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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

# *Quasi-Variational Inequality Approach to Multi-Item Single Machine Lot Scheduling Problem*

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**QUASI-VARIATIONAL INEQUALITY APPROACH  
TO MULTI-ITEM SINGLE MACHINE  
LOT SCHEDULING PROBLEM**

**Sur les Inéquations Quasi-variationnelles**

**Associées aux Problèmes de Commande Optimale de Commutation  
des Machines Multiproduits**

**Roberto L. V. González<sup>1</sup>, Kenji Muramatsu<sup>2</sup>, Edmundo Rofman<sup>3</sup>**

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## ABSTRACT

We study in this paper the optimization of a production system comprising a multi-item single machine. Taking into account the switching-costs, we analyze the optimization of the system in terms of a system of Quasi-Variational-Inequalities (QVI), which due to the presence of state restrictions, has unbounded solutions. This system represents the Bellman equation associated to this optimal switching control problem. We study the structure of this QVI system, its solutions and numerical methods to solve it.

## RÉSUMÉ

On étudie ici l'optimisation d'un système de production qui est composé d'une machine multi-produit. On montre que le coût optimal est donné par la solution d'un système d'inéquations quasi-variationnelles (QVI), avec solutions non bornées. Le système représente les équations de Bellman associées au problème de contrôle optimal de commutation avec contraintes sur l'état. On étudie les propriétés du QVI, de ces solutions et des méthodes numériques pour sa solution approchée.

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## 1. INTRODUCTION

We consider the optimization of a production system comprising a multi-item single machine. The problem consists in finding the optimal schedule of production that minimizes the addition of production and inventory costs. As it is well known (see [4], [14]), in general it is possible to find an optimal control policy when we know the optimal cost function. Taking into account the switching-costs, it is possible to find the optimal cost function solving a system of Quasi-Variational-Inequalities (QVI) (see [4], [6], [17], [19], [24]). This system is obtained using dynamic programming techniques. We study in this paper the structure of this QVI system, its solutions and numerical methods to solve it in a practical way.

### 1.1 Central Results

In general, under rather mild assumptions, it is possible to find an optimal control policy when we know the optimal cost function  $U^d$ . In consequence, this paper deals fundamentally with the study of  $U^d$  and the development of numerical methods to compute it.

The principal results obtained in our study are the followings:

- The optimal cost function is locally Lipschitz continuous and unbounded.
- The optimal cost function  $U^d$  is the maximum subsolution of the QVI system associated to the problem.
- There exists at least an optimal control policy which realizes the minimum cost. It can be defined in terms of  $U^d$  as an optimal feedback control policy.
- We present a discretization procedure for the numerical solution of the (QVI) system associated to the problem. The procedure has the following properties:
  - \* The solution of QVI is reduced to find the unique fixed point of a non-linear operator
$$P^h: \mathfrak{R}^\eta \rightarrow \mathfrak{R}^\eta.$$
  - \* The usual iteration  $u^{\nu+1} = P^h(u^\nu)$  converges from any arbitrary initial point.
  - \* There are special techniques to compute the fixed point of operator  $P^k$ . In particular, we have developed a special method that converges in a finite number of steps.
- Here we have obtained satisfactory numerical results for the case  $m = 2$  (size of discretization of the state space:  $50 \times 50$ ; memory requirements: 383 Kbytes; computational time: 1,64 sec. in a PC 486 computer). Industrial applications for  $m = 3, 4, 5$ , have been done by K. Muramatsu.

## 1.2 Description of the System

At any time the machine is either idle or producing any one of  $m$  different items. We will denote by  $d=0$  the state of not-production of the machine, and by  $d=1, \dots, m$ , when it is producing item  $d$ .

For item  $d=1, \dots, m$ ; we define the problem data as follows:

- $r_d$  the demand by unit time of item  $d$
- $p_d$  the production quantity by unit time at the machine setting  $d$
- $M_d$  the inventory capacity constraint
- $q(d, \tilde{d})$  the switching cost of the machine from state  $d$  to  $\tilde{d}$
- $\lambda$  the discount rate
- $f$  the instantaneous inventory-holding/production cost

We will always assume the following non zero loop cost condition:

- \*  $\exists q_0 > 0$  such that for any closed loop  $d_0, d_1, \dots, d_p, d_{p+1}$ ,  
with  $d_0 = d_{p+1}$ ,  $p \leq m$ , we have

$$\sum_{i=0}^p q(d_i, d_{i+1}) \geq q_0 \quad (1)$$

and we suppose that the following conditions are verified

$$* q(d, \tilde{d}) \geq 0 \quad \forall \tilde{d} \neq d, \quad q(d, d) = 0 \quad \forall d \in D = \{0, 1, \dots, m\}, \quad (2)$$

$$* q(d, \hat{d}) \leq q(d, \tilde{d}) + q(\tilde{d}, \hat{d}), \quad \forall d \neq \tilde{d} \neq \hat{d}. \quad (3)$$

In addition, we assume that the switching time is small enough to be disregarded and that the following condition, under which a feasible schedule exists, holds:

$$0 < \sum_{d=1}^m \frac{r_d}{p_d} \leq 1. \quad (4)$$

In fact, we will always assume  $\sum_{d=1}^m \frac{r_d}{p_d} < 1$ , because condition  $\sum_{d=1}^m \frac{r_d}{p_d} = 1$  forbids the machine to be in the idle state except for a total time of  $\tau = \sum_{d=1}^m \frac{x_d}{r_d}$  and this is not a natural condition for a problem with infinite horizon.

### 1.3 Formulation of the Optimization Problem

#### 1.3.1 The state space $Q$

Let us denote by  $y_d(t)$  the inventory level of item  $d$  at time  $t$ , starting at  $x_d$  at the origin, i.e.

$$y(t) = (y_1(t), \dots, y_1(t), \dots, y_m(t)), \quad t \geq 0, \quad y(t) \in \mathfrak{R}^m, \quad (5)$$

$$y(0) = x. \quad (6)$$

As neither backlogging nor production over the capacity constraints are allowed, the following restrictions hold for the inventory state  $y_d$

$$0 \leq y_d \leq M_d, \quad d = 1, \dots, m. \quad (7)$$

We will use special sets  $\Gamma \subset \mathfrak{R}^m$ , defined through the use of the following notation:

$$(a_1, \dots, a_1, \dots, a_m) \in \{0, 1, 2\}^m, \quad (8)$$

$$\Gamma(a_1, \dots, a_1, \dots, a_m) = \left\{ y = (y_1, \dots, y_m) \in \mathfrak{R}^m / (9) \right\},$$

$$\left| \begin{array}{l} a_i = 0 \Rightarrow y_i = 0, \\ a_i = 1 \Rightarrow 0 < y_i < M_i, \\ a_i = 2 \Rightarrow y_i = M_i. \end{array} \right. \quad (9)$$

If the inventory of at least two items reach zero simultaneously, the shortage of at least one item is inevitable with any admissible control policy. So, the admissible state space  $Q$  comprises only the set of points with at most one zero component, i.e.

$$Q = \bigcup_a \left\{ \Gamma(a_1, \dots, a_1, \dots, a_m) / \text{at most one component } a_i = 0 \right\}, \quad (10)$$

then

$$\bar{Q} = \bigcup_a \left\{ \Gamma(a_1, \dots, a_m), \quad \forall a = (a_1, \dots, a_1, \dots, a_m) \right\} = \prod_{i=1}^m [0, M_i] \quad (11)$$

and the corresponding interior, denoted with  $\Omega$ , is

$$\dot{Q} = \Omega = \Gamma(1, \dots, 1, \dots, 1). \quad (12)$$

The boundary  $\partial Q$  will be split in the following way

$$\partial Q = \partial Q_e \cup \partial Q^+, \quad \partial Q_e \cap \partial Q^+ = \emptyset, \quad (13)$$

where



$$\partial Q_e = \bigcup_{i=1}^m (\gamma_i^+ \cup \gamma_i^-), \quad (14)$$

being

$$\gamma_i^+ = \bigcup_a \Gamma(a_1, \dots, a_i, \dots, a_m) \cap Q, \quad a_i = 2; \quad \gamma_i^- = \bigcup_a \Gamma(a_1, \dots, a_i, \dots, a_m) \cap Q, \quad a_i = 0, \quad (15)$$

and

$$\partial Q^+ = \bigcup_a \left\{ \Gamma(a_1, \dots, a_i, \dots, a_m) / \text{at least two components } a_i = 0 \right\}. \quad (16)$$

It is clear, from the definition of  $(r_d, p_d)$  that the following equation of evolution holds

$$\frac{dy}{dt} = g(\alpha(t)), \quad y \in \mathbb{R}^m, \quad y(0) = x \in Q, \quad (17)$$

where,  $\forall d \neq 0$ , the evolution of each component is given by

$$g_d(\alpha) = \begin{cases} -r_d & \text{if } \alpha \neq d \\ p_d - r_d & \text{if } \alpha = d; \end{cases} \quad (18)$$

For simplicity of expression, we will denote

$$\begin{cases} g(\alpha) = (g_1(\alpha), \dots, g_m(\alpha)) \\ M_g = \max_{j=1, m} (r_j, p_j - r_j). \end{cases} \quad (19)$$

### 1.3.2 The set of controls

An admissible control  $\alpha(\cdot)$  is a step function with values in the finite set  $D$ . The control will be characterized by a sequence of pairs  $(\theta_\nu, d_\nu)$ , where

$$0 = \theta_0 \leq \theta_1 < \dots < \theta_\nu < \theta_{\nu+1} < \dots \quad (20)$$

are the switching times, and the step function verifies

$$\alpha(t) = d_\nu, \quad \forall t \in (\theta_\nu, \theta_{\nu+1}], \quad d_\nu \in D, \quad d_\nu \neq d_{\nu+1}, \quad \nu = 0, 1, \dots \quad (21)$$

For each  $x \in Q$ ,  $d \in D$ , we define  $\mathcal{A}_x^d$  the set of all admissible schedules with initial state  $x$  and initial machine setting  $d$ :

$$\mathcal{A}_x^d = \left\{ \alpha(\cdot) = (\theta_\nu, d_\nu)_{\nu=0}^\infty / \forall t \in \mathbb{R}^+, y(t) \in Q, d_0 = d \right\}. \quad (22)$$

In other words, we will consider sequences  $(\theta_\nu, d_\nu)$  such that the associated trajectories remain in  $Q$ ,  $\forall t \geq 0$ .

### 1.3.3 The optimal cost function $U$

We consider the following cost function corresponding to a control policy  $\alpha(\cdot)$

$$J(\alpha(\cdot)) = \sum_{i=1}^{\infty} \left( \int_{\theta_{i-1}}^{\theta_i} f(y(s), \alpha(s)) e^{-\lambda s} ds + q(d_{i-1}, d_i) e^{-\lambda \theta_i} \right). \quad (23)$$

For each  $d \in D$  and  $x \in Q$ , we define the value function

$$U_d(x) \equiv \inf \left\{ J(\alpha(\cdot)) / \alpha(\cdot) \in \mathcal{A}_x^d \right\}, \quad (24)$$

which is the minimum cost to be attained starting at  $x$  with the initial machine setting  $d$ . Our objective is to design for each  $x \in Q$ ,  $d \in D$  an optimal schedule  $\bar{\alpha}(\cdot) \in \mathcal{A}_x^d$ , such that:

$$J(\bar{\alpha}(\cdot)) = U_d(x) = \inf \left\{ J(\alpha(\cdot)) / \alpha(\cdot) \in \mathcal{A}_x^d \right\}. \quad (25)$$

We denote  $U$  the vector with components  $U_d$ . The properties of function  $U_d$  are described in the theorems of the following sections.

**Remark 1.1:** For mathematical reasons, we admit multiple instantaneous transitions. In fact, they can be eliminated; because if there is a set of multiple instantaneous transitions

$$d_1 \rightarrow d_2 \rightarrow \dots \rightarrow d_\mu,$$

all arising at  $\theta$ , it can be replaced by the direct transition

$$d_1 \rightarrow d_\mu,$$

which, by virtue of (3), has a smaller total switching cost  $q(d_1, d_\mu) \leq \sum_{i=1}^{\mu} q(d_{i-1}, d_i)$ .

**Remark 1.2:** We will suppose that  $f(x, d)$  is a bounded and Lipschitz-continuous function of  $x$ , for any  $d \in D$ . Also, we will denote

$$M_f = \sup \left\{ |f(x, d)| / x \in Q, d \in D \right\}.$$

## 2. PROPERTIES OF THE OPTIMAL COST FUNCTION U

The value function is rather regular at interior points of  $Q$ , but it becomes singular as the point  $x$  approaches the special part of the boundary designed with  $\partial Q^+$ , which definition we recall here

$$\partial Q^+ = \bigcup_a \left\{ \Gamma(a_1, \dots, a_i, \dots, a_m) / \text{at least two components } a_i = 0 \right\}. \quad (26)$$

Specifically, we have the properties established in the following sections.

### 2.1 Property of Local Boundness

**Theorem 2.1:** *The optimal cost function  $U_d$  is locally bounded in  $Q$  and uniformly bounded in the complement of any neighborhood of  $\partial Q^+$ , i.e. there is a constant  $C$  (independent of  $\lambda$ ), such that  $\forall x \in Q$  there exists a policy  $\alpha(\cdot)$  satisfying*

$$J(\alpha(\cdot)) \leq C + \frac{1}{\lambda} M(d(x, \partial Q^+)) = C \left( 1 + \frac{1}{\lambda} \left( 1 + \frac{1}{d(x, \partial Q^+)} \right) \right). \quad (27)$$

Also,  $U_d$  is uniformly bounded from below, i.e.

$$\inf_{x \in Q} U_d(x) \geq 0. \quad (28)$$

**Proof:** We will prove it only for the case  $m = 2$ . In this case,  $\partial Q^+ = \{0\}$ . Let

$$a = \min \left( \frac{M_1}{2}, \frac{M_2}{2} \frac{r_1}{(p_2 - r_2)}, \frac{M_1}{(p_1 - r_1)} \frac{r_1 s_2}{s_1} \right), \quad (29)$$

$$b = \min \left( \frac{M_2}{2}, \frac{M_1}{2} \frac{r_2}{(p_1 - r_1)}, \frac{M_2}{(p_2 - r_2)} \frac{r_2 s_1}{s_2} \right), \quad (30)$$

where

$$s_1 = \frac{\xi_2}{(\xi_1 - 1)(\xi_2 - 1) - 1}, \quad s_2 = \frac{\xi_1}{(\xi_1 - 1)(\xi_2 - 1) - 1}, \quad (31)$$

$$\xi_1 = \frac{p_1}{r_1}, \quad \xi_2 = \frac{p_2}{r_2}. \quad (32)$$

In the analysis of a first case we will suppose that  $x \in E = [0, a] \times [0, b] \setminus \{(0, 0)\}$ .

If

$$x_2 \leq x_1 \frac{r_2 s_1}{r_1 s_2},$$

we define

$$h_0 = \frac{x_1}{r_1 s_2} \quad (33)$$

and we consider the control policy defined by

$$\begin{aligned} \alpha(s) &= 2 & \forall s \in [0, s_2 h_0) \\ \alpha(s) &= 1 & \forall s \in [s_2 h_0, (s_2 + s_1) h_0) \\ \alpha(s) &= 0 & \forall s \in [(s_1 + s_2) h_0, (s_2 + s_1 + 1) h_0) \end{aligned} \quad (34)$$

and then repeated periodically. As shown in Figure 1, this policy forces the system to return at time  $t_\nu$  to the initial state  $x$ , where

$$t_\nu = (1+s_1+s_2) h_0 \nu \quad (\nu = 1, \dots), \quad (35)$$

and it is admissible because it can be verified (after lengthy calculations) that  $y(s) \in Q \forall s \geq 0$ .

For the corresponding cost we have

$$J(\alpha(\cdot)) \leq \frac{M_f}{\lambda} + \frac{3 M_q}{1 - e^{-\lambda(1+s_1+s_2) h_0}}, \quad (36)$$

then

$$J(\alpha(\cdot)) \leq \frac{M_f}{\lambda} + 3 M_q \left( \frac{1}{\lambda(1+s_1+s_2) h_0} + 1 \right), \quad (37)$$

because

$$\frac{1}{1 - e^{-\lambda(1+s_1+s_2) h_0}} \leq \left( \frac{1}{\lambda(1+s_1+s_2) h_0} + 1 \right). \quad (38)$$

If we denote  $\eta = \sqrt{1 + \left( \frac{r_2 s_1}{r_1 s_2} \right)^2}$ , from (33) we have

$$d(x, \partial Q^+) \leq \eta x_1 = \eta h_0 r_1 s_2, \quad (39)$$

then

$$\frac{1}{h_0} \leq \frac{\eta r_1 s_2}{d(x, \partial Q^+)}, \quad (40)$$

so

$$J(\alpha(\cdot)) \leq \frac{M_f}{\lambda} + \frac{3 r_1 s_2}{(1+s_1+s_2)} \frac{\eta}{d(x, \partial Q^+)} \frac{M_q}{\lambda} + 3 M_q. \quad (41)$$

For the case

$$x_2 > x_1 \frac{r_2 s_1}{r_1 s_2},$$

we can get the similar result

$$J(\alpha(\cdot)) \leq \frac{M_f}{\lambda} + \frac{3 r_2 s_1}{1+s_1+s_2} \frac{M_q}{\lambda} \frac{\eta'}{d(x, \partial Q^+)} + 3 M_q, \quad (42)$$

with  $\eta = \sqrt{1 + \left( \frac{r_1 s_2}{r_2 s_1} \right)^2}$ .

If  $x_1 > a$  or  $x_2 > b$ , and if we apply to the system a policy of type

$$0, 1, 0, 1, \dots,$$

or

$$1, 0, 1, 0, \dots,$$

or

$$0, 2, 0, 2, \dots,$$

or

$$2, 0, 2, 0, \dots,$$

with at most  $N_c$  switchings, and with the corresponding evolutions shown in Figure 2, where

$$N_c = 2 \max \left( \frac{M_2}{b r_2 \left( \frac{1}{r_1} + \frac{1}{p_1 - r_1} \right)}, \frac{M_1}{a r_1 \left( \frac{1}{r_2} + \frac{1}{p_2 - r_2} \right)} \right), \quad (43)$$

we reach the boundary of the set  $E$ , at a time  $T$ , where

$$T \leq \max \left( \frac{M_2}{r_2}, \frac{M_1}{r_1} \right). \quad (44)$$

Then

$$U_d(x) \leq M_f T + N_c M_q + e^{-\lambda T} U_d(a, s), \quad s < b; \quad (45)$$

or

$$U_d(x) \leq M_f T + N_c M_q + e^{-\lambda T} U_d(s, b), \quad s < a. \quad (46)$$

In consequence, for all cases  $\exists C$  (independent of  $x$ ) such that

$$U_d(x) \leq \frac{C}{\lambda} + C, \quad \forall x \notin E. \quad (47)$$

Then, for a suitable constant  $C$  (dependent only on problem data and not on  $x$ ), it holds

$$U_d(x) \leq C \left( 1 + \frac{1}{\lambda} \left( 1 + \frac{1}{d(x, \partial Q^+)} \right) \right). \quad (48)$$

□

**Corollary 2.1:** *The optimal cost function  $U_d$  is bounded in the following form*

$$U_d(x) \leq C \left( 1 + \frac{1}{\lambda} + (\log(d(x, \partial Q^+))) \right)^-. \quad (49)$$

**Proof:** The proof will be restricted to the case  $m = 2$ . If  $x \notin E = [0, a] \times [0, b]$ , from Theorem 2.1 we have that  $U_d(x) \leq C \left( 1 + \frac{1}{\lambda} \right)$ , with  $C$  independent of  $x$  (for  $x \notin E$ ).

Then the proof must analyze only the case  $x \in E$ . Let us suppose in a first analysis that  $x_1 = 0, x_2 < b$ .

We apply to the system the following control policy

$$d = 1 \quad \text{for } t \in [0, \theta_1], \quad \text{with } \theta_1 = \frac{x_2}{r_2}, \quad (50)$$

$$d = 2 \quad \text{for } t \in [\theta_1, \theta_2], \quad \text{with } \theta_2 = \theta_1 + \delta_1, \quad \delta_1 = \frac{\theta_1}{r_1} (p_1 - r_1), \quad (51)$$

$$d = 1 \quad \text{for } t \in [\theta_{2\nu}, \theta_{2\nu+1}], \quad (52)$$

$$d = 2 \quad \text{for } t \in [\theta_{2\nu+1}, \theta_{2\nu+2}], \quad (53)$$

$$\theta_{2\nu+1} = \theta_{2\nu} + x_2 \frac{1}{r_2} \xi^\nu, \quad (54)$$

$$\theta_{2\nu+2} = \theta_{2\nu+1} + \delta_\nu, \quad (55)$$

with

$$\delta_\nu = x_2 \frac{1}{r_2} \xi^\nu \left( \frac{p_1 - r_1}{r_1} \right), \quad (56)$$

$$\xi = \left( \frac{p_1}{r_1} - 1 \right) \left( \frac{p_2}{r_2} - 1 \right), \quad (57)$$

being  $\xi > 1$ , by virtue of (4).

It is easy to verify that the trajectory associated to this control remains in  $E$  and it reaches the boundary of  $E$ , in a time  $T < T'$ , being

$$T' = x_2 \left( \frac{1}{r_2} + \frac{1}{r_1} \left( \frac{p_1 - r_1}{r_2} \right) \right) \frac{\xi^n - 1}{\xi - 1}, \quad (58)$$

with

$$n = 1 + \frac{1}{\log \xi} \log \left( \frac{b}{x_2} \right). \quad (59)$$

We complete the definition of this control after  $T$  using the control defined in Theorem 2.1. Then, the cost corresponding to this control can be bounded in the following way

$$J(\alpha(\cdot)) \leq \left( 1 + \frac{1}{\log \xi} \cdot \log \left( \frac{b}{x_2} \right) \right) M_q + \frac{M_f}{\lambda} + \max_{x \notin E} U_d(x). \quad (60)$$

From Theorem 2.1, we get

$$J(\alpha(\cdot)) \leq C \left( 1 + \frac{1}{\lambda} + (\log(x_2))^- \right), \quad (61)$$

then

$$U_d(x) \leq C \left( 1 + \frac{1}{\lambda} + (\log(x_2))^- \right). \quad (62)$$

For the cases  $x_2 = 0$ ,  $x_1 < a$ , and the mixed case  $0 < x_1 < a$ ,  $0 < x_2 < b$ , the proof follows exactly the same argument and the result is also

$$U_d(x) \leq C \left( 1 + \frac{1}{\lambda} + (\log \|x\|)^- \right). \quad (63)$$

In the general case, we get

$$U_d(x) \leq C \left( 1 + \frac{1}{\lambda} + (\log(d(x, \partial Q^+)))^- \right). \quad (64)$$

□

**Remark 2.1:**  $U_d(\cdot)$  cannot be uniformly bounded, because  $U_d$  is unbounded at  $\partial Q^+$ ; in fact (64) is not only an upper bound of  $U_d$ , but also it gives the behavior of  $U_d(x)$  as  $x \rightarrow \partial Q^+$  (see § 2.4).

