Compilation of head and strong reduction

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Abstract: Functional language compilers implement only weak-head reduction. However, there are cases where head normal forms or full normal forms are needed. Here, we study how to use cps conversion for the compilation of head and strong reductions. We apply cps expressions to a special continuation so that their head or strong normal form can be obtained by the usual weak-head reduction. We remain within the functional framework and no special abstract machine is needed. Used as a preliminary step our method allows a standard compiler to evaluate under λ's.

Key-words: λ-calculus, Cps conversion, Head and Strong Reduction, Compilation

(Résumé : tsvp)

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Compilation de la réduction forte et de tête

Résumé : Les compilateurs de langages fonctionnels ne mettent en œuvre que la réduction faible. En certaines occasions il est nécessaire d'évaluer des formes normales ou des formes normales de tête. Nous étudions ici comment la transformation cps ("continuation passing style") peut être utilisée pour compiler la réduction forte ou de tête. Une expression cps est appliquée à une continuation particulière et sa forme normale (forte ou de tête) est évaluée par l'habituelle réduction faible. En restant dans le cadre du λ-calcul, nous n'avons pas à concevoir de machine abstraite spécialisée et cette technique permet à un compilateur standard de réduire sous les lambdas.

Mots-clé : λ-calcul, continuations, réduction forte, réduction de tête, compilation.

1 Introduction

Functional language compilers consider only weak-head reduction and the evaluation stops when a weak head normal form (whnf), that is a constant or a λ-abstraction, is reached. In practice, whnf's are considered sufficient because the printable results of programs belong to basic domains. However, there are cases where one would like to reduce under λ's to get head normal forms (hnf) or even (strong) normal forms (nf). Specifically, head/strong reduction can be of interest in:

- program transformations (like partial evaluation) which need to reduce under λ's,
- higher order logic programming like λ-prolog [14] where unification involves reducing λ-terms to normal forms,
- evaluating data structures coded in λ-expressions,
- compiling more efficient evaluation strategies.

A well known tool used to compile (weak) evaluation strategies of functional programs is continuation-passing style (cps) conversion [5][15]. This program transformation makes the evaluation ordering explicit. We see it as a compiling tool since cps expressions can be reduced without any dynamic search for the next redex. Its main advantage is that it stays within the functional framework and thus does not preclude further transformations. Several compilers for strict and non-strict functional languages integrate a cps conversion as a preliminary step [6][10].

Here, we study how to use cps conversion for the implementation of head and strong reductions. To the best of our knowledge, the application of this transformation to such reduction strategies has not been investigated for far. A key property of cps expressions is that their (weak) evaluation is order independent: there is a unique (weak) redex at each reduction step. This property does not hold with strong or head reduction ; a cps expression may have several (strong) redexes. Our approach is to simulate head/strong reductions by weak reductions. Cps expressions are applied to special continuations so that their head/strong normal form can be obtained by the usual weak-head reduction. This way, we still use the only strategy known by compilers (weak reduction), and we retain the key property of cps. The advantage of this approach is that we do not have to introduce a special abstract machine and/or particular structures. It can be used to extend an existing compiler with head/strong reduction capabilities and it enables us to use classical implementation and optimization techniques.

In the following, we assume a basic familiarity with the λ-calculus and cps. In section 2, we introduce some notations, the definitions of the different reduction strategies and cps conversion. We consider in section 3 how to use standard cps conversion to simulate head-reduction of λ-expressions. Section 4 is devoted to strong reduction which involves a minor modification of the technique used for head reduction. In section 5, we envisage a restriction of λ-calculus with a flexible notion of typing which allows a better treatment of head reduction. Section 6 describes how this method could be used to compile more efficient reduction strategies, addresses implementation issues and discusses possible extensions.

2 Preliminaries

One of the application of head reduction being to avoid duplicated or useless computations (see section 6), we will focus on call-by-name. We consider pure λ-calculus and the global λ-expression to be reduced is always assumed to be closed. Given a reduction strategy x, $E \underset{x}{\rightarrow} F$ (resp. $E \overset{i}{\underset{x}{\rightarrow}} F$) reads "E reduces to F after one (resp. i) reduction step by x". The transitive closure of $\underset{x}{\rightarrow}$ is noted $\overset{+}{\underset{x}{\rightarrow}}$ and its transitive, reflexive closure is noted $\overset{*}{\underset{x}{\rightarrow}}$. The three computation rules we are dealing with (i.e. weak head, head and strong reduction) are described in the form of deductive systems.

- Weak head reduction is noted $\underset{w}{\rightarrow}$ and is defined by

$$(\lambda x.E)\ F \underset{w}{\rightarrow} E[F/x] \qquad \frac{E \underset{w}{\rightarrow} E'}{E\ F \underset{w}{\rightarrow} E'\ F}$$

  Closed whnf's are of the form λx.E.

- Head reduction is noted $\underset{h}{\rightarrow}$ and is defined by

$$\frac{E \underset{w}{\rightarrow} E'}{\lambda x_1.\ \ldots\lambda x_n.E \underset{h}{\rightarrow} \lambda x_1.\ \ldots\lambda x_n.E'} \quad n\geq 0$$

  Closed hnf's are of the form $\lambda x_1.\ \ldots\lambda x_n.x_i\ E_1\ldots E_p$ ($1\leq i\leq n$, $p\geq 0$). x_i is called the head variable.

- Strong reduction is noted $\underset{s}{\rightarrow}$ and, with N_i's standing for nf's, is defined by

$$\frac{E \underset{h}{\rightarrow} E'}{E \underset{s}{\rightarrow} E'} \qquad \frac{E_i \underset{s}{\rightarrow} E_i'}{\lambda x_1.\ \ldots\lambda x_n.x_i\ E_1\ldots\ E_i\ N_{i+1}\ldots N_p \underset{s}{\rightarrow} \lambda x_1.\ \ldots\lambda x_n.x_i\ E_1\ldots\ E'_i\ N_{i+1}\ldots N_p}$$

Strong reduction is described as a sequence of head reductions. When a hnf is reached, the arguments of the head variable are reduced in a right to left fashion. Closed normal forms are of the form $\lambda x_1.\ \ldots\lambda x_n.x_i\ N_1\ldots N_p$ with $1\leq i\leq n$ and with $N = x_j \mid \lambda x_1.\ \ldots\lambda x_n.x_k\ N_1\ldots N_p$

$\mathcal{N}$ stands for the standard cps conversion associated with call-by-name [15] and is defined in Figure 1.

$$\mathcal{N}(x) = x$$

$$\mathcal{N}(\lambda x.E) = \lambda c.c\ (\lambda x.\mathcal{N}(E))$$

$$\mathcal{N}(E\ F) = \lambda c.\mathcal{N}(E)\ (\lambda f.f\ \mathcal{N}(F)\ c)$$

Figure 1 Standard Call-by-Name Cps

Variables c and f are supposed not to occur in the source term. The reduction of a cps term consists of a sequence of administrative reductions (i.e. reduction of redexes introduced by the transformation, here redexes of the form (λc.E) F or (λf.E) F), followed by a proper reduction (corresponding to a reduction of the source term), followed by administrative reductions, and so on. The relation induced by administrative reductions is noted $\overset{a}{\underset{w}{\rightarrow}}$, for example:

$\mathcal{N}((\lambda x.E)\ F)\ \mathbf{I} \equiv (\lambda c.(\lambda c.c\ (\lambda x.\mathcal{N}(E)))\ (\lambda f.f\ \mathcal{N}(F)\ c))\ \mathbf{I} \xrightarrow[w]{a} (\lambda x.\mathcal{N}(E))\ \mathcal{N}(F)\ \mathbf{I}$

The following property states that evaluation of cps expressions simulates the reduction of source expressions ; it is proved in [15].

Property 1 *If* $E \xrightarrow[w]{*} W$ *then* $\mathcal{N}(E)\ \mathbf{I} \xrightarrow[w]{*} X \xleftarrow[w]{a} \mathcal{N}(W)\ \mathbf{I}$ *and if W is a whnf then X is a whnf. Furthermore E does not have a whnf iff* $\mathcal{N}(E)\ \mathbf{I}$ *does not have a whnf.*

Cps conversion introduces many new λ-abstractions and affects the readability of expressions. In the remainder of the paper we use the following abbreviations

$$\lambda_c x.E \equiv \lambda c.c\ (\lambda x.E)$$

$$\lambda_c \vec{x}_n\ .\ E \equiv \lambda c.c\ (\lambda x_1.\ \ldots(\lambda c.c\ (\lambda x_n.E))\ldots)$$

$$\vec{X}_n\ \ E \equiv \lambda f.f\ X_1\ (\ldots(\lambda f.f\ X_n\ E)\ \ldots)$$

3 Head Reduction

Since we are interested in compiling, we consider only programs, i.e. closed expressions. A compiler does not know how to deal with free variables ; the expression to be reduced must remain closed throughout the evaluation. Furthermore, in order to use weak head reduction to evaluate hnf's, the leading λ's must be suppressed as soon as the whnf is reached. Our solution is to apply the whnf to combinators so that the associated variables are replaced with closed expressions. The head lambdas disappear, the expression remains closed and the evaluation can continue as before. After the body is reduced to hnf the expression must be reconstructed (i.e. the leading λ's must be reintroduced as well as their variables). We reach a hnf when the head variable is applied (a closed hnf is of the form $\lambda x_1.\ \ldots\lambda x_n.x_i\ E_1\ldots E_n$) so the combinators previously substituted for the leading variables should take care of the reconstruction process.

In general, it is not possible to know statically the number of leading λ's (sometimes called the binder length) of the hnf of an expression. We have to keep track of their number in order to eventually reintroduce them.This complicates the evaluation and reconstruction process. In section 5 we present a means of avoiding this need for counting.

