for discrete models and the proof remains the same.

Theorem 5.7 *The sequence of matrices $\{A(n)\}$ is i.i.d. or stationary and ergodic. A necessary and sufficient condition for the model to couple in finite time with a unique stationary regime is that, $\exists K$ such that*

$$P\{C = A(K-1) \dots A(0) \text{ is a scs1-cycl matrix and verifies } C^2 = \lambda \otimes C\} > 0.$$

In other words we must have patterns verifying our usual conditions with positive probability.

Remark When we have patterns satisfying our usual conditions but which are of probability 0, the direct theorems 5.3 and 5.4 still applies and give that there is a unique stationary regime which is attained with ε -coupling. Thus, ε -coupling appears as a limit case of coupling in finite time.

In discrete models, we will have only ε -coupling when there exist scs1-cyc1 patterns but of probability 0, which means that they are only accumulation points of the support. In the general case, we will have only ε -coupling when the scs1-cyc1 patterns are isolated points of the support (which implies that they are boundary points of the support). Of course, in the finite case, a pattern is always of positive probability which is the reason why there is always coupling in finite time.

Example To illustrate the previous remark, let us consider the following example.

$$A(n) = \begin{pmatrix} U(n) & e \\ e & U(n) \end{pmatrix},$$

where $U(n)$ are i.i.d. random variables of uniform law over $[-1, 1]$. It is quite clear that there is a unique stationary regime for $\pi x(n)$ which is $\pi(e, e)'$. We denote by $d(., .)$ the projective distance. For an initial condition $(y, e)'$ with $y \geq 1$, we have $d(x(n); (e, e)') = \text{Inf}_{k \leq n} |U(k)|$. Thus convergence to $\pi(e, e)'$ occurs in infinite time. We are in the case where there is no coupling but only ε -coupling with the unique stationary sequence. The matrix C defined by $C_{ij} = e, \forall i, j \in \{1, 2\}$ is the only pattern verifying our usual conditions and it is of probability 0. If we had instead that the diagonals elements were two random variables $U(n)$ and $U'(n)$ such that

$$P \{ (U(n) > U'(n)) \cap (U(n) > e) \} > 0 \text{ or } P \{ (U'(n) > U(n)) \cap (U'(n) > e) \} > 0,$$

then we have patterns with positive probability and there is coupling in finite time to the unique stationary regime.

6 Without irreducibility

We have supposed from the beginning that the matrices $\{A(n)\}$ were irreducibles. In fact we don't need such a strong assumption. Let us come back to the places where this assumption is used. First of all, it is used in proposition 2.1. But in fact the only point we need in order to prove this proposition is: " $\forall u \in \mathbb{R}^J, Au \in \mathbb{R}^J$ ", i.e. if u has only non- ε coordinates then Au has the same property. So the only assumption we need on the matrices $\{A(n)\}$ is:

$$\forall i, \exists j \text{ s.t. } A_{ij} > \varepsilon, P - a.s.$$

We will call this the condition (*). Secondly, the irreducibility is critic in the \mathbb{R}_{Max} spectral theory of section 3. A reducible matrix $A \in \mathbb{R}^{J \times J}$ may have several eigenvalues. Definition 3.2 and theorem 3.2 have to be reinterpreted by replacing the unique eigenvalue by the

maximal eigenvalue. Theorem 3.3 isn't true anymore. But as far as the main theorems (5.1, 5.2, 5.3 and 5.4) are concerned, we use results of \mathbb{R}_{Max} spectral theory only for the pattern C whose existence is critic for the proof. They are still valid, then, if we state that condition (*) is verified and that there exist a pattern C which is scs1-cyc1 **and** irreducible. The gain, in terms of modelling power, is interesting. It enables us for example to consider task graphs with random precedence like the ones defined in [5].

Dropping the irreducibility assumption do not modify the converse results. More precisely, the proof of proposition 5.3, theorem 5.5 and theorem 5.6 are still valid. In fact, if we look at the proofs carefully, we see that there are only three points which need to be verified:

1. matrices $\{A(n)\}$ verify condition (*) $P - a.s.$
2. $\exists n \mid P \{D(A(n) \cdots A(0)) < +\infty\} > 0.$
3. uniqueness condition (i.e. $\forall x_0, y_0, d((x(n, x_0), x(n, y_0)) \xrightarrow{n} 0 P - a.s.)$).

