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# ***Practical aspects of the Moreau-Yosida regularization I: theoretical properties***

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## Practical aspects of the Moreau-Yosida regularization I: theoretical properties

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**Abstract:** When computing the infimal convolution of a convex function  $f$  with the squared norm, one obtains the so-called Moreau-Yosida regularization of  $f$ . Among other things, this function has a Lipschitzian gradient. We investigate some more of its properties, relevant for optimization. Our main result concerns second-order differentiability and is as follows. Under assumptions that are quite reasonable in optimization, the Moreau-Yosida is twice differentiable if and only if  $f$  is twice differentiable as well. In the course of our development, we give

some results of general interest in convex analysis. In particular, we establish primal-dual relationship between the remainder terms in the first-order development of a convex function and its conjugate.

**Key-words:** Convex optimization, mathematical programming, proximal point, second-order differentiability.

*(Résumé : tsvp)*

# Aspects pratiques de la régularisée de Moreau-Yosida I: propriétés théoriques

**Résumé :** La régularisée de Moreau-Yosida d'une fonction convexe  $f$  est l'inf-convolution de  $f$  avec le carré de la norme. Cette régularisée a un gradient lipschitzien, nous avons étudié d'autres propriétés, d'intérêt en optimisation. Notre résultat principal concerne les dérivées secondes. Sous des hypothèses courantes en optimisation, la Moreau-Yosida est deux fois différentiable si et seulement si  $f$  l'est aussi. Au cours de notre étude, nous prouvons quelques résultats généraux d'analyse convexe. Nous établissons en particulier des relations primales-duales entre les développements au premier ordre d'une fonction convexe et de sa conjuguée.

**1. Introduction.** The motivation for this paper is to explore the possibility of introducing efficient preconditioners into the proximal-point algorithm to minimize a convex function  $f$ . This algorithm ([2], [8], [14]) is essentially an implicit (sub)gradient method. However it is much more fruitful to see it as the ordinary gradient method applied to a certain perturbation of  $f$ : the Moreau-Yosida regularization ([10], [16]), whose minima coincide with those of  $f$ . The idea of a preconditioner is thus natural, first steps into this direction were already made in [12], [3]. Naturally, such a preconditioner has to exploit the second-order properties of the perturbed objective function; a study of these properties is therefore a prerequisite to the development of any reasonable algorithm. Here, we address this last, purely theoretical, question; we study also some other properties relevant for optimization; specifically, we relate the smoothness, behaviour at infinity, strong convexity of an objective function to the corresponding properties of its Moreau-Yosida. A subsequent paper will exploit the results obtained here to develop some related algorithms, emphasizing the implementable aspect. Along these lines, we also mention the computational considerations contained in [4], [1], [6], [15], [9], [7].

Our notation follows closely that of [13] and [5]. In the space  $\mathbb{R}^N$ , the Euclidean product is denoted by  $\langle \cdot, \cdot \rangle$ , and  $\| \cdot \|$  is the associated norm;  $B(x, \rho)$  is the ball centered at  $x$  with radius  $\rho$ . Given a symmetric positive definite linear operator  $M$ , we set  $\langle \cdot, \cdot \rangle_M := \langle M \cdot, \cdot \rangle$ ; accordingly, we will shorten  $\|x\|_M^2 := \langle x, x \rangle_M$ . The smallest and largest eigenvalues of  $M$  will be denoted by  $\lambda$  and  $\Lambda$  respectively.

We denote by  $F$  the Moreau-Yosida regularization of a given closed convex function  $f$ , associated to the metric defined by  $M$ :

$$(1) \quad F(x) := \min_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2} \|y - x\|_M^2 \right\} =: \left( f \mathbin{\mathbb{J}} \frac{1}{2} \| \cdot \|_M^2 \right) (x),$$

where  $\mathbin{\mathbb{J}}$  stands for the infimal convolution.

First-order regularity of  $F$  is well known: without any further assumption,  $F$  has a Lipschitzian gradient. More precisely, for all  $x_1, x_2 \in \mathbb{R}^N$ :

$$(2) \quad \|\nabla F(x_1) - \nabla F(x_2)\|^2 \leq \Lambda \langle \nabla F(x_1) - \nabla F(x_2), x_1 - x_2 \rangle.$$

If we denote by  $p(x)$  the unique minimizer in (1), called the *proximal* point of  $x$ ,  $\nabla F(x)$  has the following expression

$$(3) \quad G := \nabla F(x) = M(x - p(x)) \in \partial f(p(x)).$$

Note in particular that  $f$  has a nonempty subdifferential at any point  $p$  of the form  $p(x)$ .

The main aim of this paper is to suggest the following: even though the Moreau-Yosida is a powerful tool to bring first-order differentiability, it can hardly go beyond. Indeed, the existence of a Hessian of  $F$  normally implies that  $f$  itself has a Hessian. This sort of rough statement will be made precise in §4, using a rather sophisticated machinery which we develop in §3. In §2, we review a few elementary results on the Moreau-Yosida  $F$  of (1), which are relevant when developing optimization algorithms. Some of them are easy and/or already known, at least for  $M = Id$ . For the sake of simplicity, we often consider finite-valued objective functions  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ . This avoids some technical difficulties and makes the reading lighter.

**2. Properties of the Moreau-Yosida.** We study here some properties which  $F$  of (1) inherits from  $f$ . For the reader's convenience, we start by recalling a few classical results in convex analysis, which will be frequently used in the sequel. They can be found in [13] or Chapter X in [5].

- (i) The conjugate of a closed (i.e. lsc) convex function  $\varphi$  is
- (4) 
$$\varphi^*(g) := \sup_{x \in \mathbb{R}^N} \{ \langle g, x \rangle - \varphi(x) \} .$$
- (ii) The conjugacy operation is an involution, i.e. the conjugate of  $\varphi^*$  is  $\varphi$  itself.
- (iii) The conjugate of a sum is the infimal convolution of the conjugates.
- (iv) If  $C$  is a closed convex set, the conjugate of its indicator function  $I_C$  (0 on  $C$ ,  $+\infty$  outside) is

$$(I_C)^*(g) = \sigma_C(g) := \sup \{ \langle g, x \rangle : x \in \mathbb{R}^N \} ,$$

the support function of  $C$ .

- (v) The subdifferential of  $\varphi^*$  is the argmax in (4). Said otherwise,

$$g \in \partial \varphi(x) \iff \langle g, x \rangle - \varphi^*(g) \geq \langle \gamma, x \rangle - \varphi^*(\gamma) \text{ for all } \gamma .$$

- (vi) The directional derivative of a finite-valued convex function  $\varphi$  is the support function of its subdifferential:  $\varphi'(x; \cdot) = \sigma_{\partial \varphi(x)}(\cdot)$ .
- (vii) Naturally, all these properties imply for example  $[I_{B(0, \varepsilon)}]^*(\cdot) = \varepsilon \|\cdot\|$ , and also  $[\varphi'(x; \cdot)]^* = I_{\partial \varphi(x)}$ .

We first show that  $f$  and  $F$  have the same behaviour at infinity. Recall that the recession (or asymptotic) function of a closed convex function  $\varphi$  is defined by

$$\varphi'_\infty(d) = \lim_{t \rightarrow +\infty} [\varphi(x + td) - \varphi(x)]/t$$

(a limit which does not depend on  $x \in \text{dom } \varphi$ ). This function is useful because  $\varphi$  has a nonempty bounded set of minima if and only if  $\varphi'_\infty(d) > 0$  for all  $d \neq 0$ .