We use the standard call-by-name cps conversion ($\mathcal{N}$). The global cps expression is applied to a recursive continuation Ω and an index $\overline{\mathbf{n}}$ such that $\Omega\ E\ \overline{\mathbf{n}} = E\ \mathbf{H}_n\ \Omega\ \overline{\mathbf{n+1}}$ (Ω, $\mathbf{H}_n$ and $\overline{\mathbf{n}}$ being combinators). Combinators $\overline{\mathbf{n}}$ represent the number of head abstractions already encountered. The weak head reduction of such expressions looks like

$\mathcal{N}(E)\ \Omega\ \overline{\mathbf{n}} \xrightarrow[w]{i} (\lambda c.c\ (\lambda x.F))\ \Omega\ \overline{\mathbf{n}}$	when a cps expression E is evaluated by wh-reduction, its whnf (if any) will be of the form λc.c (λx.F)
$\xrightarrow[w]{} \Omega\ (\lambda x.F)\ \overline{\mathbf{n}}$	the continuation Ω is applied
$\xrightarrow[w]{} (\lambda x.F)\ \mathbf{H}_n\ \Omega\ \overline{\mathbf{n+1}}$	Ω applies the whnf to combinator $\mathbf{H}_n$, Ω and the new index
$\xrightarrow[w]{} F[\mathbf{H}_n/x]\ \Omega\ \overline{\mathbf{n+1}}$	$\mathbf{H}_n$ is substituted for x

The expression remains closed and the evaluation continues, performing the same steps if other whnf's are encountered. Eventually a hnf is reached, that is, a combinator $\mathbf{H}_i$ is in head position and this combinator is responsible for reconstructing the expression.

In fact, we do not apply the global expression directly to Ω but to combinator **A** (defined by **A** E F = F E) whose task is to apply the expression to Ω. This way Ω remains outside the expression and it makes its suppression during the reconstruction process easier. This technical trick is not absolutely necessary but it simplifies things when working within the pure λ-calculus. The reduction steps that occur when a whnf is reached actually are

$$(\lambda c.c\ (\lambda x.F))\ \mathbf{A}\ \Omega\ \overline{\mathbf{n}} \underset{w}{\Rightarrow} \mathbf{A}\ (\lambda x.F)\ \Omega\ \overline{\mathbf{n}} \underset{w}{\Rightarrow} \Omega\ (\lambda x.F)\ \overline{\mathbf{n}} \underset{w}{\Rightarrow} (\lambda x.F)\ \mathbf{H}_n\ \mathbf{A}\ \Omega\ \overline{\mathbf{n+1}}$$

If E is a closed expression, its transformed form will be $\mathcal{N}(E)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ with

$$\mathbf{A}\ M\ N \underset{w}{\Rightarrow} N\ M \qquad (\mathbf{A})$$

$$\Omega\ M\ \overline{\mathbf{n}} \overset{*}{\underset{w}{\Rightarrow}} M\ \mathbf{H}_n\ \mathbf{A}\ \Omega\ \overline{\mathbf{n+1}} \qquad (\Omega)$$

The family of combinators $\mathbf{H}_i$ is defined by

$$\mathbf{H}_i\ M\ N\ \overline{\mathbf{n}} = \lambda_c \vec{x}_n\ .\ \lambda c.x_{i+1}\ (M\ (\mathbf{R}\ \overline{\mathbf{n}}\ (\lambda c.\ \vec{x}_n\ \ c))\ (\mathbf{K}\ c)) \qquad (\mathbf{H})$$

with $$\mathbf{R}\ E\ F\ G\ H\ I \underset{w}{\Rightarrow} \lambda f.f\ (\lambda c.G\ \mathbf{A}\ \Omega\ E\ (F\ c))\ (H\ (\mathbf{R}\ E\ F)\ I) \qquad (\mathbf{R})$$

When the hnf is reached the expression is of the form $\mathbf{H}_i\ (\lambda f.f\ E_1 \ldots (\lambda f.f\ E_n\ \mathbf{A}) \ldots)\ \Omega\ \overline{\mathbf{n}}$, $\overline{\mathbf{n}}$ representing the number of head abstractions of the hnf. The reduction rule of $\mathbf{H}_i$ deletes Ω, reintroduces the n leading λ's and yields

$$\lambda_c \vec{x}_n\ .\ \lambda c.x_{i+1}\ ((\lambda f.f\ E_1 \ldots (\lambda f.f\ E_n\ \mathbf{A}) \ldots)\ (\mathbf{R}\ \overline{\mathbf{n}}\ (\lambda c.\vec{x}_n\ \ c))\ (\mathbf{K}\ c)).$$

Some $\mathbf{H}_i$'s may remain in the continuation of x_{i+1} and the role of **R** is to remove them by applying suitable arguments to each E_i. The final continuation **A** calls **K** which removes the reconstructing expression $\mathbf{R}\ \overline{\mathbf{n}}\ (\lambda c.\vec{x}_n\ \ c)$.

Example: Let E ≡ λx.(λw.λy. w y x) (λz. z) x

Its head reduction is

$$\begin{aligned} E &\underset{h}{\rightarrow} \lambda x.(\lambda y.(\lambda z.\ z)\ y\ x)\ x \\ &\underset{h}{\rightarrow} \lambda x.(\lambda z.\ z)\ x\ x \\ &\underset{h}{\rightarrow} \lambda x.x\ x \end{aligned}$$

After cps conversion and simplification the expression becomes

$$\mathcal{N}(E) = \lambda_c x.\ \lambda c.(\lambda w.\ \lambda_c y.\lambda c.\ w\ (\lambda f.f\ y\ (\lambda f.f\ x\ c)))\ (\lambda_c z.z)\ (\lambda f.f\ x\ c)$$

The weak head reduction of the cps expression simulates the head reduction. Reductions corresponding to head reductions of the source expression are marked by ✣ ; the other being administrative reductions.

$$\mathcal{N}(E)\ \mathbf{A}\ \Omega\ \overline{0} \xrightarrow[w]{} \mathbf{A}\ (\lambda x.\lambda c.(\lambda w.\ \lambda_c y.\lambda c.\ w\ (\lambda f.f\ y\ (\lambda f.f\ x\ c)))\ (\lambda_c z.z)\ (\lambda f.f\ x\ c))\ \Omega\ \overline{\mathbf{0}}$$

$$\xrightarrow[w]{} \Omega\ (\lambda x.\lambda c.(\lambda w.\ \lambda_c y.\lambda c.\ w\ (\lambda f.f\ y\ (\lambda f.f\ x\ c)))\ (\lambda_c z.z)\ (\lambda f.f\ x\ c))\ \overline{\mathbf{0}}$$

$$\xrightarrow[w]{} (\lambda x.\lambda c.(\lambda w.\ \lambda_c y.\lambda c.\ w\ (\lambda f.f\ y\ (\lambda f.f\ x\ c)))\ (\lambda_c z.z)\ (\lambda f.f\ x\ c))\ \mathbf{H}_0\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}$$

$$\xrightarrow[w]{2} (\lambda w.\ \lambda_c y.\lambda c.\ w\ (\lambda f.f\ y\ (\lambda f.f\ \mathbf{H}_0\ c)))\ (\lambda_c z.z)\ (\lambda f.f\ \mathbf{H}_0\ \mathbf{A})\ \Omega\ \overline{\mathbf{1}}$$

$$\xrightarrow[w]{} (\lambda_c y.\lambda c.\ (\lambda_c z.z)\ (\lambda f.f\ y\ (\lambda f.f\ \mathbf{H}_0\ c)))\ (\lambda f.f\ \mathbf{H}_0\ \mathbf{A})\ \Omega\ \overline{\mathbf{1}} \quad ✣$$

$$\xrightarrow[w]{2} (\lambda y.\lambda c.\ (\lambda_c z.z)\ (\lambda f.f\ y\ (\lambda f.f\ \mathbf{H}_0\ c)))\ \mathbf{H}_0\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}$$

$$\xrightarrow[w]{} (\lambda c.\ (\lambda_c z.z)\ (\lambda f.f\ \mathbf{H}_0\ (\lambda f.f\ \mathbf{H}_0\ c)))\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}} \quad ✣$$

$$\xrightarrow[w]{3} (\lambda z.z)\ \mathbf{H}_0\ (\lambda f.f\ \mathbf{H}_0\ \mathbf{A})\ \Omega\ \overline{\mathbf{1}}$$

$$\xrightarrow[w]{} \mathbf{H}_0\ (\lambda f.f\ \mathbf{H}_0\ \mathbf{A})\ \Omega\ \overline{\mathbf{1}} \quad ✣$$

The hnf is reached. Using the definition of $\mathbf{H}_0$ we get

$$\mathbf{H}_0\ (\lambda f.f\ \mathbf{H}_0\ \mathbf{A})\ \Omega\ \overline{\mathbf{1}} \rightarrow \lambda_c x.\lambda c.x\ ((\lambda f.f\ \mathbf{H}_0\ \mathbf{A})\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c)) \equiv \Delta \qquad (\mathbf{H})$$

Now, we show that this hnf Δ is equivalent to (or that the reconstruction yields) the principal hnf $(\lambda x.x\ x)$ in cps form $(\lambda_c x.\lambda c.x\ (\lambda f.f\ x\ c))$.

$$\Delta \rightarrow \lambda_c x.\lambda c.x\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c)\ \mathbf{H}_0\ \mathbf{A}\ (\mathbf{K}\ c))$$

$$\rightarrow \lambda_c x.\lambda c.x\ (\lambda f.f\ (\lambda c.\mathbf{H}_0\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}\ ((\lambda c.\lambda f.f\ x\ c))\ c))\ (\mathbf{A}\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c))) \qquad (\mathbf{R})$$

Since $\lambda c.\mathbf{H}_0\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}\ ((\lambda c.\lambda f.f\ x\ c)\ c) \rightarrow \lambda c.(\lambda_c w.\lambda c.w\ c)\ ((\lambda c.\lambda f.f\ x\ c)\ c)$ (**H**),(**A**),(**K**)

$$\rightarrow \lambda c.x\ c =_\eta x$$

and $\mathbf{A}\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c) \xrightarrow[w]{} c$ (**A**),(**K**)

then $\Delta = \lambda_c x.\lambda c.x\ (\lambda f.f\ x\ c)$ ❑

All reductions taking place in the head reduction of the source expression are performed on the transformed expression by weak head reduction. Here the resulting expression is interconvertible with the principal hnf in cps form. Note that the reconstruction process is not completed by weak head reduction. In a sense, the reconstruction process is lazy ; it can take place (by wh-reduction) only when the resulting expression is applied. Only the required subexpressions will be reconstituted.

The following property states that for any closed expression E the weak head reduction of $\mathcal{N}$(E) **A** Ω $\overline{\mathbf{0}}$ simulates the head reduction of E. If E has a hnf H then the wh-reduction of $\mathcal{N}$(E) **A** Ω $\overline{\mathbf{0}}$ yields an expression equal to $\mathcal{N}$(H) **A** Ω $\overline{\mathbf{0}}$ (after administrative reductions).