Under the three conditions previously stated, we can carry a backward argument. Of course, irreducibility $P - a.s.$ is not necessary to ensure those conditions. We conclude that we can state our converse results under those minimal assumptions.

Let us discuss condition 2 a little further. First of all, we introduce a counter-example proving that this condition is necessary.

Let $\Omega = \{\omega_1, \omega_2\}$ be the probability space, $P = \{\frac{1}{2}, \frac{1}{2}\}$ the probability law, and θ the ergodic shift defined by: $\theta(\omega_1) = \omega_2$ and $\theta(\omega_2) = \omega_1$. We consider

$$A = \begin{pmatrix} \varepsilon & e \\ e & 1 \end{pmatrix}, B = \begin{pmatrix} e & e \\ 1 & \varepsilon \end{pmatrix}.$$

$$\{A(n, \omega_1)\} = A, B, A, B, \dots \quad \{A(n, \omega_2)\} = B, A, B, A, \dots$$

It is clear that all patterns have an infinite projective diameter. Nonetheless, it is easy to check that there is coupling in finite time to a unique periodic regime. More precisely, there is coupling of $\pi(x(n, x^0))$ to the periodic regime $\{\pi(1, e), \pi(e, 1)\}$ and coupling occurs for $n > \bigoplus(x_1^0 - x_2^0, x_2^0 - x_1^0)$. In this case, we conclude that there is coupling in finite time to a unique stationary regime, without the existence of a scs1-cyc1 pattern which contradicts the converse theorem 5.5.

Condition 2 is weak and will be verified in most cases. For a discrete **i.i.d.** model, for example, it is sufficient to have one pattern of finite length $A_{u_K} \cdots A_{u_1}$ which is irreducible and aperiodic to verify it. For a general **i.i.d.** model, it is sufficient to have $P \{A(0) \text{ irreducible and aperiodic}\} > 0$. In a stationary and ergodic framing, condition 2 is a little bit stronger as shown by the previous example where $P \{A(0) \text{ irreducible and aperiodic}\} = 1$ and where condition 2 is not verified.

Another class of systems where condition 2 is not verified is the class of non-autonomous systems studied by Baccelli in [3]. We will see that in this case also, theorem 5.5 is not true anymore. The model is the following.

The sequence $\{A(n), n \in \mathbb{N}\}$ is stationary and ergodic. Matrices $A(n)$ have a fixed structure, $P - a.s.$ (def 2.5). This structure is of the form:

$$A(n) = \begin{pmatrix} \tilde{U}(n) & \varepsilon \\ \tilde{B}(n) & \tilde{A}(n) \end{pmatrix}.$$

The block \tilde{U} is a square matrix of size $I \times I$, irreducible. It is interpreted as the input of our system. The block \tilde{A} is a square matrix of size $(J - I) \times (J - I)$, irreducible and aperiodic. The block \tilde{B} is the matrix of the communication between the sources (\tilde{U}) and the circuit (\tilde{A}). We suppose that the block \tilde{U} in isolation has a unique stationary regime (for example $I=1$, there is a unique source). We have the following theorem.

Theorem 6.1 *Let u and a be the Lyapunov exponents of \tilde{U} and \tilde{A} respectively (see proposition 3.1). If $a < u$, there is a unique stationary regime for the SRS $\pi(x(n))$, regardless of the initial condition. Convergence to the stationary regime occurs with coupling in finite time. If $a > u$, then the increments of the form*

$$\delta_{ji}(n, x_0) = x_j(n, x_0) - x_i(n, x_0), \quad i = 1, \dots, I, \quad j = I + 1, \dots, J,$$

tend to $+\infty$ $P - a.s.$ for all finite initial conditions.

A good way to get an intuition of this result is to consider deterministic matrices. We consider first a matrix for which $(u = 1) > (a = e)$.

$$A = \begin{pmatrix} 1 & \varepsilon \\ e & e \end{pmatrix}, \quad A^n = \begin{pmatrix} n & \varepsilon \\ n - 1 & e \end{pmatrix}.$$

We put $x_0 = (u, v)'$. We have $A^n x_0 = (nu, (n - 1)u \oplus v)'$. For n sufficiently big, we have

$$A^n x_0 = nu \otimes \begin{pmatrix} e \\ -1 \end{pmatrix} \implies \pi(A^n x_0) = \pi \begin{pmatrix} e \\ -1 \end{pmatrix}.$$

On the other hand, if we consider:

$$A = \begin{pmatrix} e & \varepsilon \\ e & 1 \end{pmatrix}, \quad A^n = \begin{pmatrix} e & \varepsilon \\ n - 1 & n \end{pmatrix},$$

we have

$$\pi(A^n x_0) = \pi \begin{pmatrix} e \\ (n - 1) \oplus (v - u)n \end{pmatrix}.$$

We check that $x_2(n) - x_1(n) = (n - 1) \oplus (v - u)n$ tends to $+\infty$ for all finite x_0 .