**THEOREM 2.1.** *The recession functions of  $f$  and  $F$  are identical.*

*Proof.* Apply Corollary 9.2.1 in [13]: the recession function of an infimal convolution is the infimal convolution of the recession functions. Then we obtain

$$F'_\infty(d) = \left( f'_\infty \mathbin{\dot{+}} I_{\{0\}} \right) (d) = \inf_{y=0} f'_\infty(d - y) = f'_\infty(d) ,$$

because the recession function of a squared norm is clearly  $I_{\{0\}}$ .  $\square$

Recall that a function  $\varphi$  is said to be *strongly convex* with modulus  $c > 0$  if and only if  $\varphi(\cdot) - \frac{1}{2}c\|\cdot\|^2$  is a convex function. This property plays the role of nondegenerate Hessians in smooth optimization; as such, it is fairly relevant for optimization algorithms. We show that strong convexity is transmitted between  $f$  and  $F$ . Dually, smoothness is likewise transmitted between  $f^*$  and  $F^*$ .

**THEOREM 2.2.** *For a finite-valued convex function  $f$ , the following statements are equivalent:*

- (i)  $f$  is strongly convex with modulus  $c$ ;
- (ii)  $f^*$  has a Lipschitzian gradient with Lipschitz constant  $1/c$ ;
- (iii)  $F^*$  has a Lipschitz continuous gradient with constant  $1/c + 1/\lambda$ ;
- (iv)  $F$  is strongly convex with modulus  $c \frac{\lambda}{\lambda+c}$ .

*Proof.* Because  $f$  and  $F$  are finite-valued, Theorems X.4.2.1 and X.4.2.2 in [5] can be applied to yield the equivalences (i)  $\iff$  (ii) and (iii)  $\iff$  (iv).

Let us prove (ii)  $\iff$  (iii). Since  $F$  is the infimal convolution of  $f$  and  $\frac{1}{2}\|\cdot\|_M^2$ , its conjugate is the sum of the respective conjugates:  $F^*(\cdot) = f^*(\cdot) + \frac{1}{2}\|\cdot\|_{M^{-1}}^2$ . Actually

$$\nabla F^*(\cdot) = \nabla f^*(\cdot) + M^{-1}(\cdot),$$

whenever one of the gradients exists.  $\square$

We now turn our attention to properties involving the proximal operator more directly. They will be useful for the study of second-order smoothness.

**PROPOSITION 2.3.** *For any  $x_1$  and  $x_2$  in  $\mathbb{R}^N$ ,*

$$(5) \quad \|p(x_1) - p(x_2)\|_M^2 \leq \langle x_1 - x_2, p(x_1) - p(x_2) \rangle_M.$$

*It follows that the mapping  $x \mapsto p(x)$  is Lipschitzian with constant  $\Lambda/\lambda$ .*

*Proof.* For arbitrary  $p_1, p_2 \in \mathbb{R}^N$  and  $G_i \in \partial f(p_i)$ , the convexity of  $f$  gives the monotonicity of the subgradients:  $\langle G_1 - G_2, p_1 - p_2 \rangle \geq 0$ . Take now  $x_1$  and  $x_2$  in  $\mathbb{R}^N$ , and write the inequality for  $p_i := p(x_i)$ ,  $G_i$  from (3):

$$\langle M(x_1 - p(x_1)) - M(x_2 - p(x_2)), p(x_1) - p(x_2) \rangle \geq 0,$$

which is (5). From this we can obtain

$$\lambda \|p(x_1) - p(x_2)\|^2 \leq \Lambda \|x_1 - x_2\| \|p(x_1) - p(x_2)\|$$

and the Lipschitz property follows immediately.  $\square$

**PROPOSITION 2.4.** *Assume  $f$  is a closed convex function. Then  $\nabla F(\cdot)$  has directional derivatives if and only if  $p(\cdot)$  has directional derivatives. Moreover, the Hessian  $\nabla^2 F(x)$  exists if and only if  $p$  has a Jacobian  $P'(x)$ :*

$$\nabla^2 F(x) = M(Id - P'(x)) \quad \text{for all } x \in \mathbb{R}^N.$$

*Proof.* Straightforward from (3).  $\square$

As observed in [9], a space decomposition reveals important when combining quasi-Newton updates with proximal-point algorithms. Along these lines, we show that the directional derivatives of  $p(\cdot)$  lie in  $N_{\partial f(p(x))}(G)$ , the normal cone to  $\partial f(p(x))$  at  $G$ , whenever they exist.

**LEMMA 2.5.** *For the closed convex function  $f$ , let  $G$  be defined by (3), and denote by  $\mathcal{N} := N_{\partial f(p(x))}(G)$  the normal cone to  $\partial f(p(x))$  at  $G$ . Then any cluster point of  $\{[p(x + td) - p(x)]/t\}_{t \downarrow 0}$  lies in  $\mathcal{N}$ .*

*Proof.* By definition,  $p(x + td)$  minimizes (1) with  $x$  replaced by  $x + td$ , therefore

$$f(p(x + td)) + \frac{1}{2}\|p(x + td) - x - td\|_M^2 \leq f(p(x)) + \frac{1}{2}\|p(x) - x - td\|_M^2,$$



that is,

$$f(p(x + td)) - f(p(x)) \leq \frac{1}{2} \|p(x) - x - td\|_M^2 - \frac{1}{2} \|p(x + td) - x - td\|_M^2.$$

Apply to the lefthand side the subgradient inequality for any  $g \in \partial f(p(x))$  and develop the righthand side to obtain

$$\langle p(x + td) - p(x), g \rangle \leq \frac{1}{2} \langle p(x) - p(x + td), p(x) + p(x + td) - 2x - 2td \rangle_M.$$

Due to (3),  $\langle p(x) - p(x + td), G \rangle = \frac{1}{2} \langle p(x) - p(x + td), 2x - 2p(x) \rangle_M$  which, added to both sides, gives:

$$\langle p(x + td) - p(x), g - G \rangle \leq -\frac{1}{2} \|p(x + td) - p(x)\|_M^2 + t \langle p(x + td) - p(x), d \rangle_M.$$

Finally, divide by  $t$ :

$$\begin{aligned} \left\langle \frac{p(x + td) - p(x)}{t}, g - G \right\rangle &\leq -\frac{\|p(x + td) - p(x)\|_M^2}{2t} + \langle p(x + td) - p(x), d \rangle_M \\ &\leq \langle p(x + td) - p(x), d \rangle_M, \end{aligned}$$

for all  $g \in \partial f(p(x))$ . Passing to the limit we obtain the desired result, because of the continuity of  $p(\cdot)$ .  $\square$

**COROLLARY 2.6.** *If  $p(\cdot)$  has a Jacobian  $P'(x)$ , then  $\text{Im } P'(x) \subset \mathcal{N}$ . When  $x \rightarrow x_0$ , all the cluster points of  $\frac{p(x) - p(x_0)}{\|x - x_0\|}$  lie in  $\mathcal{N} \cap B(0, \Lambda/\lambda)$ .*

*Proof.* The first statement is clear from Lemma 2.5. The proof of the second statement is classical for Lipschitzian functions: with  $x$  tending to  $x_0$ , extract a subsequence such that  $\frac{x - x_0}{\|x - x_0\|} \rightarrow d$ ; setting  $t := \|x - x_0\| \downarrow 0$ , we have

$$x = x_0 + td + q(t), \quad \text{with } \frac{q(t)}{t} \rightarrow 0 \text{ when } t \downarrow 0.$$

Then write

$$p(x) - p(x_0) = p(x_0 + td) - p(x_0) + p(x_0 + td + q(t)) - p(x_0 + td)$$

to obtain with Proposition 2.3

$$\left\| \frac{p(x) - p(x_0)}{\|x - x_0\|} - \frac{p(x_0 + td) - p(x_0)}{t} \right\| = \frac{\|p(x_0 + td + q(t)) - p(x_0 + td)\|}{t} \leq \frac{\Lambda}{\lambda} \frac{\|q(t)\|}{t}$$

and the result follows.  $\square$

A direct consequence of Lemma 2.5 is that  $\nabla F$  enjoys automatically some directional differentiability.