Property 2 *Let E be a closed expression, if E* $\xrightarrow[h]{*}$ *H then there exists an expression X such that* $\mathcal{N}$*(E)* **A** Ω $\overline{\mathbf{0}}$ $\xrightarrow[w]{+}$ *X* $\underline{\xleftarrow[w]{a}}$ $\mathcal{N}$*(H)* **A** Ω $\overline{\mathbf{0}}$ *and if H is a hnf then X is a whnf. Furthermore E does not have a hnf iff* $\mathcal{N}$*(E)* **A** Ω $\overline{\mathbf{0}}$ *does not have a whnf.*

Proof. The proofs are described in the annex.

In general $\mathcal{N}$(H) **A** Ω $\overline{\mathbf{0}}$ ↔ $\mathcal{N}$(H) does not hold, namely the result is not always interconvertible with the hnf in cps. This is usual with this kind of transformation ; the result is in compiled form and is convertible to its source version only under certain conditions. We still have a strong relationship between $\mathcal{N}$(H) **A** Ω $\overline{\mathbf{0}}$ and $\mathcal{N}$(H). Let H ≡ $\lambda x_1. \ldots \lambda x_n.x_i\ E_1 \ldots E_p$ then

$$\mathcal{N}(H) = \lambda_c \vec{x}_n\ .\lambda c.\ x_i\ (\mathcal{N}(\vec{E}_p)\ \ c)$$

and $\mathcal{N}$(H) **A** Ω $\overline{\mathbf{0}}$ = $\lambda_c \vec{x}_n\ \ \lambda c.\ x_i\ (\vec{X}_p\ \ E)$ with $X_i \equiv \lambda c.\mathcal{N}(\lambda \vec{x}_n\ \ .E_i)$ **A** Ω $\overline{\mathbf{0}}$ $(\vec{x}_n\ \ c)$.

So, the head variable is the same and if the sub-expressions $\mathcal{N}(E_i)$ and X_i have a hnf they will also have the same head variable. Likewise, if a sub-expression $\mathcal{N}(E_i)$ does not have a hnf then the corresponding expression $\mathcal{N}(\lambda \vec{x}_n\ \ .E_i)$ **A** Ω $\overline{\mathbf{0}}$ $(\vec{x}_n\ \ c)$ does not have a whnf and they can then be considered equivalent. However we do not have a plain equivalence since there are expressions whose subexpressions all have a hnf but have no nf themselves ; for example (λxy.y (x x)) (λxy.y (x x)) → … → (λy.y (λy.y …(λxy.y (x x)) (λxy.y (x x))). For such expressions $\mathcal{N}$(H) **A** Ω $\overline{\mathbf{0}}$ and $\mathcal{N}$(H) are not interconvertible; the $\mathbf{H}_i$'s substituted for the leading variables may never be completely removed. However, for expressions with a normal form the following result holds.

Property 3 *Let E be a closed expression with a normal form then* $\mathcal{N}$*(E)* ↔ $\mathcal{N}$*(E)* **A** Ω $\overline{\mathbf{0}}$

Here, we propose one possible definition of combinators $\overline{\mathbf{n}}$, Ω, $\mathbf{H}_i$, **R** in terms of pure λ-expressions. We do not claim it is the best one ; we just want to show that such combinators can indeed be implemented in the same language. Simpler definitions could be conceived in a less rudimentary language (e.g. λ-calculus extended with constants).

We represent $\overline{\mathbf{n}}$ by Church integers, i.e. $\overline{\mathbf{0}}$ = λfx.x and $\overline{\mathbf{n}}$ = $\lambda fx.f^n\ x$. The successor function $\mathbf{S}^+$ is defined by $\mathbf{S}^+$ = λxyz.y (x y z).

I = λx.x, **K** = λxy.x, **A** = λxy.y x and **Y** = (λxy.y (x x y))(λxy.y (x x y)) (Turing's fixed point combinator)

Ω = **Y** (λwen. e (**H** n) **A** w ($\mathbf{S}^+$ n))

The family $\mathbf{H}_i$ is represented by **H** $\bar{\mathbf{i}}$ with

H = λieon. n **L** (λac.a **I** (**W** i) (e (**R** n a) (**K** c))) **I**

where **W** = λi.i (λxyz.z x) **K**

$\mathbf{L} = \lambda ab.\lambda_c x.a\ (\lambda c.b\ (\lambda f.f\ x\ c))$

$\mathbf{R} = \mathbf{Y}\ (\lambda ruvwxy.\lambda f.f\ (\lambda c.w\ \mathbf{A}\ \Omega\ u\ (v\ c))\ (x\ (r\ u\ v)\ y)$

We can easily check that these definitions imply the reduction rules previously assumed, for example $\Omega\ E\ \overline{\mathbf{n}} \overset{*}{\underset{w}{\Rightarrow}} E\ \mathbf{H}_n\ \mathbf{A}\ \Omega\ \overline{\mathbf{n+1}}$ or $\mathbf{R}\ E\ F\ G\ H\ I \overset{*}{\underset{w}{\Rightarrow}} \lambda f.f\ (\lambda c.G\ \mathbf{A}\ \Omega\ E\ (F\ c))\ (H\ (\mathbf{R}\ E\ F)\ I)$.

4 Strong Reduction

Full normal forms are evaluated by first reducing expressions to hnf and then reducing the arguments of the head variable. We follow the same idea as for head reduction. Instead of instantiating variables by combinators $\mathbf{H}_i$ we use the family $\mathbf{S}_i$ which will carry out the evaluation before reconstructing. The recursive continuation Ω is the same as before except that it applies $\mathbf{S}_i$ instead of $\mathbf{H}_i$ to the λ-abstraction.

If M is a closed expression, its transformed form will be $\mathcal{N}(M)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ with

$$\Omega\ M\ \overline{\mathbf{n}} \underset{w}{\rightarrow} M\ \mathbf{S}_n\ \mathbf{A}\ \Omega\ \overline{\mathbf{n+1}} \qquad (\Omega)$$

with $$\mathbf{S}_i\ M\ N\ \overline{\mathbf{n}} \underset{w}{\rightarrow} M\ \mathbf{E}_n\ \mathbf{B}\ \mathbf{H}_i\ N\ \overline{\mathbf{n}} \qquad (\mathbf{S})$$

where $$\mathbf{E}_i\ M\ N\ P \underset{w}{\rightarrow} N\ \mathbf{E}_i\ P\ (M\ \mathbf{A}\ \Omega\ \overline{\mathbf{i}}\ \mathbf{C}) \qquad (\mathbf{E})$$

$$\mathbf{B}\ M\ N \underset{w}{\rightarrow} N\ \mathbf{A} \qquad (\mathbf{B})$$

$$\mathbf{C}\ M\ N\ P \underset{w}{\rightarrow} P\ (\lambda f.f\ (\lambda c.\ c\ M)\ N) \qquad (\mathbf{C})$$

When the hnf is reached the head variable previously instantiated by $\mathbf{S}_i$ is called. It triggers the evaluation of its arguments via $\mathbf{E}_i$ and insert $\mathbf{H}_i$ as last continuation. $\mathbf{E}_i$ applies the arguments to $\mathbf{A}\ \Omega\ \overline{\mathbf{i}}$ which will be evaluated in a right to left order and inserts the continuation **C** needed to put back the evaluated arguments $X_1,\ldots,X_n$ in cps form (i.e. $\lambda f.f\ X_1\ (\ldots(\lambda f.f\ X_n\ E)\ldots)$). The role of $\mathbf{H}_i$'s is still to reconstruct the expression. Combinator $\mathbf{H}_i$ keeps the same definition except for **R** which have now the simplified reduction rule

$$\mathbf{R}\ E\ F\ G\ H\ I = \lambda f.f\ (\lambda c.G\ (F\ c))\ (H\ (\mathbf{R}\ E\ F)\ I) \qquad (\mathbf{R})$$

When **R** is applied, the arguments are already evaluated and reconstructed so there is no need to apply them to $\mathbf{A}\ \Omega\ \overline{\mathbf{i}}$ as before.

Example: Let $M = \lambda x.(\lambda w.\lambda y.\ w\ y\ ((\lambda v.\ v)\ x))\ (\lambda z.\ z)\ x$

Its strong reduction is

$$\begin{aligned} E &\underset{s}{\rightarrow} \lambda x.(\lambda y.(\lambda z.\ z)\ y\ ((\lambda v.\ v)\ x))\ x \\ &\underset{s}{\rightarrow} \lambda x.(\lambda z.\ z)\ x\ ((\lambda v.\ v)\ x) \\ &\underset{s}{\rightarrow} \lambda x.x\ ((\lambda v.\ v)\ x) \end{aligned}$$

$\xrightarrow[s]{}$ λx.x x

After cps conversion and simplification the expression becomes

$\mathcal{N}(M) = \lambda_c x.\ (\lambda w.\ \lambda_c y.\lambda c.\ w\ (\lambda f.f\ y\ (\lambda f.f\ ((\lambda v.\ v)\ x))\ c))\ (\lambda_c z.z)\ (\lambda f.f\ x\ c)$

The weak head reduction of the cps expression is (reductions corresponding to strong reductions of the source expression are marked by ☩)

$\mathcal{N}(M)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ $\xrightarrow[w]{3}$ $(\lambda x.\ \lambda c.(\lambda w.\ \lambda_c y.\lambda c.\ w\ (\lambda f.f\ y\ (\lambda f.f\ ((\lambda v.\ v)\ x))\ c))\ (\lambda_c z.z)\ (\lambda f.f\ x\ c))\ \mathbf{S}_0\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{2}$ $(\lambda w.\ \lambda_c y.\lambda c.\ w\ (\lambda f.f\ y\ (\lambda f.f\ ((\lambda v.\ v)\ \mathbf{S}_0)\ c))\ (\lambda_c z.z)\ (\lambda f.f\ \mathbf{S}_0\ \mathbf{A})\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{}$ $(\lambda_c y.\lambda c.\ (\lambda_c z.z)\ (\lambda f.f\ y\ (\lambda f.f\ ((\lambda v.\ v)\ \mathbf{S}_0)\ c))\ (\lambda f.f\ \mathbf{S}_0\ \mathbf{A})\ \Omega\ \overline{\mathbf{1}}$ ☩

