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**Undiscounted zero sum differential games with
stopping times**

Mabel M. Tidball

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PROGRAMME 5

Traitement du signal,
automatique
et productique



*Rapport
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Undiscounted zero sum differential games with stopping times

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Abstract: We propose a discretization scheme for an undiscounted zero sum differential game with stopping times. The value function of the original problem satisfies an integral inequality of Isaacs type that we can discretize using finite difference or finite element techniques.

The fully discrete problem defines a stochastic game problem associated to the process, which may have, in general, multiple solutions. Among these solutions there exists one which is naturally associated with the value function of the original problem.

We completely characterize the set of solution and we describe a procedure to identify the desired solution. We present accelerated algorithms in order to compute efficiently the discrete solution.

(Résumé : tsvp)

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JEUX DIFFERENTIELS A SOMME NULLE AVEC TEMPS D'ARRET SANS TAUX D'ACTUALISATION

Résumé : On propose un schéma de discrétisation pour des jeux différentiels à somme nulle avec temps d'arrêt sans taux d'actualisation.

La fonction valeur du problème continu satisfait une inégalité de type Isaacs. On discrétise ces inégalités en utilisant des techniques de différences finies.

Le problème complètement discret définit un problème stochastique associé au processus. Ce problème peut avoir plusieurs solutions. Néanmoins, une seule de ces solutions correspond à une approximation de la fonction valeur du problème continu. On caractérise l'ensemble des solutions et on décrit une procédure pour trouver la solution cherchée.

On présente un algorithme pour calculer efficacement la solution discrète.

1 Introduction

We consider an undiscounted zero sum differential game problem with stopping times . The same problem with discount rate ($\lambda > 0$) was study in [20] and a discretization procedure in [22].

For case $\lambda > 0$ (see [20]) it is proved that the value function of the game is a Holder continuous function; but in the present case ($\lambda = 0$), we loose, in general, this regularity property. In spite of this discouraging result we can prove that the value function is always well defined and unique when the stopping costs verify a natural condition which correspond to a non trivial game; but even though we have regular data, the value function can be discontinuous. Works related with this type of phenomenon can be found in [1]. Problems without discount was also studied in [18] for ergodic control problems.

The aim of this work is to obtain a numerically computable discrete solution of this game and to precise when this solution represents the value function of our problem.

By the dynamic programming principle, we can prove that the value function satisfies an integral inequality system of Isaacs type (see [10], [11], [20], [22]). Since an exact solution of this type of problems is usually impossible, we must discretize the problem and to solved it numerically. In order to obtain a discrete problem we approximate, using finite element techniques (see [21]), these inequalities. The fully discrete problem that we obtain, may have, in general, multiple solutions. It defines a stochastic game problem with stopping time associated to the process.

Among these solutions, there exists one that is naturally associated with the value function of the original problem (remind that the solution of our problem is unique). In this work we completely characterize the set of solutions of the discrete problem and we describe a procedure to identify this particular discrete solution. We present, in order to compute it, accelerated algorithms based on the techniques given in [9] and [22].

We also present special cases where we can prove regularity properties of the value function and the existence of a unique solution of the discrete problem. In these cases we can prove the convergence, in the viscosity sense (see [8]), of the discrete solution to the value function of the original problem.

2 Description of the problem

2.1 The differential game problem

In this differential game problem, the decision variables used by the players are the stopping times for the uncontrolled system. The evolution of the system is described by an ordinary differential equation of the form:

$$\frac{dy(t)}{dt} = g(y(t)); \quad y(0) = x \quad (1)$$

with $x \in \Omega \subset \mathbb{R}^n$, an open bounded domain. We denote τ_1 and τ_2 the stopping times chosen by players 1 and 2 respectively. To those stopping times is associated a payoff J where J takes the form :

$$J(x, \tau_1, \tau_2) = \overline{\lim}_{\tau \rightarrow \infty} \int_0^{\tau_1 \wedge \tau_2 \wedge \tau \wedge \theta} f(y(s)) ds + \phi_1(y(\tau_1)) \chi_{\tau_1 < \tau_2 \wedge \theta} + \\ + \phi_2(y(\tau_2)) \chi_{\tau_2 < \tau_1 \wedge \theta} + \phi_0(y(\theta)) \chi_{\tau_2 \wedge \tau_1 > \theta} \quad (2)$$

Where in (2): χ_E is the characteristic function of set E , $\theta(x)$ is the first time when the trajectory, solution of (1) with initial condition x reaches $\partial\Omega$ ($\theta(x) = \infty$ when the trajectory always remains in Ω), that is:

$$\theta(x) = \inf\{t : y(t) \notin \Omega\}.$$

ϕ_0 is the final cost to pay if the game stops because the system leaves Ω , f is the instantaneous cost, ϕ_i , $i = 1, 2$ is the final costs if the game stops because player i decides first to stop the game, with

$$\phi_2(x) \geq \phi_1(x) \quad \forall x \in \Omega \quad (3)$$

As the game is of zero sum, the goal of the first player is to maximize the payoff J and the goal of the second player is to minimize it. Our objective is to find (if they exist) $\bar{\tau}_1$ and $\bar{\tau}_2$ such that:

$$V(x) = \sup_{\tau_1} \inf_{\tau_2} J(x, \tau_1, \tau_2) = J(x, \bar{\tau}_1, \bar{\tau}_2)$$

where V is the value function of the game and $\bar{\tau}_1$ and $\bar{\tau}_2$ are optimal strategies or saddle points of the functional J .