**PROPOSITION 2.7.** *For the closed convex function  $f$ , let  $G$  be defined by (3), and denote by  $\mathcal{T}$  the tangent cone to  $\partial f(p(x))$  at  $G$ . Then, for any  $d$  such that  $Md \in \mathcal{T}$ ,*

$$\frac{\nabla F(x + td) - \nabla F(x)}{t} \longrightarrow Md \quad \text{when } t \downarrow 0.$$

*Proof.* From (3),  $\nabla F(x + td) - \nabla F(x) = tMd - M(p(x + td) - p(x))$ , we only need to show that  $[p(x + td) - p(x)]/t$  tends to 0 when  $t \downarrow 0$ . For this, use (5):

$$\langle M(p(x + td) - p(x)), p(x + td) - p(x) \rangle \leq t \langle p(x + td) - p(x), Md \rangle .$$

Observing that the lefthand side is minorized by  $\lambda \|p(x + td) - p(x)\|^2$ , divide by  $t^2$  to obtain

$$0 \leq \lambda \frac{\|p(x + td) - p(x)\|^2}{t^2} \leq \left\langle \frac{p(x + td) - p(x)}{t}, Md \right\rangle .$$

In view of Lemma 2.5, the (bounded) righthand side cannot have any positive cluster point, it must tend to 0 and the proof is complete.  $\square$  Of course, because  $\nabla F$  is Lipschitzian, we can proceed as in Corollary 2.6 to see that, if  $x \rightarrow x_0$  in such a way that  $(x - x_0)/\|x - x_0\| \rightarrow d$ , with  $Md \in \mathcal{T}$ , then

$$\frac{\nabla F(x) - \nabla F(x_0)}{\|x - x_0\|} \longrightarrow Md .$$

As an illustration, take the bivariate function  $f(\xi, \eta) = |\xi| + \frac{1}{2}\eta^2$  and  $M = Id$ . The optimality condition for the proximal point  $(\pi, \rho)$  of  $(\xi, \eta)$  close to 0 results in

$$\pi = 0 \quad \text{if } |\xi| \leq 1, \quad \text{and} \quad \rho = \eta/2 .$$

Thus, at  $x = 0$ ,  $\partial f(x) = [-1, 1] \times \{0\}$ ,  $p(x) = 0$ ,  $G = 0$  and  $p(\cdot)$  has the Jacobian  $\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}$ . We see that the non-differentiability of  $f$  at 0 in the subspace  $\mathcal{T} = \mathbb{R} \times \{0\}$  does not affect the second-order differentiability of  $F$ .

We conclude this section with a trivial but not so well-known observation: the proximal mapping has an explicit inverse. This may be very useful when designing algorithms, see [7].

**THEOREM 2.8.** *Let  $p$  be such that  $\partial f(p) \neq \emptyset$  and take  $G \in \partial f(p)$ . Then  $p$  is the proximal point of  $x := p + M^{-1}G$ .*

*Proof.* We have  $M(x - p) \in \partial f(p)$  and this characterizes the proximal point in a unique way, see (3).  $\square$

**3. Some results in convex analysis.** In this section we gather some results of general interest concerning convex sets and functions. They will be instrumental for our second-order analysis.

**3.1. On the geometry of convex sets.** In the following Propositions we characterize those convex cones that are subspaces. For the subspace  $\mathcal{M}$  in Proposition 3.1, see Theorem 2.7 in [13].

**PROPOSITION 3.1.** *Let  $\mathcal{N}$  be a closed convex cone and call  $\mathcal{M} := \mathcal{N} \cap \{-\mathcal{N}\}$  the largest subspace contained in  $\mathcal{N}$ . Then*

$$\mathcal{N} \text{ is a subspace if and only if } \mathcal{N} \cap \mathcal{M}^\perp = \{0\} .$$

Moreover, for any  $\nu_0 \in \mathcal{M}^\perp$  and  $\nu \in \mathcal{N}$ ,

$$(6) \quad \langle \nu_0, \nu \rangle \neq 0 \implies -\nu \notin \mathcal{N}.$$

*Proof.* When the convex cone  $\mathcal{N}$  is a subspace, it is symmetric:  $\mathcal{N} = -\mathcal{N} = \mathcal{M}$ . In this case,  $\mathcal{N} \cap \mathcal{M}^\perp = \mathcal{M} \cap \mathcal{M}^\perp = \{0\}$ .

Conversely, when  $\mathcal{N} \cap \mathcal{M}^\perp = \{0\}$ , suppose for contradiction that  $\mathcal{N}$  is not a subspace; we can take  $\nu \in \mathcal{N} \setminus \mathcal{M}$ , which can be expressed as a direct sum:

$$\nu = \nu_m + \nu_0 \quad \text{with } \nu_m \in \mathcal{M} \text{ and } 0 \neq \nu_0 \in \mathcal{M}^\perp.$$

Since  $\mathcal{M}$  is a symmetric set,  $-\nu_m \in \mathcal{M} \subset \mathcal{N}$  and, since  $\mathcal{N}$  is a convex cone,

$$\nu_0 = \nu + (-\nu_m) \in \mathcal{N};$$

thus we have exhibited a nonzero  $\nu_0 \in \mathcal{N} \cap \mathcal{M}^\perp$ . This is the required contradiction.

Finally, we have to prove (6). Take  $\nu_0 \in \mathcal{M}^\perp$  and  $\nu \in \mathcal{N}$ . If  $-\nu$  were in  $\mathcal{N}$ , it would be also in  $\mathcal{M}$  and this would contradict  $\langle \nu_0, \nu \rangle \neq 0$ .  $\square$

The next result deals with  $N_C(g_0)$ , the normal cone to a closed convex set  $C$  at  $g_0 \in C$ . It will be used in Proposition 4.3 with  $C := \partial f(p(x))$  and  $g_0 := G$  of (3).

**PROPOSITION 3.2.** *Assume  $C \subset \mathbb{R}^N$  is a closed convex set and let  $g_0 \in C$ . Then*

$$\mathcal{N} := N_C(g_0) \text{ is a subspace} \iff g_0 \in \text{ri } C,$$

where  $\text{ri}$  denotes the relative interior.

*Proof.* Call  $\sigma_C$  the support function of  $C$ . By definition of normal cones,  $s \in \mathcal{N}$  exactly when  $\langle s, g_0 \rangle = \sigma_C(s)$ . The convex cone  $\mathcal{N}$  is a subspace if and only if it is symmetric. Thus,  $\mathcal{N}$  is a subspace if and only if the following property holds:

$$s \in \mathcal{N} \implies \sigma_C(s) + \sigma_C(-s) = 0, \quad [= \langle s, g_0 \rangle + \langle -s, g_0 \rangle]$$

or equivalently

$$\sigma_C(d) + \sigma_C(-d) > 0 \implies d \notin \mathcal{N},$$

i.e., applying the definition of  $\mathcal{N}$ :

$$\sigma_C(d) + \sigma_C(-d) > 0 \implies \exists g_d \in C : \langle d, g_d - g_0 \rangle > 0,$$

which in turn can be written

$$\sigma_C(d) + \sigma_C(-d) > 0 \implies \sigma_C(d) > \langle d, g_0 \rangle.$$

By Theorem 13.1 in [13] or Theorem V.2.2.3(ii) in [5], this last property just expresses  $g_0 \in \text{ri } C$ .  $\square$

We now characterize those convex sets that are simultaneously closed and relatively open.

PROPOSITION 3.3. *Assume  $C \subset \mathbb{R}^N$  is a closed convex set. Then*

$$C = \operatorname{ri} C \quad \Longleftrightarrow \quad C = \operatorname{aff} C ,$$

where  $\operatorname{aff}$  denotes the affine hull. If, in addition,  $C$  is bounded, then  $C = \operatorname{ri} C$  if and only if  $C$  is a singleton.