$\xrightarrow[w]{2}$ $(\lambda y.\lambda c.\ (\lambda_c z.z)\ (\lambda f.f\ y\ (\lambda f.f\ ((\lambda v.\ v)\ \mathbf{S}_0)\ c))\ \mathbf{S}_0\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{}$ $(\lambda c.\ (\lambda_c z.z)\ (\lambda f.f\ \mathbf{S}_0\ (\lambda f.f\ ((\lambda v.\ v)\ \mathbf{S}_0)\ c))\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}$ ☩

$\xrightarrow[w]{3}$ $(\lambda z.z)\ \mathbf{S}_0\ (\lambda f.f\ ((\lambda v.\ v)\ \mathbf{S}_0)\ \mathbf{A})\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{}$ $\mathbf{S}_0\ (\lambda f.f\ ((\lambda v.\ v)\ \mathbf{S}_0)\ \mathbf{A})\ \Omega\ \overline{\mathbf{1}}$ the hnf is reached, the reduction rule of $\mathbf{S}_0$ is used.☩

$\xrightarrow[w]{}$ $(\lambda f.f\ ((\lambda v.\ v)\ \mathbf{S}_0)\ \mathbf{A})\ \mathbf{E}_1\ \mathbf{B}\ \mathbf{H}_0\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{}$ $\mathbf{E}_1\ ((\lambda v.\ v)\ \mathbf{S}_0)\ \mathbf{A}\ \mathbf{B}\ \mathbf{H}_0\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{3}$ $((\lambda v.\ v)\ \mathbf{S}_0\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}\ \mathbf{C})\ \mathbf{A}\ \mathbf{H}_0\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{}$ $\mathbf{S}_0\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}\ \mathbf{C}\ \mathbf{A}\ \mathbf{H}_0\ \Omega\ \overline{\mathbf{1}}$ the nf is reached ; the reconstruction begins.☩

$\xrightarrow[w]{3}$ $\mathbf{H}_0\ \mathbf{A}\ \Omega\ \overline{\mathbf{1}}\ \mathbf{C}\ \mathbf{A}\ \mathbf{H}_0\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{}$ $(\lambda_c x.\lambda c.x\ (\mathbf{A}\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c)))\ \mathbf{C}\ \mathbf{A}\ \mathbf{H}_0\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{}$ $\mathbf{C}\ (\lambda x.\lambda c.x\ (\mathbf{A}\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c)))\ \mathbf{A}\ \mathbf{H}_0\ \Omega\ \overline{\mathbf{1}}$

$\xrightarrow[w]{}$ $\mathbf{H}_0\ X\ \Omega\ \overline{\mathbf{1}}$

with $X \equiv \lambda f.f\ (\lambda c.c(\lambda x.\lambda c.x\ (\mathbf{A}\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c))))\ \mathbf{A}$

$\rightarrow \lambda_c x.\lambda c.x\ (X\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c))$

The wh-reduction is completed. Now, we show that the result is equivalent to the normal form in cps form.

$X \xrightarrow{*} \lambda f.f\ (\lambda_c x.x)\ \mathbf{A}$ (**A**),(**K**)

and $X\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c) \xrightarrow{*} \mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c)\ (\lambda_c x.x)\ \mathbf{A}\ (\mathbf{K}\ c)$

$\xrightarrow{*}$ $\lambda f.f\ (\lambda c.(\lambda_c x.x)\ ((\lambda c.\lambda f.f\ x\ c)\ c)\ (\mathbf{A}\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c))$ (**R**)

$\xrightarrow{*}$ $\lambda f.f\ x\ c$ since $\mathbf{A}\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c) \xrightarrow{*} c$ (**A**),(**K**)

and $\lambda c.(\lambda_c x.x)\ (\lambda c.\lambda f.f\ x\ c)\ c \rightarrow \lambda c.x\ c \rightarrow_\eta x$

So $\lambda_c x.\lambda c.x\ (X\ (\mathbf{R}\ \overline{\mathbf{1}}\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c)) \xrightarrow{*} \lambda_c x.\lambda c.x\ (\lambda f.f\ x\ c)$ which is the normal form in cps form. ❑

All the reductions taking place during the strong reduction of the source expression are carried out by wh-reduction of the transformed expression. We do not really get the full normal form since the reconstruction can not be achieved completely by weak head reduction. As before the reconstruction is lazy. However the result is convertible to the normal form in cps and the complexity of this last step is bounded by the size of the normal form. If we were just interested in normal forms as a syntactic result, $\mathbf{H}_i$'s could be replaced by functions printing the nf instead of building a suspension representing it. In this case, the evaluation would be completely carried out by wh-reduction.

We have the analogues of Property 2 and Property 3. The following property states that for any closed expression E the weak head reduction of $\mathcal{N}(E)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ simulates the strong reduction of E.

Property 4 *Let E be a closed expression, if* $E \xrightarrow[S]{*} S$ *then there exists an expression X such that* $\mathcal{N}(E)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}} \xrightarrow[W]{*} X \xleftarrow[W]{a} \mathcal{N}(S)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ *and if S is a nf then X is a whnf. Furthermore, E does not have a nf iff* $\mathcal{N}(E)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ *does not have a whnf.*

The result of the evaluation of $\mathcal{N}(E)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ is interconvertible with the nf in cps.

Property 5 *If a closed expression E has a normal form then* $\mathcal{N}(E) \leftrightarrow \mathcal{N}(E)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$

5 Head Reduction of Typed Lambda Expressions

In the previous sections we needed to count the number of leading λ's during the evaluation. Using some form of typing it is possible to know the functionality of the expression prior to evaluation and thus get rid of this counter. We consider only head reduction ; typing does not seem to simplify the compilation of strong reduction.

Simply typed λ-calculus would suit our purposes but would harshly restrict the class of expressions. More flexible typing systems are sufficient. One candidate is reflexive reducing typing [1] which has already been used in [8] to determine the functionality of expressions. It is shown in [1] that we can restrict a language to reflexive reducing types without weakening its expressive power. Reflexive reducing types are defined by (possibly recursive) equations of the form $\sigma = \sigma_1 \rightarrow \ldots \rightarrow \sigma_n \rightarrow \alpha$, σ_1 , …, σ_n being themselves reflexive reducing types and α being a base type (not a reflexive type). This enables us to type recursive functions but not for example $(\lambda xy.xx)(\lambda xy.xx)$ (this expression has the reflexive type $\rho \rightarrow \alpha \rightarrow \sigma$ with $\sigma = \alpha \rightarrow \sigma$ which is not reducing). We do not dwell here on the details of this typing system. The important point for us is that a closed expression with type $\sigma = \sigma_1 \rightarrow \ldots \rightarrow \sigma_n \rightarrow \alpha$ has a hnf of the form $\lambda x_1. \ldots \lambda x_n.x_i\ E_1 \ldots E_p$.

If the expression to reduce to hnf has functionality n then the transformed expression is

$\mathcal{N}(E)\ (\lambda f.f\ \mathbf{X}^1_n \ldots (\lambda f.f\ \mathbf{X}^n_n\ \mathbf{L}_n)\ldots)$ and we note $\mathcal{N}(E)\ (\vec{\mathbf{X}}_n\ \ \mathbf{L}_n)$.

That is, we apply the expression to n arguments in order to remove the n leading abstractions. Combinators $\mathbf{X}_n^i$ play the same role as the combinators $\mathbf{H}_i$ introduced in section 3. They will be substituted for variables and used to start the reconstruction process.

$$\mathbf{X}_n^i \; E = \lambda_c \vec{x}_n \; . \; \lambda c.x_i \; (E \; (\mathbf{R}_n \; (\lambda c.\vec{x}_n \;\; c)) \; (\mathbf{K} \; c)) \qquad (\mathbf{X})$$

Combinator $\mathbf{R}_n$ used in the definition of $\mathbf{X}_n^i$ plays the same role as **R** in the definition of $\mathbf{H}_i$.

$$\mathbf{R}_n \; E \; F \; G \; H = \lambda f.f \; (\lambda c.F \; \mathbf{L}_n \; (E \; c)) \; (G \; (\mathbf{R}_n \; E) \; H) \qquad (\mathbf{R})$$

In the preceding sections, the reconstruction of subexpressions was based on the same technique as the reduction of the global expression: each subexpression was applied to continuation **Ω** and was rebuild after being reduced to hnf. Here, there is no type information available on the subexpressions and we cannot use the same method as for the global expression. In particular, a subexpression $(\lambda_c z.E)$ can not be reduced since we do not know its functionality. However, it may contain occurrences of combinators $\mathbf{X}_n^i$ which are to be removed. This case is treated using combinators $\mathbf{L}_n$ and $\mathbf{Z}_n$ which carry on the reconstruction inside the λ-abstraction.

$$\mathbf{L}_n \; E = \lambda_c \vec{x}_n \; . \; \lambda_c z.\lambda c. \; E \; (\mathbf{Z}_n \; z) \; \mathbf{L}_n \; (\vec{x}_n \;\; c) \qquad (\mathbf{L})$$

$$\mathbf{Z}_n \; E \; F = \lambda_c \vec{x}_n \; . \; \lambda c. \; E \; (F \; (\mathbf{R}_n \; (\lambda c.\vec{x}_n \;\; c)) \; (\mathbf{K} \; c)) \qquad (\mathbf{Z})$$

For example, if the hnf is of the form $\lambda x_1. \ldots \lambda x_n.x_i \ldots (\lambda z.E) \ldots$ then $\mathbf{R}_n$ applies each subexpression to $\mathbf{L}_n$ and $(\vec{x}_n \;\; c)$ and we will get for the λ-abstraction $(\lambda z.E)$

$$\begin{aligned} \lambda c.(\lambda_c z.E) \; \mathbf{L}_n \; (\vec{x}_n \;\; c) \;\; &\rightarrow \lambda c. \; \mathbf{L}_n (\lambda z.E) \; (\vec{x}_n \;\; c) \\ &\rightarrow \; \lambda c. \; (\lambda_c \vec{x}_n \; . \; \lambda_c z.\lambda c. \; (\lambda z.E) \; (\mathbf{Z}_n \; z) \; \mathbf{L}_n \; (\vec{x}_n \;\; c)) (\vec{x}_n \;\; c) \\ &\rightarrow \; \lambda_c z.\lambda c. \; E[\mathbf{Z}_n \; z/z] \; \mathbf{L}_n \; (\vec{x}_n \;\; c) \end{aligned}$$

The list of variables has been pushed inside the λ-abstraction and the reconstruction can continue. Variable z is replaced by $(\mathbf{Z}_n \; z)$ so that when it is applied to the list $(\vec{x}_n \;\; c)$ it returns z. Combinators $\mathbf{R}_n$, $\mathbf{L}_n$, $\mathbf{X}_n^i$, $\mathbf{Z}_n$ act very much like combinators used in abstraction algorithms. $\mathbf{X}_n^i$ is a selector (it selects the ith variable), $\mathbf{Z}_n$ is (like **K**) a destructor (it ignores the list and returns its first argument), $\mathbf{R}_n$ and $\mathbf{L}_n$ distribute the list of variables $(\lambda c.\vec{x}_n \;\; c)$ throughout the expression.