Inequality (3) indicates that the game is not a trivial game (no player can obtain advantages of priority of actions). It plays a key role in the proof of the existence of the value function but it does not represent a restriction as we can see in [20].

We suppose:

$$(H) \begin{cases} g, f, \phi_i, & i = 0, 1, 2 \text{ are bounded functions in } \Omega \\ g, f, \phi_i, & i = 0, 1, 2 \text{ are Lipschitz continuous functions in } \Omega \end{cases}$$

Remark 2.1 : *We can see that condition (3) implies that the functional (2) is well defined (in this case finite strategies $\tau_1 = \tau_2$ are not realistic strategies for both players).*

Remark 2.2 : *The limsup in (2) means that player 2 must minimize his worst situation.*

Remark 2.3 : *In definition (2), it is necessary to consider the limsup (or the liminf, in case we study the worst situation for player 1) in order to have a well defined payoff functional J . The fact that a limit need not exist can be easily seen in the following example. Let $x = (x_1, x_2)$; $\phi_2(x) = M$ and $\phi_1(x) = -M$, with M large enough such that $\tau_1 = \tau_2 = +\infty$, $f(x) = x_1$, $\Omega = (-2, 2)$ and*

$$\frac{dx_1}{dt}(t) = x_2(t); \quad x_1(0) = 1, \quad \frac{dx_2}{dt}(t) = -x_1(t); \quad x_2(0) = 0$$

in this case $\int_0^\infty f(y(s))ds$ does not exist.

2.2 Properties of the value function.

We can define the lower and upper value of the game \underline{V} and \bar{V} respectively, in the following way:

$$\underline{V}(x) = \sup_{\tau_1} \inf_{\tau_2} J(x, \tau_1, \tau_2) \quad \bar{V}(x) = \inf_{\tau_2} \sup_{\tau_1} J(x, \tau_1, \tau_2)$$

If (H) and (3) are valid we can prove, following the techniques used in [20], that $\underline{V} = \bar{V} = V$. V will be the unique value function of the game.

Remark 2.4 : In general, in terms of the value function ϵ -optimal stopping times $\bar{\tau}_1^\epsilon, \bar{\tau}_2^\epsilon$ can be found, (see [10]), i.e. $\bar{\tau}_1^\epsilon, \bar{\tau}_2^\epsilon$, such that:

$$J(x, \tau_1, \bar{\tau}_2^\epsilon) - \epsilon \leq J(x, \bar{\tau}_1^\epsilon, \bar{\tau}_2^\epsilon) \leq J(x, \bar{\tau}_1^\epsilon, \tau_2) + \epsilon \quad \forall \tau_1, \tau_2$$

They are defined in the following way (see [11]):

$$\bar{\tau}_1^\epsilon = \inf \{s/V(y(s)) \leq \phi_1(y(s)) + \epsilon\} \quad \bar{\tau}_2^\epsilon = \inf \{s/V(y(s)) \geq \phi_2(y(s)) - \epsilon\}$$

Remark 2.5 : V can be a discontinuous function as we can see in the following example.

Let $\Omega = [0, \pi]$ and the dynamic of the system given by: $\frac{dy}{dt}(t) = y(t)$, $y(0) = x$, $f(x) = 10 \sin x$, $\phi_0 = 0$, $\phi_1(x) = 2x - 1$, $\phi_2(x) = 2x + 1$; then:

$$V(x) = \begin{cases} \phi_2(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Isaacs inequality. Multiplicity of solutions.

By the dynamic programming principle we obtain that function V satisfies the following Isaacs inequality:

$$\phi_1(x) \leq V(x) \leq \phi_2(x) \quad \forall x \in \Omega$$

$$V(x) < \phi_2(x) \Rightarrow \exists \delta_x > 0 \quad / \quad \forall \delta < \delta_x, \quad V(x) \geq \int_0^\delta f(y(s))ds + V(y(\delta))$$

$$V(x) > \phi_1(x) \Rightarrow \exists \delta_x > 0 \quad / \quad \forall \delta < \delta_x, \quad V(x) \leq \int_0^\delta f(y(s))ds + V(y(\delta))$$

then for δ_x small enough we can insure that $V(x)$ is the fixed point of the following operator:

$$V(x) = (M_{\delta_x} V)(x) = P_I \left\{ \int_0^{\delta_x} f(y(s))ds + V(y(\delta)) \right\} \quad (4)$$

where P_I denotes the projection on interval $I = [\phi_1(x), \phi_2(x)]$ i.e:

$$P_I(a) = \begin{cases} \phi_1(x) & \text{if } a \leq \phi_1(x) \\ \phi_2(x) & \text{if } a \geq \phi_2(x) \\ a & \text{everywhere else} \end{cases}$$

If we consider x fixed and $y_x(t)$ the solution of (1) with initial condition x ; we have that (4) has infinite solutions (manifold of one-dimension). Indeed, we define the following equivalence relation:

$$x \sim x' \text{ if there exists } t_1 \geq 0 \text{ such that } y_x(t_1) = x' \text{ or } y_{x'}(t_1) = x.$$

Let C_x be the equivalence classes and define $\varphi : C_x \rightarrow \mathbb{R}$; then: if $V(\cdot)$ is a solution of (4), $\bar{V}(x) = V(x) + \varphi(C_x)$ is also a solution if $\bar{V}(x) \in [\phi_1(x), \phi_2(x)]$.