*Proof.* Since an affine set is relatively open, the “ $\Leftarrow$ ” part is straightforward. Our proof of the “ $\Rightarrow$ ” part is geometric. Assume  $C = \operatorname{ri} C$ ; from Theorem 6.4 in [13], this means

$$(7) \quad \forall x_1, x_2 \in C, \quad \sup\{t : tx_1 + (1-t)x_2 \in C\} > 1 .$$

Fix  $x_1, x_2$  and call  $T$  the above supremum; we claim that  $T = +\infty$ . If  $T$  were finite, the vector  $x_3 := Tx_1 + (1-T)x_2$  would be in  $C$  ( $C$  is closed). But then the optimality of  $T$  would imply

$$\sup\{t : tx_1 + (1-t)x_3 \in C\} = 1 ,$$

which is impossible: replace  $x_2$  by  $x_3$  in (7).

Exchanging  $x_1$  and  $x_2$ , we see that  $C$  contains the affine hull of  $\{x_1, x_2\}$ . We conclude  $C = \operatorname{aff} C$ , since  $x_1$  and  $x_2$  were arbitrary.

The rest of the statement follows because the only affine bounded sets are the singletons.

□

**3.2. First-order developments of convex functions.** We now present a theory analogous to that of § X.4.2(b) in [5]. Instead of finding quadratic bounds depending on a particular subgradient, we consider here the first-order expansion of a convex function  $\varphi$ :

$$\varphi(x_0 + h) = \varphi(x_0) + \varphi'(x_0; h) + o(\|h\|) .$$

Clearly, the remainder term is nonnegative, we want to estimate it more accurately. To bound it from above means to find  $\varepsilon > 0$  and some function  $r$  satisfying

$$r \text{ is convex, nonnegative, differentiable at } 0, \quad r(0) = 0, \quad \nabla r(0) = 0 ,$$

such that

$$(8) \quad \varphi(x_0 + h) \leq \varphi(x_0) + \varphi'(x_0; h) + r(h) + \mathbf{I}_{B_\varepsilon}(h) \quad \text{for all } h .$$

For a lower bound we need likewise  $\varepsilon$  and  $r$  such that

$$(9) \quad \varphi(x_0 + h) + \mathbf{I}_{B_\varepsilon}(h) \geq \varphi(x_0) + \varphi'(x_0; h) + r(h) \quad \text{for all } h .$$

Here and throughout,  $\mathbf{I}_{B_\varepsilon}$  denotes the indicator function of  $B(0, \varepsilon)$ . Note that, despite the appearances, the existence of  $\nabla r(0)$  is automatic from (8); by contrast, convexity of  $r$  is a more restrictive assumption.

Our aim is to study the duality between (8) and (9): if  $\varphi$  satisfies one of these inequalities, does  $\varphi^*$  satisfy the other? The answer is given in Theorems 3.5 and 3.7 below, which play

the role of respectively Theorems X.4.2.7 and X.4.2.6 in [5]. To establish this, we will always need the following assumption on  $r$ :

$$(10) \quad \text{there exists } c > 0 \text{ such that } r(h) \geq \frac{1}{2}c\|h\|^2 \quad \text{for all } h.$$

We will frequently compute the conjugate of  $\psi + \mathbf{I}_{B_\varepsilon}$ , i.e. the infimal convolution

$$(11) \quad \Psi_\varepsilon^*(g) := (\psi + \mathbf{I}_{B_\varepsilon})^*(g) = \min_{s \in \mathbb{R}^N} \{\psi^*(g - s) + \varepsilon\|s\|\}.$$

This is the so-called Lipschitzian regularization of  $\psi^*$ , and we start by showing some properties of this operation, illustrated by Fig. 1.

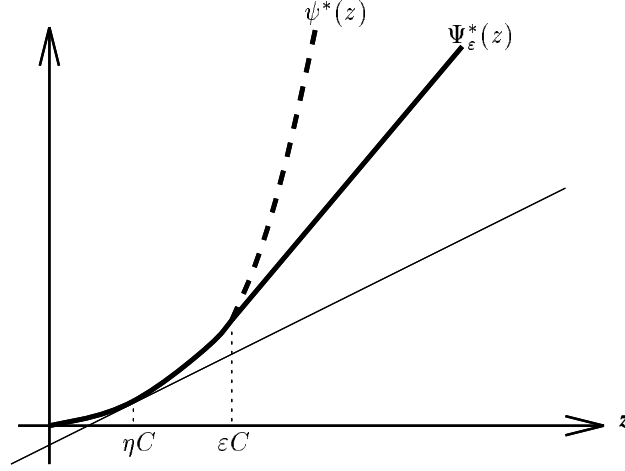


FIG. 1. Lipschitzian Regularization

LEMMA 3.4. *Let  $\psi$  be a finite-valued convex function and denote by  $\Psi_\varepsilon^*$  the Lipschitzian regularization of its conjugate  $\psi^*$ .*

(i) *If there are  $c > 0$  and  $\delta > 0$  such that*

$$(12) \quad \psi(h) \geq \psi(0) + \psi'(0; h) + \frac{1}{2}c\|h\|^2 \quad \text{for } \|h\| \leq \delta,$$

*then  $\Psi_\varepsilon^*(s) = \psi^*(s)$  for all  $\varepsilon \leq \delta$  and  $s \in \partial\psi(0) + B(0, \varepsilon c/2)$ .*

(ii) *If  $\psi$  has the form  $\psi(h) = \frac{1}{2}C\|h\|^2$  (i.e.  $\psi^* = \frac{1}{2C}\|\cdot\|^2$ ) for some  $C > 0$ , then*

$$\Psi_\varepsilon^*(z) = \begin{cases} \frac{1}{2C}\|z\|^2 & \text{if } \|z\| \leq \varepsilon C \\ -\frac{\varepsilon^2 C}{2} + \varepsilon\|z\| & \text{if not.} \end{cases}$$

(iii) *With  $\psi$  as in (ii) and  $\eta \in ]0, \varepsilon]$ , there holds*

$$\Psi_\varepsilon^*(z) \geq -\frac{\eta^2 C}{2} + \eta\|z\| \quad \text{for all } z.$$

*Proof.* [(i)] Apply Proposition XI.3.4.5 of [5]:  $\psi^*$  and its regularization  $\Psi_\varepsilon^*$  coincide on the set

$$\{s \in \mathbb{R}^N : \psi^*(s) = \Psi_\varepsilon^*(s)\} = \{s \in \mathbb{R}^N : \partial\psi^*(s) \cap B(0, \varepsilon) \neq \emptyset\}.$$

Since  $s \in \partial\psi(0)$  means  $0 \in \partial\psi^*(s)$ , this coincidence set contains in particular  $\partial\psi(0)$ .

To say that  $\partial\psi^*(s) \cap B(0, \varepsilon) \neq \emptyset$  is to say that  $\langle s, \cdot \rangle - \psi(\cdot)$  attains its maximum  $\psi^*(s)$  on  $B(0, \varepsilon)$ . Accordingly, let us find an upper bound for

$$A := \sup_{\|h\| > \varepsilon} \{\langle s, h \rangle - \psi(h)\}.$$

For this, use the convexity of  $\psi$  on  $[0, h]$ : if  $h \notin B(0, \varepsilon)$ ,

$$\psi(\varepsilon h / \|h\|) \leq \psi(0) + \frac{\varepsilon}{\|h\|} [\psi(h) - \psi(0)],$$

which, after some algebraic manipulations using (12), gives when  $\varepsilon \leq \delta$ :

$$\psi(h) \geq \psi(0) + \psi'(0; h) + \frac{1}{2}c\varepsilon\|h\|.$$

Therefore

$$\begin{aligned} A &\leq \sup_{\|h\| > \varepsilon} \{\langle s, h \rangle - \psi(0) - \psi'(0; h) - \frac{1}{2}c\varepsilon\|h\|\} \\ &\leq -\psi(0) + \sup_{\|h\| \in \mathbb{R}^N} \{\langle s, h \rangle - \psi'(0; h) - \frac{1}{2}c\varepsilon\|h\|\} \\ &= -\psi(0) + \left( \psi'(0; \cdot) + \frac{1}{2}c\varepsilon\|\cdot\| \right)^*(s) \\ &= -\psi(0) + \left( \mathbf{I}_{\partial\psi(0)} \downarrow \mathbf{I}_{B(0, \varepsilon c/2)} \right)^*(s). \end{aligned}$$

The last infimal convolution is zero when  $s \in \partial\psi(0) + B(0, \varepsilon c/2)$ . For every such  $s$  which is not in  $\partial\psi(0)$ , we thus have

$$A \leq -\psi(0) = \inf \psi^* < \psi^*(s).$$

Then  $\langle s, h \rangle - \psi(h)$  attains its maximum on  $B(0, \varepsilon)$ ; (i) is proved.