## 2.2 Existence of Optimal Control Policies

Condition (1) limits the accumulation of switching points. This property implies the existence of an optimal control policy.

### 2.2.1 Property of bounded density of switching points

From the previous result, we conclude that to find the policy that realizes the minimum value  $U_d(x)$ , it is enough to restrict the election to those policies that satisfy

$$J(\alpha(\cdot)) \leq C \left(1 + \frac{1}{\lambda} + (\log(d(x, \partial Q^+)))^-\right). \quad (65)$$

These policies, in turn, has a bounded density of switching points, in the following strict sense:

**Proposition 2.1:** *Let  $\alpha(\cdot)$  be an arbitrary policy satisfying (65),  $\tau$  a positive real number, and  $\theta_1, \theta_{i+1}, \dots$ , the switching times of  $\alpha(\cdot)$  and let us define*

$$\nu_\tau(\alpha(\cdot)) = \max \left\{ i / \theta_i \leq \tau \right\}, \quad (66)$$

then

$$\nu_\tau(\alpha(\cdot)) \leq \psi_\tau(x), \quad (67)$$

where

$$\psi_\tau(x) = \left( \frac{e^{\lambda\tau}}{q_0} C \left(1 + \frac{1}{\lambda} + (\log(d(x, \partial Q^+)))^-\right) + 1 \right) m. \quad (68)$$

Proof: As  $f \geq 0$ , then

$$J(\alpha(\cdot)) \geq e^{-\lambda\tau} \left( \sum_{i=0}^{\nu_\tau} q(d_i, d_{i+1}) \right) \geq e^{-\lambda\tau} q_0 \left[ \frac{\nu_\tau}{m} \right] \geq q_0 e^{-\lambda\tau} \left( \frac{\nu_\tau}{m} - 1 \right). \quad (69)$$

because  $\nu_\tau$  switching points determine at least  $\left[ \frac{\nu_\tau}{m} \right]$  closed cycles and because (1) holds; where  $[s]$  denotes the integer part of  $s$ . Then, by virtue of (65)

$$q_0 e^{-\lambda\tau} \left( \frac{\nu_\tau}{m} - 1 \right) \leq C \left(1 + \frac{1}{\lambda} + (\log(d(x, \partial Q^+)))^-\right). \quad (70)$$

So

$$\nu_\tau \leq \left( \frac{e^{\lambda\tau}}{q_0} C \left(1 + \frac{1}{\lambda} + (\log(d(x, \partial Q^+)))^-\right) + 1 \right) m \quad (71)$$

and then (67) holds. □

## 2.2.2 Existence of open loop optimal policies

**Theorem 2.2:** For any state  $(x, d)$ , there exists at least an optimal control policy  $\bar{\alpha}$  /

$$U_d(x) = J_x(\bar{\alpha}). \quad (72)$$

Proof: By Theorem 2.1 we have that  $\mathcal{A}_x^d \neq \emptyset$ . Let  $\alpha^\nu$  be a minimizing sequence, i.e.

$$U_d(x) = \lim_{\nu \rightarrow \infty} J(\alpha^\nu(\cdot)). \quad (73)$$

If we denote  $\theta_i^\nu$  the  $i^{\text{th}}$  switching point, we have

$$\alpha^\nu = (\theta_i^\nu, d_i^\nu). \quad (74)$$

From Proposition 2.1, we know that  $\forall T$  there are in the interval  $[0, T]$  at most  $\psi_T(x)$  switching points, i.e.

$$i \geq \psi_T(x) \Rightarrow \theta_i^\nu \geq T. \quad (75)$$

By the state restrictions, we have that

$$\theta_{i+1}^\nu \leq \theta_i^\nu + \max_d \left( \max \left( \frac{M_d}{r_d}, \frac{M_d}{p_d - r_d} \right) \right), \quad (76)$$

then, by the diagonalization procedure, we can choose a subsequence, (w.l.g. we denote it also as  $\alpha^\nu$ ) satisfying

$$\lim_{\nu \rightarrow \infty} \theta_i^\nu = \bar{\theta}_i \quad (77)$$

and such that

$$0 \leq \bar{\theta}_1 \leq \dots \leq \bar{\theta}_i \leq \bar{\theta}_{i+1} \leq i \max_{d \in D} \left( \max \left( \frac{M_d}{r_d}, \frac{M_d}{p_d - r_d} \right) \right). \quad (78)$$

Then, for any  $T > 0$ , the sequence of functions

$$\alpha^\nu \rightarrow \bar{\alpha} \quad (79)$$

converge to a step function  $\bar{\alpha}$  in  $L^1(0, T)$ , i.e.  $\bar{\alpha}$  has a finite number of switchings in the interval  $[0, T]$  and differs from  $\alpha^\nu$  in the set

$$D_T^\nu = \{t / \alpha^\nu(t) \neq \bar{\alpha}(t), t \leq T\}, \quad (80)$$

with the property

$$m(D_T^\nu) \rightarrow 0. \quad (81)$$

It is also clear that

$$\alpha^\nu(\cdot) \rightarrow \bar{\alpha} \quad \text{in } L^1(0, T), \quad (82)$$

and that

$$g(\alpha^\nu) \rightarrow g(\bar{\alpha}) \quad \text{in } L^1(0, T). \quad (83)$$

We define

$$\begin{cases} y^\nu(t) = x + \int_0^t g(\alpha^\nu(s)) ds, \\ \bar{y}(t) = x + \int_0^t g(\bar{\alpha}(s)) ds, \end{cases} \quad (84)$$



then

$$\|y^\nu(t) - \bar{y}(t)\| \leq M_g m(D_T^\nu) \rightarrow 0, \quad (85)$$

i.e.

$$y^\nu \rightarrow \bar{y} \quad \text{uniformly in } [0, T]. \quad (86)$$

If we define

$$J_f(\alpha) = \int_0^\infty e^{-\lambda t} f(y(s), \alpha(s)) ds, \quad (87)$$

then

$$\begin{aligned} |J_f(\alpha^\nu) - J_f(\bar{\alpha})| &\leq 2 \frac{M_f}{\lambda} e^{-\lambda T} + \int_0^T e^{-\lambda t} L_f \|y^\nu(s) - \bar{y}(s)\| ds + 2 M_f m(D_T^\nu) \leq \\ &\leq 2 \frac{M_f}{\lambda} e^{-\lambda T} + (L_f M_g + 2 M_f) m(D_T^\nu) \end{aligned} \quad (88)$$

and

$$J_f(\bar{\alpha}(\cdot)) \leq 2 M_f \frac{e^{-\lambda T}}{\lambda} + J_f(\alpha^\nu(\cdot)) + (L_f M_g + 2 M_f) m(D_T^\nu) \quad (89)$$

$$J_f(\bar{\alpha}(\cdot)) \leq 2 M_f \frac{e^{-\lambda T}}{\lambda} + \lim_{\nu \rightarrow \infty} J_f(\alpha^\nu(\cdot)). \quad (90)$$

So

$$J_f(\bar{\alpha}(\cdot)) \leq \lim_{\nu \rightarrow \infty} J_f(\alpha^\nu(\cdot)), \quad (91)$$

because  $T$  is arbitrary.

As for  $\bar{\alpha}$  it may appear a cluster of switching points (finite), it is possible to have

$$\bar{\theta}_i = \bar{\theta}_{i+1} = \dots = \bar{\theta}_{i+q}. \quad (92)$$

As these instantaneous switching loops leaves the system invariant, it is also possible to replace the policy  $\bar{\alpha}$  by a new policy  $\bar{\bar{\alpha}}$  having exactly the same effect on the system (see remark 1.1), but with a smaller number of switching points, i.e. we eliminate all the instantaneous switching closed loops.

As the cost associated to switchings is

$$\sum_{i=1}^{\infty} q(d_{i-1}, d_i) e^{-\lambda \theta_i}, \quad (93)$$

some part of the sum disappears from the formula corresponding to  $\bar{\bar{\alpha}}$ , leaving outside some sets of terms with a collective cost bounded from below by  $q_0 \nu_c$ , where  $\nu_c$  is the number of cycles above mentioned.

Then,

$$\sum_{i=1}^{\infty} e^{-\lambda \bar{\bar{\theta}}_i} q(d_{i-1}, d_i) \leq \sum_{i=1}^{\infty} e^{-\lambda \bar{\theta}_i} q(d_{i-1}, d_i). \quad (94)$$

If we define, for  $\alpha = (\theta_i, d_i)$

$$J_q(\alpha(\cdot)) = \sum_{i=1}^{\infty} e^{-\lambda \theta_i} q(d_{i-1}, d_i), \quad (95)$$

we have

$$J_q(\bar{\bar{\alpha}}(\cdot)) \leq J_q(\bar{\alpha}(\cdot)) = \lim_{r \rightarrow \infty} \left( \sum_{i=1}^r q(d_{i-1}, d_i) e^{-\lambda \bar{\theta}_i} \right). \quad (96)$$

But

$$\sum_{i=1}^r q(d_{i-1}, d_i) e^{-\lambda \bar{\theta}_i} = \lim_{\nu \rightarrow \infty} \left( \sum_{i=1}^r q(d_{i-1}, d_i) e^{-\lambda \bar{\theta}_i} \right) \leq \lim_{\nu \rightarrow \infty} J_q(\alpha^\nu(\cdot)), \quad (97)$$

because, w.l.g. we can suppose  $\{J(\alpha^\nu(\cdot))\}$  convergent. In consequence

$$J_q(\bar{\alpha}(\cdot)) \leq \lim_{\nu \rightarrow \infty} J_q(\alpha^\nu(\cdot)), \quad (98)$$

so

$$J(\bar{\bar{\alpha}}) \leq J(\bar{\alpha}(\cdot)) \leq \lim_{\nu \rightarrow \infty} J_f(\alpha^\nu(\cdot)) + \lim_{\nu \rightarrow \infty} J_q(\alpha^\nu(\cdot)) = \lim_{\nu \rightarrow \infty} J(x, \alpha^\nu) = U_d(x). \quad (99)$$

As  $J(\bar{\bar{\alpha}}(\cdot)) \geq U_d$ , we obtain the optimality of the control  $\bar{\bar{\alpha}}(\cdot)$ , i.e.

$$J(\bar{\bar{\alpha}}(\cdot)) = U_d(x). \quad (100)$$

□

### 2.3 U is Unbounded

In any neighborhood of  $\partial Q^+$ , due to the shortage of two or more items, the system is enforced to change its production policy with very small intervals between switching points. This behavior has the consequence that the optimal cost function becomes unbounded as  $x \rightarrow \partial Q^+$ .

**Theorem 2.3:**  $U_d$  is unbounded at the origin; moreover, there exists a positive constant  $C$  such that the following inequality holds

$$U_d(x) \geq C(1 + (\log(d(x, \partial Q^+)))^-). \quad (101)$$

**Proof:** The proof will be restricted to  $m = 2$ . Let  $\alpha(\cdot) \in A_X^d$  with switching times  $\theta^\nu$ . The first important fact to establish is that for any point  $x$ , there is at least a first switching time  $\theta(x)$  that is bounded by a linear function of  $\|x\|$ .

Let  $x_1, x_2$  be the coordinates of  $x$ . Always, for any arbitrary control policy, at least one of the functions  $y_1$  or  $y_2$  are decreasing and as they must be non-negative, a switching point  $\theta(x)$  will appear before  $\hat{\theta}_1$  or  $\hat{\theta}_2$ , where

$$\hat{\theta}_1 = \frac{x_1}{r_1} \quad (102)$$

and

$$\hat{\theta}_2 = \frac{x_2}{r_2}. \quad (103)$$

In consequence

$$\theta(x) \leq \max\left(\frac{1}{r_1}, \frac{1}{r_2}\right) \times \max(x_1, x_2), \quad (104)$$

i.e.

$$\theta(x) \leq \left\| \frac{1}{r} \right\|_\infty \times \|x\|_\infty, \quad (105)$$

where

$$\left\| \frac{1}{r} \right\|_\infty = \max\left(\frac{1}{r_1}, \frac{1}{r_2}\right) \quad (106)$$

and

$$\|x\|_\infty = \max(x_1, x_2). \quad (107)$$

As

$$\max_{i=1,m} (\max(p_i - r_i), r_i) \leq M_g, \quad (108)$$

then, the state of the system at the instant  $\theta(x)$  will verify the following inequality (for  $x^0 = x$ )

$$\|y(\theta(x))\| \leq \|x^0\| + M_g \theta^1 \leq \|x^0\| \left(1 + M_g \left\| \frac{1}{r} \right\|_\infty\right) = \xi \|x^0\|, \quad (109)$$

where

$$\xi = \left(1 + M_g \left\| \frac{1}{r} \right\|_\infty\right). \quad (110)$$

In this way, if we define recursively

$$x^{\nu+1} = y(\theta^{\nu+1}) \quad \forall \nu = 0, 1, 2, \dots, \quad (111)$$

i.e.,  $x^\nu$  is the state value at the  $\nu^{\text{th}}$  switching time  $\theta^\nu$ , we have by induction

$$\|x^\nu\|_\infty \leq \xi^\nu \|x^0\|, \quad (112)$$

$$\theta^{\nu+1} \leq \left\| \frac{1}{\Gamma} \right\|_{\infty} \times \|x^{\nu}\|_{\infty} \leq \xi^{\nu} \left\| \frac{1}{\Gamma} \right\|_{\infty} \times \|x^0\|. \quad (113)$$

For the associated discount factor we have

$$e^{-\lambda\theta_{\nu}} \geq e^{-\lambda \left\| \frac{1}{\Gamma} \right\|_{\infty} \xi^{\nu-1} \|x^0\|}. \quad (114)$$

Then, if  $x^0$ ,  $\nu$  verify

$$\|x^0\| \leq \frac{\left( \frac{\log 2}{\lambda} \right)}{\left\| \frac{1}{\Gamma} \right\|_{\infty} \xi^{\nu-1}}, \quad (115)$$

then, it is

$$e^{-\lambda\theta_{\nu}} \geq 1/2; \quad (116)$$

in consequence,

$$J(\alpha(\cdot)) \geq \left[ \frac{\nu}{m} \right] \frac{q_0}{2} \quad (117)$$

and therefore

$$U_d(x^0) \geq \frac{q_0}{2} \left[ \frac{\nu}{m} \right] \quad \forall x^0 \leq \delta(\nu). \quad (118)$$

Finally, to obtain an explicit lower estimation of  $U_d$  at the origin, we define  $\forall x^0 \in Q$

$$\hat{\nu}(x^0) = 1 + \frac{1}{\log \xi} \log \left( \frac{\log 2}{\lambda} \cdot \frac{1}{\left\| \frac{1}{\Gamma} \right\|_{\infty} \|x^0\|} \right). \quad (119)$$

In consequence,  $\forall \nu \leq \hat{\nu}(x^0)$  we have

$$e^{-\lambda\theta_{\nu}} \geq 1/2 \quad (120)$$

and therefore

$$J(\alpha(\cdot)) \geq \frac{q_0}{2} \left[ \frac{\hat{\nu}(x^0)}{m} \right] \geq \frac{q_0}{2} \left( \frac{1}{m} \left( 1 + \frac{1}{\log \xi} \log \left( \frac{\log 2}{\lambda} \cdot \frac{1}{\left\| \frac{1}{\Gamma} \right\|_{\infty} \|x^0\|} \right) \right) - 1 \right). \quad (121)$$

Then

$$U_d(x^0) \geq -C_1 \log \|x^0\| - C_2, \quad (122)$$

where

$$C_1 = \frac{q_0}{2m} \left( \frac{1}{\log \xi} \right). \quad (123)$$

and

$$C_2 = \left( \frac{1}{m} \left( 1 + \frac{1}{\log \xi} \log \left( \frac{\log 2}{\lambda} \cdot \frac{1}{\left\| \frac{1}{\Gamma} \right\|_{\infty}} \right) \right) - 1 \right) \frac{q_0}{2}. \quad (124)$$

Finally, from (122) we have that (101) holds and in consequence

$$\lim_{x^0 \rightarrow 0} U_d(x^0) = +\infty. \quad (125)$$

□

**Remark 2.2:** For the case where  $m \geq 3$ , the optimal cost function  $U_d$  is not only unbounded at the origin. In fact, it is unbounded at any point of  $\partial Q^+$ , which is a set of dimension  $m-2$ .

**Remark 2.3:** From (64) and (101), we see that the behavior of  $U_d$  in any neighborhood of  $\partial Q^+$  is of type

$$\left( \log \left( d(x, \partial Q^+) \right) \right)^-, \quad (126)$$

because this type of function is an upper and a lower bound.

**Remark 2.4:** The control problem we analyze in this paper, is one of those where boundary conditions play a key role. Many techniques have been devised to deal with this sort of problems; we can mention among them those discussed in [8], [33].

In our approach only restrictions on the value of  $U_d$  at  $\partial Q^+$  are retained and no conditions on the behavior of  $\frac{\partial U_d}{\partial x}$  are used, while in [8], [33] those type of conditions are considered. In fact, we do not impose any equality conditions on  $U_d$  at  $\partial Q^+$ ; only inequality conditions of type

$$U_d(x) \leq S^d(U)(x)$$

holds by continuity and the function  $U$  is found as the maximum element of a suitable introduced set of subsolution of the associated HJB equation (see §3).

### 2.4 U is Locally Lipschitz Continuous

To prove that the optimal cost function is Lipschitz-continuous, we use the well known technique of comparing two trajectories with different initial points and the same control (see [13], [17]). In this case, due to the state restrictions, the applied control cannot be the same one and we apply a suitable and carefully chosen control for neighboring trajectories, in such a way that after a finite time both trajectories are coincident (up to a delay that is a linear function of the initial displacement).

#### Definition

We introduce the set of  $\epsilon$ -suboptimal policies:

$$\hat{\mathcal{A}}_{x,\epsilon}^d = \left\{ \alpha(\cdot) \in \mathcal{A}_x^d / J_x(\alpha(\cdot)) \leq U_d(x) + \epsilon \right\}.$$

**Lemma 2.1:** *There exist  $T > 0$ , and a pair of functions*

- $H: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  non-decreasing,  $\lim_{\rho \rightarrow 0} H(\rho) = 0$ , (127)
- $Z: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  non-increasing

such that, if

$$\|x - x'\| \leq H(d(x, \partial Q^+)), \quad (128)$$

then

$$\forall \alpha(\cdot) \in \hat{\mathcal{A}}_{x,o}^d, \exists \alpha'(\cdot) \in \mathcal{A}_{x'}^d, \quad (129)$$

and  $\tau, \tau'$  with the following properties:

$$\bullet \tau \leq T \quad (130)$$

$$\bullet \tau' \leq T \quad (131)$$

$$\bullet |\tau - \tau'| \leq Z(d(x, \partial Q^+)) \|x - x'\| \quad (132)$$

$$\bullet \|y(t) - y'(t)\| \leq Z(d(x, \partial Q^+)) \|x - x'\| \quad \forall t \leq T \quad (133)$$

$$\bullet y(\tau) = y'(\tau') \quad (134)$$

$$\bullet \alpha'(t) = \alpha(t - (\tau' - \tau)) \quad \forall t \geq \tau' \quad (135)$$

$$\bullet m \left\{ t / t \leq \tau', \alpha(t) \neq \alpha'(t) \right\} \leq Z(d(x, \partial Q^+)) \|x - x'\|. \quad (136)$$

The proof of this property is contained in the Appendix.

**Theorem 2.4:** *U is locally Lipschitz-continuous, i.e. there is a function  $L: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  non-increasing such that  $\forall$  compact subset  $G$  of  $Q$ ,  $\forall x \in G$ ,  $\forall x' \in G$*

$$|U_d(x) - U_d(x')| \leq L(\rho) \|x - x'\|, \quad (137)$$

where

$$\rho = \min_{x \in G} (d(x, \partial Q^+)). \quad (138)$$

**Proof:** Let  $x \in G$ , then  $d(x, \partial Q^+) \geq \rho$ . In consequence, from Lemma 2.1 we have that  $\exists Z: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  non-increasing such that, if  $x' / \|x - x'\| \leq H(\rho)$  and  $\bar{\alpha}(\cdot) \in \hat{\mathcal{A}}_{x,0}^d$ , there exist  $\alpha'(\cdot) \in \mathcal{A}_x^d$ , such that

$$|J(\bar{\alpha}(\cdot)) - J(\alpha'(\cdot))| \leq Z(\rho) \|x - x'\|. \quad (139)$$

Then

$$U_d(x') - J(\bar{\alpha}(\cdot)) \leq J(\alpha'(\cdot)) - J(\bar{\alpha}(\cdot)) \leq Z(\rho) \|x - x'\|. \quad (140)$$

In consequence, we have

$$U_d(x') - U_d(x) = (U_d(x') - J(\bar{\alpha}(\cdot))) \leq Z(\rho) \|x - x'\|. \quad (141)$$

Finally, interchanging  $x'$ ,  $x$ , we obtain  $\forall x' / \|x - x'\| \leq H(\rho)$

$$|U_d(x') - U_d(x)| = (U_d(x') - J(\bar{\alpha}(\cdot))) \leq Z(\rho) \|x - x'\|. \quad (141)$$

As  $G$  is compact, it results

$$|U_d(x') - U_d(x)| \leq \frac{\max_{x \in G} (\|x - x'\|) + 1}{H(\rho)} Z(\rho) \|x - x'\|.$$

$\forall x \in G$ ,  $\forall x' \in G$ . So, (137) holds if we define

$$L(\rho) = \frac{\max_{d \in D} (M_d) + 1}{H(\rho)} Z(\rho).$$

□

**Remark 2.4:** From Theorem 2.4 it results that  $U_d(\cdot)$  is uniformly Lipschitz continuous outside any neighborhood of  $\partial Q^+$ . The coefficient  $L$  is independent of the discount rate  $\lambda$ .

