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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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## Circular Separability of Polygons

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Programme 4 — Robotique, image et vision  
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**Abstract:** Two planar sets are circularly separable if there exists a circle enclosing one of the set and whose open interior disk does not intersect the other set. This paper studies two problems related to circular separability. A linear-time algorithm is proposed to decide if two polygons are circularly separable. The algorithm outputs the smallest separating circle. The second problem asks for the largest circle included in a preprocessed, convex polygon, under some point and/or line constraints. The resulting circle must contain the query points and it must lie in the halfplanes delimited by the query lines.

**Key-words:** Computational geometry, Parallel robots

*(Résumé : tsvp)*

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## Séparabilité de polygones par des cercles

**Résumé :** Deux ensembles plans sont séparables par un cercle s'il existe un disque dont l'intérieur est disjoint de l'un et dont l'adhérence contient l'autre. Cet article s'intéresse à deux problèmes liés à la séparabilité par un cercle. Il propose un algorithme en temps linéaire pour décider si deux polygones sont séparables par un cercle. L'algorithme produit le plus petit cercle séparateur. Le deuxième problème consiste à trouver le plus grand cercle inclus dans un polygone convexe prétraité satisfaisant certaines requêtes. Les requêtes sont liées à des points : le cercle doit contenir les points, ou à des droites : le cercle doit être inclus dans certains demi-espaces limités par ces droites.

**Mots-clé :** Géométrie algorithmique, Robots parallèles.

## 1 Introduction

Let  $\mathcal{C}$  denote a family of orientable surfaces in the Euclidean space  $E^d$ . We say that  $P \subset E^d$  and  $Q \subset E^d$  are  $\mathcal{C}$ -separable, if there exists  $\Sigma \in \mathcal{C}$ , such that every point of  $P$  lies on one side of  $\Sigma$  and every point of  $Q$  lies on the other side. In the last decade, divers aspects of the separability problem attracted the research interest, with  $\mathcal{C}$  most often being considered as the families of hyperplanes, spheres and polyhedra. For  $P$  and  $Q$  being two finite sets of points, the hyperplane separability may be solved by linear programming [Meg84]. In the case of  $P$  and  $Q$  being two convex polyhedra, this problem is efficiently solved in [DK85].

The problem of finding a polygon with minimum number of vertices, separating two given sets was studied in numerous papers: [EP88], [ABO<sup>+</sup>89], [DJ90], [Mou92], [MS92], [BG94].

The interest in circular separability was fuelled by applications in pattern recognition and image processing, [KA84] [Fis86]. Notice that for two finite sets of points, following the idea of Lay [Lay71], an instance of a spherical separability problem in  $E^d$  may be transformed into a linear separability problem, using a stereographic projection. Kim and Anderson [KA84] presented a quadratic algorithm solving the circular separability problem for two finite sets of points. Bhattacharya [Bha88] improves this bound to  $O(n \log n)$ , computing the entire region at which may be centered all the circles separating the two point sets. O'Rourke, Kosaraju and Megiddo [OKM86] proposed optimal algorithms, finding in  $O(n)$  time the smallest separating circle, and in  $O(n \log n)$  time all largest separating circles for two sets of points. They use the paraboloid transformation to get an instance of a convex, quadratic minimization problem in three dimensions.

In this paper we study two types of problems related to circular separability. In section 3, we propose a linear algorithm determining whether two given simple polygons are separable by a circle. The algorithm simultaneously scans two structures: (1) the list of edges of one polygon, and (2) a path in the furthest point Voronoi diagram of the vertices of the other polygon. The resulting separating circle, which is the smallest possible, is always centered on this path. In section 4, we address a dynamic version of another circular separability problem. We preprocess a convex polygon  $P$ , so that the largest circle inscribed in  $P$ , subject to some query points and/or line constraints may be found efficiently. The resulting circle must contain the query points, and/or it must lie in the halfplanes delimited by the query lines. Our interest in the problem was motivated by an application in motion planning, where convex paths of bounded curvature inside a convex polygon were to be computed [BCD<sup>+</sup>94].