For a stochastic model the idea remains the same. If $u > a$, the sources which are slower impose their dynamic. If $u < a$, everything happens asymptotically as if \tilde{A} was in isolation.

Of course in such models, there are no scs1-cyc1 patterns! If we look more carefully it is even clear that we can not exhibit a type of pattern which would decide of the uniqueness of the stationary regimes. The good criteria is, here, the comparison between Lyapunov exponents. The computation of such exponents involves the whole structure of the stochastic sequences $\{A(n)\}$, and not only an extracted pattern. In fact, theorem 6.1 is more a first order result than a second order one.

Remark In the previous theorem, we made the assumption that \tilde{U} in isolation has a unique stationary regime. But the problem of knowing if \tilde{U} has a unique stationary regime is precisely the one which has been addressed in this paper. Then, to determine if there is a unique stationary regime for $\pi(x(n))$, we have in fact to use the results of section 5 (applied to \tilde{U}) together with the comparison of Lyapunov exponents (of \tilde{U} and \tilde{A}).

We have seen that we need only three conditions in order to get converse results. There are anyway more precise results when we make stronger assumptions on the matrices of the model. For example in theorem 5.5, under the stronger condition that the matrices $\{A_i, i = 1, \dots, I\}$ are irreducible and aperiodic, we can obtain moreover that the unique stationary regime has a bounded state space. Let us prove it.

We make the natural definition: $Im(A_i) = \{v \in \mathbb{R}^J \mid \exists u \in \mathbb{R}^J, v = A_i u\}$. We get very easily that $\pi(Im(A_i))$ is bounded in $\mathbb{P}\mathbb{R}^J$. In the same way, $\Pi = \pi(Im(A_1)) \cup \dots \cup \pi(Im(A_I))$ is bounded and we have $\forall x_0 \in \mathbb{R}^J, \pi(x(n, x_0)) \in \Pi$ for $n \geq 1$. Then the support of the stationary measure is included in Π . ■

This is not the case anymore under the previous minimal assumptions. Let us consider an i.i.d. model with 2 matrices, verifying condition (*). We write $p_1 = P(A(0) = A_1)$ and $p_2 = 1 - p_1 = P(A(0) = A_2)$.

$$A_1 = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & e \end{pmatrix}, A_2 = \begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix}.$$

We identify $\mathbb{P}\mathbb{R}^2$ and \mathbb{R} in the following way: $\pi(a, b)' \simeq a - b$. It is clear that after a transient regime, $\pi(x(n, x_0))$ is a MC on $\mathbb{N} \setminus \{0\}$ whose transition probability is given by:

$$p(i, i + 1) = p_2, p(i, 1) = p_1, \forall i \in \mathbb{N} \setminus \{0\}.$$

This is a classical recurrent positive Markov chain whose invariant distribution is:

$$\mu(1) = 1 - p_1, \mu(n) = (1 - p_1)p_1^{n-1}, \forall n \geq 2.$$

So the state space is not bounded. The pattern C is here for example $C = A_1 A_2$. There exist also examples of models where all the matrices are irreducible and where the state space of the unique stationary regime is non-bounded. We can consider for example:

$$B_1 = \begin{pmatrix} \varepsilon & 1 \\ e & \varepsilon \end{pmatrix}, B_2 = \begin{pmatrix} \varepsilon & e \\ 1 & \varepsilon \end{pmatrix}, B_3 = \begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix}.$$

A possible pattern to prove the uniqueness of stationary regimes is $C = B_1 B_2 B_3$. We show that the state space is unbounded by remarking that $B_1 B_2 = A_1$ and $B_3 = A_2$.

The problem of determining if the state space of the MC $x(n, x_0)$ is finite or bounded is close to the problem of knowing if the semigroup of matrices generated by $\{A_i, i \in \{1, \dots, I\}\}$ is finite or bounded. This problem has been addressed by Gaubert in [11].