## 2.5 Some properties of optimal control policies

### 2.5.1 Basic properties of optimal control policies

We will prove here the following technical (and clearly intuitive) result about optimal policies.

**Proposition 2.3:**  $\forall T > 0, \forall x \in Q, \exists b > 0$  / if  $\bar{\alpha}(\cdot)$  is an optimal control policy and  $y(\cdot)$  is the corresponding trajectory, then  $\exists t(\bar{\alpha}(\cdot)) \in [\frac{T}{2}, T]$ , such that

$$\|y(t)\| > b. \quad (142)$$

**Proof:** We will proof it by reductio ad absurdum. If  $\|y(t)\| \leq b \forall t \in [\frac{T}{2}, T]$ , then, due to the presence of forced switching times (because in the interval  $[T/2, T]$  it will be necessary to exist  $\hat{C} T/b$  switching times), we will have

$$U_d(x) = J(\hat{\alpha}(\cdot)) > \bar{C} e^{-\lambda T} T/b,$$

where  $\hat{C}, \bar{C}$  are suitable positive constants.

By using the same arguments as those employed in Corollary 2.1, we know that always exists a control policy  $\alpha'(\cdot)$  with an associated cost

$$J(\alpha'(\cdot)) \leq C' \left(1 + \frac{1}{\lambda} + (\log(d(x, \partial Q^+)))^-\right). \quad (143)$$

Then the following inequality must be verified

$$C' \left(1 + \frac{1}{\lambda} + (\log(d(x, \partial Q^+)))^-\right) \geq \frac{\bar{C} T}{b} e^{-\lambda T}, \quad (144)$$

but if

$$b < \frac{\bar{C} T e^{-\lambda T}}{C' \left(1 + \frac{1}{\lambda} + (\log(d(x, \partial Q^+)))^-\right)} \quad (145)$$

we get a contradiction which proves that (144) does not hold.

□



**Proposition 2.4:** *The following properties hold for the optimal policies*

- If  $\bar{\alpha}(x, d)$  is an optimal feedback policy, then  $\exists \epsilon > 0$  such that

$$\bar{\alpha}(x, d) = d \quad \forall x/\|x\| \leq \epsilon, \quad d \neq 0; \quad (146)$$

$$\bar{\alpha}(x, 0) \neq 0 \quad \forall x/\|x\| \leq \epsilon. \quad (147)$$

- $\forall x$ , if  $y(\cdot)$  is the trajectory corresponding to an optimal feedback policy, then

$$\lim_{t \rightarrow \infty} \|y(t)\| > 0. \quad (148)$$

Proof: The proof will be restricted to the case  $m=2$ . Let  $x$  be such that  $\|x\| \leq \epsilon$  and  $d = 1$ . We will prove that if  $\alpha(\cdot)$  is an optimal policy, the switching point is located at the boundary, i.e.  $\alpha(t) = 1 \quad \forall t \leq \tau_1 = x_2/r_2$ .

If we suppose that the switching time is  $\theta_1 < \tau_1$  (the maximum admissible first stopping time), we can consider the control with switching times  $\theta'_1 = \theta_1 + \delta$ ,  $\theta'_2 = \theta_2 + \delta$ , where the associated trajectories are shown in Figure 3. For the corresponding costs we have

$$J(\alpha(\cdot)) = I + \left( e^{-\lambda\theta_1} q(1, 2) + e^{-\lambda\theta_2} q(2, 1) \right) + e^{-\lambda\theta'_2} U_1(y(\theta'_2)), \quad (149)$$

$$\begin{aligned} J(\alpha'(\cdot)) &= I' + e^{-\lambda\theta'_1} q(1, 2) + e^{-\lambda\theta'_2} q(2, 1) + e^{-\lambda\theta'_2} U_1(y'(\theta'_2)) = \\ &= I' + e^{-\lambda(\theta'_1 - \theta_1)} \left( e^{-\lambda\theta_1} q(1, 2) + e^{-\lambda\theta_2} q(2, 1) \right) + e^{-\lambda\theta'_2} U_1(y(\theta'_2)), \end{aligned} \quad (150)$$

because

$$y(\theta'_2) = y'(\theta'_2)). \quad (151)$$

$I$  and  $I'$  are defined respectively by

$$I = \int_0^{\theta_1} e^{-\lambda s} f(y(s), 1) ds + \int_{\theta_1}^{\theta_2} e^{-\lambda s} f(y(s), 2) ds + \int_{\theta_2}^{\theta'_2} e^{-\lambda s} f(y(s), 1) ds, \quad (152)$$

$$I' = \int_0^{\theta'_1} e^{-\lambda s} f(y'(s), 1) ds + \int_{\theta'_1}^{\theta'_2} e^{-\lambda s} f(y'(s), 1) ds + \int_{\theta'_2}^{\theta'_2} e^{-\lambda s} f(y'(s), 2) ds. \quad (153)$$

Then,  $I - I' = I_1 + I_2$ , where

$$\begin{aligned} I_1 &= \int_{\theta_2}^{\theta'_2} e^{-\lambda s} f(y(s), 1) ds - \int_{\theta_1}^{\theta'_1} e^{-\lambda s} f(y'(s), 1) ds = \\ &= \int_0^{\theta'_1 - \theta_1} \left( e^{-\lambda(\theta_2 + s)} f(y(\theta_2 + s), 1) - e^{-\lambda(\theta_1 + s)} f(y'(\theta_1 + s), 1) \right) ds = \\ &= \int_0^{\theta'_1 - \theta_1} e^{-\lambda(\theta_2 + s)} \left( f(y(\theta_2 + s), 1) - f(y'(\theta_1 + s), 1) \right) ds + \\ &\quad + \int_0^{\theta'_1 - \theta_1} \left( e^{-\lambda(\theta_2 + s)} - e^{-\lambda(\theta_1 + s)} \right) f(y'(\theta_1 + s), 1) ds. \end{aligned} \quad (154)$$

In consequence

$$\begin{aligned}
 |I_1| &\leq \int_0^{\theta'_1 - \theta_1} \left( L_f \|y(\theta_2 + s) - y'(\theta_1 + s)\| + \lambda M_f (\theta_2 - \theta_1) \right) ds \leq \\
 &\leq \int_0^{\theta'_1 - \theta_1} (L_f M_g + \lambda M_f) (\theta_2 - \theta_1) ds = (L_f M_g + \lambda M_f) (\theta_2 - \theta_1) (\theta'_1 - \theta_1), \quad (155)
 \end{aligned}$$

$$\begin{aligned}
 I_2 &= \int_{\theta_1}^{\theta_2} e^{-\lambda s} f(y(s), 2) ds - \int_{\theta'_1}^{\theta'_2} e^{-\lambda s} f(y'(s), 2) ds = \\
 &= \int_0^{\theta_2 - \theta_1} \left( e^{-\lambda(\theta_1 + s)} f(y(\theta_1 + s), 2) - e^{-\lambda(\theta'_1 + s)} f(y'(\theta'_1 + s), 2) \right) ds = \\
 &= \int_0^{\theta_2 - \theta_1} e^{-\lambda(\theta_1 + s)} \left( f(y(\theta_1 + s), 2) - f(y'(\theta'_1 + s), 2) \right) ds + \\
 &\quad + \int_0^{\theta_2 - \theta_1} \left( e^{-\lambda(\theta_1 + s)} - e^{-\lambda(\theta'_1 + s)} \right) f(y'(\theta'_1 + s), 2) ds.
 \end{aligned}$$

Then

$$\begin{aligned}
 |I_2| &\leq \int_0^{\theta_2 - \theta_1} \left( L_f \|y(\theta_1 + s) - y'(\theta'_1 + s)\| + \lambda M_f (\theta'_1 - \theta_1) \right) ds \leq \\
 &\leq \int_0^{\theta_2 - \theta_1} (L_f M_g + \lambda M_f) (\theta'_1 - \theta_1) ds = (L_f M_g + \lambda M_f) (\theta_2 - \theta_1) (\theta'_1 - \theta_1). \quad (156)
 \end{aligned}$$

Taking into account that

$$\theta_2 - \theta_1 \leq \left( x_1 + (p_1 - r_1) \frac{x_2}{r_2} \right) \frac{1}{r_1}$$

and the inequality

$$\left( e^{-\lambda \theta_1} q(1, 2) + e^{-\lambda \theta_2} q(2, 1) \right) \geq \frac{q_0}{2},$$

we have

$$J(\alpha'(\cdot)) \leq J(\alpha(\cdot)) + |I_1 - I_2| + \left( e^{-\lambda(\theta'_1 - \theta_1)} - 1 \right) \frac{q_0}{2}.$$

Then, if we take  $(\theta'_1 - \theta_1) \leq 1/\lambda$ , we obtain

$$\begin{aligned}
 J(\alpha'(\cdot)) - J(\alpha(\cdot)) &\leq 2(L_f M_g + \lambda M_f) (\theta_2 - \theta_1) (\theta'_1 - \theta_1) - \lambda (\theta'_1 - \theta_1) \frac{q_0}{4} \leq \\
 &\leq \left( 2(L_f M_g + \lambda M_f) (\theta_2 - \theta_1) - \lambda \frac{q_0}{4} \right) (\theta'_1 - \theta_1) < 0,
 \end{aligned}$$

when

$$(\theta_2 - \theta_1) < \frac{q_0 \lambda}{4(L_f M_g + \lambda M_f)}$$

and that relation is verified when

$$\epsilon < \epsilon_1 = \frac{q_0 \lambda}{4(L_f M_g + \lambda M_f) \left( \frac{1}{r_1} + \left( \frac{p_1}{r_1} - 1 \right) \frac{1}{r_2} \right)};$$

then, when  $\epsilon < \epsilon_1$ .

$$J(x, \alpha'(\cdot)) - J(x, \alpha(\cdot)) < 0.$$

Then, if  $\epsilon < \epsilon_1$ , for  $\alpha(\cdot)$  to be optimal, it is necessary that  $\theta_1$  be maximal, i.e.

$$y_1(\theta_1) = 0.$$

The meaning of this property is clear, the production of any item is continued until the boundary of  $Q$  is reached. At that point, the item with zero stock is chosen for production and that policy is maintained till a new boundary is reached. Then, for  $\epsilon$  small enough, once any state of production is reached, it is not leaved till the system leaves  $B_\epsilon(0)$ .

If the state of production is 0, the system switches instantaneously to an active production state. The proof of this result, although technically more involved than the previous one, follows essentially the same arguments used above to prove the maximality of the operation under control "1", "2". It is here omitted for sake of brevity. We have proved in this form (146), (147), (148) follows immediately by virtue of (18).

□

### 2.5.2 Existence of a minimum distance between switching points

**Theorem 2.5:** *If the following strengthened modification of hypothesis (3) holds*

$$q(d, \hat{d}) + q(\hat{d}, \bar{d}) \geq q(d, \bar{d}) + \hat{q}_0, \quad \forall \bar{d} \neq d, \quad (157)$$

*being  $\hat{q}_0 > 0$ ; then there exists a function  $\sigma(x) > 0$  such that the switching times of any optimal policy verify*

$$\theta_i \geq \theta_{i-1} + \sigma(x) > 0 \quad \forall i=2, \dots, \forall d_i \neq 0. \quad (158)$$

**Proof:** By virtue of the results of Proposition 2.4, we have that the trajectory corresponding to the optimal control remains in a set where  $U_d(\cdot)$  is uniformly Lipschitz continuous, i.e.  $\exists L$  such that,  $\forall t, t', \forall d \in D$

$$|U_d(y(t)) - U_d(y(t'))| \leq L \|y(t) - y(t')\|. \quad (159)$$

Let  $\theta_{i-1}, \theta_i$  be two successive switching times for an optimal policy  $\alpha(\cdot)$ . By dynamic programming optimality principle, we have:

$$U_{d_{i-1}}(y(\theta_i)) = q(d_{i-1}, d_i) + U_{d_i}(y(\theta_i)), \quad (160)$$

$$U_{d_{i-2}}(y(\theta_{i-1})) = q(d_{i-2}, d_{i-1}) + U_{d_{i-1}}(y(\theta_{i-1})) \quad (161)$$

and

$$U_{d_{i-2}}(y(\theta_{i-1})) \leq q(d_{i-2}, d_i) + U_{d_i}(y(\theta_{i-1})). \quad (162)$$

From (159) and (17), we have by (19)

$$\left| U_{d_i}(y(\theta_i)) - U_{d_i}(y(\theta_{i-1})) \right| \leq L M_g (\theta_i - \theta_{i-1}) \quad (163)$$

and

$$\left| U_{d_{i-1}}(y(\theta_i)) - U_{d_{i-1}}(y(\theta_{i-1})) \right| \leq L M_g (\theta_i - \theta_{i-1}). \quad (164)$$

In this form, taking into account (160)–(164), we arrive to the inequality

$$\begin{aligned} L M_g (\theta_i - \theta_{i-1}) + U_{d_i}(y(\theta_i)) + q(d_{i-2}, d_i) &\geq U_{d_i}(y(\theta_{i-1})) + q(d_{i-2}, d_i) \geq U_{d_{i-2}}(y(\theta_{i-1})) = \\ &= q(d_{i-2}, d_{i-1}) + U_{d_{i-1}}(y(\theta_{i-1})) \geq q(d_{i-2}, d_{i-1}) + U_{d_{i-1}}(y(\theta_i)) - L M_g (\theta_i - \theta_{i-1}) = \\ &= q(d_{i-2}, d_{i-1}) + q(d_{i-1}, d_i) + U_{d_i}(y(\theta_i)) - L M_g (\theta_i - \theta_{i-1}) \geq \\ &\geq \eta + q(d_{i-2}, d_i) + U_{d_i}(y(\theta_i)) - L M_g (\theta_i - \theta_{i-1}), \end{aligned} \quad (165)$$

where

$$\eta = \min_{d, \hat{d}, \tilde{d}} (q(d, \hat{d}) + q(\hat{d}, \tilde{d}) - q(d, \tilde{d})) > 0. \quad (166)$$

Then

$$\theta_i - \theta_{i-1} \geq \frac{1}{2 M_g L} \eta. \quad (167)$$

□

### 3. DYNAMIC PROGRAMMING SOLUTION

#### 3.1 The Hamilton–Jacobi Equation in Integral Form

In this section we describe the (QVI) system originated by the differential (or integral) relations verified by the optimal cost function  $U$ . A similar system had been considered in [6] for an unconstrained problem in  $\mathbb{R}^m$ . The problem which we consider here differs from that one, in that here there are state constraints which impose boundary conditions to the value function.

For simplicity, we will use the following notation:

$$S^d(U)(x) = \min_{\tilde{d} \neq d} \left( q(d, \tilde{d}) + U_{\tilde{d}}(x) \right) \quad x \in Q, d \in D. \quad (168)$$

**Theorem 3.1:** *For each  $d=0, 1, \dots$ , and  $x \in \Omega$ , the following conditions are verified*

$$a) \quad U_d(x) \leq S^d(U)(x) \quad x \in \Omega, d \in D. \quad (169)$$

$$b) \quad U_d(x) \leq \int_0^t f(y(s), d) e^{-\lambda s} ds + U_d(y(t)) e^{-\lambda t},$$

$$\forall t > 0 / y(t) = x + t g(d) \in \Omega. \quad (170)$$

(c) *If furthermore, for some point  $x \in \Omega$ , a strict inequality holds in (a), then there exists  $t_x > 0$  such that*

$$U_d(x) = \int_0^t f(y(s), d) e^{-\lambda s} ds + U_d(y(t)) e^{-\lambda t} \quad \forall t / 0 \leq t \leq t_x. \quad (171)$$

**Proof:** The proof is classical and uses the basic tools of dynamic programming theory. It is here omitted for sake of simplicity, see e.g. ([4], [6]), [14], [17]).

### 3.2 Boundary Conditions for the Hamilton–Jacobi Equation

To obtain the boundary conditions originated by the state constraints, we must consider the behavior of  $U$  at

$$\partial Q_e = \bigcup_{d=1}^m (\gamma_d^+ \cup \gamma_d^-) \quad (172)$$

where

$$\gamma_i^+ = \bigcup_a \Gamma(a_1, \dots, a_i, \dots, a_m) \cap Q, \quad a_i = 2; \quad \gamma_i^- = \bigcup_a \Gamma(a_1, \dots, a_i, \dots, a_m) \cap Q, \quad a_i = 0. \quad (173)$$

Once the inventory level of item  $d$  reaches zero, i.e.  $x$  attains  $\gamma_d^-$ , instantaneously we must switch the machine to setting  $d$ . Otherwise, inventory shortage of item  $d$  would be inevitable. Then, it holds:

$$U_{\tilde{d}}(x) = q(\tilde{d}, d) + U_d(x), \quad \forall \tilde{d} \neq d, \quad \forall x \in \gamma_d^-. \quad (174)$$

If  $x \in \gamma_d^+$ , it is not possible to continue with the production of item  $d$  and, in consequence we get, by virtue of the dynamic programming optimality principle:

$$U_d(x) = S^d(U)(x). \quad (175)$$

In this way we get the boundary conditions for the  $m$ -item general case:

**Theorem 3.2:** *For any  $x$  on the boundary  $\partial Q_e$  the following boundary conditions are verified:*

$$U_{\tilde{d}}(x) = (S^{\tilde{d}}(U))(x) \quad \forall x \in \gamma_d^-, \quad \forall \tilde{d} \neq d, \quad (176)$$

$$U_d(x) = (S^d(U))(x) \quad \forall d \neq 0, \quad \forall x \in \gamma_d^+. \quad (177)$$

**Note:** It can be easily seen that by virtue of (1)–(3), condition (174) is equivalent to (176) in the following sense:

$$(U_{\tilde{d}}(x) = q(\tilde{d}, d) + U_d(x), \quad \forall \tilde{d} \neq d) \Leftrightarrow (U_{\tilde{d}}(x) = (S^{\tilde{d}}(U))(x), \quad \forall \tilde{d} \neq d).$$

### 3.3 The Hamilton–Jacobi Equation in Differential Form

From Theorem 2.3 we know that the value function  $U$  is Lipschitz continuous; then it is differentiable a.e. and in consequence we can associate to the value function the following Hamilton–Jacobi inequality in terms of its derivatives:

**Theorem 3.3:** *For each  $d \in D$ , the following relations are verified*

$$U_d(x) \leq S^d(U)(x) \quad \forall x \in Q \quad (178)$$

$$\lambda U_d(x) - g(d) \frac{\partial}{\partial x} U_d(x) - f(x, d) \leq 0 \quad \text{a.e. } x \text{ in } \Omega \quad (179)$$

$$\left( U_d(x) - S^d(U)(x) \right) \left( \lambda U_d(x) - g(d) \frac{\partial}{\partial x} U_d(x) - f(x, d) \right) = 0 \quad \text{a.e. } x \text{ in } \Omega \quad (180)$$

*i.e.  $\forall x$  at which  $U_d(\cdot)$  is differentiable, at least one of (178) or (179) is verified with equality.*

$$U_{\tilde{d}}(x) = S^{\tilde{d}}(U)(x), \quad \forall \tilde{d} \neq d \quad \text{if } x \in \gamma_d^- \quad (d \neq 0), \quad (181)$$

$$U_d(x) = S^d(U)(x), \quad \text{if } x \in \gamma_d^+ \quad (d \neq 0). \quad (182)$$

### 3.4 Characterization of $U$ as the Maximum Subsolution of Hamilton-Jacobi Equation

In this section, we will characterize the value function  $U$  as the maximum element of the set of subsolutions  $W$  defined below:

**Theorem 3.4:** *Let*

$$W = \left\{ w(\cdot): D \times Q \rightarrow \mathbb{R} / w_d(\cdot) \in W_{loc}^{1,\infty}(Q), (178), (179) \right\}. \quad (183)$$

*Then  $U$  is the maximum element of the set  $W$ , i.e.  $U \in W$  and*

$$U_d(x) \geq w_d(x) \quad \forall x \in Q, \forall d \in D, \forall w \in W. \quad (184)$$

**Proof:** Let  $w \in W$ . Let  $x \in \Omega$ ,  $x' \in \Omega$ ,  $d \in D$ ,  $h > 0$  such that

$$x' = x + hg(d); \quad (185)$$

then, for  $\epsilon$  small enough the set

$$S_\epsilon = \left\{ x + tg(d) + p / 0 \leq t \leq h, p \in B(\epsilon) \right\} \subset \Omega. \quad (186)$$

$w_d$  is a Lipschitz function in  $S_\epsilon$ , then by virtue of (179) and by the regularization technique (see [17], [18]) we obtain

$$w_d(x) \leq e^{-\lambda h} w_d(x') + \int_0^h e^{-\lambda s} f(y(s), d(s)) ds \quad (187)$$

By continuity, this inequality holds  $\forall x, x' \in Q$ , related by (185).

Let  $\alpha(\cdot)$  be a policy and  $y(\cdot)$  its associated trajectory  $y(\cdot)$ . We have, by virtue of (187) and (178)

$$w_d(x) \leq \int_0^\tau f(y(s), d(s)) e^{-\lambda s} ds + e^{-\lambda \tau} w_{\alpha(\tau)}(y(\tau)) + \sum_{\theta_i \leq \tau} e^{-\lambda \theta_i} q(d_i, d_{i+1}). \quad (188)$$

Then

$$w_d(x) \leq J(\alpha(\cdot)) + \overline{\lim}_{t \rightarrow \infty} \left( e^{-\lambda t} w_{\alpha(t)}(y(t)) \right). \quad (189)$$

By Proposition 2.4, we have

$$U_d(x) = \inf_{\tilde{\mathcal{A}}_x^d} J(\alpha(\cdot)), \quad (190)$$

where

$$\tilde{\mathcal{A}}_x^d = \left\{ \alpha(\cdot) \in \mathcal{A}_x^d / \lim_{t \rightarrow \infty} (d(y(t), \partial Q^+)) > 0 \right\}; \quad (191)$$

then, if  $\alpha(\cdot) \in \tilde{\mathcal{A}}_x^d$ ,

$$w_d(x) \leq J(\alpha(\cdot)), \quad (192)$$

because



$$\lim_{t \rightarrow \infty} (d(y(t), \partial Q^+)) = \rho > 0 \Rightarrow \overline{\lim}_{t \rightarrow \infty} w_d(y(t)) \leq C \left(1 + \frac{1}{\lambda} + (\log(\rho))^{-}\right); \quad (193)$$

so

$$\overline{\lim}_{\tau \rightarrow \infty} \left( e^{-\lambda \tau} w_{\alpha(\tau)}(y(\tau)) \right) = 0. \quad (194)$$

In consequence, from (190) and (192) we obtain

$$w_d(x) \leq U_d(x) \quad \forall x \in Q. \quad (195)$$

□

**Note:** This result is an extension of the methodology described in [17], [18].

By this theorem, conditions (180–182) are deleted and in its place the concept of maximum element is introduced. In this form, the computation of  $U$  is transformed into problem:

$$P: \text{Find the maximum element } U \text{ in the set } W \quad (196)$$

which can be solved with the numerical procedures described in [2], [5], [7], [20], [26], [27], [28], [30], [34].