## 2 Preliminaries.

Suppose that we are given a set  $S$  of obstacles in the plane, and we are looking for circles which do not intersect the interior of any of the obstacles. The largest such circle, centered at a query point  $p$ , may be found quickly, if the Voronoi diagram of  $S$  has been precomputed. When a query point  $p$  is localised in a Voronoi cell, the obstacle closest to  $p$  is determined, and the largest circle centered at  $p$  may be easily found.

When the set of obstacles are edges of a convex polygon  $P$ , its Voronoi diagram, called also *skeleton*, partitions polygon  $P$  into convex polygonal cells. As each cell of this partition is adjacent to an edge of  $P$ , the skeleton is a tree. This tree, rooted at the vertex which is the center of the largest circle inscribed in  $P$ , will be called *skeleton tree* and denoted  $SkT(P)$ . A useful way to represent  $SkT(P)$  is by means of a convex polyhedral surface obtained in the following way. For each edge  $e$  of  $P$  consider a plane containing  $e$ , having 45 degrees angle with the plane of  $P$ , and such that  $P$  lies below this plane. Take the lower envelope of the arrangement of all planes obtained this way. It forms a convex polyhedral surface which will be denoted  $Skel(P)$ . Obviously,  $SkT(P)$  is the projection of the edges of  $Skel(P)$  onto the plane of  $P$ .

In the following, a circle is said to be internal to a polygon  $P$  if it is included in the closure of the region which is the interior of  $P$ . There exists a standard mapping  $\phi$  from circles lying in the  $xy$ -plane to points of the three-dimensional space. A circle  $\Sigma$  of radius  $r$ , centered at  $(x_0, y_0)$  is mapped to the point  $\phi(\Sigma) = (x_0, y_0, r)$ . The points on the vertical line, passing through  $(x_0, y_0)$ , are images of the circles centered at  $(x_0, y_0)$ . As each such vertical line intersects  $Skel(P)$  in a single point  $(x_0, y_0, z_0)$ , points below  $z_0$  represent circles internal to  $P$ , and points above  $z_0$  represent circles intersecting or enclosing  $P$ . In consequence, the question of finding the largest internal circle centered at a query point  $(x_0, y_0)$  is equivalent to vertical ray-shooting from  $(x_0, y_0, 0)$  to  $Skel(P)$ .

Take a cone originating at  $(x_0, y_0, 0)$  with vertical axis and 45 degrees apex angle. The points on the surface of such cone are images of the circles passing through  $(x_0, y_0)$ . The image of the largest circle internal to  $P$  and passing through  $(x_0, y_0)$  is the point with the largest  $z$ -coordinate of the intersection of this cone with  $Skel(P)$ .

The *furthest site Voronoi diagram* for a set  $S$  of  $m$  given sites  $s_1, s_2, \dots, s_m$  is a partition of the plane into convex regions  $FSV(s_1), FSV(s_2), \dots, FSV(s_m)$ , such that any point in  $FSV(s_i)$  is farther from  $s_i$  than from any other site. The region  $FSV(s_i)$  is non empty if and only if site  $s_i$  is a vertex of the convex hull of set  $S$ , all non empty regions  $FSV(s_i)$  are unbounded and their boundaries form a tree. Each

vertices of this tree is the center of a minimum circle enclosing  $S$  which hereafter is called a further site Voronoi circle or an FS-Voronoi circle for short. Except for the smallest circle enclosing  $S$  which may pass through two or more points of  $S$ , each FS-Voronoi circle passes through at least three points of  $S$ .