7 Appendix

Proof of proposition 5.3 and of theorem 5.5

We are going to prove both results at the same time. We suppose that that we have ε -coupling, i.e. $\forall x_0, y_0 \in \mathbb{R}^J$:

$$d(A(n)A(n-1) \cdots A(0)x_0, A(n)A(n-1) \cdots A(0)y_0) \xrightarrow{n \rightarrow +\infty} c = 0, P - a.s.$$

We want to prove that this convergence occurs in finite time. We suppose it is not the case, i.e. there exist $x_0, y_0 \in \mathbb{R}^J$ such that $P(d(x(n, x_0), x(n, y_0)) \neq 0, \forall n) > 0$. By using theorems 5.1 and 5.2, it implies that there exists no pattern C which is a scs1-cyc1 matrix. We are going to show that the sequence $\{A(-1) \cdots A(-n), n \in \mathbb{N}\}$ has a simple limit in $\mathbb{P}\mathbb{R}^{J \times J}$. The argument is an analog of the famous backward scheme proposed by Loynes in [13] for G/G/1 queues. We consider the following distance on $\mathbb{P}\mathbb{R}_{Max}^{J \times J}$:

$$d(A, B) = \sup_{u \in \mathbb{R}^J} d(Au, Bu).$$

We recall that $\mathbb{D}(A)$ denotes the projective diameter of a matrix (see definition 2.8). We have

$$d(A(-1) \cdots A(-n), A(-1) \cdots A(-n) \cdots A(-n-p)) \leq \mathbb{D}(A(-1) \cdots A(-n)).$$

We want to show that $\mathbb{D}(A(n) \cdots A(0)) \xrightarrow{n} 0, P - a.s.$

First, we make the assumption that there exist N such that $\forall n \geq N, \mathbb{D}(A(n) \cdots A(0))$ is finite $P - a.s.$ It is easy to see that the event $\mathcal{A} = \{\omega \mid \exists N, \mathbb{D}(A(n) \cdots A(0)) < +\infty\} = \{\omega \mid \exists N, \forall n \geq N, \mathbb{D}(A(n) \cdots A(0)) < +\infty\}$ is invariant by the translation shift. Then by the ergodic lemma, it is of probability 0 or 1. In the i.i.d. case, a sufficient condition to get $P(\mathcal{A}) = 1$ is that there exists a pattern which is irreducible and aperiodic. It is under

that very weak assumption that we are going to carry the proof (see section 6 for more details).

So we have $P(\mathcal{A}) = 1$. Using the stationarity of the sequence $\{A(n), n \in \mathbb{Z}\}$, we have that $\exists N(\omega)$ such that $\mathbb{D}(A(-1) \cdots A(-N)) < +\infty$. Then we can define the projective image of $A(-1) \cdots A(-N)$ which is a bounded subset of $\mathbb{P}\mathbb{R}^J$. We will need the following lemma.

Lemma 7.1 *Let K be a projectively bounded subset of \mathbb{R}^J . There exist vectors $u, v \in \mathbb{R}^J$ such that*

$$\forall k \in K, d(u, k) \leq d(u, v) \text{ and } d(k, v) \leq d(u, v).$$

Proof Let $k_1, k_2 \in K$ be such that

$$d(k_1, k_2) = \sup_{k, k' \in K} d(k, k').$$

We consider $u = k_1 + \alpha \times (k_1 - k_2)$ and $v = k_2 + \alpha \times (k_2 - k_1)$, $\alpha > 0$. If we had $u \in K$, it would imply $d(u, k_2) = d(k_1 + \alpha \times (k_1 - k_2), k_2) = (1 + \alpha) \times d(k_1, k_2)$ which is impossible. We prove in the same way that $v \notin K$. We have $d(u, v) = (1 + 2\alpha) \times d(k_1, k_2)$. We consider $k \in K$, we have:

$$d(u, k) \leq d(u, k_1) + d(k_1, k) \leq \alpha d(k_1, k_2) + d(k_1, k_2) < d(u, v),$$

$$d(v, k) \leq d(v, k_2) + d(k_2, k) \leq \alpha d(k_1, k_2) + d(k_1, k_2) < d(u, v).$$