The head normal form (if any) of E will be of the form $(\lambda x_1.\ldots\lambda x_n.x_i \; E_1 \ldots E_p)$ so $\mathcal{N}(E) \; (\vec{\mathbf{X}}_n \;\; \mathbf{L}_n)$ will be reduced (by weak head reduction) to $\mathbf{X}_n^i \; (\vec{E}_p \;\; \mathbf{L}_n)$ and then, according to the definition of combinators $\mathbf{X}_n^i$, to $\lambda_c \vec{x}_n \;\; .\lambda c.x_i \; ((\vec{E}_p \;\; \mathbf{L}_n) \; (\mathbf{R}_n \; (\lambda c.\vec{x}_n \;\; c)) \; (\mathbf{K} \; c))$. As before the continuation (**K** c) removes reconstructing expressions and returns the final continuation.

Example. Let us take an example to illustrate the reconstruction process.

Let $E = \lambda x.(\lambda w.w \; (\lambda y.w)) \; x \xrightarrow[h]{} \lambda x.x \; (\lambda y.x)$

The transformed expression after simplification is

$\mathcal{N}(E) = (\lambda_c x. \; \lambda c. \; (\lambda w. \; w \; (\lambda f.f \; (\lambda_c y.w) \; c)) \; x)$

Since the expression has a type of the form $\sigma \to \alpha$ its hnf has one leading λ and the transformed expression is applied to one argument $\mathbf{X}_1^1$. The weak head reduction of this expression goes as follow

$\mathcal{N}(E)\ (\lambda f.f\ \mathbf{X}_1^1\ \mathbf{L}_1)$

$\underset{w}{\to}\ (\lambda x.\ \lambda c.\ (\lambda w.\ w\ (\lambda f.f\ (\lambda_c y.w)\ c))\ x)\ \mathbf{X}_1^1\ \mathbf{L}_1$

$\underset{w}{\to}\ (\lambda w.\ w\ (\lambda f.f\ (\lambda_c y.w)\ \mathbf{L}_1))\ \mathbf{X}_1^1$ $\mathbf{X}_1^1$ (resp. $\mathbf{L}_1$) is substituted for x (resp. c)

$\underset{w}{\to}\ \mathbf{X}_1^1\ (\lambda f.f\ (\lambda_c y.\mathbf{X}_1)\ \mathbf{L}_1)$ this step corresponds to the source head-reduction

$\underset{w}{\to}\ \lambda_c x.\ \lambda c.x\ ((\lambda f.f\ (\lambda_c y.\mathbf{X}_1^1)\ \mathbf{L}_1)\ (\mathbf{R}_1\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c))$ the hnf is reached

We now check that this hnf can be reconstructed to $\lambda_c x.\ \lambda c.x\ (\lambda f.f\ (\lambda_c y.x)\ c)$ which is the principal hnf in cps.

$(\lambda f.f\ (\lambda_c y.\mathbf{X}_1^1)\ \mathbf{L}_1)\ (\mathbf{R}_1\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c) \to \mathbf{R}_1\ (\lambda c.\lambda f.f\ x\ c)\ (\lambda_c y.\mathbf{X}_1^1)\ \mathbf{L}_1\ (\mathbf{K}\ c)$

$\to \lambda f.f\ (\lambda c.(\lambda_c y.\mathbf{X}_1^1)\ \mathbf{L}_1\ ((\lambda c.\lambda f.f\ x\ c)\ c))\ (\mathbf{L}_1\ (\mathbf{R}_1\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c))$ (**R**)

$\overset{*}{\to}\ \lambda f.f\ (\lambda c.\mathbf{L}_1\ (\lambda y.\mathbf{X}_1^1)\ (\lambda f.f\ x\ c))\ c$ (**L**),(**K**)

$\overset{*}{\to}\ \lambda f.f\ (\lambda c.(\lambda_c x.\lambda_c y.\lambda c.(\lambda y.\mathbf{X}_1^1)\ (\mathbf{Z}_1\ y)\ \mathbf{L}_1\ (\lambda f.f\ x\ c))(\lambda f.f\ x\ c))\ c$ (**L**)

$\overset{*}{\to}\ \lambda f.f\ (\lambda_c y.\lambda c.\mathbf{X}_1^1\ \mathbf{L}_1\ (\lambda f.f\ x\ c))\ c$

$\mathbf{X}_1^1\ \mathbf{L}_1 \overset{*}{\to} \lambda_c x.\lambda c.x\ (\mathbf{L}_1\ (\mathbf{R}_1\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c)) \overset{*}{\to} \lambda_c x.x$ (**K**),(**X**),(**L**),(η)

$\overset{*}{\to}\ \lambda f.f\ (\lambda_c y.\lambda c.(\lambda_c x.x)\ (\lambda f.f\ x\ c))\ c \to \lambda f.f\ (\lambda_c y.x)\ c$ (η)

So $\lambda_c x.\ \lambda c.x\ ((\lambda f.f\ (\lambda_c y.\mathbf{X}_1^1)\ \mathbf{L}_1)\ (\mathbf{R}_1\ (\lambda c.\lambda f.f\ x\ c))\ (\mathbf{K}\ c)) \overset{*}{\to} \lambda_c x.\ \lambda c.x\ (\lambda f.f\ (\lambda_c y.x)\ c)$ ❑

The following property states that for any closed expression E of type $\sigma_1 \to \ldots \to \sigma_n \to \alpha$ the weak head reduction of $\mathcal{N}(E)\ (\vec{X}_n\ \ \mathbf{L}_n)$ simulates the head reduction of E.

Property 6 *Let E be a closed expression of functionality n, if* $E \underset{h}{\overset{*}{\to}} H$ *then there exists an expression X such that* $\mathcal{N}(E)\ (\vec{\mathbf{X}}_n\ \ \mathbf{L}_n) \underset{w}{\overset{*}{\to}} X \underset{w}{\overset{a}{\leftarrow}} \mathcal{N}(H)\ (\vec{\mathbf{X}}_n\ \ \mathbf{L}_n)$ *and if H is a hnf then X is a whnf. Furthermore, E does not have a hnf iff* $\mathcal{N}(E)\ (\vec{\mathbf{X}}_n\ \ \mathbf{L}_n)$ *does not have a whnf.*

If E has a normal form the result of the evaluation of $\mathcal{N}(E)\ (\vec{\mathbf{X}}_n\ \ \mathbf{L}_n)$ is interconvertible with the principal hnf in cps form.

Property 7 *If E, a closed expression of functionality n, has a normal form then* $\mathcal{N}(E) \leftrightarrow \mathcal{N}(E)(\vec{\mathbf{X}}_n\ \ \mathbf{L}_n)$

This approach may also be useful to finely control the evaluation. For example, applying an expression of functionality n to p (< n) arguments will stop the evaluation when the expression has p

leading lambdas. The normal form obtained will be a whnf but of a more evaluated form than the principal one.

6 Applications

Among practical applications of head reduction listed in the introduction, one is to compile more efficient evaluation strategies. We describe better this question in the next section and suggest in section 6.2 how our approach can be used to compile such strategies. Implementation issues are discussed in 6.3.

6.1 Spine Strategies

Even when evaluating weak-head-normal forms it is sometimes better to reduce sub-terms in head normal forms. For example, in lazy graph reduction, the implementation of β-reduction $(\lambda x.E)F \rightarrow_\beta E[F/x]$ implies making a copy of the body E before the substitution. It is well known that this may lose sharing and work may be duplicated [17]. Program transformations, such as fully lazy lambda-lifting [9], aim at maximizing sharing but duplication of work can still occur. Another approach used to avoid recomputation is to consider alternative evaluation strategies. If the expression to reduce is $(\lambda x.E)F$ we know that the whnf of the body E will be needed and so it is safe to reduce E prior to the β-reduction. This computation rule belongs to the so-called spine-strategies [3]. It never takes more reductions than normal order and may save duplication of work.

A revealing example, taken from [7], is the reduction of A_n **I** where the family of λ-expressions A_i is defined by $A_0 = \lambda x.x$ **I** and $A_n = \lambda h.(\lambda w.w\ h\ (w\ w))\ A_{n-1}$.

A_n **I** is reduced using the call-by-name weak head graph reduction as follows:

A_n **I** $= (\lambda h.(\lambda w.w\ h\ (w\ w))\ A_{n-1})$ **I**

$\rightarrow (\lambda w.w$ **I** $(w\ w))\ A_{n-1}$

$\rightarrow A_{n-1}$ **I** $(\bullet\ \bullet) \equiv (\lambda h.(\lambda w.w\ h\ (w\ w))\ A_{n-2})$ **I** $(\bullet\ \bullet)$ ($\bullet$ representing the sharing of A_{n-1})

$\rightarrow (\lambda w.w$ **I** $(w\ w))\ A_{n-2}\ (A_{n-1}\ \bullet)$

The sharing is lost and the redexes inside A_{n-1} are duplicated. The complexity of the evaluation is $O(2^n)$. On the other hand, by reducing λ-abstractions to hnf before β-reductions the evaluation sequence becomes

A_n **I** $= (\lambda h.(\lambda w.w\ h\ (w\ w))\ A_{n-1})$ **I**

$\rightarrow \quad (\lambda h.A_{n-1}\ h\ (\bullet\ \bullet))$ **I**

$\overset{4(n-1)}{\rightarrow} \quad (\lambda h.A_0\ h\ (\bullet\ \bullet))$ **I** — A_{n-1} reduces to A_0 in 4(n-1) steps

$\rightarrow \quad (\lambda h.$ **I** $(A_0\ \bullet))$ **I** $\rightarrow (\lambda h.\ A_0\ \bullet)$ **I** $\rightarrow (\lambda h.$ **I**$)$ **I** $\rightarrow$ **I**

and the A_i's remain shared until they are reduced to their hnf A_0. The complexity of the evaluation drops from exponential to linear.