In this paper, the furthest site Voronoi diagram will be represented by a forest  $FSArcs(S)$  in the following way. The vertices of  $FSArcs(S)$  are in one to one correspondance with the arcs of the FS-Voronoi circles extending between two consecutive points of  $S$  and smaller than  $\pi$ . The roots of  $FSArcs(S)$  are the arcs of the smallest circle enclosing  $S$ . Let us consider an edge  $E$  of *furthest site Voronoi diagram* which is the common boudary of two cells  $FSV(s_i)$  and  $FSV(s_j)$ . Edge  $E$  is the locus of the centers of circles enclosing  $S$  and passing through  $s_i$  and  $s_j$ . The endpoints of  $E$  are the center of two FS-Voronoi circles  $C_-$  and  $C_+$  which are respectively the smallest and the largest circles passing through  $s_i$  and  $s_j$  and enclosing  $S$ . If segment  $s_i s_j$  is a diameter of  $C_-$ , we assume w.l.o.g. that the arc  $s_i s_j$  of  $C_-$  joining counterclockwisely  $s_i$  and  $s_j$  is smaller than  $\pi$ . If segment  $s_i s_j$  is not an egde of the convex hull of  $S$ , the arc  $s_i s_j$  of  $C_+$  includes at least a point  $s_k$  of  $S$  and, in the forest  $FSArcs(S)$  the arcs  $s_i s_k$  and  $s_k s_j$  of  $C_+$  are the children of the arc  $s_i s_j$  of  $C_-$ . If segment  $s_i s_j$  is an edge of the convex hull of  $S$ ,  $C_+$  is the line through  $s_i$  and  $s_j$  and a terminal node corresponding to the segment  $s_i s_j$  is the child of the arc  $s_i s_j$  of  $C_-$ . Observe that the arcs of a descending path of  $FSArcs(S)$  have monotonically increasing radii. Obviously,  $FSArcs(S)$  has  $O(m)$  complexity.

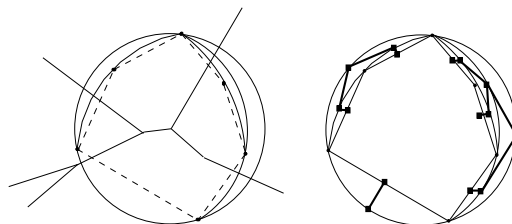


Figure 1: The furthest point Voronoi diagram of  $S$  and the associated forest  $FSArcs(S)$

We will use the *hierarchical representation* of convex polygons introduced in [DK85]. A hierarchical representation of convex polyhedron  $D$  is a nested sequence  $D_0 \supset D_1 \supset \dots \supset D_k$  of convex polyhedra, such that (i)  $D_0$  is a tetrahedron and  $D_k$  is the polyhedron  $D$  and (ii) the set of faces  $F_i$  of  $D_i$  is obtained from  $F_{i+1}$  by removing a subset  $I_{i+1}$  of pairwise non adjacent faces of  $D_{i+1}$ . Polyhedron  $D_i$  is



then formed from  $D_{i+1}$  by extending remaining faces  $F_{i+1} \setminus I_{i+1}$ . It may be proved, that in any polygon  $D_{i+1}$  it is always possible to find a set  $I_{i+1}$  of  $O(|F_{i+1}|)$  faces of bounded degree. Computing of a hierarchical representation of a convex polyhedron with  $n$  vertices may be done within  $O(n \log n)$  time and  $O(n)$  space. The hierarchical representation supports line intersection queries in  $O(\log n)$  time.

### 3 Circles Separating Simple Polygons.

Let  $P$  and  $Q$  be two simple polygons. We called interior of  $P$  and  $Q$  respectively the regions bounded by  $P$  and  $Q$ , and note respectively  $Int(P)$  and  $Int(Q)$  these regions. The regions  $Int(P)$  and  $Int(Q)$  are considered as open regions. Let us assume that  $P$  and  $Q$  have disjoint interiors. We say that circle  $\Sigma$  *separates*  $P$  from  $Q$  if the open disk which is the interior of  $\Sigma$  includes the  $Int(P)$  and no point of  $Int(Q)$  or vice versa. In this section, we propose an efficient algorithm to find a circle which separates two given polygons. The algorithm is designed in such a way, that it outputs the smallest such circle, or it stops deciding that no separating circle exists. In some cases, it is possible that the smallest separating circle has an infinite radius, that is when the polygons are separable by a line, but not by any finite circle. The following lemmas specify the condition for two polygons to be separable by a circle.