[(ii)] For our quadratic function  $\psi$ , apply the dual definition (11):

$$\begin{aligned} \Psi_\varepsilon^*(z) &= \left( \frac{1}{2}C\|\cdot\|^2 + \mathbf{I}_{B_\varepsilon} \right)^*(z) = \sup_{x \in B(0, \varepsilon)} \{\langle z, x \rangle - \frac{1}{2}C\|x\|^2\} \\ &= C \sup_{x \in B(0, \varepsilon)} \{\langle z/C, x \rangle - \frac{1}{2}\|x\|^2\} \\ &= C \sup_{x \in B(0, \varepsilon)} \left\{ \frac{1}{2}\|z/C\|^2 - \frac{1}{2}\|x - z/C\|^2 \right\} \\ &= \frac{1}{2C}\|z\|^2 - \frac{C}{2} \inf_{x \in B(0, \varepsilon)} \|x - z/C\|^2 \\ &= \frac{1}{2C}\|z\|^2 - \frac{C}{2} d_{B(0, \varepsilon)}^2(z/C). \end{aligned}$$

To finish the proof observe that the distance function  $d_{B(0, \varepsilon)}(z/C)$  is equal to  $\varepsilon - \|z\|/C$  whenever  $z/C \notin B(0, \varepsilon)$ .

[(iii)] We consider two cases. If  $\|z\| \leq \varepsilon C$ , the result follows from

$$\Psi_\varepsilon^*(z) - \left( -\eta^2 C/2 + \eta\|z\| \right) = \frac{1}{2C}\|z\|^2 + \frac{1}{2}\eta^2 C - \eta\|z\| = \frac{1}{2C}(\|z\| - \eta C)^2.$$

In the other case, set  $q(\eta) := -\eta^2 C/2 + \eta\|z\|$  (so  $q = \Psi_\varepsilon^*$  when  $\eta = \varepsilon$ ). Compute  $q'(\eta) = -\eta C + \|z\|$ . We see that  $q$  is an increasing function when  $-\eta C + \|z\| \geq 0$ , in particular when  $\eta \leq \varepsilon < \|z\|/C$ .  $\square$

We are now in a position to establish the “duality” between (8) and (9).

**THEOREM 3.5.** *Let  $\varphi$  be a finite-valued convex function satisfying (8). Assume also that  $r$  satisfies (10) as well as*

$$(13) \quad r(h) \leq \frac{1}{2}C\|h\|^2 \quad \text{for all } h.$$

*Then, for all  $g_0 \in \partial\varphi(x_0)$  and  $s \in B(0, \frac{\varepsilon c^2}{2C})$ , we have*

$$(14) \quad \varphi^*(g_0 + s) \geq \varphi^*(g_0) + \langle s, x_0 \rangle + \min_{\gamma \in \partial\varphi(x_0)} r^*(g_0 + s - \gamma).$$

*Proof.* We conjugate both sides in (8): for all  $g \in \mathbb{R}^N$

$$(15) \quad \begin{aligned} \varphi^*(g) - \langle g, x_0 \rangle &\geq [\varphi(x_0) + \varphi'(x_0; \cdot) + r + \mathbf{I}_{B_\varepsilon}]^*(g) \\ &= -\varphi(x_0) + \left( \mathbf{I}_{\partial\varphi(x_0)} \downarrow (r + \mathbf{I}_{B_\varepsilon})^* \right)(g) \\ &= -\varphi(x_0) + \left( \mathbf{I}_{\partial\varphi(x_0)} \downarrow R_\varepsilon^* \right)(g) \\ &= -\varphi(x_0) + \min_{\gamma \in \partial\varphi(x_0)} R_\varepsilon^*(g - \gamma), \end{aligned}$$

where  $R_\varepsilon^*$  denotes the Lipschitzian regularization of  $r^*$ , see (11). We will prove that, for  $g$  close enough to  $\partial\varphi(x_0)$ ,  $R_\varepsilon^*$  can be replaced by  $r^*$  in (15).

First, observe that (10) and (13) are transformed in the dual space to

$$(16) \quad \frac{1}{2C}\|\cdot\|^2 \leq r^* \leq \frac{1}{2c}\|\cdot\|^2.$$

These inequalities are transmitted to the Lipschitzian regularizations; by Lemma 3.4(iii) we therefore have, for all  $\eta \in ]0, \varepsilon]$ ,

$$R_\varepsilon^*(z) \geq \frac{1}{2} \left( -\eta^2 C + 2\eta\|z\| \right) \quad \text{for all } z.$$

Apply (16) to obtain by division

$$(17) \quad \frac{r^*(z)}{R_\varepsilon^*(z)} \leq \frac{\|z\|^2}{-\eta^2 cC + 2\eta c\|z\|} \quad \text{for all } z.$$

Let  $\gamma_g$  denote an optimal  $\gamma$  in (15). From (16) and (17),

$$\frac{1}{2C}\|g - \gamma_g\|^2 \leq r^*(g - \gamma_g) \leq \frac{\|g - \gamma_g\|^2}{-\eta^2 cC + 2\eta c\|g - \gamma_g\|} R_\varepsilon^*(g - \gamma_g).$$

We append to this chain additional upper bounds, using the optimality of  $\gamma_g$ ,  $R_\varepsilon^* \leq r^*$  and (16): for all  $\gamma \in \partial\varphi(x_0)$ ,

$$\frac{1}{2C} \leq \frac{1}{-\eta^2 cC + 2\eta c\|g - \gamma_g\|} \frac{\|g - \gamma\|^2}{2c}.$$

After some algebra this results in

$$(18) \quad \|g - \gamma_g\| \leq \frac{C}{2\eta c^2} \|g - \gamma\|^2 + \frac{\eta C}{2} \quad \begin{array}{l} \text{for } \gamma_g \text{ optimal in (15), } \eta \leq \varepsilon \\ \text{and } \gamma \text{ arbitrary in } \partial\varphi(x_0). \end{array}$$

To obtain (14), take  $\eta = \frac{\varepsilon c}{2C}$ ,  $g_0 \in \partial\varphi(x_0)$  and  $s \in B(0, \frac{\varepsilon c^2}{2C})$ . For  $g := g_0 + s$ , let  $\gamma$  in (18) be the projection of  $g$  onto  $\partial\varphi(x_0)$ : we do have  $\|g - \gamma_g\| \leq \varepsilon c/2$ . Then Lemma 3.4(i) allows us to replace  $R_\varepsilon^*$  by  $r^*$  in (15). The result follows, since  $\varphi(x_0) + \varphi^*(g_0) = \langle g_0, x_0 \rangle$ .  $\square$

The computation of the remainder term in (14) may be deemed abstract. However it can be made explicit when using a quadratic  $r$  in (8).

**COROLLARY 3.6.** *Let  $\varphi$  be a finite-valued convex function satisfying (8), with  $r = \frac{1}{2}C\|\cdot\|^2$ . Then, for all  $g_0 \in \partial\varphi(x_0)$  and  $s \in B(0, \varepsilon C/2)$ , we have*

$$\varphi^*(g_0 + s) \geq \varphi^*(g_0) + \langle s, x_0 \rangle + \frac{1}{2C} \|g_0 + s - \mathcal{P}(g_0 + s)\|^2,$$

where  $\mathcal{P}$  is the projection onto  $\partial\varphi(x_0)$ .