■

Let $x_0(\omega), y_0(\omega)$ be two vectors verifying the previous lemma for the projective image of $A(-1) \cdots A(-N)$. We fix $\varepsilon > 0$. Using the ε -coupling assumption, we have that the random variable $N'(\omega)$ is P -a.s. infinite, where N' is defined by:

$$N' = \inf\{n \mid d(x(n), x_0), x(n), y_0) \leq \varepsilon\}.$$

As both N and N' are P -a.s. finite, we have

$$\forall \eta > 0, \exists K, K' \text{ such that } P\{N < K, N' < K'\} \geq 1 - \eta.$$

On the event $\{N \leq K, N' \leq K'\}$, we have

$$\mathbb{D}(A(K' - 1) \otimes \cdots \otimes A(0)A(-1) \otimes \cdots \otimes A(-K)) \leq \varepsilon.$$

We deduce, using the stationarity of $\{A(n)\}$, that

$$\begin{aligned} P\{\mathbb{D}(A(-1) \cdots A(-K - K')) \leq \varepsilon\} &= P\{\mathbb{D}(A(K' - 1) \cdots A(-K)) \leq \varepsilon\} \\ &\geq 1 - \eta. \end{aligned}$$

It implies that the random variables $\mathbb{D}(A(-1) \cdots A(-n))$ converge in probability to 0. But as $\mathbb{D}(\cdot)$ is pathwise decreasing, the convergence occurs also P -a.s.

We have in particular $d(A(-1) \cdots A(-n), A(-1) \cdots A(-n-p)) \rightarrow 0$, P -a.s. It implies in turn that the sequence $\{A(-1) \cdots A(-n)\}$ is Cauchy and converges. We write A_∞ its limit. It should be remarked that A_∞ depends on $\omega \in \Omega$ and has no reason to be unique. By continuity of the projective distance, we have that $\forall u, v \in \mathbb{R}^J$, $d(A_\infty u, A_\infty v) = 0$. Then A_∞ is a scs1-cyc1 matrix verifying $A_\infty^2 = A_\infty$ (P -a.s.). We put $Z(\omega) = A_\infty u$ (independent of $u \in \mathbb{R}^J$). We have:

$$\begin{aligned} Z \circ \theta &= A_\infty(\theta\omega)u \\ &= A(0)A_\infty u = A(0)Z. \end{aligned}$$

The sequence $\{Z \circ \theta^n\}$ is a stationary regime. We need now a characterization of scs1-cyc1 matrices.

Lemma 7.2 *Let A be a normalized matrix, considered as a linear operator on $\mathbb{P}\mathbb{R}^J$. It is a scs1-cyc1 matrix iff there exist a new basis (obtained by permutation of the coordinates and change of the origin) in which we have:*

$$\exists M \mid A^M = \left(\begin{array}{ccc|c} e & \cdots & e & < e \\ \vdots & \vdots & \vdots & \\ e & \cdots & e & \end{array} \right),$$

where the block composed of e is of size $J \times K$ with $1 \leq K \leq J$.

Proof We modify (by a permutation) the basis in order to get one whose first coordinates are those involved in the critical graph. Let K be the number of coordinates involved in the critical graph. We choose the origin in order for A to have $(e, \dots, e)'$ as its unique eigenvector. The integer M of the enonce accounts for the possibility of an initial transient regime of the powers of A (see theorem 3.3). We have, using cyclicity one assumption, $A^{M+1} = A^M$. It implies $(A^M)^+ = A^M$. We set $B = A^M$. We make now use of theorem 3.2. The columns B_1, \dots, B_K are equivalent to $(e, \dots, e)'$, the unique eigenvector of A , i.e. are equal to $(\alpha_i, \dots, \alpha_i)$. But lemma 5.1 tells us that in each of those columns, B_i , $i = 1, \dots, K$, there are terms which are equal to e . We conclude that $B_i = (e, \dots, e)'$, $i = 1, \dots, K$. Let us suppose there exist (i, j) , $j > K$ such that $B_{i,j} = e$. Then by left-multiplication of B by A , we obtain that $B_{i,j} = (e, \dots, e)'$ which is contradiction with the assumption that there are only K coordinates involved in the critical graph. We conclude that $B_{i,j} < e$, $\forall j > K$. ■