Of course this strategy alone is not optimal (optimal reduction of λ-expressions is more complex [11][12]) and work can still be duplicated. But in [16] Staples proposes a similar evaluation strategy with the additional rule that substitutions are not carried out inside redexes (they are suspended until the redex is needed and reduced to hnf). This reduction has been shown to be optimal for a λ-calculus with explicit substitutions.

6.2 Sharing Hnf's

We saw that evaluating the λ-abstraction to hnf before the β-reduction can save work by sharing hnf's instead of whnf's. We study in this section how to make use of our approach to implement such a strategy. The straightforward idea of applying the continuation Ω to each λ-expression not in hnf does not work. Our previous results were relying on the fact that the expression to be reduced was closed. Here, even if the global expression is closed, we may have to reduce to hnf sub-expressions containing free variables.

For example, if (λw.w **I**) (λx. **I** (λy. **I** x) **I**) is cps converted and the two λ-abstractions not in hnf (λ_cx.…) and (λ_cy.…) are applied to **A** Ω $\overline{\mathbf{0}}$ then during the reduction of (λ_cx. …) we will have to reduce to hnf (λ_cy.…) **A** Ω $\overline{\mathbf{0}}$ but x is already instantiated by $\mathbf{H}_0$ and we will get (λ_cy.$\mathbf{H}_0$) **A** Ω $\overline{\mathbf{0}}$ → $\mathbf{H}_0$ **A** Ω $\overline{\mathbf{1}}$ → (λ_cy.y) which is false ; the cps hnf of (λy. **I** x) should have been (λ_cy. $\mathbf{H}_0$) and the enclosing evaluation of (λ_cx. …) can continue. The problem comes from free variables already instantiated when a new head reduction begins.

The simplest solution would be to transform the λ-abstractions into supercombinators using λ-lifting [9]. The supercombinators (not already in hnf) will be applied to **A** Ω $\overline{\mathbf{0}}$ and their hnfs will be shared naturally. However, it is not clear whether we share the same computations by (spine-)reducing an expression and its supercombinator form. In order to validate this approach, we would have to prove that this kind of transformation does not change the complexity of a spine reduction.

Another solution to the free variable problem is to change the rule of cps conversion for applications by

$\mathcal{N}^*$(E F) = λc.$\mathcal{N}^*$(E) (λf.f (**Z** $\mathcal{N}^*$(F)) c)

with **Z** E C Ω $\overline{\mathbf{n}}$ → E **A** Ω $\overline{\mathbf{n}}$ ($\vec{\mathbf{H}}_n$ C) Ω $\overline{\mathbf{n}}$

A sub-expression is evaluated to hnf using **Z** which triggers it with index $\overline{\mathbf{n}}$ instead of $\overline{\mathbf{0}}$. It amounts to closing the expression since we know at this point that the source expression has at most n free variables and E **A** Ω $\overline{\mathbf{n}}$ ↔ ($\lambda_c\vec{x}_n$. E) **A** Ω $\overline{\mathbf{0}}$. The result will be of the form ($\lambda_c\vec{x}_p$. H) (p≥n) and ($\lambda_c\vec{x}_p$. H) ($\vec{\mathbf{H}}_n$ C) reduces to ($\lambda_c x_{n+1}$. …($\lambda_c x_p$.H[$\vec{\mathbf{H}}_n$ /$\vec{x}_n$])…), that is, the hnf of E with its free variables still instantiated by their corresponding $\mathbf{H}_i$. Returning to the previous example, the reduction now looks like

Z (λ_cy.$\mathbf{H}_0$) C Ω $\overline{\mathbf{1}}$ → (λ_cy.$\mathbf{H}_0$) **A** Ω $\overline{\mathbf{1}}$ (λf.f $\mathbf{H}_0$ C) Ω $\overline{\mathbf{1}}$ → $\mathbf{H}_0$ **A** Ω $\overline{\mathbf{2}}$ (λf.f $\mathbf{H}_0$ C) Ω $\overline{\mathbf{1}}$

→ (λ_cx. λ_cy. x) (λf.f $\mathbf{H}_0$ C) Ω $\overline{\mathbf{1}}$ → (λ_cy. $\mathbf{H}_0$) C Ω $\overline{\mathbf{1}}$ → …

6.3 Implementation issues

The most obvious way to implement our approach is to transform expressions as previously described and give the result to a compiler. The combinators **A**, **Ω**, …are compiled like other functions and the reconstruction is naturally implemented by closure building. However, with compilers which already integrate a cps conversion, a more efficient way would be to directly use the cps phase. This is less trivial since the following steps expect only cps expressions and we have to introduce special combinators which are not in cps. One solution is to implement those combinators by hand so that the compiler uses them like primitive functions. We plan such an integration in our cps-based compiler. Further work is still needed on different extensions:

- So far we have only considered call by name. As cps conversion can be used to compile different computation rules (call-by-value, call-by-name with strictness annotations, …) it is likely that our method could be extended to treat those strategies as well.
- This method should be extended to a λ-calculus with constants and primitive operators.

If we just aim at reducing a program to hnf/nf and print the result then our approach will be very efficient. The whole evaluation is a weak reduction which can be completely compiled. The only slight overhead will be a few more reductions for each leading lambda and printing the result which should be proportional to the size of the expression.

The costly part of head/strong reduction is the reconstruction of expressions which happens when we actually use (i.e. apply) the hnf/nf. In particular, reconstructing uses a lot of memory space. In order to implement efficient evaluation strategies as described in section 6.1, it would be useful to develop the following points:

- Several analyses can detect expressions for which wh-reduction is better and should be implemented as well. For example, one policy could be that a λ-abstraction will be reduced to hnf prior to β-reduction only if it is shared (using a sharing analysis), complex enough (using a complexity analysis) and of course not already in hnf.
- Computation can still be duplicated by performing substitutions inside redexes. It would be interesting to extend our work to compile Staples' method [16] which avoids this loss of sharing.

We did a few experiments using the trivial way (i.e. transforming source expressions before giving them to our compiler). We transformed the family of expressions A_n defined in section 6.1 into supercombinators and into cps form. The evaluation of A_{15} **I** takes around 1s using standard reduction and around 1ms when each supercombinator is applied to **A Ω $\overline{\mathbf{0}}$**. This result is not surprising since the theoretical complexity is exponential in one case and linear in the other. More interestingly, we redefined the family A_n by $A_0 = \lambda x.x$ **I** and $A_n = \lambda h.(\lambda w.w\ h\ (A_0\ w))\ A_{n-1}$. Here, the wh-reduction of A_n **I** does not duplicate work (the second occurrence of w is not needed) and nothing is saved by using head reduction. We found that the head reduction of supercombinators made the evaluation 3 to 4 times slower than the standard wh-reduction.

This example indicates the cost of reconstructing expressions. This cost is acceptable when the final goal is to implement symbolic evaluation. When the goal is to evaluate whnf's more efficiently by sharing hnf's then such examples should be avoided using analyses. In [4] Crégut gives a function for which the reduction takes n^2 steps when sharing hnf's whereas it takes only n steps using standard wh-reduction. It is also shown that this is the worst case.

7 Conclusion

Implementation of head and strong reduction has also been studied by Crégut [4] and Nadathur and Wilson [13]. Crégut's abstract machine is based on De Bruijn's notation. Two versions have been developed. The first one evaluates the head or full normal form of the global expression. The second one implements a spine strategy and shares head normal forms. Terms are extended with formal variables and the machine state includes two indexes. One plays the role of our binder level as in section 3 and 4, the other one is needed (only in the second version of the machine) to deal with the problem of free variables in subexpressions exposed in section 6.2. The algorithm presented in [13] was motivated by the implementation of λProlog [14]. It evaluates terms to hnf and expressed as an abstract machine, this technique resembles Crégut's. It is also based on De Bruijn notation and the machine state includes two indexes.

We described in this paper how to use cps conversion to compile head and strong reduction. The hnf's or nf's of cps-expressions are evaluated by weak head reduction and at each step the unique (weak) redex is the leftmost application. The technique does not require to modify the standard cps cbn conversion. The cps expression is just applied to a special continuation and an index to keep track of the binder length. We presented a way to get rid of this index and suggested applications for our technique. Compared to [4] and [13] we do not have a second index or special structures like formal variables. But the main difference is that we proceed by program transformations and stay within the functional framework. Used as a preliminary step our technique allows a standard compiler to evaluate under λ's. Thus we can take advantage of all the classical compiling tools like analyses, transformations or simplifications. As already emphasized in [6], another advantage of this approach is that we do not have to introduce an abstract machine which makes correctness proofs simpler. Furthermore, optimizations of this compilation step can be easily expressed and justified in the functional framework.

Apart from the practical issues discussed in section 6.3, several others research directions like the application of this approach to partial evaluation or to the compilation of λ-prolog should be explored.

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Annex

The proofs of Property 2 and Property 3 are described here in details. They are elementary even if the expressions involved (usually of the general form $\lambda x_1. \ldots \lambda x_n.F_0\ F_1 \ldots F_p$) and cps conversion affect the readability. The proofs of the corresponding properties for strong reduction and head reduction of typed λ-expressions are very similar and are just sketched. We use the convention that in an expression all bound variables are different from the free variables (it can be seen as an implicit use of α-conversion).

A Proof of Property 2

In the following $\vec{x}_n$ stands for variables $x_1,\ldots,x_n$ whereas $\vec{H}_n$ stands for combinators $\mathbf{H}_0,\ldots,\mathbf{H}_{n-1}$. We will need the following lemmas.

Lemma 8 *$\mathcal{N}(E)[\mathcal{N}(F)/x] \equiv \mathcal{N}(E[F/x])$*

Lemma 9 *if $x \not\equiv y$ and x does not occur free in G then $E[F/x][G/y] \equiv E[G/y][F[G/y]/x]$*

Lemma 8 is shown in [15] and Lemma 9 is shown in [2] (2.1.16 pp. 27).