As a result

$$(19) \quad \langle s, x - x_0 \rangle \geq \frac{\|s\|^2}{2C} \quad \text{for all } s \in N_{\partial\varphi(x_0)}(g_0) \cap B(0, \varepsilon C/2) \text{ and } x \in \partial\varphi^*(g_0 + s).$$

*Proof.* Use Theorem 3.5 with  $r^* = \frac{1}{2C}\|\cdot\|^2 = \frac{1}{2c}\|\cdot\|^2$ . To prove (19), use the subgradient inequality  $\varphi^*(g_0) \geq \varphi^*(g_0 + s) - \langle s, x \rangle$  for  $x \in \partial\varphi^*(g_0 + s)$ , and observe that  $g_0 + s$  is projected onto  $g_0$ .  $\square$

This result reveals a sort of continuity of *particular* subgradients of a convex function. More precisely, take  $x$  close to  $x_0$ ; because  $\varphi$  is finite-valued, any  $g \in \partial\varphi(x)$  is close to  $\partial\varphi(x_0)$  (closedness of the subdifferential mapping). Then use the Cauchy-Schwarz inequality in (19) to see that  $g$  has actually a Lipschitzian behaviour with respect to its projection  $g_0$  onto  $\partial\varphi(x_0)$ .

It is not clear whether our assumptions (10) and (13) are really essential for the above results. At least, (13) is natural: useful lower bounds for  $\varphi^*$  need nontrivial upper bounds for  $\varphi$ ; but the role of (10) is more obscure. By contrast, the assumptions in the following dual counterpart to Theorem 3.5 seem rather minimal.

**THEOREM 3.7.** *Let  $\varphi$  be a finite-valued convex function satisfying (9). Assume also that  $r$  satisfies (10). Then, for all  $g_0 \in \partial\varphi(x_0)$ ,  $\varphi^*$  is differentiable at  $g_0$  ( $\nabla\varphi^*(g_0) = x_0$ ) and, for all  $s \in B(0, \varepsilon c/2)$ , we have*

$$(20) \quad \varphi^*(g_0 + s) \leq \varphi^*(g_0) + \langle s, x_0 \rangle + \min_{\gamma \in \partial\varphi(x_0)} r^*(g_0 + s - \gamma).$$

*Proof.* Proceed as in the proof of Theorem 3.5. Clearly (12) holds with  $\psi = \varphi$  and  $\delta = \varepsilon$ , so Lemma 3.4(i) can be applied to conjugate the lefthand side of (9):

$$\varphi^*(g) - \langle g, x_0 \rangle \leq -\varphi(x_0) + \min_{\gamma \in \partial\varphi(x_0)} r^*(g - \gamma) \quad \text{for } g \in \partial\varphi(x_0) + B(0, \varepsilon c/2).$$



Set  $g = g_0 + s$ , with  $s$  as stated, and observe that  $\varphi(x_0) + \varphi^*(g_0) = \langle g_0, x_0 \rangle$  to obtain (20).

Now, in view of (10),  $r^* \leq \frac{1}{2c} \|\cdot\|^2$ , so (20) gives

$$\varphi^*(g_0 + s) - \varphi^*(g_0) - \langle s, x_0 \rangle \leq \frac{1}{2c} \min_{\gamma \in \partial \varphi(x_0)} \|g_0 + s - \gamma\|^2 \leq \frac{1}{2c} \|s\|^2.$$

Because the lefthand side is nonnegative, this clearly implies  $\nabla \varphi^*(g_0) = x_0$ .  $\square$

This result explains why full duality cannot hold between (8) and (9): indeed, if (8) implied (9) for  $\varphi^*$ , then Theorem 3.7 applied to  $\varphi$  would imply the existence of  $\nabla \varphi(x_0)$ .

**4. Second-order analysis.** The aim of this section is to study the statement “ $F$  has second-order derivatives if and only if  $f$  has second-order derivatives”. We start with the easy part.

**PROPOSITION 4.1.** *Let  $f$  be a finite-valued convex function and  $x_0$  such that  $\nabla^2 f(p(x_0))$  exists. Then the Hessian of  $F$  exists at  $x_0$ , more precisely*

$$\nabla^2 F(x_0) = M - M[\nabla^2 f(p(x_0)) + M]^{-1}M$$

(here  $\nabla^2$  means the Hessian in the classical sense).

*Proof.* The assumptions imply the existence of  $\nabla f$  in a neighbourhood of  $p(x_0)$ . Then (3) shows that  $p(x)$  is implicitly defined by

$$\Phi(x, p) := \nabla f(p) + M(p - x) = 0.$$

Since  $\Phi$  has Jacobians  $\nabla_x \Phi = -M$  and  $\nabla_p \Phi(x_0, p(x_0)) = \nabla^2 f(p(x_0)) + M$ , the Implicit Function Theorem applies:  $P'(x_0) = [\nabla^2 f(p(x_0)) + M]^{-1}M$ . The result follows from Proposition 2.4.  $\square$

**COROLLARY 4.2.** *Let  $f$  be a finite-valued convex function and  $x_0$  such that  $\nabla^2 f(p(x_0))$  exists. Then*

$$\ker \nabla^2 F(x_0) = \ker \nabla^2 f(p(x_0)).$$

*Proof.* Use the notation  $A := \nabla^2 f(p(x_0))$  and  $A' := \nabla^2 F(x_0)$ . From Proposition 4.1,  $M^{-1}A' = Id - [A + M]^{-1}M = Id - [M^{-1}A + Id]^{-1}$ , hence

$$Id - M^{-1}A' = (Id + M^{-1}A)^{-1}.$$

If  $A'v = 0$ , then  $(Id + M^{-1}A)v = v$  and  $Av = 0$ . Taking inverses,  $(Id - M^{-1}A')^{-1} = Id + M^{-1}A$  and we show likewise that  $Av = 0$  implies  $A'v = 0$ .  $\square$

The converse part of Theorem 4.1 is not so simple; it will be stated in Theorems 4.7 and 4.8 below. First the next geometrical result is crucial.

**PROPOSITION 4.3.** *Let  $f$  be a finite-valued strongly convex function such that, given  $x$ , (8) holds with  $x_0 = p(x)$  and  $r = \frac{1}{2}C\|\cdot\|^2$ . More precisely,*

$$(21)f(p(x) + h) \leq f(p(x)) + f'(p(x); h) + \frac{1}{2}C\|h\|^2 + \mathbf{I}_{B_\varepsilon}(h) \quad \text{for some } C > 0 \text{ and all } h.$$

If  $\nabla^2 F(x)$  exists, then  $G$  of (3) lies in the relative interior of  $\partial f(p(x))$ .

*Proof.* Let  $F$  have a Hessian at  $x$ , hence, by Proposition 2.4,  $P'(x)$  exists.

Assume for contradiction  $G \in \text{rbd } \partial f(p(x))$ ; by Proposition 3.2, the normal cone  $\mathcal{N} = N_{\partial f(p(x))}(G)$  is not a subspace. In view of Proposition 3.1, we can take a unitary  $\nu_0 \in \mathcal{N} \cap \mathcal{M}^\perp$ ; then take  $G_t := G + t\nu_0$ , with  $t > 0$ . Calling  $c$  the modulus of strong convexity of  $f$ , Theorem 2.2 guarantees that  $p_t := \nabla f^*(G_t)$  satisfies the Lipschitz condition

$$(22) \quad \|p_t - p(x)\| \leq \frac{1}{c} \|G_t - G\| = \frac{1}{c} t.$$

By Theorem 2.8 this  $p_t$  is the proximal point of  $x_t := p_t + M^{-1}G_t$  which, because of (22), satisfies

$$(23) \quad \|x_t - x\| \leq \|p_t - p(x)\| + \frac{1}{\lambda} \|G_t - G\| \leq \left(\frac{1}{c} + \frac{1}{\lambda}\right)t.$$

Furthermore,  $G_t$  is projected onto  $G$ :  $\mathcal{P}(G_t) = G$  and, whenever  $t \in ]0, \varepsilon C/2]$ , (19) applied to  $f$  with  $x_0 = p(x)$ ,  $x = p_t$  and  $g_0 = G$  gives

$$\langle G_t - G, p_t - p(x) \rangle \geq \frac{1}{2C} \|G_t - G\|^2 = \frac{1}{2C} t^2.$$

Combine this with (23):

$$\frac{\lambda c}{2C(c + \lambda)} \leq \left\langle \nu_0, \frac{p_t - p(x)}{\|x_t - x\|} \right\rangle.$$

Let  $t \downarrow 0$ . By (23),  $x_t \rightarrow x$ ; by Corollary 2.6 (extract a subsequence if necessary)  $[p_t - p(x)]/\|x_t - x\| \rightarrow \nu$ ; clearly,  $\langle \nu_0, \nu \rangle > 0$ . By Lemma 2.5,  $\nu \in \mathcal{N}$  and, by (6),  $-\nu \notin \mathcal{N}$ . This shows that  $\text{Im } P'(x)$  is not a symmetric set;  $P'(x)$  cannot be a linear operator, and this is the required contradiction.  $\square$

We can now establish a local and a global second-order result, valid for strongly convex functions.