**Note:** In ([21], [22]), the analysis of the QVI system is done in terms of viscosity solutions (see [11], [12] for a description of this concept). In that sense, the optimal cost is the maximum subsolution and the minimum supersolution.

### 3.5 Construction of an Optimal Feedback Policy

The optimal cost function can be used to define optimal control policies in the following way:

**Theorem 3.5:** For any state  $(x, d)$ , there exists at least an optimal control policy  $\bar{\alpha}$ , i.e.

$$U_d(x) = J(\bar{\alpha}). \quad (197)$$

An optimal feedback control policy  $\bar{\alpha} \in \mathcal{A}_x^d$  can be obtained in terms of  $U^d$  in the following way:

We define  $\theta_0 = 0$ ,  $d_0 = d$  and recursively the switching times  $\theta_i$  and the controls  $d_i$ ,  $i=1,2,\dots$

$$\theta_i = \min \left\{ t \geq \theta_{i-1} / U_{d_{i-1}}(y(t)) = (S^{d_{i-1}}(U))(y(t)) \right\}, \quad (198)$$

$$d_i \in \left\{ d \in D / d \neq d_{i-1}, (S^{d_{i-1}}(U))(y(\theta_i)) = U_d(y(\theta_i)) + q(d_{i-1}, d) \right\}. \quad (199)$$

**Proof:** The proof is classical (see [4]) and it is here omitted for sake of simplicity.

## 4. THE DISCRETE PROCEDURE

## 4.1 Elements of the Discrete Problem

To define the discrete problem, it is necessary to discretize the set  $W$  using an approximate set  $W_k$ . This approximation comprises a discretization of the space  $W_{loc}^{1,\infty}(Q)$  and a discretization of conditions (178), (179) that verifies a discrete maximum principle (DMP), (see [9]–[10] for a discussion of this concept and its role in the convergence of approximation methods).

4.1.1 Approximation of domain  $Q$ 

We will identify the discretization of space variables with the parameter  $k$ , which also indicate the size of the discretization.

We will approximate  $Q$  with  $Q_k = \bigcup_j S_j^k$ .  $\{S_j^k\}$  is a finite set of simplices and then, as it is shown in Figure 4,  $Q_k$  is a polyhedron of  $\mathbb{R}^n$ .  $\{S_j^k\}$  is a family of regular triangulations (see [15], [16], [27], [31], [35]), i.e. a finite set of simplices with the following properties:

- $Q_k = \bigcup_j S_j^k$  is a polyhedron of  $\mathbb{R}^m$
- $\max_j (\text{diám } S_j^k) = k$

(200)

- $Q_k \rightarrow Q$ , when  $k \rightarrow 0$ , in the following sense:

$$* Q_k \subset Q$$

$$* \forall Z \text{ compact} \subset \overset{\circ}{Q}, \exists \bar{k}(Z) / Z \subset Q_k \quad \forall k < \bar{k}(Z)$$

(201)

- If  $d_j$  is the diameter of the simplex  $S_j^k$ , then

$\exists \chi_1 > 0$  such that for any simplex of  $Q_k$ , there exists in the interior of the simplex a sphere of radius  $r$

$$r \geq \chi_1 d_j$$

(202)

- Moreover there exists  $\chi_2$ , independent of the discretization such that:

$$\frac{k}{d_i} \leq \chi_2, \quad \forall i.$$

(203)

We denote with  $V_k = \{x^i, i=1, \dots, N\}$  the set of vertices of  $Q_k$  (nodes), being  $N$  the cardinal of  $V_k$ .

#### 4.1.2 Approximation of the boundary

We define,  $\forall d=1, \dots, m$

$$\gamma_{k,d}^+ = \left\{ x^i \in V_k / x^i + h g(d) \notin Q_k, \forall h > 0 \right\} \quad (204)$$

$$\gamma_{k,d}^- = \left\{ x^i \in V_k / x^i + h g(\tilde{d}) \notin Q_k, \forall h > 0 \vee \tilde{d} \neq d \right\} \quad (205)$$

#### 4.1.3 Definition of the approximation space $F_k$

We consider the set  $F_k$  of functions  $w: Q_k \rightarrow \mathbb{R}$ ,  $w \in W^{1,\infty}(Q_k)$ ,  $\frac{\partial w}{\partial x}$  constant in the interior of each simplex of  $Q_k$  (i.e.  $w$  are linear finite elements). It is obvious that any  $w \in F_k$  is entirely characterized by the values  $w(x^i)$ ,  $x^i \in V_k$ ,  $i = 1, \dots, N$ .

We denote  $\bar{F}_k = (F_k)^{m+1}$ .

#### 4.1.4 Discretization of Hamilton-Jacobi inequalities

The approximate set  $W_k$  is a subset of  $\bar{F}_k$ . It comprises a discretization of conditions (178), (179); they take the following form:

$$w_d(x^i) \leq S^d(w)(x^i) \quad \forall x^i \in V_k \quad (206)$$

$$w_d(x^i) \leq D_d^k(w_d)(x^i) \quad \forall x^i \in V_k \quad (207)$$

$D_d^k(w_d)(x^i)$  is defined by:

$$D_d^k(w_d)(x^i) = \frac{1}{1+\lambda h(i, k, d)} \left( w_d(x^i + h(i, k, d) g(d)) + h(i, k, d) f(x^i, d) \right) \quad (208)$$

$$\forall x^i \in V_k \cap c_{\gamma_{k,d}^+} \cap c \left( \bigcup_{r \neq d} \gamma_{k,r}^- \right) \quad (209)$$

$$D_d^k(w_d)(x^i) = +\infty \quad \forall x^i \in \gamma_{k,d}^+ \cup \left( \bigcup_{r \neq d} \gamma_{k,r}^- \right) \quad (210)$$

where, for  $x^i \in V_k \cap c_{\gamma_{k,d}^+} \cap c \left( \bigcup_{r \neq d} \gamma_{k,r}^- \right)$ ,  $h(i, k, d)$  is a positive scalar such that

$$x^i + h(i, k, d) g(d) \in Q_k \quad (211)$$

**Note:**  $w_d(x^i) \leq D_d^k(w_d)(x^i)$  is the natural discretization of (179) (see [17], [20], [27], for other uses of this type of discretization).

**Remark 4.1:** we will suppose that the following inequality is always satisfied:  $\exists c > 0, C > 0 /$

$$c k \leq h(i, k, d) \leq C k, \forall d \in D, \forall x^i \in V_k \quad (212)$$

#### 4.1.5 The solution as a fixed point of operator $P_k$

We define operator  $P_k : (\bar{F}_k) \rightarrow (\bar{F}_k)$  as

$$(P_k w)_d(x^i) = \min \left( (D_d^k w_d)(x^i), S^d(w)(x^i) \right) \quad \forall x^i \in V_k, \forall d=0, \dots, m. \quad (213)$$

#### 4.1.6 Definition of discrete subsolutions and supersolutions

The set  $W_k$  of discrete subsolutions is defined:

$$W_k = \left\{ w(\cdot) \in \bar{F}_k \quad / \quad w_d(x^i) \leq (P_k w)_d(x^i) \quad \forall d=0, \dots, m \quad \forall x^i \in V_k \right\} \quad (214)$$

In a similar form we define the set  $S_k$  of discrete supersolutions

$$S_k = \left\{ s(\cdot) \in \bar{F}_k \quad / \quad s_d(x^i) \geq (P_k s)_d(x^i) \quad \forall d=0, \dots, m, \quad \forall x^i \in V_k \right\} \quad (215)$$

## 4.2 Three Equivalent Discrete Problems

In relation to the original problem  $P$  we introduce the discrete problems:

$$\text{Problem } P_k^1: \text{Find the maximum element of } W_k \quad (216)$$

$$\text{Problem } P_k^2: \text{Find the fixed point of operator } P_k \quad (217)$$

$$\text{Problem } P_k^3: \text{Find the minimum element of } S_k \quad (218)$$

Using basically the techniques introduced in [29], we can prove the following results:

- Problems  $P_k^1$ ,  $P_k^2$  and  $P_k^3$  are equivalent in the sense that they have the same unique solution  $U^k$ .
- The solution  $U^k$  can be computed with the following naïve iteration which converges from any starting point  $U^0$ .

**Algorithm  $A_0$ :**

Step 0: Set  $\nu = 0$ ,  $U^0 \in \bar{F}_k$

Step 1:  $U^{\nu+1} = P_k U^\nu$

Step 2: If  $U^{\nu+1} = U^\nu$ , stop; else, set  $\nu = \nu + 1$ , and go to step 1.

**Remark 4.2:** The real computation of  $U_k$  is done in practice with more efficient algorithms, like those presented in 4.3.

#### 4.2.1 The discrete problems $P_k^1, P_k^3$ : existence and uniqueness of solutions

The existence of unique solutions of problems  $P_k^1, P_k^3$  is given by Theorem 4.2. Those results are stated using the properties established in the following Lemmas.

**Lemma 4.1:**  $P_k$  is monotone, i.e.

$$w \leq \hat{w} \quad \text{implies} \quad P_k w \leq P_k \hat{w} \quad (219)$$

**Proof:** By definition, it is:

$$(P_k w)_d(x^i) = \min\left((D_d^k w)_d(x^i), S^d(w)(x^i)\right) \quad \forall x^i \in V_k, \quad \forall d=0, \dots, m. \quad (220)$$

Let be  $w \leq \hat{w}$ . As operators  $D_d^k, S^d$  are monotone, we have

$$D_d^k w_d \leq D_d^k \hat{w}_d \quad (221)$$

and also

$$S^d(w) \leq S^d(\hat{w}) \quad (222)$$

then,

$$(P_k w)_d(x^i) = \min\left((D_d^k w)_d(x^i), S^d(w)(x^i)\right) \leq \min\left((D_d^k \hat{w}_d)(x^i), S^d(\hat{w})(x^i)\right) = (P_k \hat{w})_d(x^i) \quad (223)$$

□

**Note:** In order to simplify the presentation, the proof of Lemmas 4.2, 4.4 and 4.5, Theorem 4.3 and Corollary 4.1, will be done with the following strengthened modification of hypothesis (2):

$$q(d, \tilde{d}) \geq \hat{q}_0 > 0, \quad \forall \tilde{d} \neq d. \quad (224)$$

The validity of those results is nevertheless general, only with the assumption (1).

**Lemma 4.2:**

- $S_k \neq \emptyset$  in particular,  $\bar{s} \in S_k$ , where

$$\bar{s}_d(x_i) = \begin{cases} \frac{M_f}{\lambda} + \frac{M_q}{\lambda\eta} & \forall x^i \in V_k \cap c\gamma_{k,d}^+ \cap c\left(\bigcup_{r \neq d} \bar{\gamma}_{k,r}\right) \\ \frac{M_f}{\lambda} + \frac{M_q}{\lambda\eta} + M_q & \forall x^i \in \gamma_{k,d}^+ \cup \left(\bigcup_{r \neq d} \bar{\gamma}_{k,r}\right) \end{cases} \quad (225)$$

and

$$\eta = \min \left\{ h(i, k, d) / \forall i \in V_k^*, x^i \in V_k \cap c\gamma_{k,d}^+ \cap c\left(\bigcup_{r \neq d} \bar{\gamma}_{k,r}\right) \right\} \quad (226)$$

- $W_k \neq \emptyset$

In particular,  $\hat{w} \in W_k$ , where  $\hat{w}$  is the constant element of  $\bar{F}_k$  such that

$$\hat{w}_d(x^i) = -\frac{M_f}{\lambda}, \quad \forall x^i \quad \forall d \quad (227)$$

- $W_k$  is uniformly upper-bounded, i.e.

$$\bar{s} \geq w, \quad \forall w \in W_k \quad (228)$$

- $S_k$  is uniformly lower-bounded, i.e.

$$s \geq \hat{w}, \quad \forall s \in S_k \quad (229)$$

$$w, w' \in W_k \Rightarrow \max(w, w') \in W_k \quad (230)$$

$$s, s' \in S_k \Rightarrow \min(s, s') \in S_k \quad (231)$$

Proof: We must prove that  $\tilde{s} \geq P_k \tilde{s}$ .

- Let be  $x^i$  an arbitrary point of  $V_k$  and  $d \in D$ . There are only two cases to analyze:

$$a) x^i \in V_k \cap c_{\gamma_{k,d}^+} \cap c\left(\bigcup_{r \neq d} \gamma_{k,r}^-\right) \quad (232)$$

$$b) x^i \in \gamma_{k,d}^+ \cup \left(\bigcup_{r \neq d} \gamma_{k,r}^-\right) \quad (233)$$

a) In this case,

$$\begin{aligned} (P_k \tilde{s})_d(x^i) &\leq (D_d^k(\tilde{s}_d))(x^i) = \frac{1}{1+\lambda h(i, k, d)} \left( \tilde{s}_d(x^i + h(i, k, d)g(d)) + h(i, k, d) f(x^i, d) \right) \\ &\leq \frac{1}{1+\lambda h(i, k, d)} \left( \frac{M_f}{\lambda} + \frac{M_q}{\lambda \eta} + M_q + h(i, k, d) M_f \right) \leq \frac{M_f}{\lambda} + \frac{M_q}{\lambda \eta} = \tilde{s}_d(x^i) \end{aligned} \quad (234)$$

b) in this case,

$$(P_k \tilde{s})_d(x^i) = (S^d(\tilde{s}))(x^i) = q(d, d') + \tilde{s}_{d'}(x^i) \leq M_q + \left(\frac{M_f}{\lambda} + \frac{M_q}{\lambda \eta}\right) = \tilde{s}_d(x^i) \quad (235)$$

because  $x^i \notin V_k \cap c_{\gamma_{k,d}^+} \cap c\left(\bigcup_{r \neq d} \gamma_{k,r}^-\right)$

As a consequence,  $\tilde{s} \geq P_k \tilde{s}$ , and then  $S_k \neq \emptyset$ .

- As  $q(d, \hat{d}) \geq 0 \quad \forall d, \hat{d}$ , by definition of  $S^d$ , it is verified that

$$S^d(\hat{w}) \geq \hat{w}. \quad (236)$$

Then, if  $x^i \in \gamma_{k,d}^+ \cup \left(\bigcup_{r \neq d} \gamma_{k,r}^-\right)$ , we have

$$\hat{w}_d(x^i) \leq (S^d(\hat{w}))(x^i) = \min((S^d(\hat{w}))(x^i), +\infty) = \min\left((S^d(\hat{w}))(x^i), (D_d^k(\hat{w}_d))(x^i)\right) = (P_k \hat{w})_d(x^i) \quad (237)$$

If  $x^i \in V_k \cap c_{\gamma_{k,d}^+} \cap c\left(\bigcup_{r \neq d} \gamma_{k,r}^-\right)$  then

$$\begin{aligned} (D_d^k \hat{w}_d)(x^i) &= \frac{1}{1+\lambda h(i, k, d)} \left( \hat{w}_d(x^i + h(i, k, d)g(d)) + h(i, k, d) f(x^i) \right) \geq \\ &\geq \frac{1}{1+\lambda h(i, k, d)} \left( -\frac{M_f}{\lambda} - h(i, k, d) M_f \right) = -\frac{M_f}{\lambda} = \hat{w}_d(x^i) \end{aligned} \quad (238)$$

Then, by (236), we have:

$$\hat{w}_d(x^i) \leq \min\left((S^d(\hat{w}))(x^i), (D_d^k(\hat{w}_d))(x^i)\right) = (P_k \hat{w})_d(x^i) \quad (239)$$

and then,  $\hat{w} \in W_k$ .



- We will prove now that any subsolution is smaller than  $\tilde{s}$ .

Let  $w$  be any element of  $W_k$ , then let us denote

$$A_1^d = \gamma_{k,d}^+ \cup \left( \bigcup_{r \neq d} \gamma_{k,r}^- \right) \quad (240)$$

$$A_2^d = V_k \cap c\gamma_{k,d}^+ \cap c\left( \bigcup_{r \neq d} \gamma_{k,r}^- \right) \quad (241)$$

and

$$a_1^d = \max_{A_1^d} (w_d(x^i)) \quad \text{and} \quad a_2^d = \max_{A_2^d} (w_d(x^i)) \quad (242)$$

$$a^d = \max(a_1^d, a_2^d)$$

Then, from (208), (210), (213), (214), we have (using the short notation  $h=h(i, k, d)$ ):

$$a_2^d \leq \frac{1}{1+\lambda h} (a^d + M_f h)$$

and from (210), (213), (214), we have:

$$a_1^d \leq a_2^d + M_q.$$

In consequence it is verified that:

$$a^d \leq a_2^d + M_q.$$

Then,

$$a_2^d \leq \frac{1}{1+\lambda h} (a_2^d + M_q + h M_f), \quad (243)$$

so

$$a_2^d \left( \frac{\lambda h}{1+\lambda h} \right) \leq \frac{M_q + h M_f}{1+\lambda h}. \quad (244)$$

In consequence

$$a_2^d \leq \frac{M_q}{\lambda \eta} + \frac{M_f}{\lambda} \quad (245)$$

and finally:

$$\max_{x^i \in V_k} (w_d(x^i)) = a^d \leq \frac{M_f}{\lambda} + \frac{M_q}{\lambda \eta} + M_q. \quad (246)$$

- We will prove now that any supersolution is lower bounded by  $\hat{w} = -\frac{M_f}{\lambda}$ .

Let  $s$  be any element of  $S_k$ , then

$$s_d(x^i) \geq -\frac{M_f}{\lambda} \quad \forall d, x^i. \quad (247)$$

To prove (247), let us suppose that

$$\underline{s} = \min_{d, x^i} (s_d(x^i)) < -\frac{M_f}{\lambda}$$

and let  $(\hat{d}, x^{\hat{i}})$  such that  $s_{\hat{d}}(x^{\hat{i}}) = \underline{s}$ .

We will see that

$$P_k(s_{\hat{d}})(x^{\hat{i}}) > s_{\hat{d}}(x^{\hat{i}}) \quad (248)$$

and that inequality will be a contradiction to the definition of  $S_k$ .

By (208), we have,  $\forall (d, x^i)$

$$(D_d^k s)(x^i) \geq \frac{1}{1+\lambda h(i, k, d)} \left( \underline{s} - \frac{M_f}{\lambda} \lambda h(i, k, d) \right) = \underline{s} + \frac{\lambda h(i, k, d)}{1+\lambda h(i, k, d)} \left( -\underline{s} - \frac{M_f}{\lambda} \right) > \underline{s}.$$

Also

$$(S^d(s))(x^i) \geq \underline{s} + \hat{a}_0 > \underline{s},$$

then

$$(P_k s)(x^i) \geq \min((S^d(s))(x^i), (D_d^k s)(x^i)) > \underline{s}$$

and we obtain (248)

- By (219) and if  $w, w' \in W_k$ , we have

$$P_k(\max(w, w')) \geq P_k(w) \geq w, \quad (249)$$

$$P_k(\max(w, w')) \geq P_k(w') \geq w', \quad (250)$$

then

$$P_k(\max(w, w')) \geq \max(w, w'). \quad (251)$$

As a consequence,  $\max(w, w') \in W^k$ .

The proof of (231) is entirely similar and it is omitted.

□

**Theorem 4.2:**

- $\exists$  unique maximum element (the maximum subsolution) of  $W_k$  defined by

$$(\bar{U}^k)_d(x^i) = \sup_{w_d \in W_k} (w_d(x^i)) \quad (252)$$

- $\bar{U}^k$  is a fixed point of  $P_k$ , i.e.

$$\bar{U}^k = P_k \bar{U}^k \quad (253)$$

- $\exists$  unique minimum element (the minimum supersolution) of  $S_k$  defined by

$$(\underline{U}^k)_d(x^i) = \inf_{s_d \in S_k} (s_d(x^i)) \quad (254)$$

- $\underline{U}^k$  is a fixed point of  $P_k$ , i.e.

$$\underline{U}^k = P_k \underline{U}^k. \quad (255)$$

**Proof:** From Lemma 4.2 we have that

$$w \leq \bar{s} \quad \forall w \in W_k \quad (256)$$

then

$$(\bar{U}^k)_d(x^i) = \sup_{w \in W_k} (w_d(x^i)) \quad (257)$$

is well defined.

By the monotony of  $P_k$ , we have

$$w \leq P_k w \leq P_k \bar{U}^k \quad (258)$$

then

$$\bar{U}^k(x^i) = \sup_{w \in W_k} (w_d(x^i)) \leq \sup_{w \in W_k} (P_k w)(x^i) \leq (P_k \bar{U}^k)(x^i). \quad (259)$$

In consequence,  $\bar{U}^k \in W_k$  and then it is the maximum element of  $W_k$ .

From (259), we have

$$P_k \bar{U}^k \leq P_k (P_k \bar{U}^k) \quad (260)$$

and then  $P_k \bar{U}^k \in W_k$ . This property implies, in turns,

$$P_k \bar{U}^k \leq \bar{U}^k. \quad (261)$$

Then, by virtue of (259),  $\bar{U}^k \in W_k \cap S_k$  and it is a fixed point of  $P_k$ , i.e.

$$\bar{U}^k = P_k \bar{U}^k.$$

By virtue of (247),

$$(\underline{U}^k)_d(x^i) = \inf_{s \in S_k} (s_d(x^i)) \quad (262)$$

is well defined.

By the monotony of  $P_k$ , we have

$$s \geq P_k s \geq P_k \underline{U}^k, \quad (263)$$

then

$$\underline{U}^k(x^i) = \inf_{s \in S_k} (s_d(x^i)) \geq \inf_{s \in S_k} (P_k s)(x^i) \geq (P_k \underline{U}^k)(x^i). \quad (264)$$

In consequence,  $\underline{U}^k \in S_k$ . From (264), we have

$$P_k \underline{U}^k \geq P_k (P_k \underline{U}^k) \quad (265)$$

and then  $P_k \underline{U}^k \in S_k$ . This property implies, in turns,

$$P_k \underline{U}^k \geq \underline{U}^k. \quad (266)$$

Then, by virtue of (264),

$$\underline{U}^k = P_k \underline{U}^k$$

i.e.,  $\underline{U}^k$  is a fixed point of  $P_k$ . In consequence,  $\underline{U}^k \in W_k \cap S_k$  and is a fixed point of  $P_k$ .