We place ourselves in a basis satisfying the previous lemma for A_∞ . We can apply the lemma with $M = 1$ as $A_\infty^2 = A_\infty$. We denote by K the number of columns of A_∞ composed of e 's. The existence of the limit A_∞ tells us that there exist a real $\eta > 0$ and a rank N

such that $\forall n \geq N$, we have:

$$A(n-1) \cdots A(-n) = \left(\begin{array}{ccc|c} \geq e & \cdots & \geq e & \\ \vdots & & \vdots & \\ \geq e & \cdots & \geq e & < e - \eta \end{array} \right),$$

and at least one of the terms of the left block of size K is exactly equal to e (it is always possible to impose this last condition as we are working in $\mathbb{P}\mathbb{R}^J$ and are free to choose the representative of A in its parallelism class). We take η sufficiently small, for example $\eta = \frac{1}{2} \times \sup_{i,j} (A_\infty)_{ij}$ where the sup is taken over coordinates outside of the left block. As $A(n)$ can only take a finite number of values (A^i , $i \in \{1, \dots, I\}$), we can define

$$a = \bigwedge_{i=1}^I \bigwedge_{(j,k);(j',k')} |(A^i)_{jk} - (A^i)_{j'k'}|,$$

where \wedge denotes the minimum. This minimum is taken only over non- e and non- ε terms. The finiteness implies that $a > 0$. We set $\otimes \mathbf{A}(k) = A(-1)A(-2) \cdots A(-k)$. We are interested by the differences between two coordinates of $\otimes \mathbf{A}(k)$, i.e. by quantities $|\otimes \mathbf{A}(k)_{ij} - \otimes \mathbf{A}(k)_{i'j'}|$. We want to compare those quantities for $\otimes \mathbf{A}(k+1)$ and $\otimes \mathbf{A}(k)$. we can show immediately that:

$$|| \otimes \mathbf{A}(n+1)_{ij} - \otimes \mathbf{A}(n+1)_{i'j'} | - | \otimes \mathbf{A}(n)_{ij} - \otimes \mathbf{A}(n)_{i'j'} | | \geq a,$$

But there exists N' such that $\forall n \geq N'$, the left block of matrices $A(-1) \cdots A(-n)$ verifies

$$\forall i \in \{1, \dots, J\}, \forall j \in \{1, \dots, K\}, e \leq A(-1) \cdots A(-n)_{ij} \leq e + \frac{a}{2}.$$

Comparing those two statements, we obtain that all terms of the left block are equal.

$$\forall i \in \{1, \dots, J\}, \forall j \in \{1, \dots, K\}, A(-1) \cdots A(-n)_{ij} = e$$

As a conclusion, we have proven that, $\forall n \geq N'$, the matrix $A(-1) \cdots A(-n)$ is of the form of lemma 7.2 and those is a scs1-cycl matrix. We have proved that there exist a pattern of finite length which is a scs1-cycl matrix. Theorems 5.1 and 5.2 can be applied, we conclude that there is coupling in **finite** time with a unique stationary regime, uniformly over initial conditions, for the SRS $\pi x(n, x_0)$. It achieves the proof. \blacksquare

Remark The first part of the proof, i.e. the existence of the backward limit A_∞ doesn't require that we have a finite model and is true for a general model. We use this remark in the proof of proposition 5.6.

Proof of proposition 5.4

If our model is i.i.d., the sequence $\{x(n, x_0)\}$ is a denumerable Markov Chain (MC) for a

fixed and deterministic initial condition x_0 . When a denumerable MC is ergodic, i.e. has only one class of recurrence which is attained with probability one, we can use a result of decomposition into cyclic classes (see for example: sections 6.2 and 6.3 of Revuz [16]). We denote by p the transition probability of the MC.

Proposition 7.1 *There exists a finite integer d (the period) and a finite sequence of disjoint sets D_1, D_2, \dots, D_d such that:*

1. $D = \bigcup_{i=1}^d D_i$ where D is the unique class of recurrence of the MC;
2. $p(\cdot, D_{i+1}) = 1$ on D_i for $i = 1, 2, \dots, d-1$ and $p(\cdot, D_1) = 1$ on D_d .

We suppose there exists a finite number ($k > 1$) of stationary or periodic regimes, uniformly over deterministic initial conditions in \mathbb{R}^J . We write C_1, \dots, C_k the countable state spaces of those stationary or periodic MC.

We decompose the state spaces C_i of periodic regimes into their cyclic classes, we keep those of the aperiodic stationary regimes unchanged. We get k' state spaces $C_1, \dots, C_{k'}$. They are disjoint, as an easy consequence of the Markov property and of proposition 7.1. We consider two of these state spaces, for example C_1 and C_2 . We define the following function on $\mathcal{P}(\mathbb{P}\mathbb{R}^J) \times \mathcal{P}(\mathbb{P}\mathbb{R}^J)$.

$$\mathbf{d}(B, C) = \inf_{b \in B, c \in C} d(b, c), \quad B, C \in \mathcal{P}(\mathbb{P}\mathbb{R}^J),$$

where $d(\cdot, \cdot)$ is the projective distance.