**PROPOSITION 4.4.** *Let  $f$  be a finite-valued strongly convex function such that (21) holds for a given  $x$ . If  $\nabla^2 F(x)$  and  $\nabla f(p(x))$  exist, then  $\nabla^2 f(p(x))$  exists.*

*Proof.* We have from (3)  $p(x) = x - M^{-1}G$  with  $G = \nabla f(p(x))$ . Apply Corollary 3.6 with  $\varphi = f$ ,  $x_0 = p(x)$  and  $g_0 = G$ ; since  $\partial f(p(x)) = \{G\}$ ,  $G + s$  is projected onto  $G$  for all  $s$ :

$$(24) \quad f^*(G + s) \geq f^*(G) + \langle s, p(x) \rangle + \frac{1}{2C} \|s\|^2 \quad \text{for } \|s\| \leq \varepsilon C/2.$$

By Corollary X.4.2.9 in [5], the existence of  $\nabla^2 F(x)$  (positive definite, recall Theorem 2.2) implies the existence of  $\nabla^2 F^*(G) = \nabla^2 f^*(G) + M$ . Therefore  $\nabla^2 f^*(G)$  exists and, by (24), is positive definite. Again by Corollary X.4.2.9 in [5],  $f$  is twice differentiable at  $p(x)$ .  $\square$

**PROPOSITION 4.5.** *Let  $f$  be a finite-valued strongly convex function such that (21) holds for all  $x$ . If  $\nabla^2 F$  exists on the whole of  $\mathbb{R}^N$ , then  $\nabla^2 f$  exists on the whole of  $\mathbb{R}^N$ .*

*Proof.* We claim that  $f$  is differentiable at every  $p \in \mathbb{R}^N$ . Indeed, if  $\partial f(p)$  is not a singleton, take a subgradient  $G$  in the relative boundary of  $\partial f(p)$  (Proposition 3.3). Because

of Theorem 2.8,  $p$  is the proximal point of  $x := p + M^{-1}G$  and, by assumption,  $\nabla^2 F(x)$  exists. From Proposition 4.3 we get the desired contradiction:  $G$  lies in the relative interior of  $\partial f(p)$ .

Then  $\nabla^2 F$  and  $\nabla f$  exist on the whole of  $\mathbb{R}^N$  and Proposition 4.4 applies.  $\square$

Our final goal will be to eliminate the strong convexity assumption in these last two results. For this we perturb  $f$  to a strongly convex function  $f + \frac{1}{2}\tau\|\cdot\|_M^2$  and we study the effect of this perturbation on the proximal point.

**PROPOSITION 4.6.** *Let  $f$  be a finite-valued convex function satisfying (21) for a given  $x$ , and define  $f_\tau := f + \frac{1}{2}\tau\|\cdot\|_M^2$ , for  $\tau \in [0, 1[$ . Consider the Moreau-Yosida of  $f_\tau$  associated to the metric defined by  $(1 - \tau)M$ :*

$$(25) \quad F_\tau(x) := \min_{y \in \mathbb{R}^n} \left\{ f_\tau(y) + \frac{1}{2}(1 - \tau)\|y - x\|_M^2 \right\}$$

and denote by  $q_\tau(x)$  the unique minimizer of (25).

Then the following statements hold

- (i) The function  $f_\tau$  is strongly convex and satisfies (21) with  $C$  replaced by  $C + \tau\Lambda$ ;
- (ii) for all  $x$ ,  $p(x) = q_\tau(\frac{x}{1-\tau})$ .

*Proof.* The strong convexity of  $f_\tau$  is clear. To prove that (21) holds for  $f_\tau$ , add  $\frac{1}{2}\tau\|x_0 + h\|_M^2$  to both sides of (21) applied to  $f$ . Then use the properties  $f'_\tau(x_0; h) = f'(x_0; h) + \frac{1}{2}\tau\langle Mx_0, h \rangle$  and  $\frac{1}{2}\tau\|\cdot\|_M^2 \leq \frac{1}{2}\tau\Lambda\|\cdot\|^2$ .

For proving (ii), write the optimality condition for  $p(x)$  and  $q_\tau(\frac{x}{1-\tau})$ :

$$\begin{aligned} p(x) \text{ solves } & M(x - p) \in \partial f(p) \quad \text{and} \\ q_\tau(\frac{x}{1-\tau}) \text{ solves } & (1 - \tau)M(\frac{x}{1-\tau} - p) \in \partial f_\tau(p). \end{aligned}$$

Since  $\partial f_\tau(p) = \partial f(p) + \{\tau Mp\}$ , we are done.  $\square$

Thus, passing from  $f$  to  $f_\tau$  can be absorbed by perturbing  $M$  to  $(1 - \tau)M$  in the Moreau-Yosida of (1). Then Proposition 4.5 can be applied to the perturbed data. Our global and local results become as follows.

**THEOREM 4.7.** *Let  $f$  be a finite-valued convex function such that (21) holds for all  $x$ . Then  $\nabla^2 F$  exists on the whole of  $\mathbb{R}^N$  if and only if  $\nabla^2 f$  exists on the whole of  $\mathbb{R}^N$ .*

*Proof.* The “only if” part is Theorem 4.1. As for the “if” part, suppose  $\nabla^2 F$  exists everywhere; hence, from Proposition 2.4,  $p(\cdot)$  has a Jacobian everywhere. Then consider  $f_\tau$  as in Proposition 4.6; by virtue of Proposition 4.6(ii),  $q_\tau$  also has a Jacobian  $Q'_\tau(\frac{x}{1-\tau}) = (1 - \tau)P'(x)$  for all  $x$ . Apply now Proposition 2.4 to  $F_\tau$ :  $\nabla^2 F_\tau = (1 - \tau)M(Id - Q'_\tau)$ , so  $F_\tau$  has a Hessian everywhere. Since  $f_\tau$  is strongly convex and satisfies (21), Proposition 4.5 holds:  $\nabla^2 f_\tau$  exists everywhere. Then  $\nabla^2 f = \nabla^2 f_\tau - \tau M$  exists everywhere.  $\square$

**THEOREM 4.8.** *Let  $f$  be a finite-valued convex function such that (21) holds for an optimal  $x = \bar{x}$ . Assume  $\nabla f(\bar{x})$  exists. Then  $\nabla^2 F(\bar{x})$  exists if and only if  $\nabla^2 f(\bar{x})$  exists.*

*Proof.* Without loss of generality assume that  $\bar{x} = 0$ ; hence  $p(\bar{x}) = \bar{x} = 0$ . Copy the proof of Theorem 4.7, replacing “everywhere” by “at  $0 = \bar{x} = p(\bar{x}) = q_\tau(\bar{x})$ ” and using the relation  $Q'(0) = (1 - \tau)P'(0)$ .  $\square$

**5. Concluding Remarks.** The very first motivation for the Moreau-Yosida regularization was to solve ill-conditioned systems of linear equations ([2], Chap.V). In fact, suppose  $f$  is quadratic, its Hessian  $A$  having extreme eigenvalues  $c$  and  $C$ . From Proposition 4.1,  $F$  is also quadratic with Hessian  $M + M(A + M)^{-1}M$ . Taking  $M = \lambda Id$ , a quick calculation shows that the condition number  $C/c$  of  $A$  is divided by  $(\lambda + C)/(\lambda + c)$ . This is a clear beneficial effect of the Moreau-Yosida.