Let us show that the property holds for one reduction step. The expression must be of the form $\lambda x_1. \ldots \lambda x_n.(\lambda y.E)\ F_0\ F_1 \ldots F_p$, with $n \geq 0$, $p \geq 0$ and

$$\lambda\vec{x}_n\ .(\lambda y.E)\ F_0\ F_1 \ldots F_p \xrightarrow[h]{} \lambda\vec{x}_n\ .E[F_0/y]\ F_1 \ldots F_p$$

$$\mathcal{N}(\lambda\vec{x}_n\ .(\lambda y.E)\ F_0\ F_1 \ldots F_p)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$$

$$\equiv \lambda_c\vec{x}_n\ .(\lambda c.\ldots(\lambda c.(\lambda_c y.\mathcal{N}(E))\ (\lambda f.f\ \mathcal{N}(F_0)\ c)) \ldots(\lambda f.f\ \mathcal{N}(F_p)\ c))\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$$

$$\xrightarrow[w]{*} (\lambda c.\ldots(\lambda c.(\lambda_c y.\mathcal{N}(E))\ (\lambda f.f\ \mathcal{N}(F_0)\ c)) \ldots(\lambda f.f\ \mathcal{N}(F_p)\ c))[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ \mathbf{A}\ \Omega\ \overline{\mathbf{n}}$$

$$\xrightarrow[w]{*} (\lambda y.\mathcal{N}(E)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ])\ \mathcal{N}(F_0)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ (\lambda f.f\ \mathcal{N}(F_1)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ldots$$

$$\ldots\ (\lambda f.f\ \mathcal{N}(F_p)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ \mathbf{A}))\ \Omega\ \overline{\mathbf{n}}$$

Administrative reductions are now completed ; the proper reduction takes place

$$\xrightarrow[w]{} \mathcal{N}(E)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ [\mathcal{N}(F_0)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]/y]\ (\lambda f.f\ \mathcal{N}(F_1)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ldots$$

$$\ldots\ (\lambda f.f\ \mathcal{N}(F_p)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ \mathbf{A}))\ \Omega\ \overline{\mathbf{n}}$$

On the other hand

$$\mathcal{N}(\lambda\vec{x}_n\ .E[F_0/y]\ F_1 \ldots F_p)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$$

$$\equiv \lambda_c\vec{x}_n\ .(\lambda c.\ldots(\lambda c.(\lambda_c y.\mathcal{N}(E[F_0/y]))\ (\lambda f.f\ \mathcal{N}(F_1)\ c)) \ldots(\lambda f.f\ \mathcal{N}(F_p)\ c))\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$$

$$\xrightarrow[w]{a} \mathcal{N}(E[F_0/y])[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ (\lambda f.f\ \mathcal{N}(F_1)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ldots(\lambda f.f\ \mathcal{N}(F_p)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]\ \mathbf{A}))\ \Omega\ \overline{\mathbf{n}}$$

$\mathcal{N}(E[F_0/y])\ [\vec{\mathbf{H}}_n\ /\vec{x}_n\ ] \equiv \mathcal{N}(E)[\mathcal{N}(F_0)/y]\ [\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]$ Lemma 8

since the $\mathbf{H}_i$'s are closed we can apply Lemma 9 (actually a straightforward generalization of it) to get

$$\mathcal{N}(E)[\mathcal{N}(F_0)/y]\ [\vec{\mathbf{H}}_n\ /\vec{x}_n\ ] \equiv \mathcal{N}(E)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ][\mathcal{N}(F_0)[\vec{\mathbf{H}}_n\ /\vec{x}_n\ ]/y]$$

So both expressions are identical and the property holds for one reduction step.

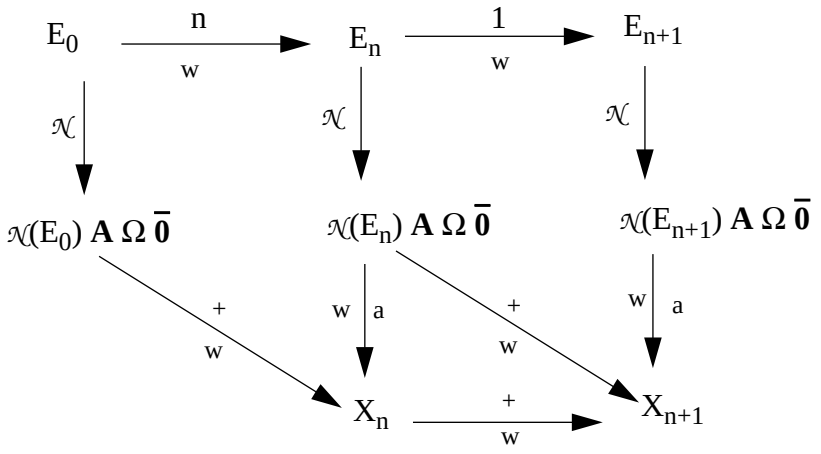
We can now show Property 2. The base case (no reduction step) is trivial. Let the property hold for n ($n \geq 0$) reduction steps, that is

$E_0 \xrightarrow[w]{n} E_n$ and $\mathcal{N}(E_0)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}} \xrightarrow[w]{+} X_n \xleftarrow[w]{a} \mathcal{N}(E_n)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$

then if $E_n \xrightarrow[w]{1} E_{n+1}$ we have shown that there is an expression X_{n+1} such that

$\mathcal{N}(E_n)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}} \xrightarrow[w]{+} X_{n+1} \xleftarrow[w]{a} \mathcal{N}(E_{n+1})\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ and the reduction $\xrightarrow[w]{+}$ contains the administrative reductions followed by a proper reduction, so $\mathcal{N}(E_n)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}} \xrightarrow[w]{a} X_n \xrightarrow[w]{} X_{n+1}$. Thus

$$\mathcal{N}(E_0)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}} \xrightarrow[w]{+} X_{n+1} \xleftarrow[w]{a} \mathcal{N}(E_{n+1})\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$$



Concerning the second part of the property, if an expression E_0 does not have a hnf there is an infinite reduction sequence $E_o \xrightarrow[h]{} E_1 \xrightarrow[h]{} \ldots$. It is clear from the proof above that the corresponding weak head reduction on $\mathcal{N}(E_0)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ will also be infinite, so this expression does not have a whnf. Concerning the reverse implication, if E_0 has a hnf H, there is an integer n such that $E_0 \xrightarrow[h]{n} H$ so there is a X such that $\mathcal{N}(E_0)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}} \xrightarrow[w]{+} X \xleftarrow[w]{a} \mathcal{N}(H)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$. H is of the form $\lambda x_1. \ldots \lambda x_n.x_i\ E_1 \ldots E_p$ and, after administrative reductions, $\mathcal{N}(H)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ is of the form $\mathbf{H}_i\ C\ \Omega\ \overline{\mathbf{n}}$. Then the reduction rule of $\mathbf{H}_i$ yields a whnf. ❑

B Proof of Property 3

It is clear that if $E \rightarrow F$ then $\mathcal{N}(E) \xrightarrow{*} \mathcal{N}(F)$: the redex $(\lambda x.M)\ N$ in E which appears as $\lambda c.(\lambda_c x.\mathcal{N}(M))\ (\lambda f.f\ \mathcal{N}(N)\ c)$ in $\mathcal{N}(E)$ can be reduced in $\lambda c.\mathcal{N}(M)[\mathcal{N}(N)/x]\ c) \rightarrow_\eta \mathcal{N}(M)[\mathcal{N}(N)/x] \equiv \mathcal{N}(M[N/x])$ (Lemma 8) in order to get $\mathcal{N}(F)$. If $\mathcal{N}(E) \rightarrow \mathcal{N}(F)$ then obviously $\mathcal{N}(E)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}} \rightarrow \mathcal{N}(F)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$ (just pick up

the same redex). If E has a normal form S then E $\xrightarrow{*}$ S and $\mathcal{N}$(E) $\xrightarrow{*}$ $\mathcal{N}$(S) and $\mathcal{N}$(E) **A** Ω $\overline{\mathbf{0}}$ $\xrightarrow{*}$ $\mathcal{N}$(S) **A** Ω $\overline{\mathbf{0}}$. So we just have to show that for any closed normal form S, $\mathcal{N}$(S) ↔ $\mathcal{N}$(S) **A** Ω $\overline{\mathbf{0}}$.

Actually we prove the stronger property: for every set of variables $\{x_1,\ldots,x_n\}$ containing the free variables of a normal form S, $\mathcal{N}(S) \leftrightarrow \lambda c.\mathcal{N}(\lambda\vec{x}_n\ .\ S)$ **A** Ω $\overline{\mathbf{0}}$ $(\vec{x}_n\ \ c)$. This is proved by structural induction over the set of normal forms which is defined by $S = x \mid \lambda x_1.\ \ldots\lambda x_n.\ x_i\ S_1\ldots S_m$.