Remark 2.6 : *Properties of problems with boundary.*

In order to avoid discontinuous solutions, we analyze the case where the following condition holds (see [11] and [19]):

$$\begin{aligned} \theta(x) < +\infty \quad \forall x \\ \exists a > 0, \gamma > 0 \quad / \quad d(y(s), \partial\Omega) \leq a \Rightarrow \frac{d}{ds}d(y(s), \partial\Omega) \leq -\gamma \end{aligned} \tag{5}$$

If (5) is valid we can prove that there exists a unique Lipschitz continuous value function V and that there exists optimal strategies (see [10], [11]). This property arises because the exit time is a Lipschitz continuous function of x .

Remark 2.7 : *Properties of some ergodic processes.*

It is possible to give a precise characterization of the solution of our problem for a process such that the system remains in a compact manifold. In this case the solution of (1) defines in Ω a semigroup of translations $S(t)$. If we assume that this semigroup has a unique invariant measure that satisfies:

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau f(y_x(s)) ds = \int_\Omega f(z) d(\mu(z)) =: \langle f, \mu \rangle \tag{6}$$

we say that the process is ergodic. Moreover if we suppose

$$\int_\Omega f(z) d(\mu(z)) \neq 0$$

then

$$\lim_{\tau \rightarrow \infty} \int_0^\tau f(y_x(s)) ds = \infty \quad \text{sign} \left\{ \int_\Omega f(z) d(\mu(z)) \right\}$$

We can prove that this condition implies the value function is uniformly Lipschitz continuous and that there exists optimal strategies.

Remark 2.8 : *Characterization of the set where V is a discontinuous function.*

Define, $\tau(x) = \bar{\tau}_1 \wedge \bar{\tau}_2 \wedge \theta$ if there exists finite optimal strategies; $\tau(x) = +\infty \wedge \theta$ if there does not exist optimal strategies; $A = \{x \in \Omega : V(x) \text{ is discontinuous}\}$ and $\Omega_\tau = \{x \in \Omega : \tau(x) \leq \tau\}$.

Then, it is easy to see that:

$$A \subset \overline{\left(\bigcup_{\tau > 0} \Omega_\tau \right)^c}.$$

In order to obtain a numerical solution of the original problem we analyze in the following two sections a time discretization and a space discretization (using finite element techniques, (see [7], [13], [14], [15], [16] and [21])).

3 Time discretization.

3.1 The problem with discrete time

To obtain a time discretization (see [7] for control problems and [1] for games problems), we consider $h > 0$, an uniform partition of $[0, +\infty)$ in intervals of same length h and Euler approximation for the dynamics. Then, the Isaacs equation for time discretization becomes in this case:

$$V^h(x) = (M^h V^h)(x) = \begin{cases} P_I [hf(x) + V^h(x + hg(x))] & \text{if } x + hg(x) \in \Omega \\ P_I [\phi_0(x)] & \text{if } x + hg(x) \notin \Omega \end{cases} \quad (7)$$

It is easy to see that $V^h \in B(\Omega)$ (the set of bounded functions in Ω).

We can define a game problem with discrete stopping time associated to our real problem. The discrete time value function has the following form:

$$J^h(x, \tau_1, \tau_2) = \overline{\lim}_{\nu \rightarrow \infty} \sum_{\nu=1}^{\tau_1 \wedge \tau_2 \wedge \nu h \wedge \theta} f(y_x^h(\nu)) + \phi_1(y(\tau_1)) \chi_{\tau_1 < \tau_2 \wedge \theta} + \phi_2(y(\tau_2)) \chi_{\tau_2 < \tau_1 \wedge \theta} + \phi_0(y(\theta)) \chi_{\tau_2 \wedge \tau_1 > \theta}$$

where $\theta, \tau_1, \tau_2 \in hZ^+$ and the discrete dynamic is:

$$y_x^h(0) = x; \quad y_x^h(\nu + 1) = y_x^h(\nu) + hg(y_x^h(\nu)) \quad (8)$$

For sake of simplicity we denote:

$$y^\nu = y_x^h(\nu) = \left(1 - \frac{t - \nu h}{h}\right) y_\nu + \frac{t - \nu h}{h} y_{\nu+1} \quad \text{if } t \in [\nu h, (\nu + 1)h) \quad (9)$$

The time discrete value function is:

$$V^h(x) = \inf_{\tau_1} \sup_{\tau_2} J^h(x, \tau_1, \tau_2) \quad (10)$$

The following theorem characterizes the set of all solutions of (7).

Theorem 3.1 *Equation (7) has at least one solution. Moreover:*

- i) V^h given by (10) is one of them.
- ii) The following iteration

$$w^{n+1} = M^h w^n, \quad w^0 \geq \phi_2$$

converges to the maximal solution that we call \bar{w} .

iii) $w^{n+1} = M^h w^n, \quad w^0 \leq \phi_1$ converges to the minimal solution, \underline{w} .

iv) If $w(\cdot)$ is a solution, $C_x^h = \{x' = y_x^h(t), t \geq 0\}$, $\phi : C_x^h \rightarrow \mathbb{R}$, then $\bar{w}(x) = w(x) + \phi(C_x^h)$ is also a solution if $\bar{w}(x) \in [\phi_1(x), \phi_2(x)]$.