□

4.2.2 The second discrete problem  $P_k^2$  : Find the fixed point of  $P^k$

We have seen above the existence of solution of  $P_k^2$ . The uniqueness of the solution is the consequence of the properties established in the following Lemmas. Theorem 4.3 follows essentially the arguments introduced in [29] to prove the geometric rate of convergence of Bensoussan–Lions Algorithm for QVI.

**Lemma 4.3:**  $P_k$  is concave, i.e. for  $\theta \in [0, 1]$

$$u = \theta w + (1 - \theta) v \quad (267)$$

implies

$$P_k u \geq \theta (P_k w) + (1 - \theta) P_k v \quad (268)$$

Proof: As  $(D_d^k w_d)$  is a linear function, we have

$$D_d^k(\theta w + (1 - \theta)v) = \theta D_d^k w_d + (1 - \theta) D_d^k v_d. \quad (269)$$

$S^d$  is the minimum of linear functions, then it is a concave function, i.e.

$$S^d(\theta w + (1 - \theta)v) \geq \theta(S^d w) + (1 - \theta)(S^d v), \quad (270)$$

then,

$$\begin{aligned} P_k(\theta w + (1 - \theta)v) &= \min\left(\theta D_d^k w_d + (1 - \theta) D_d^k v_d, S^d(\theta w + (1 - \theta)v)\right) \geq \\ &\geq \min\left(\theta D_d^k w_d + (1 - \theta) D_d^k v_d, \theta(S^d w) + (1 - \theta)(S^d v)\right) \geq \theta P_k w + (1 - \theta) P_k v \end{aligned} \quad (271)$$

□

Lemma 4.4:

$$\forall \rho > 0 \exists \underline{w}(\rho) \in \bar{F}_k, \bar{w}(\rho) \in \bar{F}_k, 0 < \mu(\rho) \leq 1, \quad (272)$$

such that

$$\forall u, v \in \bar{F}_k / \|u\| \leq \rho, \|v\| \leq \rho, \quad (273)$$

$$\underline{w} \leq u \leq \bar{w}, \quad (274)$$

$$\underline{w} \leq v \leq \bar{w}, \quad (275)$$

$$\underline{w} + \mu(\bar{w} - \underline{w}) \leq P_k \underline{w}, \quad (276)$$

$$\bar{w} \geq P_k \bar{w}. \quad (277)$$

Proof: Let be

$$\underline{w}_d(x^i) = \min \left( -\rho, -\frac{M_f}{\lambda} \right) \quad \forall d, x^i \quad (278)$$

$$\bar{w}_d(x^i) = \hat{s}_d(x^i) + \rho. \quad (279)$$

It is obvious that (274) and (275) are verified.

By definition of  $\hat{s}_d(x^i)$ , we have

$$P_k(\hat{s} + \rho) = \min \left( D_d^k(\hat{s} + \rho), S_d^k(\hat{s} + \rho) \right) \quad (280)$$

but

$$S_d^k(\hat{s} + \rho) = S_d^k(\hat{s}) + \rho \quad (281)$$

and

$$D_d^k(\hat{s} + \rho) < D_d^k(\hat{s}) + \rho \quad (282)$$

then

$$P_k(\hat{s} + \rho) \leq \min \left( D_d^k(\hat{s}) + \rho, S_d^k(\hat{s}) + \rho \right) = P_k(\hat{s}) + \rho \leq \hat{s} + \rho = \bar{w} \quad (283)$$

For the proof of (276), we have:

$$(P_k \underline{w}_d)(x^i) \geq \hat{q}_0 + \underline{w}_d(x^i) \quad (284)$$

then, taking

$$0 < \mu \leq \frac{\hat{q}_0}{2 \frac{M_f}{\lambda} + 2\rho + M_q \left( 1 + \frac{1}{\lambda \eta} \right)} \quad (285)$$

we have

$$P_k \underline{w} \geq \underline{w} + \mu(\bar{w} - \underline{w}) \quad (286)$$

□

**Lemma 4.5:** Let  $v, u \in \bar{F}_k$  and  $\underline{w}, \bar{w}, \mu$  given by Lemma 4.4, i.e.

$$\underline{w} \leq u \leq \bar{w} \quad (287)$$

$$\underline{w} \leq v \leq \bar{w} \quad (288)$$

and let  $\theta, \tau$  be such that  $\theta \in [0, 1], \tau \in [0, 1]$  and

$$\tau(\underline{w} - u) \leq v - u \leq \theta(v - \underline{w}), \quad (289)$$

then

$$(1 - \mu) \tau(\underline{w} - P_k u) \leq P_k v - P_k u \leq (1 - \mu) \theta(P_k v - \underline{w}). \quad (290)$$

**Proof:**

$$u \geq v - \theta(v - \underline{w}) = (1 - \theta)v + \theta \underline{w} \quad (291)$$

By the properties of monotony and concavity (219)–(267), we have

$$P_k u \geq (1 - \theta) P_k v + \theta P_k \underline{w}. \quad (292)$$

By (276), we have

$$P_k u \geq (1 - \theta) P_k v + \theta (\underline{w} + \mu(\bar{w} - \underline{w})), \quad (293)$$

then:

$$\begin{aligned} P_k u - P_k v &\geq -\theta P_k v + \theta (\underline{w} + \mu(\bar{w} - \underline{w})) \geq \\ &\geq -\theta P_k v + (\underline{w} + \mu P_k v - \mu \underline{w}) - \theta(-\mu + 1) P_k v + \theta(1 - \mu) \underline{w} \geq (1 - \mu) \theta (\underline{w} - P_k v), \end{aligned} \quad (294)$$

in consequence

$$P_k v - P_k u \leq \theta(1 - \mu) (P_k v - \underline{w}). \quad (295)$$

In the same form, as

$$v \geq u + \tau(\underline{w} - u) = (1 - \tau)u + \tau \underline{w}, \quad (296)$$

we obtain

$$P_k v \geq (1 - \tau) P_k u + \tau P_k \underline{w} \quad (297)$$

and from here, it results

$$(1 - \mu) \tau(\underline{w} - P_k u) \leq P_k v - P_k u. \quad (298)$$

Finally, (295) and (298) prove (290).

□

Theorem 4.3:

$$\bullet \exists \text{ unique } U^k / U^k = P_k U^k \quad (299)$$

$$\bullet \forall w \in \bar{F}_k \quad \lim_{\nu \rightarrow \infty} P_k^\nu w = U^k \quad (300)$$

• the following rate of convergence holds

$$\|P_k^\nu w - U^k\| \leq K(\rho) (1 - \mu(\rho))^\nu \quad (301)$$

where  $0 < \mu(\rho) \leq 1$  and  $K(\rho) = (\|\underline{w}(\rho)\| + \|\bar{w}(\rho)\|)$  are given by Lemma 4.4, for  $\rho = \|w\|$ .

Proof: Let  $u, v$  be arbitrary elements of  $\bar{F}_k$ . Let  $\rho = \max(\|u\|, \|v\|)$  and  $\underline{w}, \bar{w}$  given by Lemma 4.5, i.e.

$$\underline{w} \leq u \leq \bar{w}, \quad \underline{w} \leq v \leq \bar{w}. \quad (302)$$

If we define

$$v' = P_k v, \quad u' = P_k u, \quad \theta' = \theta(1 - \mu), \quad (303)$$

by Lemma 4.4 we have:

$$v' - u' \leq (1 - \mu) \theta (v' - \underline{w}), \quad (304)$$

$$v' - u' \leq \theta' (v' - \underline{w}). \quad (305)$$

In consequence, again by Lemma 4.5 we have

$$P_k v' - P_k u' \leq (1 - \mu) \theta' (P_k v' - \underline{w}), \quad (306)$$

in other words

$$P_k^2 v - P_k^2 u \leq (1 - \mu)^2 \theta (P_k^2 v - \underline{w}); \quad (307)$$

finally by induction we obtain

$$P_k^\nu v - P_k^\nu u \leq (1 - \mu)^\nu \theta (P_k^\nu v - \underline{w}). \quad (308)$$

As  $P_k^\nu v \leq \bar{w}$ , the following inequality holds

$$\overline{\lim} (P_k^\nu v - P_k^\nu u) \leq 0; \quad (309)$$

in a similar way, we can obtain

$$0 \leq \underline{\lim} (P_k^\nu v - P_k^\nu u), \quad (310)$$

then

$$\lim_{\nu \rightarrow \infty} (P_k^\nu v - P_k^\nu u) = 0. \quad (311)$$



We shall prove now that each sequence  $v^\nu, u^\nu$  is convergent. If we take, in particular

$$\hat{v} = \bar{w}, \quad \hat{u} = \underline{w} \quad (312)$$

and we define

$$\hat{v}^\nu = P_k^\nu \bar{w} \quad (313)$$

$$\hat{u}^\nu = P_k^\nu \underline{w}. \quad (314)$$

As

$$\bar{w} \geq P_k \bar{w}, \quad (315)$$

$\hat{v}^\nu$  is decreasing because by the monotony of  $P_k$  and by induction we have

$$\bar{w} \geq P_k^\nu \bar{w} \geq P_k^{\nu+1} \bar{w}. \quad (316)$$

Also, it follows from (276) that  $\hat{u}^\nu$  is increasing and in consequence it is verified that

$$\underline{w} \leq \hat{u}^\nu \leq \hat{u}^{\nu+1} \leq \hat{v}^{\nu+1} \leq \hat{v}^\nu \leq \bar{w}. \quad (317)$$

Then  $\hat{u}^\nu$  and  $\hat{v}^\nu$  are convergent to the same point  $\bar{u}$  because

$$0 \leq \hat{v}^\nu - \hat{u}^\nu \leq (1 - \mu)^\nu (\|\bar{w}\| + \|\underline{w}\|). \quad (318)$$

As  $\left| (v^\nu)_d(x^i) - (u^\nu)_d(x^i) \right| \leq \|\hat{v}^\nu - \hat{u}^\nu\|$ , the rate (301) of convergence holds.

As  $u, v$  are arbitrary, we obtain the uniqueness of the limit and of the fixed point of  $P_k$ .

From (255), we have that  $U^k = P_k U^k$ , then

$$\underline{U}^k = P_k \underline{U}^k = \lim_{\nu \rightarrow \infty} (P_k^\nu \underline{U}^k) = \lim_{\nu \rightarrow \infty} (P_k^\nu \bar{U}^k) = P_k \bar{U}^k = \bar{U}^k \quad (319)$$

and we obtain that the naïve algorithm  $A_0$  converges from any arbitrary initial point  $v \in \bar{F}_k$ .

□

### 4.2.3 Equivalence of Problems $P_k^1$ , $P_k^2$ , $P_k^3$ .

In summary, from the previous theorem, it holds the following result, which established the equivalence of problems  $P_k^1$ ,  $P_k^2$ ,  $P_k^3$ .

**Corollary 4.1:**

1)  $\forall w \in W_k$ ,  $w^\nu$  defined by  $w^{\nu+1} = P_k w^\nu$  verifies

$$w^{\nu+1} \geq w^\nu, \quad (320)$$

$$w^\nu \rightarrow U^k. \quad (321)$$

2)  $\forall s \in S_k$ ,  $s^\nu$  defined by  $s^{\nu+1} = P_k s^\nu$  verifies

$$s^{\nu+1} \leq s^\nu, \quad (322)$$

$$s^\nu \rightarrow U^k. \quad (323)$$

3)  $U^k$  is the minimum supersolution and the maximum subsolution, i.e.  $S_k \cap W_k = U^k$  and

$$\underline{U}^k = \bar{U}^k = U^k = P_k U^k.$$

### 4.3 Numerical Algorithms

#### 4.3.1 Definition of the accelerated algorithm $A_1$

Algorithm  $A_0$ , although convergent, may be dismally slow and in consequence (see [3], [19], [23], [25], [30], [32] for related discussions), generally remains a theoretical one. To accelerate the convergence we define here an algorithm that employs the ideas introduced in [23], extended in [25] and also used also in [24] to compute the solution of production-inventory problems.

Algorithm  $A_1$  essentially consists of a suitable combination of Jacobi or value iteration, and Newton (or policy iteration) methods.

#### Definition

$\epsilon$ -suboptimal discrete controls associated to  $w$

$$A_\epsilon: \bar{F}_k \rightarrow (P(D))^{(m+1) \times N}$$

$$(A_\epsilon(w))_d(x^i) = (B_\epsilon(w))_d(x^i) \cup (C_\epsilon(w))_d(x^i) \quad \forall d \in D, \forall x^i \in V_k \quad (324)$$

where

$$(B_\epsilon(w))_d(x^i) = \{ \bar{d} / (P_k(w))_d(x^i) + \epsilon \geq q(d, \bar{d}) + w_{\bar{d}}(x^i) \} \quad (325)$$

$$(C_\epsilon(w))_d(x^i) = \begin{cases} \{d\} & \text{if } (P_k(w))_d(x^i) + \epsilon \geq (D_d^k(w_d))(x^i) \\ \{\emptyset\} & \text{if } (P_k(w))_d(x^i) + \epsilon < (D_d^k(w_d))(x^i) \end{cases} \quad (326)$$

#### Definition

For any  $A \in (P(D))^{(m+1) \times N}$  such that  $\text{card}((A)_d(x^i))=1$ , we consider the system of linear equations:

$$L(A)u = b(A), \quad (327)$$

defined by the relations:

$$\left| \begin{array}{ll} u_d(x^i) = (D_d^k u_d)(x^i) & \text{if } d \in (A)_d(x^i), \end{array} \right. \quad (328)$$

$$\left| \begin{array}{ll} u_d(x^i) = q(d, \bar{d}) + u_{\bar{d}}(x^i) & \text{if } d \notin (A)_d(x^i), \text{ with } \bar{d} \in (A)_d(x^i). \end{array} \right. \quad (329)$$

**Algorithm A<sub>1</sub>**

Step 0: Set  $\nu = 0$ ,  $\mu = 0$ ,  $w^{0,0} \in \bar{F}_k$ ,  $\epsilon^{0,1} > 0$ ,  $\bar{r} > 0$ ,  $0 < \gamma < 1$ ,  $A^\nu = \{\emptyset\}^{(m+1) \times N}$ .

Step 1: Set  $\mu = \mu + 1$ , compute  $A^{\nu,\mu} = A_{\epsilon^{\nu,\mu}}(w^{\nu,\mu})$  and  $w^{\nu,\mu} = P_k(w^{\nu,\mu-1})$ .

Step 2: If  $A^{\nu,\mu} \neq A^\nu$ , set  $r = 0$ ,  $A^\nu = A^{\nu,\mu}$ ,  $\epsilon^{\nu,\mu+1} = \gamma \epsilon^{\nu,\mu}$  and go to step 1;  
 else, set  $r = r + 1$ ; if  $r < \bar{r}$ , set  $\epsilon^{\nu,\mu+1} = \epsilon^{\nu,\mu}$  and go to step 1;  
 else, go to step 3.

Step 3: Choose any selection  $\hat{A}^{\nu,\mu}$  such that  $(\hat{A}^{\nu,\mu})_d(x^i) \subset (A^{\nu,\mu})_d(x^i)$  and  $\text{card}((\hat{A}^{\nu,\mu})_d(x^i)) = 1$ ,  
 form  
 the system  $L(\hat{A}^{\nu,\mu})u = b(\hat{A}^{\nu,\mu})$ .

Step 4: If  $\det(L(\hat{A}^{\nu,\mu})) = 0$ , set  $A^\nu = \{\emptyset\}^{(m+1) \times N}$  and go to step 1;  
 else, solve the system  $L(\hat{A}^{\nu,\mu})u = b(\hat{A}^{\nu,\mu})$ , and go to step 5.

Step 5: If  $u = P_k u$ , stop, ( $U^k = u$ );  
 else; if  $(\nu=0 \text{ or } u \leq w^{\nu,\mu})$ ; set  $w^{\nu+1,0} = u$ ,  $\epsilon^{\nu+1,0} = \epsilon^{\nu,\mu}$ ,  
 $\nu = \nu + 1$ ,  $\mu = 0$ ,  $A^\nu = \{\emptyset\}^{(m+1) \times N}$ ;  
 else,  $\epsilon^{\nu,\mu+1} = \gamma \epsilon^{\nu,\mu}$ .

Step 6: Set  $r = 0$  and go to step 1.

### 4.3.2 Convergence of the accelerated algorithm $A_1$

**Theorem 4.5:** *Algorithm  $A_1$  converges in a finite number of steps.*

*Proof:* The proof is essentially the same used in [23], [25].

Let us consider in a first place that for any  $\nu$ , the sequence  $w^{\nu, \mu}$  is finite. By virtue of the condition verified in Step 5,

$$u \leq w^{\nu, \mu},$$

functions  $w^{\nu, \mu}$  verify

$$w^{\nu, 0} \leq w^{\nu-1, \mu}$$

because  $w^{\nu, 0} = u$ . If  $w^{\nu, 0}$  is not the fixed point  $U^k$  of  $P_k$ , then by (215) and (219) and taking into account that  $w^{\nu, 0}$  is a supersolution, we have by induction:

$$w^{\nu, 0} > w^{\nu, 1} \geq w^{\nu, 2} \geq \dots \geq w^{\nu, \mu}.$$

Then the sequence  $w^{\nu, 0}$  verifies:

$$w^{1, 0} > w^{2, 0} > \dots > w^{\nu, 0} > \dots$$

The element  $w^{\nu, 0}$  results from solving the linear system  $L(\hat{A}^{\nu, \mu})u = b(\hat{A}^{\nu, \mu})$ . As the set of possible systems (associated to selections  $\hat{A}^{\nu, \mu}$ ) is of finite cardinality, due to the fact that the set  $(P(D))^{(m+1) \times N}$  is finite, the set of possible solutions  $w^{\nu, 0}$  is also finite and the procedure must finish in a finite number of steps, finding the fixed point  $U^k$  of  $P_k$ .

Let us now suppose that for a particular  $\hat{\nu}$ , the sequence  $w^{\hat{\nu}, \mu}$  is infinite. In that case, from Theorem 4.3, the following properties holds:

$$w^{\nu, \mu} \rightarrow U^k \quad \text{if } \mu \rightarrow \infty, \quad (330)$$

and also

$$\epsilon^{\nu, \mu} \rightarrow 0 \quad \text{if } \mu \rightarrow \infty, \quad (331)$$

because if  $\mu \rightarrow \infty$ , the parameter  $\epsilon^{\nu, \mu}$  is decreased by factor  $\gamma$  an infinite number of times.

It can be easily seen that (330) and (331) imply

$$A_{\epsilon^{\nu, \mu}}(w^{\nu, \mu}) \rightarrow A_0(U^k), \quad (332)$$

then,  $\exists \tilde{\mu} / \forall \mu \geq \tilde{\mu}$ , it is

$$A_{\epsilon^{\nu, \mu}}(w^{\nu, \mu}) = A_0(U^k). \quad (333)$$

In that case, the first test in Step 2 would be verified negatively and the algorithm would go through Step 3 and Step 4 because any selection  $A_{\epsilon^{\nu, \mu}}(w^{\nu, \mu})$  of  $A_0(U^k)$  generates an invertible matrix  $L(A_{\epsilon^{\nu, \mu}})$ . Finally, in Step 5, the test  $u = P_k u$  would be positively verified.

In this form, we have seen that the generation of an infinite sequence  $w^{\nu, \mu}$  is impossible and in consequence, algorithm  $A_1$  finishes in a finite number of steps, bringing the fixed point  $U^k$  of  $P_k$ .

□

#### 4.4 Convergence of the Discretization Procedure

For the convergence of the discretization procedure to the solution of the original continuous problem, we have the following result.

**Theorem 4.5:** *Solutions  $U^k$  converge to the value function  $U$ , also the following estimation of the error of approximation holds.*

$$\|U^k - U\| \leq C\sqrt{k}. \quad (318)$$

The proof uses the techniques employed in [23], [25] and is here omitted for sake of brevity. In [1]–[2], more tight results of convergence are proved.

#### 4.5 Applications

We have applied the above presented numerical procedure to an example with  $m=2$  items and  $\lambda=0.0198/\text{month}$ .

The remaining data are:

$r_1 = 100$	$r_2 = 400$
$p_1 = 600$	$p_2 = 600$
$h_1 = 0.1$	$h_2 = 0.2$
$M_1 = 200$	$M_2 = 200$
$q_{01} = 15$	$q_{02} = 6$
$q_{10} = 0$	$q_{12} = 6$
$q_{20} = 0$	$q_{21} = 15$

The discretization of the set  $Q$  comprises  $30 \times 30$  nodes. The results for the optimal trajectory obtained are shown in Figure 5. These results were obtained in a PC 486 using the special procedure introduced in [2] and a computational time of 1,64 sec. Comparison between computational times corresponding to algorithms  $A_0$  and  $A_1$  (times obtained with a supercomputer NEC SX-1E are shown in the following Table.

N	$A_0$	$A_1$
$30 \times 30$	5'34"	10.8"
$50 \times 50$	19'52"	13.7"
$100 \times 100$	1 <sup>h</sup> 55'	1'16"
$200 \times 200$	—	7'25"
$290 \times 290$	—	22'10"

## CONCLUSIONS

This paper has analyzed the problem of the optimal schedule of a production system comprising a multi-item single machine.

As a first result, we show that there exists at least an optimal control policy which realizes the minimum cost.

The solution has been found in terms of the optimal cost function. This function has some peculiar regularity properties. We have established that the optimal cost function is locally Lipschitz continuous and unbounded, being the Lipschitz coefficient independent of the discount rate.

Using dynamic programming techniques, we have studied the optimal cost function  $U^d$  through a system of Quasi-Variational-Inequalities (QVI), we have characterized it as the maximum subsolution of this QVI system. We have analyzed in this paper the structure of this QVI system, its solutions and numerical methods to solve it in a practical way.

We have presented a discretization procedure for the numerical solution of the (QVI) system associated to the problem. We have given explicit error estimates and we have shown that the solution of the discrete QVI is reduced to find the unique fixed point of a non-linear operator  $P^h: \mathcal{R}^\eta \rightarrow \mathcal{R}^\eta$ . There are special techniques to compute the fixed point of operator  $P^k$ . In particular, we have developed a special method that converges in a finite number of steps. Other numerical methods with higher performance have been devised in [1], [2].