**Lemma 1** *Consider two polygons  $P$  and  $Q$  with disjoint interiors, such that  $Int(P) \cap CH(Q) \neq \emptyset$  and  $Int(Q) \cap CH(P) \neq \emptyset$ . There exist a line  $l$  and four points  $x_1, x_2, x_3$  and  $x_4$ , lying in that order  $l$ , such that  $x_1, x_3 \in Int(P)$ , and  $x_2, x_4 \in Int(Q)$  (see Figure 2).*

**Proof :** If  $Int(P) \cap CH(Q) \neq \emptyset$ , there exist a line  $l_1$  and three points  $q_1, p_3, q_2$  in that order on  $l_1$  such that  $q_1, q_2 \in Int(Q)$  and  $p_3 \in Int(P)$ . Indeed,  $Int(P)$  has to intersect at least one of the pockets  $\mathcal{R}$  of  $Q$ , where a pocket is a region of  $CH(Q) \setminus Q$ , limited by an edge  $E$  of  $CH(Q)$  which is not an edge of  $Q$  and a part of  $Q$  joining the endpoints of  $E$ . Then any line going through a point  $p_3 \in Int(P) \cap \mathcal{R}$  and parallel to  $E$  intersects  $int(Q)$  on both sides of  $p_3$ . In the same way, there is a line  $l_2$  and three points  $p_1, q_3, p_2$  in that order on  $l_2$  such that  $p_1, p_2 \in Int(P)$  and  $q_3 \in Int(Q)$ .

Let  $l_3$  be the line through  $p_3$  and  $q_3$  (see Figure 3). We show now that at least one of the three lines  $l_1, l_2$  or  $l_3$  meets the requirement of the lemma. We note  $[q_1, \infty]$  the infinite part of  $l_1$  originating in  $q_1$  and not including  $q_2$ . In the same way, we note  $[q_2, \infty]$ ,  $[p_1, \infty]$  and  $[p_2, \infty]$  the infinite parts of  $l_1$  and  $l_2$ . Let  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ . There is a path  $\gamma_{q_i, q_j}$  included in  $Int(Q)$  and joining  $q_i$  to  $q_j$ . In the same way, we

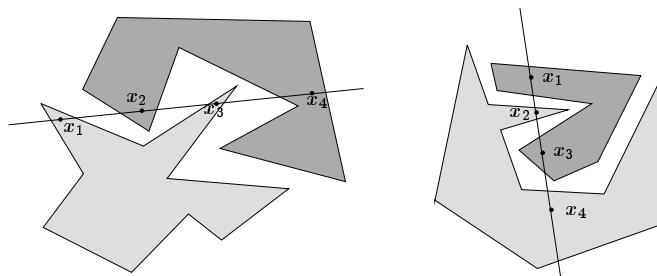


Figure 2: There exist four points  $x_1, x_2, x_3$  and  $x_4$ , lying in that order on a line, such that  $x_1, x_3 \in \text{Int}(P)$ , and  $x_2, x_4 \in \text{Int}(Q)$

shall note  $\gamma_{p_i, p_j}$  a path included in  $\text{Int}(P)$  and joining  $p_i$  to  $p_j$ . Let us assume that neither  $l_1$  nor  $l_2$  meet the requirement of the lemma and show that in that case  $l_3$  will do. Since  $l_2$  does not meet this requirement,  $\gamma_{q_1, q_2}$  does not intersect  $[p_1, \infty]$  nor  $[p_2, \infty]$ . Then, we claim that  $\gamma_{q_1, q_2}$  has to intersect  $[p_3, \infty]$ , the infinite part of  $l_3$  originating in  $q_3$  and not including  $p_3$ . Indeed, let  $[p_i, \infty]$  with  $i = 1$  or  $2$  be one of the infinite portion of  $l_2$  which does not intersect  $l_1$ . The concatenation of  $[p_3, \infty]$ ,  $\gamma_{p_3, p_i}$  and  $[p_i, \infty]$  intersects line  $l_1$  in the single point  $p_3$  and thus separates  $q_1$  from  $q_2$ . As  $\gamma_{q_1, q_2}$  can't intersect  $\gamma_{p_3, p_i}$  nor  $[p_i, \infty]$  it has to intersect  $[p_3, \infty]$ . In the same way  $\gamma_{p_1, p_2}$  has to intersect  $[q_3, \infty]$ , the other infinite part of  $l_3$ , and  $l_3$  meets the requirement of the lemma.  $\square$