Proof: We are going to prove item ii) (the others are similar). With condition $w^0 \geq \phi_2$ the sequence $M^h w$ is monotonously decreasing. In effect $w^1 = M^h w^0 \leq \phi_2 \leq w^0$ and as P_I is a monotonous operator, we have $w^{n+1} = M^h w^n \leq M^h w^{n-1} = w^n$. Moreover the sequence is bounded from below by $-\|\phi_1\|$. Then w^n converges to \bar{w} when $n \rightarrow \infty$ and it is a solution of (7). Moreover it is the maximal solution because if \hat{w} is a solution of (7) $\hat{w}(x) \leq \phi_2(x) \quad \forall x$ then $\hat{w} = M^h \hat{w} \leq \phi_2 \leq w_0$ and $\hat{w} \leq (M^h)^n w_0 = w^n \quad \forall n$, that implies $\hat{w} \leq \bar{w}$. ■

Remark 3.1 : *Boundary problems.*

For the time discretization of problem presented in Remark 2.6 we obtain that the discrete solution is:

$$V^h(x) = P_I[\phi_0(x)] \quad \forall x \in \partial\Omega_h^+ = \{x \in \Omega / x + hg(x) \notin \Omega\}$$

Remark 3.2 : *a) Ergodic case with non zero mean, ($\langle \mu, f \rangle \neq 0$).*

In this case we can identify the unique time discrete solution of the ergodic problem presented in Remark 2.7 because it is easy to see that:

$$\langle \mu, f \rangle \neq 0 \implies$$

$$\exists \bar{n}(x) / V^h(x) = \sum_{n=0}^{\bar{n}-1} f(y^n) + V^h(y^{\bar{n}}); \quad V^h(y^{\bar{n}}) = \phi_i(y^{\bar{n}}) \quad \text{with } i = 1 \text{ or } 2.$$

This result and iv) of theorem (3.1) prove the unicity of the discrete solution, moreover it is given by the usual iteration:

$$w^{n+1} = M^h w^n, \quad w^0 \text{ given}$$

b) Ergodic case with zero mean, ($\langle \mu, f \rangle = 0$)..

In this case, it is easy to see that:

$$\text{If } \phi_1(x) < \sum_{n=0}^{\nu-1} f(y^n) + V^h(y^\nu) < \phi_2(x) \quad \forall \nu \quad \text{then } \underline{\lim}_\nu V^h(y^\nu) = 0.$$

4 Space discretization

4.1 Elements of the discrete problem

To define a fully discrete problem it is necessary to discretize set Ω , space $B(\Omega)$ and operator (7).

We approach Ω with $\Omega_k = \bigcup_j S_j^k$, where S_j^k is a finite set of simplices; that is, $\Omega_k \subset \mathbb{R}^n$ is a polyhedron with $\max_j(\text{diam } S_j^k) = k$.

We denote $X = \{x_i, i = 1, \dots, N\}$ the set of vertices of Ω_k (nodes), N the cardinality of X and $\partial\Omega_k^+ = \{x_i \in X / x_i + hg(x_i) \notin \Omega_k\}$.

We consider as approximation of $B(\Omega)$ the set W_k of functions $w : \Omega_k \rightarrow \mathbb{R}$, w continuous in Ω_k , with $\frac{\partial w}{\partial x}$ constant in the interior of each simplex of Ω_k (that is, w is a linear finite element).

Then, by definition of W_k , to obtain a solution of the discrete problem it is only necessary to obtain a solution in X . With the same methodology used in [15] and [16] we obtain a formulation of the **fully discrete problem**; (problem P_k^h); that is, to find a fixed point of the operator M^k by the transformation: $V(x) \rightarrow P_I [hf(x) + V(x + hg(x))]$ through the natural isomorphism $W_k \leftrightarrow \mathbb{R}^N$. $M^k : \mathbb{R}^N \rightarrow \mathbb{R}^N$, with $M^k w(x_i) = P_I (Pw + F)(x_i)$.

If we call V_k^h a solution of the fully discrete problem we have:

$$V_k^h(i) = P_I (PV_k^h + F)(i) \quad (11)$$

where P_I denotes the projection on interval $[\phi_1(i), \phi_2(i)]$; P is a matrix with components $P_{i,j}$ given by:

$$\begin{cases} x_i + hg(x_i) = \sum_{j=1}^N P_{i,j} x_j, & \text{if } x_i \notin \partial\Omega_k^+; x_i \in \Omega_k \\ P_{i,j} = 0 & \text{if } x_i \in \partial\Omega_k^+ \end{cases} \quad (12)$$

and F is a vector of components:

$$F(x_i) = \begin{cases} hf(x_i) & \text{if } x_i \notin \partial\Omega_k^+, x_i \in \Omega_k \\ P_I[\phi_0(x_i)] & \text{if } x_i \in \partial\Omega_k^+ \end{cases} \quad (13)$$

Because $\|P\| = 1$ the operator M is not a contractive operator and in the general case we can not insure a unique solution of (11).

Markov chain associated to the process.

We consider a random process with finite states $y_i^{(\nu)}$, where ν is the time discrete variable, i the initial state and the probability evolution is given by the matrix P defined in (12) ($P_{i,j}$ is the probability transition from node i to node j).