We have obtained satisfactory numerical results for the case  $m = 2$  (size of discretization of the state space:  $50 \times 50$ ; memory requirements: 383 Kbytes; computational time: 1,64 sec. in a PC 486 computer).

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### Preliminaries for the proof of Lemma 2.1

To prove Lemma 2.1 we need some auxiliary results established in the following Propositions.

**Proposition A1:** *Let be*

$$\zeta = \left(1 - \frac{r_1}{p_1} - \frac{r_2}{p_2}\right), \quad (334)$$

$$T = \max \left( \frac{2}{\zeta} \left( \frac{M_1}{p_1} + \frac{M_2}{p_2} \right), 2 \left( \max_i \left( \frac{M_i}{r_i} \right) \right) \right), \quad (335)$$

$x \in Q$ ,  $d \in D$  and  $\alpha(\cdot) \in \mathcal{A}_x^d$ , any admissible control policy. Then,  $\forall \hat{d} \in D$  it is verified:

$$m \left\{ t \in [0, T] / \alpha(t) = \hat{d} \right\} \geq \min \left( \frac{M_1}{p_1} + \frac{M_2}{p_2}, \min_{i=1,2} \left( \frac{M_i}{r_i} \right) \right). \quad (336)$$

Proof: We denote,  $\forall i \in D$

$$\tau_i = m \left\{ t \in [0, T] / \alpha_{x,d}(t) = i \right\}.$$

The state  $y$  must always verify,  $\forall i \neq 0$

$$0 \leq y_i \leq M_i. \quad (337)$$

But for the  $i^{\text{th}}$  component, we have

$$y_i = x_i + (p_i - r_i) \tau_i - r_i (T - \tau_i);$$

then, by (337) we get

$$\tau_i p_i \geq T r_i - x_i.$$

Now, taking into account that

$$0 \leq x_i \leq M_i,$$

we get

$$\tau_i \geq \frac{r_i}{p_i} T - \frac{M_i}{p_i} = \frac{r_i}{p_i} \left( T - \frac{M_i}{r_i} \right). \quad (338)$$

By virtue of (345) we have

$$T \geq 2 \frac{M_i}{r_i}, \quad (339)$$

then by (338)–(339) there exists at least a set  $\left\{ t \in [0, T] / \alpha(t) = i \right\}$  with measure  $\tau_i \geq \frac{M_i}{r_i}$ .

If we consider the auxiliary variable  $z = \frac{x_1}{p_1} + \frac{x_2}{p_2}$ , we can easily verify that if the control  $d = 1$  is applied, we have

$$\dot{z} = \frac{p_1 - r_1}{p_1} - \frac{r_2}{p_2} = \zeta.$$

If the control  $d = 2$  is applied, we have

$$\dot{z} = -\frac{r_1}{p_1} + \frac{(p_2 - r_2)}{p_2} = \zeta, \quad (340)$$

then  $\dot{z}$  is constant when the control applied is  $d \neq 0$ .

If the control applied is  $d = 0$ , we have

$$\dot{z} = -\frac{r_1}{p_1} - \frac{r_2}{p_2} = \zeta - 1 < 0. \quad (341)$$



By (337),  $0 \leq z \leq \frac{M_1}{p_1} + \frac{M_2}{p_2}$ , then, from (340), (341) we get

$$\frac{M_1}{p_1} + \frac{M_2}{p_2} \geq z(t) = z(0) + \zeta(T - \tau_0) + (\zeta - 1)\tau_0 \geq \zeta(T - \tau_0) + (\zeta - 1)\tau_0 \geq \zeta T - \tau_0.$$

Finally, taking into account that by (345) it is verified

$$T \geq \frac{2}{\zeta} \left( \frac{M_1}{p_1} + \frac{M_2}{p_2} \right),$$

it results

$$\tau_0 \geq \left( \frac{M_1}{p_1} + \frac{M_2}{p_2} \right). \quad (342)$$

In consequence the inequality (336) is verified.

□

**Proposition A2:** Let us define

$$\hat{e} = (M_1, M_2, \dots, M_m), \quad (343)$$

$$E_a = \left\{ x \in Q / \|x - \hat{e}\| < a \right\}. \quad (344)$$

Let  $\bar{\alpha}(\cdot)$  be an optimal control for the initial conditions  $(x, d)$  and  $y(\cdot)$  the corresponding trajectory. If

$$T = \max \left( \frac{2}{\zeta} \left( \frac{M_1}{p_1} + \frac{M_2}{p_2} \right), 2 \left( \max_i \left( \frac{M_i}{r_i} \right) \right) \right), \quad (345)$$

then  $\exists a > 0$ , such that  $\exists t \in [T, 2T]$ , verifying  $y(t) \notin E_a$ .

**Proof:** The proof will be done by reductio ad absurdum. Let us then suppose that

$$\|y(t) - \hat{e}\| < a, \quad \forall t \in [T, 2T]. \quad (346)$$

Let be  $\gamma < \frac{1}{2}$ . We define,  $\forall \nu \in \mathbb{N}$

$$I_\nu = [T + T(1 - 2^{-\nu+1}), T + T(1 - 2^{-\nu})], \quad (347)$$

$$I_\nu^- = T + T(1 - 2^{-\nu+1}), \quad I_\nu^+ = T + T(1 - 2^{-\nu}). \quad (348)$$

We will proof in a first place that  $\exists \hat{\nu} \in \mathbb{N}$  and  $\hat{t} \in I_{\hat{\nu}}$ , such that

$$\|y(\hat{t}) - \hat{e}\| \geq \gamma^{\hat{\nu}-1} a. \quad (349)$$

From assumption (346), it must be  $\hat{\nu} \geq 2$ .

If property (349) is not verified, we would have:

$$\|y(s) - \hat{e}\| < \gamma^{\nu-1} a \quad \forall s \in I_\nu, \quad \forall \nu = 1, 2, \dots \quad (350)$$

It can be easily verified for that case, that each interval  $I_\nu$  provides a partial cost bounded from below in the form:

$$\int_{I_\nu^-}^{I_\nu^+} f(y(s), \bar{\alpha}(\cdot)) e^{-\lambda s} ds + \sum_{\theta_p \in I_\nu} e^{-\lambda \theta_p} q(\theta_{p-1}, \theta_p) \geq \frac{C}{2^{\nu-1} \gamma^{\nu-1} a}. \quad (351)$$

In that case, the total cost would be unbounded, because by virtue of (351):

$$J(\bar{\alpha}(\cdot)) \geq \sum_{\nu=1}^{\infty} \int_{I_{\nu}^{-}}^{I_{\nu}^{+}} f(y(s), \bar{\alpha}(\cdot)) e^{-\lambda s} ds + \sum_{p=1}^{\infty} e^{-\lambda \theta_p} q(\theta_{p-1}, \theta_p) \geq \sum_{\nu=1}^{\infty} \frac{C}{(2\gamma)^{\nu-1} a} \geq \sum_{\nu=1}^{\infty} \frac{C}{a} = +\infty. \quad (352)$$

Then, (349) is verified for a pair  $(\hat{t}, \hat{\nu})$  and we can modify the original control policy  $\bar{\alpha}(\cdot)$  in the form shown in Figure A1, to obtain a new policy  $\alpha'(\cdot)$  with associated trajectory  $y'(\cdot)$  having the properties:

$$\bullet y'(\hat{t}) = y(\hat{t}) \quad (353)$$

$$\bullet \alpha'(s) = \bar{\alpha}(\cdot) \quad \forall s \geq \hat{t} \quad (354)$$

$$\bullet \int_{\hat{T}}^{\hat{t}} f(y'(s), \alpha'(s)) e^{-\lambda s} ds + \sum_{T \leq \theta'_p \leq \hat{t}} e^{-\lambda \theta'_p} q(\theta'_{p-1}, \theta'_p) \leq c (1 + \hat{\nu} (\log a)^{-}). \quad (355)$$

From (351), we have for the original policy  $\bar{\alpha}(\cdot)$

$$\begin{aligned} J(\bar{\alpha}(\cdot)) &\geq \int_0^T f(y(s), \bar{\alpha}(\cdot)) e^{-\lambda s} ds + \sum_{\theta_p \leq T} e^{-\lambda \theta_p} q(\theta_{p-1}, \theta_p) + \\ &\quad + \int_0^{\hat{t}} f(y(s), \bar{\alpha}(\cdot)) e^{-\lambda s} ds + \sum_{\theta_p \leq \hat{t}} e^{-\lambda \theta_p} q(\theta_{p-1}, \theta_p) \geq \\ &\geq \int_0^T f(y(s), \bar{\alpha}(\cdot)) e^{-\lambda s} ds + \sum_{\theta_p \leq T} e^{-\lambda \theta_p} q(\theta_{p-1}, \theta_p) + \sum_{\nu=1}^{\hat{\nu}-1} \frac{C}{2^{\nu-1} \gamma^{\nu-1} a} \geq \\ &\geq \int_0^T f(y(s), \bar{\alpha}(\cdot)) e^{-\lambda s} ds + \sum_{\theta_p \leq T} e^{-\lambda \theta_p} q(\theta_{p-1}, \theta_p) + (\hat{\nu}-1) \frac{C}{a}. \end{aligned} \quad (356)$$

As  $\bar{\alpha}(\cdot)$  is optimal, we have  $J(\bar{\alpha}(\cdot)) \leq J(\alpha'(\cdot))$ . Now, taking into account (353), (354), (355), (356), we get

$$(\hat{\nu}-1) \frac{C}{a} \leq c (1 + (\hat{\nu}-1) (\log a)^{-}). \quad (357)$$

Then

$$1 \leq \frac{c}{C} \left( (a ((\log a)^{-}) - \frac{a}{(\hat{\nu}-1)}) \right)$$

and this inequality is not true if we take "a" small enough such that

$$a (\log a)^{-} < \frac{1}{2} \frac{C}{c}.$$

The absurd comes from (346), then  $\exists t \in [T, 2T] / \|y(t) - \hat{e}\| < a$ .

□

**Proposition A3:** *Let be*

$$T = \max \left( \frac{2}{\zeta} \left( \frac{M_1}{p_1} + \frac{M_2}{p_2} \right), 2 \left( \max_i \left( \frac{M_i}{r_i} \right) \right) \right), \quad (358)$$

*then  $\forall x \in Q, \forall d \in D$  and for any optimal control policy  $\bar{\alpha}(\cdot)$ , the number of switching points in  $[0, T]$  of  $\bar{\alpha}(\cdot)$ , denoted with  $\nu_c$ , verifies*

$$\nu_c \leq \frac{C}{q_0} \left( 1 + (\log \|x\|)^- \right). \quad (359)$$

**Proof:** Let  $x \in Q, d \in D$  and  $\bar{\alpha}(\cdot)$  an optimal policy corresponding to those initial conditions with  $y(\cdot)$  the associated trajectory. From the dynamic programming principle, we have  $\forall \tau$

$$U_d(x) = J(\bar{\alpha}(\cdot)) = I + S + e^{-\lambda\tau} U_d(y(\tau)) \quad (360)$$

where

$$I = \int_0^\tau e^{-\lambda s} f(y(s), \bar{\alpha}(s)) ds \quad (361)$$

$$S = \sum_{\nu=1}^{\nu(\tau)} e^{-\lambda \bar{\theta}^\nu} q(d_{\nu-1}, d_\nu). \quad (362)$$

Let  $\tau \in [T, 2T]$ , such that  $y(\tau) \notin E_a$  ( $\tau$  exists by virtue of Proposition A2) then,

$$U_d(y(\tau)) \geq \inf_{z \in \mathbb{C} E_a} U_d(z). \quad (363)$$

In consequence, it is possible to find  $\hat{z} \notin E_a$  such that

$$U_d(y(\tau)) \geq U_d(\hat{z}). \quad (364)$$

In addition, it is easy to see that there exists a suitable chosen control policy  $\alpha'(\cdot)$  with a number  $\nu'_c$  of switching points in the interval  $[0, \tau]$  bounded by

$$\nu'_c \leq \omega \left( 1 + (\log \|x\|)^- \right), \quad (365)$$

that carries the system from  $x$  to  $\hat{z}$  at time  $\tau$ .

From the optimality of  $\bar{\alpha}(\cdot)$ , we have

$$U_d(x) \leq \hat{I} + \hat{S} + e^{-\lambda\tau} U_d(\hat{z}), \quad (366)$$

where

$$\hat{I} = \int_0^\tau e^{-\lambda s} f(y'(s), \alpha'(s)) ds, \quad (367)$$

$$\hat{S} = \sum_{\nu=1}^{\nu'(\tau)} e^{-\lambda \theta^\nu} q(d_{\nu-1}, d_\nu), \quad (368)$$

are the integral and switching costs associated to  $\alpha'(\cdot)$ .

Then

$$0 \leq (\hat{I} - I) + (\hat{S} - S) + e^{-\lambda \tau} (U_d(\hat{z}) - U_d(y(\tau))) \leq (\hat{I} - I) + (\hat{S} - S).$$

For  $I, \hat{I}$ , it is verified

$$\hat{I} \leq I + 4 M_f T,$$

also

$$\hat{S} \leq M_q \omega \left( 1 + (\log \|x\|)^- \right)$$

and

$$S \geq \nu_c \frac{q_0}{2} e^{-2\lambda T}.$$

In consequence,

$$\nu_c \leq \frac{2}{q_0} e^{2\lambda T} \left( M_q \omega \left( 1 + (\log \|x\|)^- \right) + 4 M_f T \right)$$

and as  $\lambda \leq \lambda_0$ , (359) holds.

□

**Proposition A4:** *Let be*

$$T = \max \left( \frac{2}{\zeta} \left( \frac{M_1}{p_1} + \frac{M_2}{p_2} \right), 2 \left( \max_i \left( \frac{M_i}{r_i} \right) \right) \right), \quad (369)$$

*then, there exists  $L > 0$ , such that there are  $m+1$  intervals  $I_i$ ,  $i = 0, 1, \dots, m$ , verifying:*

$$I_i \subseteq \left\{ t \in [0, T] / \bar{\alpha}(t) = i \right\}, \quad m(I_i) \geq L. \quad (370)$$

**Proof:** From Proposition A1,

$$m \left\{ t \in [0, T] / \bar{\alpha}(t) = \hat{d} \right\} \geq \min \left( \left( \frac{M_1}{p_1} + \frac{M_2}{p_2} \right), \min_{i=1,2} \left( \frac{M_i}{r_i} \right) \right) = \hat{L}$$

and from Proposition A3, the total number of switchings in the interval  $[0, T]$  is bounded by:

$$\nu_c \leq \frac{C}{q_0} \left( 1 + (\log \|x\|)^- \right).$$

For each  $d$ , the set where  $\bar{\alpha}(t) = d$  is a finite union of intervals; then if for each  $d$ , we pick up the interval with maximal length (we denote it  $\hat{I}_i$ ), we have:

$$m(\hat{I}_i) \geq \hat{L} / \nu_c \quad (371)$$

and then Proposition A4 is proved if we define  $L = \hat{L} / \nu_c$ .

□

**Proposition A5:** *Let  $x \in Q$ ,  $d \in D$  and  $\bar{\alpha}(\cdot)$  an optimal policy with  $y(\cdot)$  the associated trajectory. Then,  $\forall x' \in Q$ , there is a modified control policy  $\alpha'(\cdot)$  with associated trajectory  $y'(\cdot)$  such that*

$$\|y'(s) - y(s)\| \leq \kappa(\|x\|) \|x - x'\|, \quad \forall s \in [0, T], \quad (372)$$

$$m \left\{ s \in [0, T] / \alpha'(s) \neq \bar{\alpha}(s) \right\} \leq \hat{\kappa}(\|x\|) \|x - x'\|, \quad \forall s \in [0, T], \quad (373)$$

where

$$\kappa(\|x\|) = (1 + \eta)^{\hat{\nu}}, \quad (374)$$

$$\hat{\kappa}(\|x\|) = \frac{1}{\hat{\eta}} (1 + \hat{\eta})^{\hat{\nu}}; \quad (375)$$

being

$$\hat{\nu} = \frac{C}{q_0} \left( 1 + (\log \|x\|)^- \right), \quad (376)$$

$$\eta = \max_{i, d} \left| \frac{1}{g_i(d)} \right|, \quad (377)$$

$$\hat{\eta} = M_g \eta. \quad (378)$$

If  $\theta'_\nu$  are the switching points of  $\alpha'$ , then

$$\theta'_\nu \leq \theta_\nu,$$

$$\theta_\nu - \theta'_\nu \leq \frac{1}{M_g} (1 + \hat{\eta})^{\hat{\nu}} \|x - x'\|. \quad (379)$$

Proof: It will be done for  $m=2$ . Let be

$$\tau < T = \max \left( \frac{2}{\zeta} \left( \frac{M_1}{p_1} + \frac{M_2}{p_2} \right), 2 \left( \max_i \left( \frac{M_i}{r_i} \right) \right) \right).$$

From Proposition A3, the control policy  $\bar{\alpha}(\cdot)$  has at most  $\nu_c$  switching points  $\theta_1, \dots, \theta_{\nu_c}$ , in  $[0, T]$ , with  $\nu_c \leq \hat{\nu}$  (376), being this bound independent of the optimal policy  $\bar{\alpha}$  (it depends only on  $\|x\|$ ). In a similar manner, for any  $x \in Q$ , we define the modified control policy  $\alpha'(\cdot)$  through the selection of its switching points  $\theta'_1, \dots, \theta'_{\nu_c}$ , in  $[0, T]$ .

This choice is made in such a way that the corresponding trajectory obtained is admissible, i.e. it verifies the condition  $y'(s) \in Q$ ,  $\forall s \in [0, T]$ . To alleviate the arguments, we will denote

$$\delta_\nu = \max_i |y_i(\theta_\nu) - y'_i(\theta'_\nu)|, \quad (380)$$

$$\mu_\nu = \theta_\nu - \theta'_\nu. \quad (381)$$

We define the switching points  $\theta'_\nu$ ,  $\nu=0,1,\dots$ , in the following way

$$\theta'_0 = \theta_0 = 0, \quad (382)$$

$$\theta'_{\nu+1} = \theta'_\nu + ((\theta_{\nu+1} - \theta_\nu) - \Delta_\nu)^+, \quad (383)$$

where

$$\Delta_\nu = \eta \delta_\nu. \quad (384)$$

We must prove here that  $\theta'_{\nu+1}$  is admissible, i.e.

$$0 \leq y'_i(\theta'_{\nu+1}) \leq M_i. \quad (385)$$

If we denote with  $y'(t)$  the trajectory with initial condition  $x'$ , we have

$$y'(\theta'_{\nu+1}) = y'(\theta'_\nu) + g(d_\nu)(\theta'_{\nu+1} - \theta'_\nu). \quad (386)$$

We have to analyze some different cases

$$1) \quad ((\theta_{\nu+1} - \theta_{\nu}) - \Delta_{\nu})^+ = 0$$

In this case,  $\theta'_{\nu+1} = \theta'_{\nu}$  and by (386), (385) is verified.

$$2) \quad ((\theta_{\nu+1} - \theta_{\nu}) - \Delta_{\nu})^+ > 0$$

$$\bullet \quad y_i(\theta_{\nu}) \geq y'_i(\theta'_{\nu}) \text{ and } g_i(d_{\nu}) \geq 0$$

In this case:

$$0 \leq y'_i(\theta'_{\nu}) \leq y'_i(\theta'_{\nu+1}) \leq y_i(\theta_{\nu}) + g_i(d_{\nu})(\theta_{\nu+1} - \theta_{\nu}) = y_i(\theta_{\nu+1}) \leq M_i$$

Then, (385) is verified.

$$\bullet \quad y_i(\theta_{\nu}) \geq y'_i(\theta'_{\nu}) \text{ and } g_i(d_{\nu}) < 0$$

In this case, obviously it is:

$$y'_i(\theta'_{\nu+1}) \leq M_i \text{ and } (\theta_{\nu+1} - \theta_{\nu}) > \Delta_{\nu},$$

then

$$y'_i(\theta'_{\nu+1}) = y'_i(\theta'_{\nu}) + g_i(d_{\nu})(\theta_{\nu+1} - \theta_{\nu} - \Delta_{\nu}) =$$

$$= y'_i(\theta'_{\nu}) - g_i(d_{\nu}) \eta \delta_{\nu} + g_i(d_{\nu})(\theta_{\nu+1} - \theta_{\nu}) \geq$$

$$\geq y'_i(\theta'_{\nu}) + \delta_{\nu} + g_i(d_{\nu})(\theta_{\nu+1} - \theta_{\nu}) \geq$$

$$y_i(\theta_{\nu}) + g_i(d_{\nu})(\theta_{\nu+1} - \theta_{\nu}) = y_i(\theta_{\nu+1}).$$

The other cases can be proved in a similar way and we get an admissible control policy  $\alpha'(\cdot)$ .

We will obtain now a recursive estimate for  $\delta_{\nu}$ ,  $\mu_{\nu}$ . Obviously, the initial values are

$$\delta_0 = \max_i |x_i' - x_i|,$$

$$\mu_0 = 0.$$

**Analysis of the differences.**

We define

$$\hat{\eta} = M_g \eta;$$

obviously, it results  $\hat{\eta} > 1$ .