**Lemma 2** *Two polygons  $P$  and  $Q$  with disjoint interiors cannot be separated by a circle, if and only if there exists a circle  $C$  and four points  $x_1, x_2, x_3$  and  $x_4$ , in that order on the boundary of  $C$ , such that  $x_1, x_3 \in \text{Int}(Q)$ , and  $x_2, x_4 \in \text{Int}(P)$  (see Fig. 4).*

**Proof :** We prove first that the existence of a circle  $C$  satisfying the above condition implies that the two polygons are not separable by a circle. Circle  $C$  is split by the points  $x_1, x_2, x_3$  and  $x_4$  into four arcs. Observe that any Jordan curve  $\zeta$  separating  $P$  and  $Q$  must intersect each of these four arcs. As any two non-identical circles intersect at two points at most,  $\zeta$  cannot be a circle.

Assume now that there exist no circle  $C$  satisfying the above property, we prove that  $P$  and  $Q$  are separable by a circle. Observe first, that either  $\text{Int}(Q) \cap CH(P) = \emptyset$  or  $\text{Int}(P) \cap CH(Q) = \emptyset$ , otherwise, by lemma 1, there would exist four points  $x_1,$

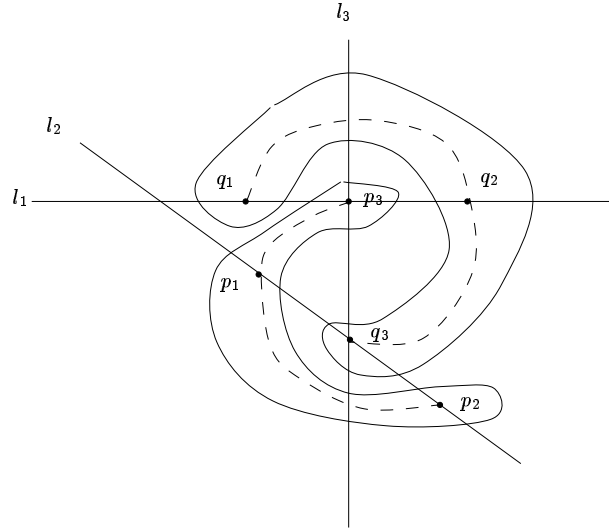


Figure 3: For the proof of lemma 1

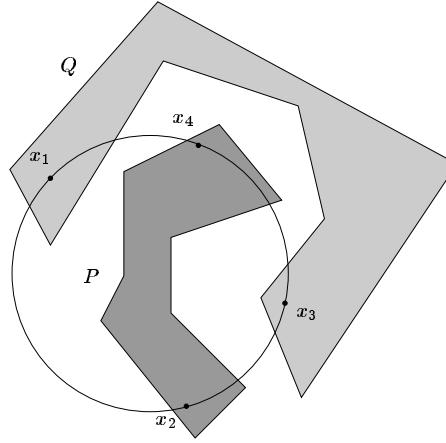


Figure 4: No circle separates  $P$  and  $Q$ .