In this Markovian system we can define a stochastic game problem with stopping times where the functional cost becomes:

$$J^k(i, \tau_1, \tau_2) = \overline{\lim}_{\nu \rightarrow \infty} E \left(\sum_{\nu=1}^{\tau_1 \wedge \tau_2 \wedge \nu h \wedge \theta} F(y_i^\nu) + \phi_1(y(\tau_1))\chi_{\tau_1 < \tau_2 \wedge \theta} + \phi_2(y(\tau_2))\chi_{\tau_2 < \tau_1 \wedge \theta} + \phi_0(y(\theta))\chi_{\tau_2 \wedge \tau_1 > \theta} \right)$$

where, $\theta, \tau_1, \tau_2 \in hZ^+$ are the exit and stopping times associated to the process and F is defined in (13). For this discrete process the value function is:

$$V_h^k(i) = \inf_{\tau_2} \sup_{\tau_1} J^k(i, \tau_1, \tau_2) \quad (14)$$

Theorem 4.1 Equation (11) has at least one solution. V_h^k , given by (14) is one of its solutions.

Equation (11) has in general more than one solution. We call S the set of its solutions. We are going to characterize S taking in mind the fundamental structure of the graph associated to the Markov chain related with the discrete problem. We are going to identify in this set of solution the solution which represent the probabilistic formulation of the original problem.

We introduce the following equivalence relation in the set of nodes:

$i \sim j$ if $i = j$ or if there exists $i_1, \dots, i_\nu, \nu > 1$ such that $P(i_m, i_{m+1}) > 0 \forall m = 1, \dots, \nu$ $i_1 = i_\nu = i$ and there exists $0 < k < \nu$ with $i_k = j$.

This equivalence relation defines equivalence classes, $C^r, r = 1, \dots, \hat{N}$. We are going to define an order relation between these classes. So we define $S^\nu, \nu = 1, \dots, \bar{\nu}$ such that:

$C^r \in S^1$ if it does not exist $\hat{r} \neq r$ such that $P(i, j) > 0$, for any $i \in C^r, j \in C^{\hat{r}}$.
 $C^r \in S^\nu$ $\nu > 1$ if there exists $\hat{r} \neq r$ such that $P(i, j) > 0, i \in C^r, j \in C^{\hat{r}}, C^{\hat{r}} \in S^{\nu-1}$.
 If $C^r \in S^1, r = 1, \dots, n_1$, we say that $C^r =: C_1^r$ is a **final class**. We call **transitory** all other class that is not contained in S^1 and with H the set of transitory elements.
 If $C^r \in S^\nu$ we set $C^r = C_\nu^r$. Without loss of generality we order the classes of S^1 putting in the first position classes $C_1^r = \{x_j\}$ corresponding to points $x_j \in \partial\Omega_k^+$. If $C_\nu^r \in S^\nu$ we denote cardinality of $C_\nu^r = m_\nu^r$.
 The analysis of (11) can be restricted to the analysis in each final class C_1^r (taking in account that values in transitory classes are a function of values in elements of S^1). In effect the following theorem holds:

Theorem 4.2 *If $i \in C_\nu^r$ $\nu > 1$ there exists a function $Q : \mathbb{R}^t \rightarrow \mathbb{R}$, with $t = \cup_{r=1}^{n_1} m_1^r$, where $V_k^h(i) = Q(V_k^h(j) \ j \in C_1^r)$.*

Proof: For each $C_1^r \in S^1$ $r = 1, \dots, n_1$ we obtain $v_1 \in M_{m_1^r \times 1}$ with

$$v_1(i) = P_{[\phi_{11}(i), \phi_{21}(i)]} (P_1 V_1 + F_1) (i)$$

where $P_1 \in M_{m_1^r \times m_1^r}$ is the restriction of matrix P considering only its action in $C_1^r \in S^1$. In the same way $F_1, \phi_{11}, \phi_{21} \in M_{m_1^r \times 1}$ are the restriction of F, ϕ_1 and ϕ_2 respectively.

To compute the solution for $i \in C_\nu^r \in S^\nu, \nu > 1$ $r = 1, \dots, n_\nu$ we compute in each class $C_\nu^r \in S^\nu, v^\nu \in M_{m_\nu^r \times 1}$,

$$v_\nu(i) = P_{[\phi_{1\nu}(i), \phi_{2\nu}(i)]} (P_\nu V_\nu + \bar{F}_\nu) (i)$$

where $P_\nu \in M_{m_\nu^r \times m_\nu^r}$ is the restriction of P considering only its action in $C_\nu^r \in S^\nu, F_\nu, \phi_\nu, \phi_{2\nu} \in M_{m_\nu^r \times 1}$, with $\bar{F}_\nu = \sum_{j=1}^{\nu-1} P_j v_j + F^\nu$. ■

By Theorem(4.2) we obtain:

Theorem 4.3 *Let v, w be solutions of (11); if $v(x_i) = w(x_i) \ \forall x_i \in C_1^r \subset S^1$ then $v = w$*

Study of equation (11) in final classes

1) **Case $x_j \in \partial\Omega_k^+$** . This case corresponds to the space discretization of time discrete problem presented in Remark 3.1. Here the solution is obviously:

$$V_k^h(x_j) = P_I [\phi_0(x_j)] = \min [\phi_2(x_j), \max(\phi_1(x_j), \phi_0(x_j))].$$

2) **Case** $x_j \notin \partial\Omega_k^+$. We consider the restriction of matrix P to the final class C_1^r . Without loss of clarity we denote also by P this matrix. By definition of final class, we know that P is an irreducible stochastic matrix. We call d the greatest common divisor of the length of all the circuits (closed path) determined by P . We define the following equivalence relation:

$$i \sim j \iff \exists \text{ a path between } i \text{ and } j \text{ with a length multiple of } d$$

We denote $c_i, i = 1, \dots, d$, the d subclasses of equivalence. Matrix P has the following structure:

$$P = \begin{bmatrix} 0 & P^{1,2} & 0 & 0 \\ 0 & 0 & P^{2,3} & 0 \\ 0 & 0 & 0 & P^{d-1,d} \\ P^{d,1} & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

where we denote $P^{i,j} = P^{c_i,c_j}$. Matrix P^d define an ergodic Markov chain in each subclass c_i , then (see [18]) there exists a unique invariant measure $\mu^{c_i} = \mu^i$ of $(P^d)^{i,i}$, that is, $\mu^i(P^d)^{i,i} = \mu^i \quad \forall i = 1, \dots, d$.