Then, from (383) it results

$$\begin{aligned}\theta'_{\nu+1} &\leq \theta_{\nu+1}, \\ \theta_{\nu+1} - \theta'_{\nu+1} &= (\theta_{\nu+1} - \theta_{\nu}) + (\theta'_{\nu} - \theta'_{\nu+1}) + (\theta_{\nu} - \theta'_{\nu}) \leq \mu_{\nu} + \eta \delta_{\nu}\end{aligned}$$

and in consequence

$$\mu_{\nu+1} \leq \mu_{\nu} + \eta_{\nu} \delta_{\nu}. \quad (387)$$

For a generic component "i", we have

$$y_i(\theta_{\nu+1}) = y_i(\theta_{\nu}) + g_i(d_{\nu})(\theta_{\nu+1} - \theta_{\nu}),$$

$$y'_i(\theta'_{\nu+1}) = y'_i(\theta'_{\nu}) + g_i(d_{\nu})(\theta'_{\nu+1} - \theta'_{\nu});$$

then

$$\|y_i(\theta_{\nu+1}) - y'_i(\theta'_{\nu+1})\| \leq \|y_i(\theta_{\nu}) - y'_i(\theta'_{\nu})\| + M_g |(\theta'_{\nu+1} - \theta'_{\nu}) - (\theta_{\nu+1} - \theta_{\nu})|.$$

As

$$|(\theta'_{\nu+1} - \theta'_{\nu}) - (\theta_{\nu+1} - \theta_{\nu})| \leq \eta \delta_{\nu},$$

we have

$$\delta_{\nu+1} \leq \delta_{\nu}(1 + \hat{\eta}).$$

In consequence

$$\delta_{\nu} \leq \delta_0(1 + \hat{\eta})^{\nu} \quad (388)$$

and

$$\delta_{\nu} \leq (1 + \hat{\eta})^{\nu} \|x' - x\|_{\infty}. \quad (389)$$

From here, taking into account (388), (387), it results

$$\mu_{\nu+1} \leq \eta \delta_0(1 + \hat{\eta})^{\nu} + \eta \delta_0(1 + \hat{\eta})^{\nu-1} + \dots + 0 = \eta \delta_0 \frac{(1 + \hat{\eta})^{\nu+1} - 1}{\hat{\eta}} \leq \frac{\delta_0}{M_g} (1 + \hat{\eta})^{\nu+1}.$$

Then

$$\mu_{\nu} \leq \frac{1}{M_g} (1 + \hat{\eta})^{\nu-1} \|x' - x\|_{\infty}. \quad (390)$$

Finally, we have that

$$\left\{s \in [0, T] / \alpha'(s) \neq \bar{\alpha}(s)\right\} = \bigcup_{\nu=1}^{\nu_c} [\theta'_{\nu}, \theta_{\nu}],$$

then

$$m\left\{s \in [0, T] / \alpha'(s) \neq \bar{\alpha}(s)\right\} \leq \sum_{\nu=1}^{\nu_c} \mu_{\nu} \leq \sum_{\nu=1}^{\nu_c} \frac{(1 + \hat{\eta})^{\nu-1}}{M_g} \|x' - x\|_{\infty} \leq \frac{(1 + \hat{\eta})^{\hat{\nu}}}{\eta} \|x' - x\|_{\infty}.$$

We have seen in this way that in a fixed time  $T$ , it is possible to modify the controls applied to the system starting at neighboring points in such a way that the difference between trajectories is a Lipschitz function of the initial position and that inequalities (372)–(379) hold. The trajectories obtained with the perturbation procedure are shown in Figure A2.

□

**Proposition A6:** Let  $x \in Q$  and  $\bar{\alpha}(\cdot)$  an optimal policy for the initial conditions  $(x, 0)$  and  $y(\cdot)$  the associated trajectory. Then,  $\forall x' / \exists j \in \{1, 2\}$  verifying  $x'_i > x_i$  if  $i \neq j$ ,  $x'_j = x_j$ , it is possible to modify the policy  $\bar{\alpha}(\cdot)$ , obtaining a new policy  $\alpha'(\cdot) \in A_x^0$ , with the following properties

$$\exists \delta \leq \left( \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) + \frac{\left( \max_i \frac{1}{p_i} \right)}{\zeta} \right) \|x' - x\|_\infty,$$

$$\exists \tau \leq T = \max \left( \frac{2}{\zeta} \left( \frac{M_1}{p_1} + \frac{M_2}{p_2} \right), 2 \left( \max_i \left( \frac{M_i}{r_i} \right) \right) \right), \quad (391)$$

such that

$$\bullet \quad y'(\tau - \delta) = y(\tau), \quad (392)$$

$$\bullet \quad \alpha'(t) = \alpha(t + \delta) \quad \forall t \geq \tau - \delta, \quad (393)$$

$$\bullet \quad m \left\{ t / t \leq \tau - \delta, \alpha(t) \neq \alpha'(t) \right\} \leq (\bar{\nu} + 1) \left( \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) + \frac{\left( \max_i \frac{1}{p_i} \right)}{\zeta} \right) \|x' - x\|_\infty \quad (394)$$

$$\bullet \quad \|y(t) - y'(t)\| \leq \|x - x'\| \quad \forall t \leq \tau - \delta. \quad (395)$$

Proof: By hypothesis, for the control policy  $\bar{\alpha}(\cdot)$  the set of controls  $d_0, d_1, \dots, d_\nu, \dots$ , starts at  $d_0 = 0$ . We will analyze the different transformations to be done in order to obtain the desired policy  $\alpha'(\cdot)$  with new switching points  $\theta'_\nu$ , in terms of the sequence of controls  $d_\nu$ .

The result of this Proposition is achieved defining for the trajectory starting at  $(x', 0)$  a control policy similar to  $\bar{\alpha}(\cdot)$ , except that the switching points  $\theta'_\nu$  are slightly perturbed to obtain, in a time  $\tau < T$ , a trajectory  $y'(\cdot)$  coincident with  $y(\cdot)$  (up to a delay  $\delta$ ).

We will analyze only the case  $x'_2 > x_2$ ,  $x'_1 = x_1$  (situation depicted in Figure A3); obviously, for the other case the proof is entirely similar. We define  $\delta_\nu = y'_2(\theta'_\nu) - y_2(\theta_\nu)$ .

#### Transformations.

**[0]** The control  $d = 0$  is admissible in an interval  $[0, \theta_1]$  for the trajectory starting at  $x$  and for the trajectory starting at  $x'$ . In consequence the same policy is used until the first switching time  $\theta_1$ .

There are two possibilities:

$$\text{[01]} \quad d(\theta_{1+}) = 1$$

$$\text{[02]} \quad d(\theta_{1+}) = 2$$



[01] In this case, the policy  $d = 1$  is admissible for the trajectories with starting points  $y(\theta_1)$  and  $y'(\theta_1)$ .

Moreover, the distance between both trajectories remains invariant.

We define  $\theta'_1 = \theta_1$  and the same policy is used until  $\theta_2$ , where there can be two cases too: [010], [012].

$$[010] \quad d(\theta_{2+}) = 0$$

This case is identical to the initial one, and we return to the case [0], defining

$$\theta'_2 = \theta'_1 + (\theta_2 - \theta_1). \quad (396)$$

$$[012] \quad d(\theta_{2+}) = 2$$

Here, the analysis must be divided into four cases, shown in Figure A4.

[012-a] In this case, we have:

$$\theta_3 - \theta_2 < \frac{(\theta_2 - \theta_1)(p_1 - r_1)}{r_1}. \quad (397)$$

$$\theta_3 - \theta_2 \geq \frac{\delta_1}{\zeta \left( \frac{p_1 p_2}{p_1 - r_1} \right)}, \quad (398)$$

The trajectory  $y'$  is modified, continuing with the control  $d = 1$ , until a time

$$\begin{aligned} \theta'_2 &= \theta'_1 + (\theta_2 - \theta_1) - \frac{\delta_1}{\zeta \left( \frac{p_1 p_2}{p_1 - r_1} \right)} \\ \theta'_3 &= \theta'_2 + (\theta_3 - \theta_2) - \frac{\delta_1}{\zeta \left( \frac{p_1 p_2}{p_1 - r_1} \right)} \end{aligned}$$

From that point on, the control  $d = 2$  is used and the trajectories are coincident (up to a delay  $\theta_3 - \theta'_3$ ), because

$$\delta_3 = y'_2(\theta'_3) - y_2(\theta_3) = y'_2(\theta'_2) - y_2\left(\theta_2 + \frac{\delta_1}{\zeta \left( \frac{p_1 p_2}{p_1 - r_1} \right)}\right) = \delta_1 - \frac{\delta_1}{\zeta \left( \frac{p_1 p_2}{p_1 - r_1} \right)} \zeta \left( \frac{p_2 p_1}{p_1 - r_1} \right) = 0$$

The delay between the switching points has been incremented in the following way:

$$\theta_2 - \theta'_2 = \theta_1 - \theta'_1 + \frac{\delta_1}{\zeta \left( \frac{p_1 p_2}{p_1 - r_1} \right)}$$

[012-b] In this case, we have:

$$\theta_3 - \theta_2 < \frac{(\theta_2 - \theta_1)(p_1 - r_1)}{r_1}. \quad (399)$$

$$\theta_3 - \theta_2 < \frac{\delta_1(p_1 - r_1)}{\zeta(p_1 p_2)}, \quad (400)$$

We eliminate for the trajectory  $y'$  the interval  $[\theta_2, \theta_3]$  with control  $d = 2$ . We define

$$\theta'_3 = \theta'_2 = \theta'_1 = \theta'_1 + (\theta_2 - \theta_1) - (\theta_3 - \theta_2) \frac{r_1}{(p_1 - r_1)}.$$

For these values we have:

$$y_2(\theta_3) = y_2(\theta_2) + (p_2 - r_2)(\theta_3 - \theta_2) = y'_2(\theta'_3) - \delta_1 + (p_2 - r_2)(\theta_3 - \theta_2) - \frac{r_2 r_1}{(p_1 - r_1)}(\theta_3 - \theta_2)$$

then

$$\delta_3 = y'_2(\theta'_3) - y_2(\theta_3) = \delta_1 - (\theta_3 - \theta_2) \frac{p_1 p_2}{(p_1 - r_1)} \left(1 - \frac{r_2}{p_2} - \frac{r_1}{p_1}\right)$$

The system  $y'$  is in  $\theta'_2$  in a similar state as the system " $y$ " do, i.e.

$$y'_1(\theta'_3) = y_1(\theta_3),$$

but now the difference of level  $\delta_3 < \delta_1$  is smaller than before.

The delay between the switching points has been incremented in the following way:

$$\theta_3 - \theta'_3 = \theta_1 - \theta'_1 + (\theta_3 - \theta_2) \frac{r_1}{p_1 - r_1}$$

**[012-c]** In this case, we have:

$$\theta_3 - \theta_2 > \frac{(\theta_2 - \theta_1)(p_1 - r_1)}{r_1}. \quad (401)$$

$$\delta_1 > \left((\theta_3 - \theta_2) - (\theta_2 - \theta_1) \left(\frac{p_1}{r_1} - 1\right)\right) p_2 \quad (402)$$

Here the control  $d = 0$  is used for trajectory  $y'$  until the time  $\theta'_1$ , redefined in the following form

$$\theta'_1 = \theta'_1 + (\theta_3 - \theta_2) - (\theta_2 - \theta_1) \left(\frac{p_1}{r_1} - 1\right)$$

and we define also  $\theta'_3 = \theta'_2 = \theta'_1$ .

For these values we have

$$y'_1(\theta'_3) = y_1(\theta_3),$$

$$y_2(\theta_3) = y'_2(\theta'_3) - \delta_3,$$

with

$$\begin{aligned} \delta_3 &= \delta_1 - \left( \left( (\theta_3 - \theta_2) - (\theta_2 - \theta_1) \left( \frac{p_1}{r_1} - 1 \right) \right) p_2 + (\theta_2 - \theta_1) \left( \frac{p_1}{r_1} - 1 \right) \right) \frac{p_1 p_2}{p_1 - r_1} \zeta < \\ &< \delta_1 - (\theta_3 - \theta_2) \frac{p_1 p_2}{p_1 - r_1} \zeta. \end{aligned}$$

Therefore, we are again in the same case of the analysis **[0]**, but with a smaller difference of position  $\delta_3$ .

The delay between the switching points has been incremented in the following way:

$$\begin{aligned}
 \theta_3 - \theta'_3 &= (\theta_1 - \theta'_1) + (\theta_3 - \theta_2) + (\theta_2 - \theta_1) - \left( (\theta_3 - \theta_2) - (\theta_2 - \theta_1) \left( \frac{p_1}{r_1} - 1 \right) \right) = \\
 &= (\theta_1 - \theta'_1) + (\theta_2 - \theta_1) \left( \frac{p_1}{r_1} \right) < (\theta_1 - \theta'_1) + (\theta_3 - \theta_2) \frac{p_1}{p_1 - r_1}.
 \end{aligned}$$

[012-d] In this case, we have:

$$\theta_3 - \theta_2 \geq \frac{(\theta_2 - \theta_1)(p_1 - r_1)}{r_1}, \quad (403)$$

$$\delta_1 \leq \left( (\theta_3 - \theta_2) - (\theta_2 - \theta_1) \left( \frac{p_1}{r_1} - 1 \right) \right) p_2. \quad (404)$$

The trajectory  $y'$  is modified, continuing with the control  $d = 0$ , until a time

$$\theta'_1 = \theta'_1 + \frac{\delta_1}{p_2}.$$

We define

$$\theta'_2 = \theta'_1,$$

from that point on, the control  $d = 2$  is used and the trajectories are coincident (up to a delay  $\theta_3 - \theta'_3$ ), where we have also defined

$$\theta'_3 = \theta'_2 + (\theta_3 - \theta_2) - (\theta_2 - \theta_1) \frac{(p_1 - r_1)}{r_1} - \frac{\delta_1}{p_2}.$$

Estimation of  $\theta_3 - \theta'_3$ .

The delay between the switching points has been incremented in the following way:

$$\theta_3 - \theta'_3 = (\theta_1 - \theta'_1) + (\theta_2 - \theta_1) \frac{p_1}{r_1} < (\theta_1 - \theta'_1) + (\theta_3 - \theta_2) \frac{p_1}{p_1 - r_1}.$$

which means that modifying the trajectory  $y'$  in the form indicated, we obtain a coincident trajectory with  $y$  except for a delay less than  $\theta_3 - \theta'_3$ .

It remains to analyze the case:

$$[02] d(\theta_1^+) = 2$$

There are two possibilities:

[02-a]

$$\theta_2 - \theta_1 < \frac{\delta_1}{p_2},$$

[02-b]

$$\theta_2 - \theta_1 \geq \frac{\delta_1}{p_2}.$$

These cases are shown in Figures A5a, A5b.

[02-a] for trajectory  $y'$ , the control  $d = 0$  is used until  $\theta'_1 = \theta'_1 + (\theta_2 - \theta_1)$ .

We define  $\theta'_2 = \theta'_1$  and then the analysis starts again like in case [01] but with a difference

$$\delta_2 = y'_2(\theta'_2) - y_2(\theta_2) = \delta_1 - (\theta_2 - \theta_1) p_2 < \delta_1.$$

The delay between the switching points has been incremented in the following way:

$$\theta_2 - \theta'_2 = (\theta_1 - \theta'_1) + (\theta_2 - \theta_1).$$

[02-b] In this case, the control  $d = 0$  is used for  $y'$  until  $\theta'_1 = \theta'_1 + \frac{\delta_1}{p_2}$ .

From that point on, the control  $d = 2$  is used and the trajectories are coincident.

The delay between the switching points has been incremented in the following way:

$$\theta_2 - \theta'_2 = (\theta_1 - \theta'_1) + \frac{\delta_1}{p_2}.$$

Final summary:

We have seen that each time the delay  $(\theta_\nu - \theta'_\nu)$  is incremented by a quantity  $\mu_\nu$ , the difference  $\delta_\nu$  is decremented at least by a quantity  $p_2 \zeta \mu_\nu$ .

The fact that the intervals  $I_\nu$ , where control "2" is applied in the interval  $[T, 2T]$ , cover a set of measure  $L \geq \min\left(\left(\frac{M_1}{p_1} + \frac{M_2}{p_2}\right), \min_{i=1,2}\left(\frac{M_i}{r_i}\right)\right)$ , implies that the construction of the sequence  $\theta_\nu$  finishes at a time  $\tau < T$ , if

$$\|x - x'\| \leq \frac{L}{\left(\max_i \frac{1}{p_i}\right) \frac{1}{\zeta}}.$$

The total delay between the switching points can be estimated in the following way

$$|\theta_\nu - \theta'_\nu| \leq \left(\max_i \frac{1}{p_i}\right) \frac{\|x - x'\|}{\zeta}.$$

□

### Proof of Lemma 2.1

**Lemma 2.1:** Let  $\hat{\eta}$  be given by (378),  $\nu_c$  defined by (359),  $T$  given by (395). If

$$\|x - x'\| \leq \xi^{-\nu_c} \left( L_0 \left( \min_i (r_i) \right), \zeta \frac{L}{\left( \max_i \frac{1}{p_i} \right)} \right),$$

then

$$\forall \alpha(\cdot) \in \hat{\mathcal{A}}_{x,0}^d \quad \exists \alpha'(\cdot) \in \mathcal{A}_x^d, \quad (405)$$

and  $\tau, \tau'$  with the following properties:

$$\bullet \tau \leq T, \quad (406)$$

$$\bullet \tau' \leq T, \quad (407)$$

$$\bullet |\tau - \tau'| \leq \hat{K}(\|x\|) \|x - x'\|, \quad (408)$$

$$\bullet \|y(t) - y'(t)\| \leq \kappa(\|x\|) \|x - x'\| \quad \forall t \leq T, \quad (409)$$

$$\bullet y(\tau) = y'(\tau'), \quad (410)$$

$$\bullet \alpha'(t) = \alpha(t - (\tau' - \tau)) \quad \forall t \geq \tau', \quad (411)$$

$$\bullet m \left\{ t / t \leq \tau', \alpha(t) \neq \alpha'(t) \right\} \leq K(\|x\|) \|x - x'\|, \quad (412)$$

being

$$\bullet K(r) = (\hat{\nu} + 1) \left( \frac{1}{M_g} (1 + \hat{\eta})^{\hat{\nu}} + \left( \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) + \frac{\left( \max_i \frac{1}{p_i} \right)}{\zeta} \right) \kappa(r) \right) + \hat{\kappa}(r),$$

$$\bullet \hat{K}(r) = \left( \frac{1}{M_g} (1 + \hat{\eta})^{\hat{\nu}} + \left( \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) + \frac{\left( \max_i \frac{1}{p_i} \right)}{\zeta} \right) \kappa(r) \right),$$

where  $\kappa(r), \hat{\kappa}(r)$  have been defined in (374)–(375).

Proof: Let  $\alpha(\cdot) \in \hat{\mathcal{A}}_{x,0}^d$  (i.e.  $\alpha(\cdot)$  is an optimal policy) and  $x'$  such that

$$\|x - x'\| \leq \frac{\min_i(r_i) L}{\kappa(\|x\|)}.$$

Using the modification of policies described in Proposition A5, we apply to the system starting at  $x'$  a policy  $\alpha'(\cdot)$  with the following properties:

$$m\left\{s \in [0, \theta'_{\nu}] / \alpha(s) \neq \alpha(s)\right\} \leq \hat{\kappa}(\|x\|) \|x - x'\|, \quad \forall \nu / \theta'_{\nu} \leq T$$

$$\|y'(s) - y(s)\| \leq \kappa(\|x\|) \|x - x'\|, \quad \forall s \in [0, \theta'_{\nu}], \quad \forall \nu / \theta'_{\nu} \leq T$$

$$\theta'_{\nu} \leq \theta_{\nu}, \quad \forall \nu / \theta_{\nu} \leq T$$

$$\theta_{\nu} - \theta'_{\nu} \leq \frac{1}{M_g} (1 + \hat{\eta})^{\hat{\nu}} \|x' - x\|_{\infty}.$$

By virtue of Proposition A4, we can choose  $\theta_{\nu_0} \leq T$  such that

$$\alpha(s) = 0, \quad \forall s \in (\theta_{\nu_0}, \theta_{\nu_0+1}),$$

where  $\theta_{\nu_0+1} = \theta_{\nu_0} + \delta_0$ , with  $\delta_0 \geq L$ .

We apply to the system  $y'$  the control  $\alpha' \equiv 0$  in the interval  $(\theta'_{\nu_0}, \theta'_{\nu_0} + \delta_0 + \delta_x)$ , where

$$\delta_x = \min \left( \frac{y_i(\theta_{\nu_0}) - y'_i(\theta'_{\nu_0})}{g_i(0)} \right).$$

As shown in Figure A6, at time  $\bar{\tau}' = \theta'_{\nu_0} + \delta_0 + \delta_x$  we have

$$y'_2(\bar{\tau}') > y_2(\theta_{\nu_0+1}) \text{ and } y'_1(\bar{\tau}') = y_1(\theta_{\nu_0+1}),$$

or

$$y'_1(\bar{\tau}') > y_1(\theta_{\nu_0}) \text{ and } y'_2(\bar{\tau}') = y_2(\theta_{\nu_0+1}).$$

For the difference of final times we have the estimation

$$\left| \bar{\tau}' - \theta_{\nu_0+1} \right| \leq (\theta_{\nu} - \theta'_{\nu}) + \delta_x \leq \left( \frac{1}{M_g} (1 + \hat{\eta})^{\hat{\nu}} + \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) \kappa(\|x\|) \right) \|x' - x\|_{\infty}.$$

For the difference of trajectories we have the estimation

$$\|y'(s) - y(s)\| \leq \kappa(\|x\|) \|x - x'\|, \quad \forall s \in [\theta'_{\nu_0}, \theta'_{\nu_0} + \delta + \delta_x],$$

because that difference is not incremented in  $(\theta'_{\nu_0}, \theta'_{\nu_0} + \delta_0 + \delta_x)$ .

Also

$$m\left\{s \in [\theta'_{\nu_0}, \theta'_{\nu_0} + \delta + \delta_x] / \alpha'(s) \neq \alpha(s)\right\} \leq \left( \frac{1}{M_g} (1 + \hat{\eta})^{\hat{\nu}} + \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) \kappa(\|x\|) \right) \|x' - x\|_{\infty}.$$

At time  $\bar{\tau}'$ , we are in the conditions of Proposition A6. We apply in consequence the results of

Proposition A6, and we obtain in the interval  $(\theta'_{\nu_0} + \delta_0 + \delta_x, \tau')$  a control  $\alpha'$  with the following properties

- $m\left\{s \in [\bar{\tau}', \tau'] / \alpha(s) \neq \alpha(s)\right\} \leq$   

$$\leq (\bar{\nu}+1) \left( \frac{1}{M_g} (1+\hat{\eta})^{\bar{\nu}} + \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) + \frac{\left( \max_i \frac{1}{p_i} \right)}{\zeta} \right) \kappa(\|x\|)$$
- $\|y(s) - y'(s)\| \leq \kappa(\|x\|) \|x - x'\|, \forall s \in [\bar{\tau}', \tau'],$
- $|\tau - \tau'| \leq \left( \frac{1}{M_g} (1+\hat{\eta})^{\bar{\nu}} + \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) + \frac{\left( \max_i \frac{1}{p_i} \right)}{\zeta} \right) \kappa(\|x\|) \|x' - x\|_{\infty},$
- $y'(\tau') = y(\tau).$

(415)

If we make the concatenation of controls

$$\alpha'|_{(0, \theta'_{\nu_0})} \quad \alpha'|_{(\theta'_{\nu_0}, \theta'_{\nu_0} + \delta_0 + \delta_x)} \quad \alpha'|_{(\theta'_{\nu_0} + \delta_0 + \delta_x, \tau')}$$

we get a global control  $\alpha'|_{(0, \tau')}$  with the desired properties described in (406)-(414):

- $y'(\tau') = y(\tau)$

(392)

- $\alpha'(t) = \alpha(t + \tau - \tau') \quad \forall t \geq \tau',$

(393)

- $m\left\{t / t \leq \tau', \alpha(t) \neq \alpha'(t)\right\} \leq$   

$$\leq \left( (\bar{\nu}+1) \left( \frac{1}{M_g} (1+\hat{\eta})^{\bar{\nu}} + \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) + \frac{\left( \max_i \frac{1}{p_i} \right)}{\zeta} \right) \kappa(\|x\|) + \kappa(\|x\|) \right) \|x' - x\|_{\infty}.$$

(393)

- $\|y(t) - y'(t)\| \leq \kappa(\|x\|) \|x - x'\| \quad \forall t \leq \tau',$

(394)

- $|\tau - \tau'| \leq \left( \frac{1}{M_g} (1+\hat{\eta})^{\bar{\nu}} + \left( \max_i \left| \frac{1}{g_i(0)} \right| \right) + \frac{\left( \max_i \frac{1}{p_i} \right)}{\zeta} \right) \kappa(\|x\|) \|x' - x\|_{\infty} \quad \forall t \leq \tau'.$

The trajectory  $y'(\cdot)$  obtained is shown in Figure A7.