$x_2$ ,  $x_3$  and  $x_4$  on a line  $l$  contradicting our hypothesis. Suppose, that  $\text{Int}(Q) \cap \text{CH}(P) = \emptyset$ , the other case being symmetrical. Let  $\Sigma$  denote the smallest circle enclosing  $P$ . Consider  $P_1, P_2, \dots, P_k$ , the sequence of points of tangency of  $\Sigma$  and

$P$ , in the counterclockwise order around  $\Sigma$ . Denote by  $P_{P_i P_j}$  the part of boundary of  $P$ , extending counterclockwise from  $P_i$  to  $P_j$ , and denote by  $\Sigma_{P_i P_j}$  the arc of  $\Sigma$  extending counterclockwise from  $P_i$  to  $P_j$ . As  $\Sigma$  is the smallest circle enclosing  $P$ , each arc  $\Sigma_{P_i P_{i+1}}$  is not greater than  $\pi$  (see Fig. 5(a)).

Denote by  $\mathfrak{R}_{i,i+1}$  the region bounded by  $P_{P_i P_{i+1}}$  and  $\Sigma_{P_i P_{i+1}}$ ,  $i = 1, 2, \dots, n$ . The set of regions  $\{\mathfrak{R}_{i,i+1}, i = 1, 2, \dots, n\}$  constitutes a partition of  $\text{Int}(\Sigma) \setminus P$ . If  $\Sigma$  does not separate  $P$  and  $Q$ , one of  $\mathfrak{R}_{i,i+1}$  must intersect  $\text{Int}(Q)$ . Let  $P_r$  and  $P_s$  denote two consecutive points of tangency of  $P$  and  $\Sigma$ , such that  $\mathfrak{R}_{r,s}$  intersects  $\text{Int}(Q)$ . Observe that no other region  $\mathfrak{R}_{i,i+1}$  intersects  $\text{Int}(Q)$ , otherwise, after shrinking  $\Sigma$ , we obtain a circle  $C$  having the property mentioned in the lemma.

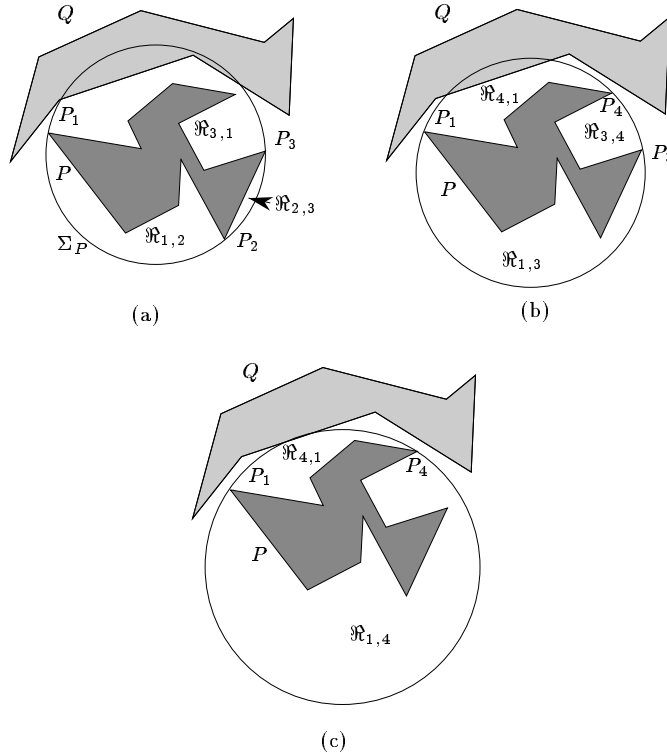


Figure 5: Illustrating existence of the separating circle

Increase continuously circle  $\Sigma$ , keeping it tangent to  $P_r$  and  $P_s$ , until either some new vertex  $P_q$  of  $P$  becomes tangent to  $\Sigma_{P_r P_s}$  or until region  $\mathfrak{R}_{r,s}$  no longer meets  $\text{Int}(Q)$ . In the latter case, observe that at the moment  $Q$  is externally