We want to prove that $\mathbf{d}(C_1, C_2) > 0$. We suppose it is not true. For all $\eta > 0$, there exist $x^1 \in C_1$ and $x^2 \in C_2$ such that $d(x^1, x^2) \leq \eta$. By irreducibility and recurrence of C_1 for the MC $x(n, x^1)$ (if C_1 is a cyclic class of a periodic regime of period d , we consider $\{x(nd, x^1), n \in \mathbb{N}\}$ which is a recurrent MC on C_1), we have that:

$$\forall x \in C_1, \exists n \in \mathbb{N} \text{ s.t. } P(x(n, x^1) = x) > 0 \Leftrightarrow P(A(n-1) \cdots A(0)x^1 = x) > 0.$$

We set $\mathcal{A} = \{\omega \mid A(n-1, \omega) \cdots A(0, \omega)x^1 = x\}$. On \mathcal{A} , we have using proposition 2.1:

$$d(A(n-1) \cdots A(0)x^1, A(n-1) \cdots A(0)x^2) \leq \eta, \quad P - a.s.$$

We deduce that $\forall \eta > 0, \forall x \in C_1, \exists y \in C_2$ such that $d(x, y) \leq \eta$. Let us consider x^1 fixed in C_1 . We can exhibit a sequence $\{x_n^2, n \in \mathbb{N}\}$ of distinct elements of C_2 verifying $d(x^1, x_n^2) \leq \frac{1}{n}$. In the same way, we can build for each element $x_{n_0}^2$ a sequence of distinct elements of C_1 , $\{x_n^1(x_{n_0}^2), n \in \mathbb{N}\}$ such that $d(x_{n_0}^2, x_n^1(x_{n_0}^2)) \leq \frac{1}{n}$. It is also easy to see that it is possible to choose those sequences $x_n^1(x_{n_0}^2)$ recursively in order to have

$$k \neq k' \implies \{x_n^1(x_k^2), n \in \mathbb{N}\} \cap \{x_n^1(x_{k'}^2), n \in \mathbb{N}\} = \emptyset.$$

We iterate the procedure, for each point $x_n^1(x_k^2)$, we build a sequence of distinct elements of C_2 converging to $x_n^1(x_k^2)$ (for $d(.,.)$) and so on. By this way, we have proven in fact that both C_1 and C_2 are isomorphic to

$$\mathbb{N}^{\mathbb{N}} \stackrel{Def}{=} \underbrace{\mathbb{N} \times \mathbb{N} \times \dots}_{\mathbb{N}}.$$

But $\mathbb{N}^{\mathbb{N}}$ is not countable which brings us to a contradiction. We conclude that $d(C_1, C_2) > 0$.

We define $\underline{c} = \inf_{i,j} d(C_i, C_j)$. We have $\underline{c} > 0$. We consider $x_0 \in \mathbb{R}^J$ and $0 < \eta < \underline{c}$. We consider the ball $\mathcal{B}(x_0, \eta)$ for the projective distance. We have:

$$\forall x \in \mathcal{B}, d(A(n) \cdots A(0)x_0, A(n) \cdots A(0)x) \leq \eta < \underline{c}.$$

We conclude that:

$$d(A(n) \cdots A(0)x_0, A(n) \cdots A(0)x) \xrightarrow{n \rightarrow +\infty} 0, \forall x \in \mathcal{B}(x_0, \eta).$$

But then we can apply the same argument to balls $\mathcal{B}(x_1, \eta)$, $x_1 \in \mathcal{B}(x_0, \eta)$. It is automatic that $\forall x_0, y_0$, there exists a finite sequence x_1, \dots, x_p such that $x_i \in \mathcal{B}(x_{i-1}, \eta)$, $i \in \{1, \dots, p\}$ and $y_0 \in \mathcal{B}(x_p, \eta)$. We conclude that:

$$\forall x_0, y_0, d(A(n) \cdots A(0)x_0, A(n) \cdots A(0)y_0) \rightarrow 0,$$

which implies a unique stationary regime and a contradiction with our initial assumption. ■

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