□

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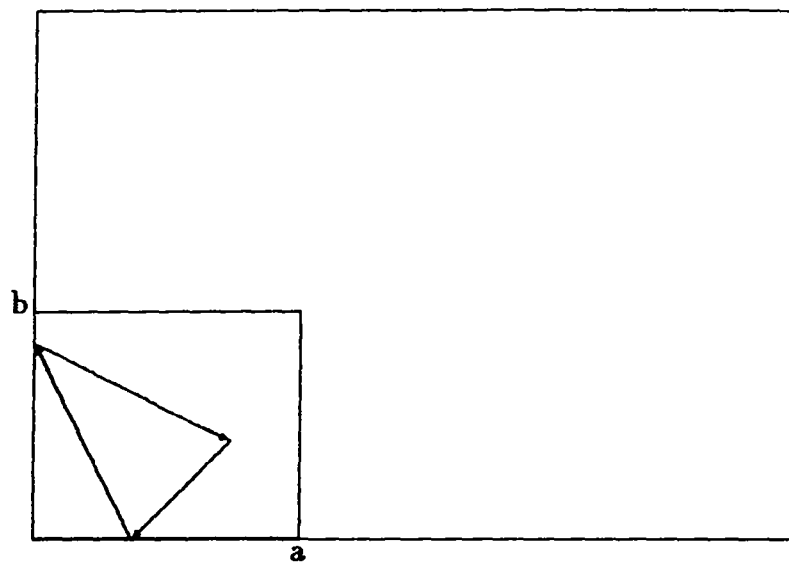


FIGURE 1

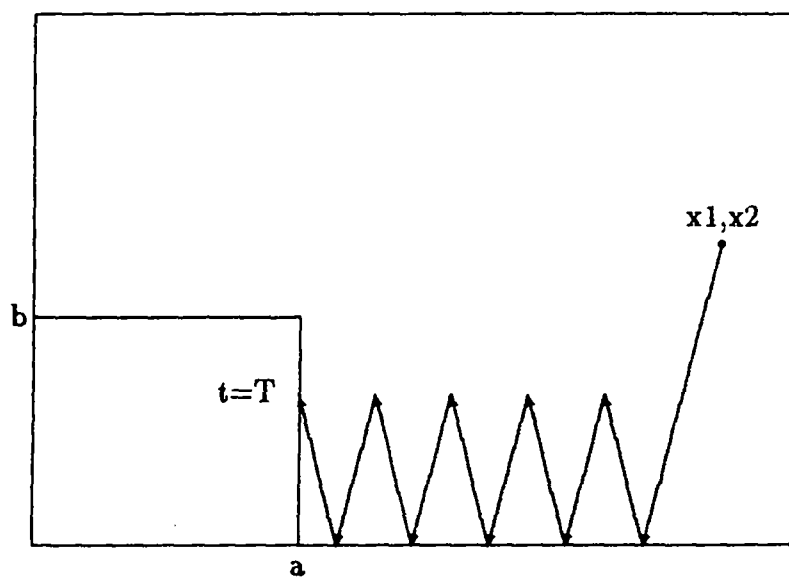


FIGURE 2

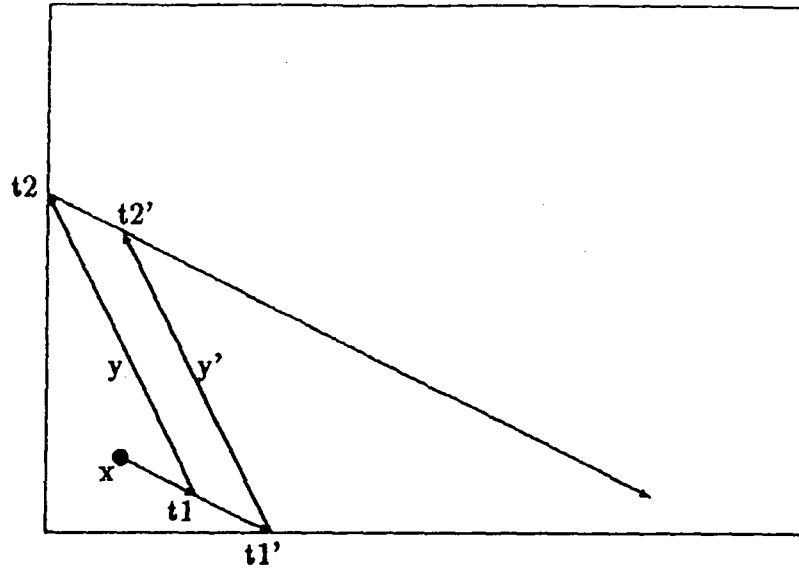


FIGURE 3

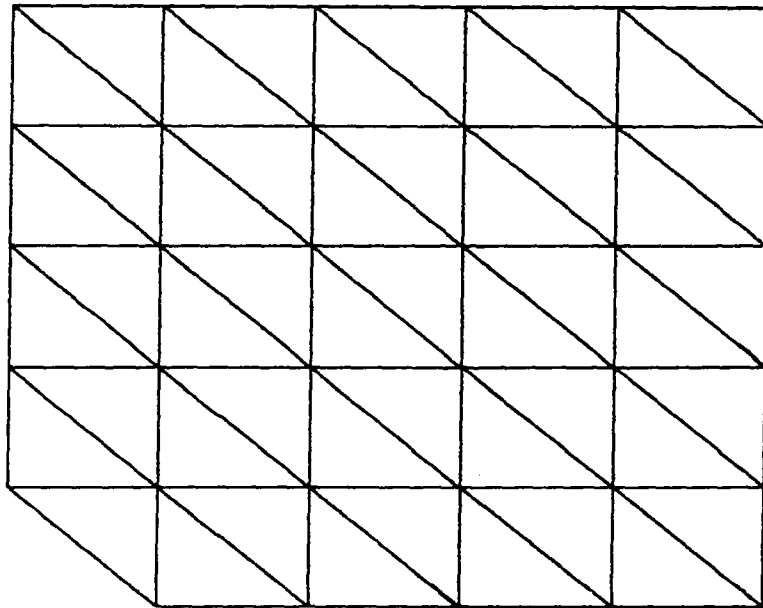


FIGURE 4

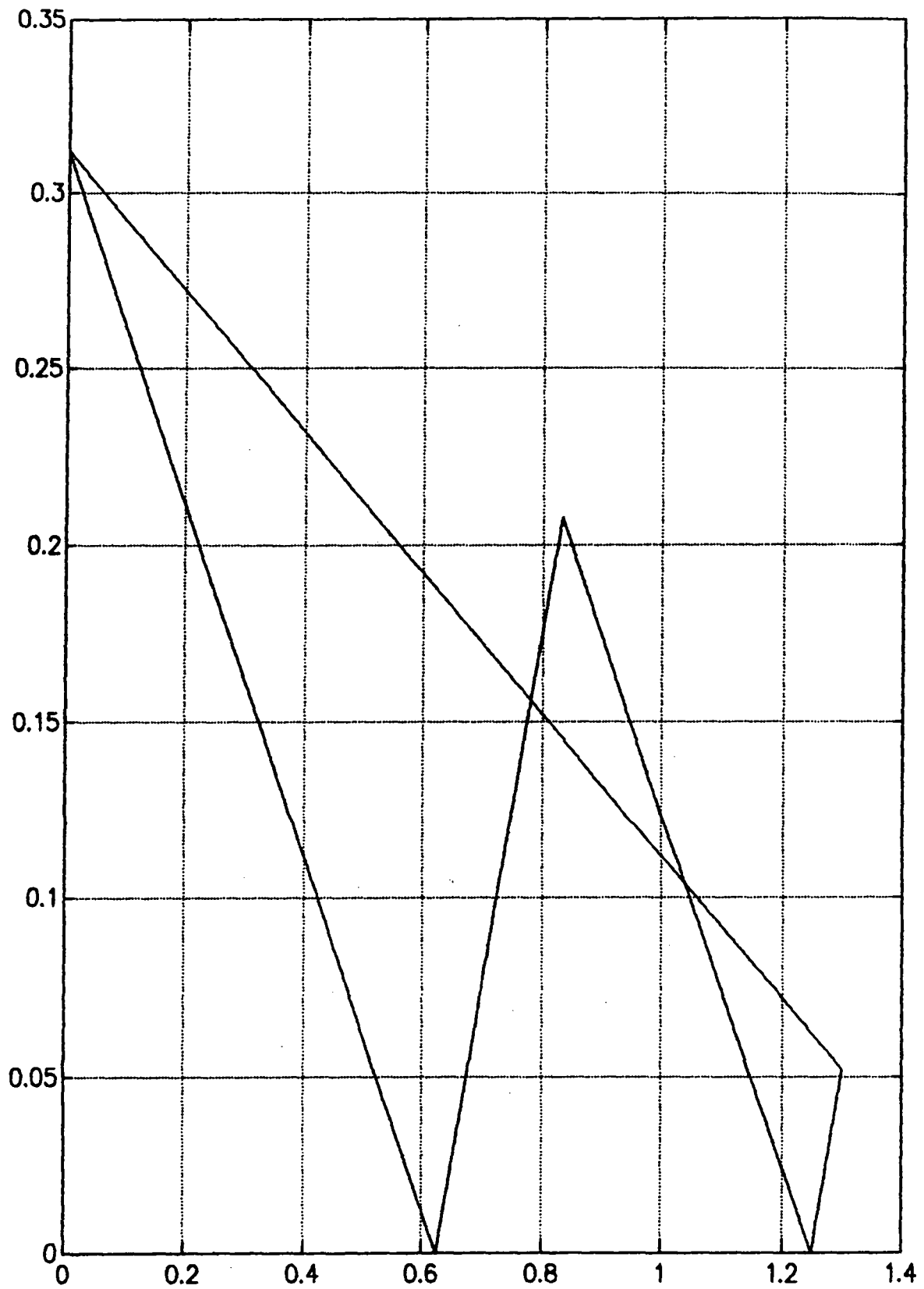


FIGURE 5

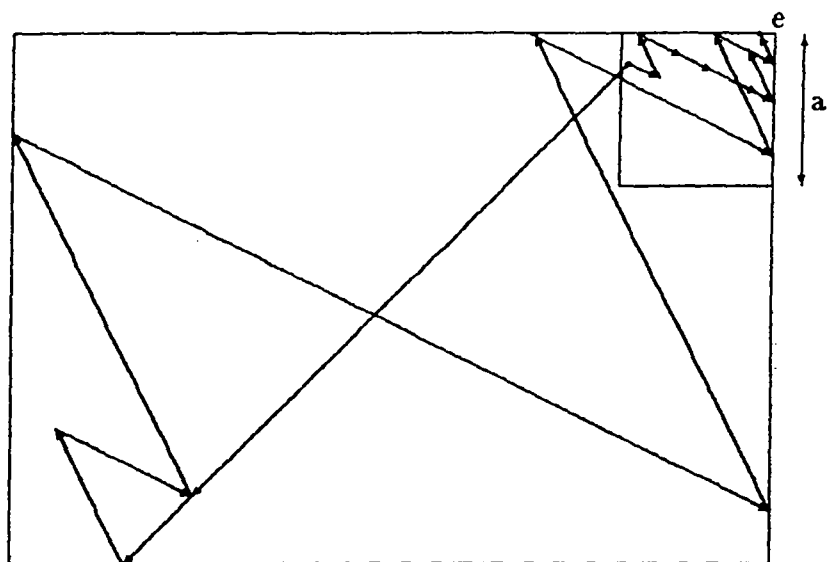


FIGURE A1

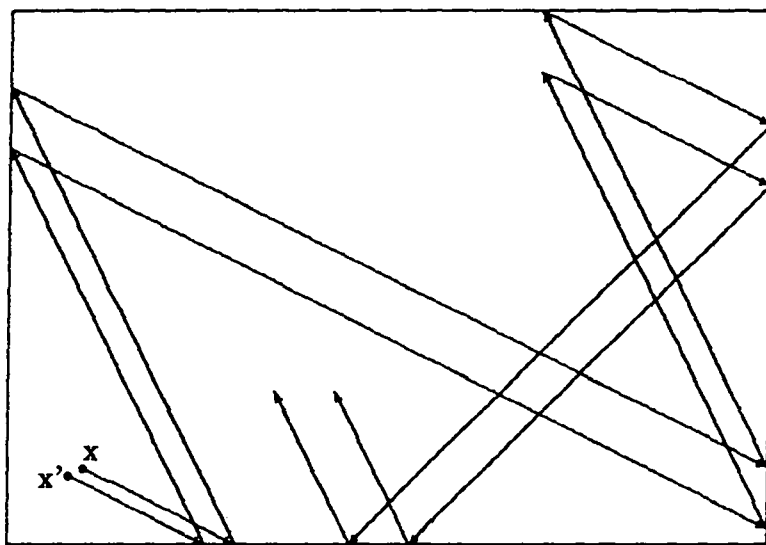


FIGURE A2

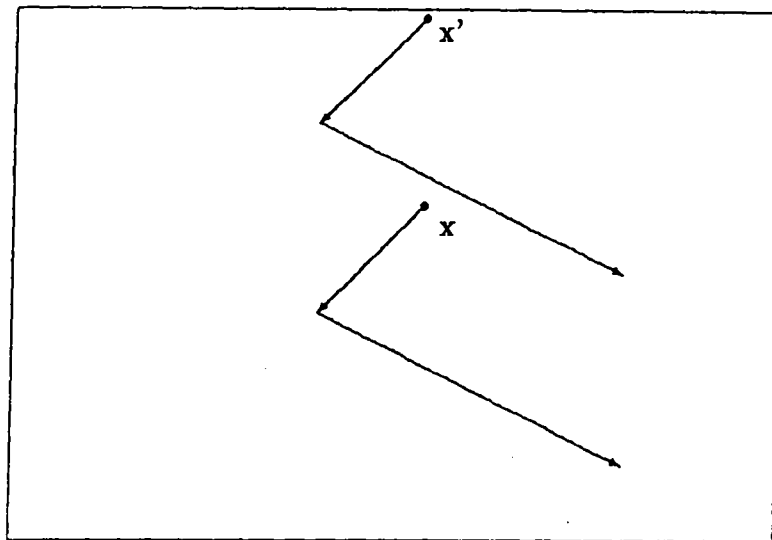


FIGURE A3

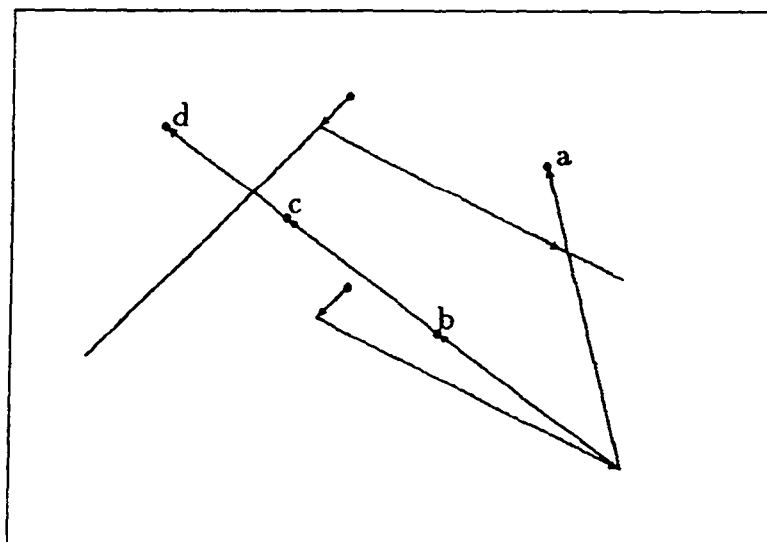


FIGURE A4

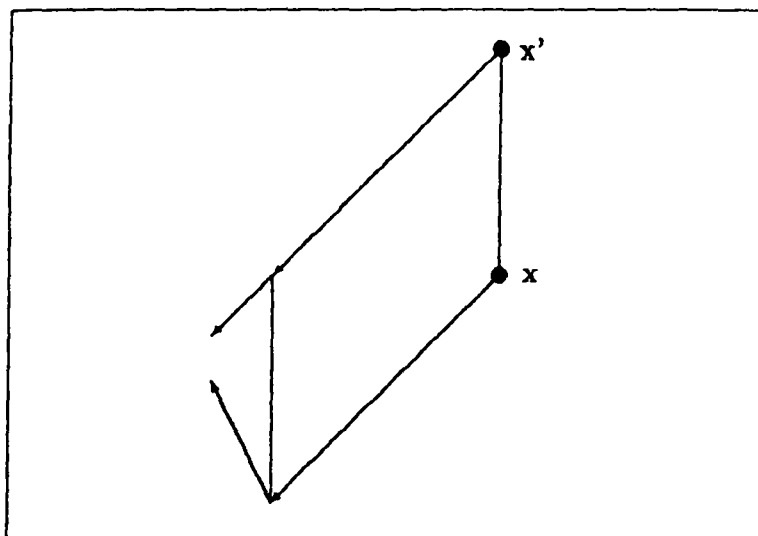


FIGURE A5a

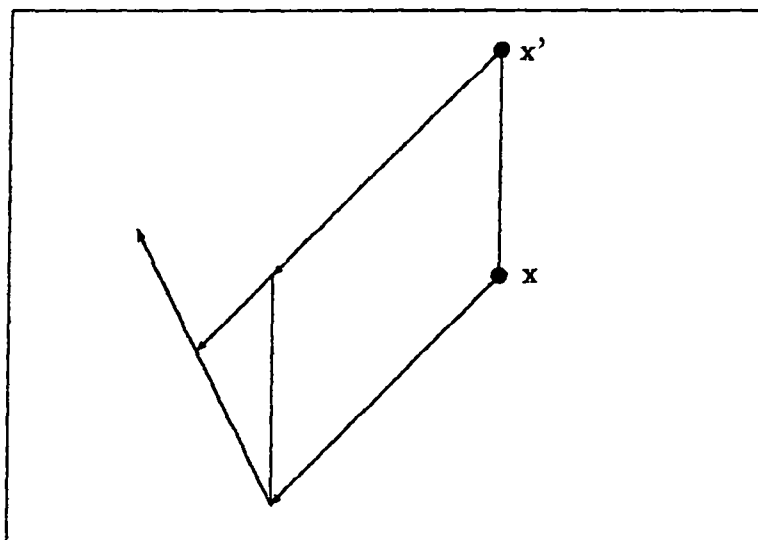


FIGURE A5b

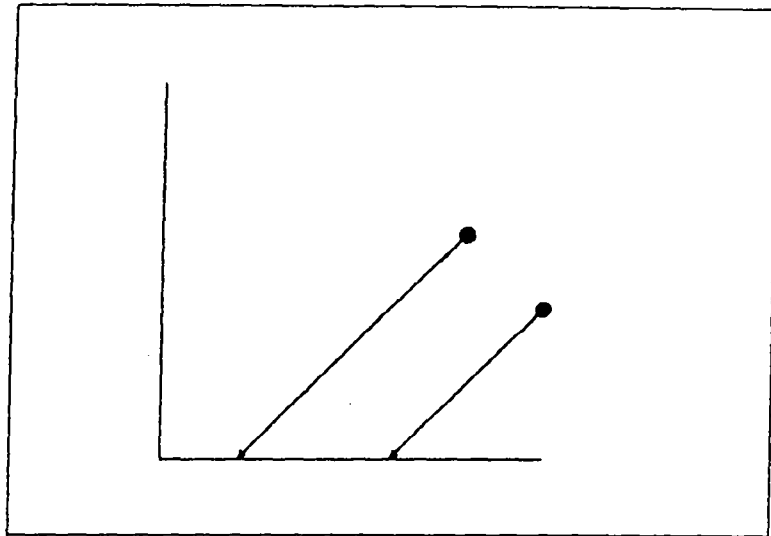


FIGURE A6

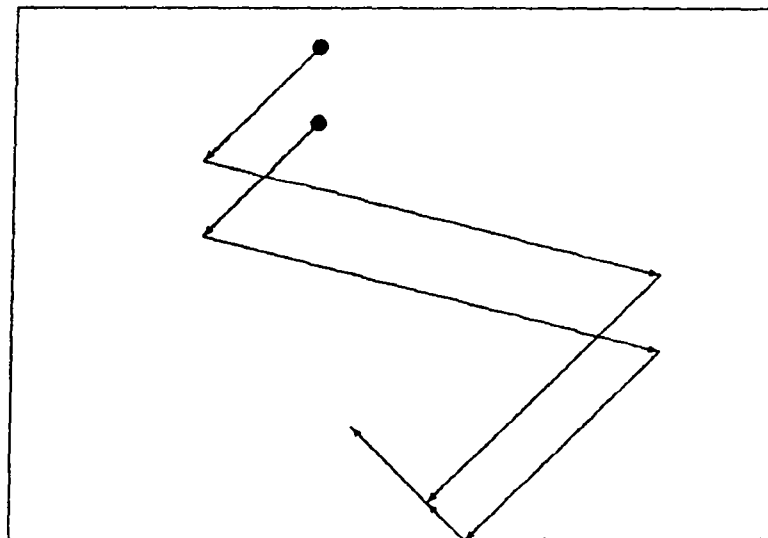


FIGURE A7





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