These invariant measures are related each other in the following way:

$$\mu^i P^{i,i+1} = \mu^{i+1} \quad \forall i = 1, \dots, d \quad (16)$$

Then $\mu = \frac{1}{d}(\mu^1, \dots, \mu^d)$ is the unique invariant measure of the Markov process P , i.e., $\mu = \mu P$.

Remark 4.1 : *Measure μ determines the characterization of the solutions of (11).*

Case $\langle \mu, f \rangle \neq 0$

Theorem 4.4 *If $\langle \mu, f \rangle \neq 0$ then the solution of (11) is unique and it is given by the limit point of the following convergent iteration,*

$$v^{\nu+1} = M^k v^\nu; \quad \forall v^0 \text{ initial} \quad (17)$$

Outline of the proof: In this case we can proof that there exists $x_i \in X$ such that $V(x_i) = \phi_j(x_i)$ with $j = 1$ or 2 and then as we can insure that the game finishes in a finite time we can prove that there exists \hat{n} such that $(M^k)^{\hat{n}}$ is a contractive operator. ■

Remark 4.2 : *To find numerically the solution of (17) we can use an adaptation of the accelerated algorithms presented in [9] and in [17].*

Case $\langle \mu, f \rangle = 0$

In this case (11) can have more than one solution.

First, to prove the existence of solution of (11) (always restricted to a final class) we are going to study the solution of the linear system: $w(x_i) = (Pw + F)(x_i)$.

Let $F = ((F^1)^t, \dots, (F^d)^t)^t$; $F^i = F^{c_i}$ (where F^t means the transpose matrix), we are going to analyze the asymptotic behavior of functions:

$$\phi_n(i) = E \left(\sum_{j=1}^n F(y_j) \right) = \left(\sum_{j=1}^n (P^{j-1} F)(i) \right) \quad (18)$$

Sequence $\phi_n(i)$ has the following properties

$$i) \phi_{md}(i) \rightarrow \eta_0(i) \quad \forall i \text{ if } m \rightarrow +\infty$$

$$ii) \phi_{md+j}(i) \rightarrow \eta_j(i) \quad \forall i \text{ if } m \rightarrow +\infty$$

Proof of i)

We have:

$$\phi_{md} = \sum_{q=1}^d P^{q-1} F + P^d \phi_{(m-1)d} = \hat{F} + P^d \phi_{(m-1)d} \quad (19)$$

where $\hat{F} = \sum_{j=1}^{d-1} P^j F$.

As P^d is block diagonal and each block defines a finite ergodic Markov chain we have that (19) geometrically converges to a finite value. Then there exists $\eta_0(i) \quad \forall i$.

ii) is obvious by i) and by the relation $\phi_{md+j} = F + P\phi_{md+j-1}$ ■

Theorem 4.5 *Characterization of solutions. The set of solution of the linear system $\phi = P\phi + F$ is a one-dimensional manifold.*

Computation of a discrete solution of $\phi = P\phi + F$

We want to compute $\overline{\lim}_{n \rightarrow +\infty} \phi_n = \max_j \eta_j(i) = \bar{\eta}(i)$ because it represents the solution of the linear system we want.

As $P^{md} \rightarrow G$ when $m \rightarrow +\infty$, (where $G = \sum_j \hat{e}^j \hat{\mu}^j$, $\hat{e}^j = (0, \dots, 1, \dots, 1, 0, \dots, 0)$ with 1 in the j -th cycle and $\hat{\mu}^j = (0, \dots, \mu^j, \dots, 0)$ with μ^j the invariant measure vector restrict to subclass c_j).

By (18) and the properties of sequence ϕ_n we have η_j satisfies the following equation:

$$\eta_j = \eta_0 + \left(\sum_{r=1}^d \hat{e}^r \hat{\mu}^r \right) \left(\sum_{q=1}^{j-1} P^q F \right) = \eta_0 + \sum_{r=1}^d \hat{e}^r \sum_{q=1}^{j-1} \hat{\mu}^{r+q} F$$

We have, in each element of the cycle, calling $w_\rho = \hat{\mu}^\rho F$, that:

$$\eta_j(s) = \eta_0(s) + \sum_{\rho=s}^{s+j-1} w_\rho$$

Let $\bar{\rho}$ such that $\sum_{r=1}^{\bar{\rho}} w_r = \max_\rho \sum_{r=1}^\rho w_r$ then we obtain $\overline{\lim}_{n \rightarrow \infty} \phi_n(i) = \max_j \eta_j(i) = \eta_{\bar{\rho}}(i) \quad \forall i$

Proposition 4.1 $\langle \mu^j, \eta_0^j \rangle = 0, i = 1, \dots, d$ where $\eta_0 = ((\eta_0^1)^t, \dots, (\eta_0^d)^t)^t$.