Consider now a general objective function. Barring all implementation considerations, assume that the proximal point  $p(x)$  can be computed for each  $x$  (perhaps approximately, but for a negligible computation cost). Then the question arises whether such a computation is of any use for a superlinearly convergent algorithm minimizing  $F$  (i.e.  $f$ ).

When  $f$  is differentiable on the whole space, Theorems 4.7 and 4.2 tell us that a combination Moreau-Yosida & Newton brings essentially nothing. Either  $F$  still does not enjoy the necessary properties of smoothness and non-degeneracy, or a superlinear algorithm could have been applied to  $f$  at the first place (ordinary Newton, quasi-Newton, or nonsmooth Newton as in [11]). For example, take an augmented Lagrangian:

$$f(x) := f_0(x) + \frac{\pi}{2} \left[ \max \left( 0, f_1(x) + \frac{\mu}{\pi} \right) \right]^2,$$

which is associated to the NLP: minimize  $f_0$  subject to  $f_1 \leq 0$ . The minimization of the corresponding  $F$  (for given  $\mu$ ,  $\pi$  and  $M$ ) is not easier than the minimization of  $f$ :  $\nabla^2 F$  exists and is positive definite only when  $\nabla^2 f$  enjoys the same properties.

On the other hand, suppose  $f$  is a non-differentiable function. Then our results are no longer so conclusive: as far as superlinear convergence is concerned, the important property is the second-order differentiability *at a minimum point*. Theorem 4.7 does not single out such an optimal point, while Theorem 4.8 does not apply at an optimal (kinky) point. To illustrate this deficiency, take the bivariate function  $f(x) := \max\{\frac{1}{2}\|x\|^2 - \alpha \langle e, x \rangle, \langle e, x \rangle\}$ , for  $e := (0, 1)^T$  and  $\alpha$  a nonnegative parameter. The kinks of  $f$  form a circle, denoted by  $C$  in Fig. 2. The subdifferential of  $f$  at 0 is the segment  $[-\alpha e, e]$  hence  $f$  is minimized at  $0 = p(0)$ , for all  $\alpha \geq 0$ . For  $M = Id$ , let us compute the proximal point of  $x \neq 0$ :

$$x - p = \begin{cases} p - \alpha e & \text{if } \frac{1}{2}\|p\|^2 - \alpha \langle e, p \rangle > \langle e, p \rangle \\ \mu(p - \alpha e) + (1 - \mu)e & \text{for some } \mu \in [0, 1] \text{ if } \frac{1}{2}\|p\|^2 - \alpha \langle e, p \rangle = \langle e, p \rangle \\ e & \text{if } \frac{1}{2}\|p\|^2 - \alpha \langle e, p \rangle < \langle e, p \rangle. \end{cases}$$

Working out the calculations, we find that

$$p(x) = \begin{cases} \frac{x + \alpha e}{2} & \text{if } \|x - (\alpha + 2)e\| > 2(\alpha + 1) \\ \frac{x + (\alpha\mu + \mu - 1)e}{1 + \mu} & \text{if } \alpha + 1 \leq \|x - (\alpha + 2)e\| \leq 2(\alpha + 1) \\ x - e & \text{if } \|x - (\alpha + 2)e\| < \alpha + 1, \end{cases}$$

where

$$(26) \quad \mu = \mu(x) := \frac{\|x - (\alpha + 2)e\|}{\alpha + 1} - 1.$$

In a condensed form,

$$(27) \quad p(x) = \frac{x - e + (\alpha + 1)\nu(x)e}{1 + \nu(x)},$$

where  $\nu(x)$  is the projection of  $\mu(x)$  in (26) onto  $[0, 1]$ .

In Fig. 2,  $C_1$  (resp.  $C_2$ ) is the boundary of the region where the first (resp. second) function prevails at  $p(x)$ . The dashed crown is the locus of those  $x$  such that  $p(x)$  is a kink. The point is that  $C_2$  is always far from 0, while  $C_1$  does contain 0 when  $\alpha = 0$ . As a result,  $\nabla^2 F(0)$  exists if  $\alpha > 0$  but not if  $\alpha = 0$ . To show this, we consider two cases.

(i) When  $\alpha > 0$ , we have  $\mu \in [0, 1]$  in (26) for small  $\|x\|$ . This comes from  $\mu(0) = 1/(\alpha + 1)$ , together with the continuity of  $\mu(\cdot)$ . In this region, which includes the origin in its interior,

$$p(x) = (\alpha + 1) \frac{x - (\alpha + 2)e}{\|x - (\alpha + 2)e\|} + (\alpha + 1)e.$$

A mere differentiation gives

$$P'(0) = \frac{\alpha + 1}{\alpha + 2} (Id - ee^T) = \frac{\alpha + 1}{\alpha + 2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(ii) When  $\alpha = 0$ , the origin is on  $C_1$ . Analytically,  $\nu(x) = 1$  in (27) whenever  $\|x - 2e\| > 2$ . From this observation, the directional derivatives are easy to compute:

$$P'(0; d) = \begin{cases} \frac{1}{2}d_1 & \text{for } d_2 > 0 \\ \frac{1}{2}d_1 + \frac{1}{2}d_2 & \text{for } d_2 \leq 0. \end{cases}$$

Here, the nonexistence of  $P'(0)$  illustrates Theorem 4.3:  $G = 0$  is on the relative boundary of  $\partial f(0) = [0, 1]$ .

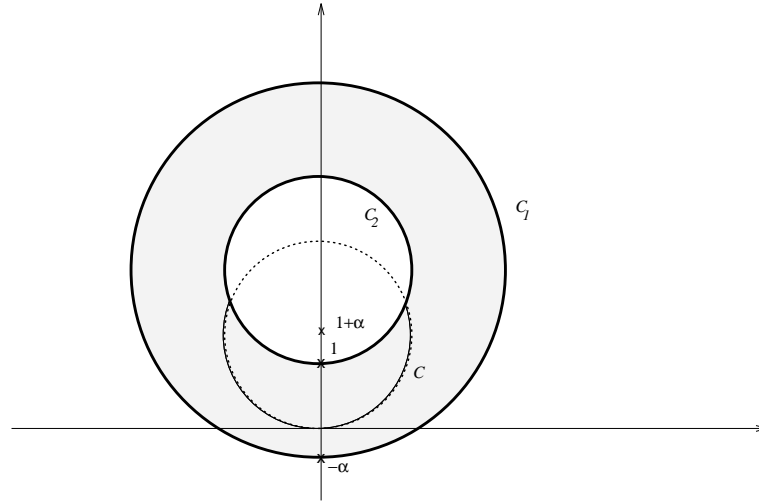


FIG. 2. Moreau-Yosida without Hessian

Let us sum up our results: as far as minimization algorithms are concerned, efficient combinations of Moreau-Yosida with Newtonian schemes might not be so straightforward. For instance, the algorithmic pattern (AP1) in [3] seems of little relevance. At present we can foresee two ways round this difficulty:

(i) A metric  $M$  in (1) that varies along the minimizing algorithm (see (AP2) and (AP3) in [3] or Algorithm 13 in [7]).

(ii) A space decomposition. From Proposition 2.7,  $F$  has a nice second-order behaviour in the tangent cone  $\mathcal{T}$ ; as for the normal cone  $\mathcal{N}$ , an approximation scheme is proposed in [9]. Thanks to the property  $\mathbb{R}^n = \mathcal{T} \oplus \mathcal{N}$  (Theorem III.3.2.5 in [5]), second-order information can thus be collected in the whole space.

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