- $S \equiv x_i$ $1\le i\le n$

  $\mathcal{N}(S) \equiv x_i$

  $\lambda c.\mathcal{N}(\lambda\vec{x}_n\ .\ S)$ **A** Ω $\overline{\mathbf{0}}$ $(\vec{x}_n\ \ c) \equiv \lambda c.(\lambda_c\vec{x}_n\ .\ x_i)$ **A** Ω $\overline{\mathbf{0}}$ $(\vec{x}_n\ \ c)$

  $\xrightarrow{*} \lambda c.\mathbf{H}_{i-1}$ **A** Ω $\overline{\mathbf{n}}$ $(\vec{x}_n\ \ c)$

  $\xrightarrow{*} \lambda c.(\lambda_c\vec{x}_n\ \ .\lambda c.x_i$ (**A** (**R** $\overline{\mathbf{n}}$ $(\lambda c.\vec{x}_n\ \ c))$ (**K** c))) $(\vec{x}_n\ \ c)$

  $\xrightarrow{*} \lambda c.(\lambda_c\vec{x}_n\ \ .\lambda c.x_i\ c) (\vec{x}_n\ \ c) \rightarrow_\eta \lambda c.(\lambda_c\vec{x}_n\ \ .x_i) (\vec{x}_n\ \ c)$

  $\equiv \lambda c.(\lambda c.c\ (\lambda x_1.\ \ldots(\lambda c.c\ (\lambda x_n.x_i))\ldots)\ (\lambda f.f\ x_1\ (\ \ldots(\lambda f.f\ x_n\ c)\ \ldots))$

  $\xrightarrow{*} \lambda c.x_i\ c \rightarrow_\eta x_i$

- $S \equiv \lambda x_{n+1}.\ \ldots\lambda x_p.\ x_i\ S_1\ldots S_m$ $\quad 1\le i\le p$ and $p\ge n$

  $\mathcal{N}(S) \equiv \lambda_c x_{n+1}.\ \ldots\lambda_c x_p.\lambda c.\ x_i\ (\mathcal{N}(\vec{S}_m)\quad c)$

  $\lambda c.\mathcal{N}(\lambda\vec{x}_n\ .\ S)$ **A** Ω $\overline{\mathbf{0}}$ $(\vec{x}_n\ \ c) \equiv \lambda c.(\lambda_c\vec{x}_p\ .\ \lambda c.x_i\ (\mathcal{N}(\vec{S}_m)\quad c))$ **A** Ω $\overline{\mathbf{0}}$ $(\vec{x}_n\ \ c)$

  $\xrightarrow{*} \lambda c.\mathbf{H}_{i-1}$ Cont Ω $\overline{\mathbf{p}}$ $(\vec{x}_n\ \ c)$

with Cont = $\lambda f.f\ \mathcal{N}(S_1)[\vec{\mathbf{H}}_p\ \ /\vec{x}_p\ \ ]\ \ldots(\lambda f.f\ \mathcal{N}(S_m)[\vec{\mathbf{H}}_p\ \ /\vec{x}_p\ \ ]$ **A**$)\ \ldots))$

$\xrightarrow{*} \lambda c.(\lambda_c\vec{x}_p\ \ .\lambda c.x_i$ (Cont (**R** $\overline{\mathbf{p}}$ $(\lambda c.\vec{x}_p\ \ c))$ (**K** c))) $(\vec{x}_n\ \ c)$

$\xrightarrow{*} \lambda_c x_{n+1}.\ \ldots\lambda_c x_p.\ \lambda c.x_i$ (Cont (**R** $\overline{\mathbf{p}}$ $(\lambda c.\vec{x}_p\ \ c))$ (**K** c)))

Cont (**R** $\overline{\mathbf{p}}$ $(\lambda c.\vec{x}_p\ \ c))$ (**K** c))=$\lambda f.f\ (\lambda c.\mathcal{N}(S_1)[\vec{\mathbf{H}}_p\ \ /\vec{x}_p\ \ ]$**A** Ω $\overline{\mathbf{p}}$ $(\vec{x}_p\ \ c))(\ldots($**A** (**R** $(\lambda c.\vec{x}_n\ \ c))$ (**K** c)) $\ldots)$

We know that $\mathcal{N}(\lambda\vec{x}_p\ \ .E)$ **A** Ω $\overline{\mathbf{0}} \equiv(\lambda_c\vec{x}_p\ \ .\mathcal{N}(E))$ **A** Ω $\overline{\mathbf{0}}$ $\xrightarrow{*} \mathcal{N}(E[\vec{\mathbf{H}}_p\ \ /\vec{x}_p\ \ ])$ **A** Ω $\overline{\mathbf{p}}$, so

$\lambda c.\mathcal{N}(S_i)[\vec{\mathbf{H}}_p\ \ /\vec{x}_p\ \ ]$ **A** Ω $\overline{\mathbf{p}}$ $(\vec{x}_p\ \ c) \leftrightarrow \lambda c.\mathcal{N}(\lambda\vec{x}_p\ .\ S_i)$ **A** Ω $\overline{\mathbf{0}}$ $(\vec{x}_p\ \ c)$

$\leftrightarrow \mathcal{N}(S_i)$ by induction hypothesis

and since **A** (**R** $(\lambda c.\vec{x}_n\ \ c))$ (**K** c) $\xrightarrow{*}$ c, we have Cont (**R** $(\lambda c.\vec{x}_p\ \ c))$ (**K** c)) ↔ $\mathcal{N}(\vec{S}_m)\quad$ c, hence

$\lambda c.\mathcal{N}(\lambda x_1.\ \ldots\lambda x_n.\lambda x_{n+1}.\ \ldots\lambda x_p.\ x_i\ S_1\ldots S_m)\ (\vec{x}_n\ \ c) \leftrightarrow \lambda_c x_{n+1}.\ \ldots\lambda_c x_p.\ \lambda c.x_i\ (\mathcal{N}(\vec{S}_m)\quad c) \equiv \mathcal{N}(S)$ ❑

C Proof of Property 4

We prove the property for expressions with free variables: if E has its free variables included in $\{x_1,\ldots,x_p\}$ then $E \xrightarrow[s]{*} S$ implies $\mathcal{N}(\lambda\vec{x}_p\ .\ E)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}} \xrightarrow[w]{*} X \xleftarrow[w]{a} \mathcal{N}(\lambda\vec{x}_p\ .\ S)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$. The proof of the property for one reduction step is done by induction on the definition of strong reduction (section 2) and is otherwise similar to the proof of Property 2. ❑

D Proof of Property 5

As with Property 3 we show that if S is a normal form with a set of free variables included in $\{x_1,\ldots,x_n\}$ then $\mathcal{N}(\lambda\vec{x}_n\ .\ S) \leftrightarrow \mathcal{N}(\lambda\vec{x}_n\ .\ S)\ \mathbf{A}\ \Omega\ \overline{\mathbf{0}}$. This is proved by structural induction over the set of normal forms which is defined by $S = x \mid \lambda x_1.\ \ldots\lambda x_n.\ x_i\ S_1\ldots S_m$. ❑

E Proof of Property 6

The expression must be of the form $\lambda x_1.\ \ldots\lambda x_m.(\lambda y.E)\ F_0\ F_1\ldots F_p$, where $p\geq 0$ and $0\leq m\leq n$; n being the functionality of the expression.The proof is similar to the proof of Property 2 still using Lemma 8 and Lemma 9. ❑

F Proof of Property 7

As with Property 3 we just show that for any closed normal form S of functionality n, $\mathcal{N}(S) \leftrightarrow \mathcal{N}(S)\ (\vec{\mathbf{X}}_n\ \ \mathbf{L}_n)$. The normal form must be of the form $S \equiv \lambda x_1.\ \ldots\lambda x_n.\ x_i\ S_1\ldots S_m$.

$\mathcal{N}(S) \xrightarrow{*} \lambda_c\vec{x}_n\ .\ \lambda c.\ x_i\ (\mathcal{N}(\vec{S_m})\ \ c)$

$\mathcal{N}(S)\ (\vec{\mathbf{X}}_n\ \ \mathbf{L}_n) \xrightarrow{*} \mathbf{X}_n^i\ (\mathcal{N}(\vec{S_m})\ \ [\vec{\mathbf{X}}_n\ \ /\vec{x}_n\ \ ]\ \mathbf{L}_n)$

$\xrightarrow{*} \lambda_c\vec{x}_n\ .\ \lambda c.\ x_i\ ((\mathcal{N}(\vec{S_m})\ \ [\vec{\mathbf{X}}_n\ \ /\vec{x}_n\ \ ]\ \mathbf{L}_n)\ (\mathbf{R}_n\ (\lambda c.\ \vec{x}_n\ \ c))\ (\mathbf{K}\ c))$

$\xrightarrow{*} \lambda_c\vec{x}_n\ .\ \lambda c.\ x_i\ (\lambda f.f\ (\lambda c.\mathcal{N}(S_1)[\vec{\mathbf{X}}_n\ \ /\vec{x}_n\ \ ]\ \mathbf{L}_n\ (\vec{x}_n\ \ c))\ \ldots(\mathbf{L}_n\ (\mathbf{R}_n\ (\lambda c.\ \vec{x}_n\ \ c))(\mathbf{K}\ c))\ldots)$

and since $\mathbf{L}_n\ (\mathbf{R}_n\ (\lambda c.\ \vec{x}_n\ \ c))\ (\mathbf{K}\ c) \xrightarrow{*} (\lambda_c\vec{x}_n\ .\ \ldots)(\mathbf{K}\ c) \rightarrow \mathbf{K}\ c\ (\lambda x_1.\lambda_c x_2\ \ldots) \rightarrow c$

$\xrightarrow{*} \lambda_c\vec{x}_n\ .\ \lambda c.\ x_i\ (\lambda f.f\ (\lambda c.\mathcal{N}(S_1)[\vec{\mathbf{X}}_n\ \ /\vec{x}_n\ \ ]\ \mathbf{L}_n\ (\vec{x}_n\ \ c))\ \ldots c)\ldots)$

If we show that for every normal form S_i, $\mathcal{N}(S_i)[\vec{\mathbf{X}}_n\ \ /\vec{x}_n\ \ ]\ \mathbf{L}_n \leftrightarrow \mathcal{N}(\lambda\vec{x}_n\ \ .S)$ then

$\leftrightarrow \lambda_c\vec{x}_n\ .\ \lambda c.\ x_i\ (\lambda f.f\ (\lambda c.\mathcal{N}(\lambda\vec{x}_n\ \ .S_1)\ (\vec{x}_n\ \ c))\ \ldots(\lambda f.f\ (\lambda c.\mathcal{N}(\lambda\vec{x}_n\ \ .S_m)\ (\vec{x}_n\ \ c))\ c)\ldots)$

$\xrightarrow{*} \lambda_c\vec{x}_n\ .\ \lambda c.\ x_i\ (\mathcal{N}(\vec{S_m})\ \ c)$

We prove the stronger property: If S is a normal form with a set of free variables included in $\{x_1,\ldots,x_n,z_1,\ldots,z_p\}$ then $\mathcal{N}(\vec{S})\ \ [\vec{\mathbf{X}}_n\ \ /\vec{x}_n\ \ ]\ [\mathbf{Z}_n\ \vec{z}_p\ \ /\vec{z}_p\ \ ]\ \mathbf{L}_n \leftrightarrow \mathcal{N}(\lambda\vec{x}_n\ .\ S)$. This is proved by structural induction over the set of normal forms defined by $S = y \mid \lambda x_1.\ \ldots\lambda x_n.\ y\ S_1\ldots S_m$. ❑

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