Procedure to compute $\bar{\eta}$

1. Consider c_p , compute $n_p =$ cardinality of c_p and determine \hat{p} such that $n_{\hat{p}} = \min_p n_p$.
2. Compute the asymptotic probability $\mu^{\hat{p}}$ and recursively μ^p for all p as a function of $\mu^{\hat{p}}$.
3. Compute $w_\rho = \langle \mu^\rho, F^\rho \rangle$ and determine $\bar{\rho}$ such that $\sum_{r=1}^{\bar{\rho}} w_r = \max_\rho \sum_{r=1}^\rho w_r$.
4. Compute $\eta_0^{\bar{\rho}}$ trough equation: $\eta_0^{\bar{\rho}} = (P^d)^{\bar{\rho}}, \bar{\rho} \eta_0^{\bar{\rho}} + \tilde{F}^{\bar{\rho}}$ where $\tilde{F}^{\bar{\rho}} = \sum_{j=0}^{\bar{\rho}-1} P^j F$ with additional condition $\langle \mu^j, \eta_0^j \rangle = 0$.
5. Compute $\eta_{\bar{\rho}}(s) = \eta_0(s) + \sum_{\rho=s}^{s+\bar{\rho}-1} w_\rho$.

Accelerated Algorithm to compute the solution of (11)

We are going to present an algorithm to obtain the solution of (11). The procedure finds in a finite number of steps, $\bar{\eta}$, the limit point of iterations $v^{\nu+1} = M^k v^\nu$, with $v^0 = \bar{\eta}$, where $\bar{\eta}$ is the solution of $\bar{\eta} = P\bar{\eta} + F$ that not satisfies $\phi_1 \leq \bar{\eta} \leq \phi_2$ (if $\bar{\eta}$ satisfies this condition it is just the solution of (11)). The methodology is an extension of the one presented in [9] to the case where operator M^k is not a contractive operator. As a matter of fact, we denote V the solution of our continuous problem and:

$$S_1 = \{j \in X \mid V(j) = \phi_1(j)\}$$

$$S_2 = \{j \in X \mid V(j) = \phi_2(j)\}$$

$$\tilde{S}_2^0 = \left\{ j \mid \bar{\eta}(j) - \phi_2(j) = \max_i (\bar{\eta}(i) - \phi_2(i)) \right\} \cap \{j \mid \bar{\eta}(j) > \phi_2(j)\}$$

$$\tilde{S}_1^0 = \left\{ j \mid \bar{\eta}(j) - \phi_1(j) = \max_i |\bar{\eta}(i) - \phi_1(i)| \right\} \cap \{j \mid \bar{\eta}(j) < \phi_1(j)\}$$

Then, we can prove:

Lemma 4.1 $\tilde{S}_i^0 \subset S_i$ with $i = 1$ or 2

Proof: Let $\tilde{S}_2^0 \neq \emptyset$, we denote $\Delta = \max_i (\bar{\eta}(i) - \phi_2(i))$, in the same way the vector with components equals to Δ and P the matrix defined in (11). Then, taking in mind that $P\Delta = \Delta$, we have:

$$\bar{\eta} - \Delta \leq \phi_2 \quad \forall j = 1, \dots, N$$

$$\bar{\eta} - \Delta = \phi_2 \quad \forall j \in \tilde{S}_2^0$$

$$P(\bar{\eta} - \Delta) + F = P\bar{\eta} + F - P\Delta = \bar{\eta} - \Delta$$

As M^k is a monotone operator:

$$P_{[\phi_1, \phi_2]}(P(\bar{\eta} - \Delta) + F) = P_{[\phi_1, \phi_2]}(\bar{\eta} - \Delta) \geq \bar{\eta} - \Delta$$

$$M^k(\bar{\eta} - \Delta) \geq \bar{\eta} - \Delta$$

by induction

$$(M^k)^m(\bar{\eta} - \delta) \geq M^k(\bar{\eta} - \Delta) \geq \bar{\eta} - \Delta$$

but

$$(M^k)^m(\bar{\eta} - \Delta)(i) \rightarrow \bar{w}(i), \text{ when } m \rightarrow \infty$$

then

$$\bar{w}(i) \geq (M^k)^m(\bar{\eta} - \Delta)(i) \geq (M^k)(\bar{\eta} - \Delta)(i) \geq (\bar{\eta} - \Delta)(i) = \phi_2(i) \quad \forall i \in \tilde{S}_2^0$$

that implies $i \in S_2$. Analogously, we obtain $\tilde{S}_1^0 \subset S_1$. ■

By Lemma (4.1) we can deduce that **we know the solution in** $\tilde{S}_1^0 \cup \tilde{S}_2^0$. We are going to use this information to find the discrete solution. We redefine matrix P and F in the following way: $P^0 = P$, $F^0 = F$, $w^0 = \bar{\eta}$ and $\forall n \geq 1$ w^n is the solution of

$$w^n = P^n w^n + F^n \tag{20}$$

where

$$P^n(i, j) = \begin{cases} 0 & \forall i \in \tilde{S}_1^{n-1} \cup \tilde{S}_2^{n-1} \\ P^{n-1}(i, j) & \text{if not} \end{cases}$$

$$F^n(i) = \begin{cases} \phi_1(i) & \forall i \in \tilde{S}_1^{n-1} \\ \phi_2(i) & \forall i \in \tilde{S}_2^{n-1} \\ F^{n-1}(i) & \text{if not} \end{cases}$$

with

$$\tilde{S}_2^n = \{j \mid w^n(j) - \phi_2(j) = \max_i (w^n(i) - \phi_2(i))\} \cap \{j \mid w^n(j) > \phi_2(j)\}$$

$$\tilde{S}_1^n = \{j \mid w^n(j) - \phi_1(j) = \max_i |w^n(i) - \phi_1(i)|\} \cap \{j \mid w^n(j) < \phi_1(j)\}$$

$$S_2^n = \bigcup_{m=0}^n \tilde{S}_2^m; \quad S_1^n = \bigcup_{m=0}^n \tilde{S}_1^m; \quad C^n = C(S_1^n \cup S_2^n).$$

Remark 4.3 : We know that there exists unique solution of (20) because P^n has a submatrix of norm smaller than 1.

Lemma 4.2 It is valid:

$$w^n(i) = \begin{cases} \phi_1(i) & \text{if } i \in \tilde{S}_1^{n-1} \\ \phi_2(i) & \text{if } i \in \tilde{S}_2^{n-1} \end{cases}$$

The proof is obvious by definitions of P^n and F^n given in (20).

We consider the following operator $M_n^k(w)$ defined by:

$$(M_n^k w)(j) = P_I (P^n w + F^n)(j)$$

with P^n and F^n defined in (20). It is easy to see that our solution is also the fixed point of operator M_n^k that is:

$$\bar{w} = M_n^k \bar{w} \quad (21)$$

The following Lemma holds:

Lemma 4.3 $\tilde{S}_1^n \cup \tilde{S}_2^n \neq \emptyset$ or $w^n = \bar{w}$

Proof: If $\tilde{S}_1^n \cup \tilde{S}_2^n \neq \emptyset$ and if $i \notin S_1^{n-1} \cup S_2^{n-1}$, we have:

$$w^n(i) \in [\phi_1(i), \phi_2(i)]$$

then

$$(M_n^k w^n)(i) = P_I (P^n w^n + F^n)(i) = P_I (w^n)(i) = w^n(i) \quad (22)$$

so, by (21) and (22) we have:

$$w^n = \bar{w}$$

■

ALGORITHM

Step 0: $n=0$

Step 1: Compute w^n as in (20). Obtain \tilde{S}_1^n and \tilde{S}_2^n

Step 2: $\tilde{S}_1^n \cup \tilde{S}_2^n \neq \emptyset$ then stop.

If not, obtain $P^n(i, j)$ and $F^n(i)$ defined in (20). Let $n = n + 1$ and go to step 1.

Theorem 4.6 The algorithm finishes in a finite number of steps smaller than N .

Proof: $\tilde{S}_1^n \cup \tilde{S}_2^n \neq \emptyset$ implies cardinality of $C^n < \text{cardinality } C^{n-1}$. But as cardinality of $C^1 \leq N$, there exists an index \tilde{n} such that cardinality of $C^{\tilde{n}} = \text{cardinality } C^{\tilde{n}-1}$, then $\tilde{S}_1^{\tilde{n}} \cup \tilde{S}_2^{\tilde{n}} = \emptyset$ and by Lemma (4.3) $w^{\tilde{n}} = \tilde{w}$. ■

Remark 4.4 : *We can modify this algorithm taking $w^0 = \phi_2$ as the initial vector (see [9]).*

Structure of the set of solutions

We are going to characterize the structure of all the solutions of (11) in final classes . In the last sections we have proved that we must take in account the following cases:

1. there exists $\bar{\eta}$ (the limit point of sequence ϕ_n) such that $\phi_1 \leq \bar{\eta} \leq \phi_2$. In this case $w = \bar{\eta} + ce$ is also solution if $w(x) \in [\phi_1(x), \phi_2(x)] \forall c \in [cmin, cmax]$ where $c, cmin, cmax$ are constants and e the identity vector.
2. $\bar{\eta}$ does not satisfy the last condition, then the iterative algorithm

$$v^{n+1} = M^k v^n; \quad v^0 = \bar{\eta}$$

converges to the solution of problem (11) and the following situations can occur:

- $\bar{\eta}$ is solution of the linear system, then there exists a constant $a > 0$ such that $\bar{\eta} + ce$ is a solution $\forall c \in [0, a]$ or $\forall c \in [-a, 0]$.
- $\bar{\eta}$ is not solution of the linear system, then is the “unique” solution of (11).

5 Convergence properties

When we can insure that the problem finishes in a finite time, we can easily prove the convergence property (of type \sqrt{k}) of the unique discrete solution of our problem. That is the case of boundary problems or ergodic problems with mean different to zero.

In this cases we can reduce the problem to a finite time problem (see [1], [2], [3], [4] and [5]) and we can prove the convergence, in the viscosity sense) with the techniques using in [3], [14], [15] and [16] for this type of problem.

6 Conclusion

We have studied an undiscounted zero sum differential game problem with stopping time. In this case we lose in general the regularity properties of the value function. We have showed an easy example where the value function is discontinuous. This fact implies the difficulty of its numerical resolution because the discrete problem may have infinite numerical solutions and it might not converge.

We have totally characterized the set of solutions of the discrete problem. We have obtained the solution that represent the probabilistic formulation of the original problem and we have presented an algorithm to compute it.

In regular cases we have proved the convergence of the discrete problem to the real solution because in these cases the problem can be associated to a finite time problem and we can use usual techniques to prove convergence properties.

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