tangent to  $\Sigma_{P_r P_s}$  (cf. Fig. 5(c)), and  $Int(Q)$  cannot intersect the opposite region  $\mathfrak{R}_{s,r}$ , otherwise the conditions of existence of circle  $C$  would be met. Thus, at that moment the current position of  $\Sigma$  must separate  $P$  and  $Q$ . In the former case, the point  $P_q$  splits the arc  $\Sigma_{P_r P_s}$  into two subarcs  $\Sigma_{P_r P_q}$  and  $\Sigma_{P_q P_s}$ . Region  $\mathfrak{R}_{r,s}$  is thus split into two subregions  $\mathfrak{R}_{r,q}$  and  $\mathfrak{R}_{q,s}$ . As only one region among  $\mathfrak{R}_{r,q}$  and  $\mathfrak{R}_{q,s}$ , say  $\mathfrak{R}_{r,q}$ , still intersects  $Int(Q)$ , replace  $\mathfrak{R}_{r,s}$  by  $\mathfrak{R}_{r,q}$  and continue the process (cf. Fig. 5(b)). Observe that the radius of  $\Sigma$  increases continuously,  $\Sigma$  encloses  $P$  being tangent in  $P_r$  and  $P_s$ , and arc  $\Sigma_{P_r P_s}$  remains smaller than  $\pi$ . As  $CH(P) \cap Int(Q) = \emptyset$ , at some point  $\mathfrak{R}_{r,s}$  will no longer intersect  $Int(Q)$ .  $\square$

Note that in a special case, when some point of the boundary of  $Q$  intersects the interior of some edge  $P_r P_s$  of  $CH(P)$ , the process of increasing  $\Sigma$  stops when the radius of  $\Sigma$  reaches infinity. The only circle separating  $P$  and  $Q$  will then be a circle of infinite radius, being the line of segment  $P_r P_s$ . The following lemma states that, in any case, the separating circle found in lemma 2 will be the smallest possible.

**Lemma 3** *If Circle  $C$  of radius  $r$  intersects polygon  $P$  in two points  $p_1$  and  $p_2$  and polygon  $Q$  in point  $q$ , such that arc  $p_1 q p_2$  is smaller than  $\pi$ , any circle enclosing  $P$  and separating  $P$  from  $Q$  must have its radius greater than  $r$ .*

**Proof :** obvious.  $\square$

### 3.1 The Algorithm.

To determine the separability of two polygons  $P$  and  $Q$ , the algorithm first looks for the smallest circle enclosing  $P$  and whose interior disk does not intersect  $Int(Q)$ , then tries the reverse. For the first purpose, the algorithm uses two data structures : the list  $\mathcal{Q}$  of edges of polygon  $Q$  and the forest  $FSArcs(P)$ , of arcs of the furthest site Voronoi circles for the set of vertices of  $P$ . For any arc  $s_p s_q$  of  $FSArcs(P)$  and a planar figure  $F$  we say that  $A$  cuts  $F$ , if the convex hull of arc  $s_p s_q$  intersects the interior of  $F$ .

The algorithm follows the idea of the proof of Lemma 2. We first determine an arc  $A$  of the smallest circle enclosing  $P$  which cuts  $Q$ . The list  $\mathcal{Q}$  of edges of  $Q$  is then scanned until an edge  $E$  of  $Q$  which actually cuts  $A$  is found. A path of a tree of  $FSArcs(P)$  is now traversed until the current arc  $A$  admits no children cutting the current edge  $E$ . This traversal of  $FSArcs(P)$  corresponds to the process of increasing the radius of the circle enclosing  $P$ , until edge  $E$  no longer intersects the circle. Then the scanning of list  $\mathcal{Q}$  resumes alternatively with the traversal of a path of  $FSArcs(P)$  until an arc  $A$  is found which intersects  $Q$  and whose children

don't. Then, let  $Arc$  be the arc extending between the endpoints of  $A$  and externally tangent to  $Q$ . If the circle of  $Arc$  does not intersect  $Int(Q)$ , we are done, otherwise there is no circle containing  $Q$  and separating  $Q$  from  $P$ .

### Algorithm Smallest Separating Circle

*Input:* A simple polygon  $P$  of  $m$  vertices and a simple polygon  $Q$  of  $n$  vertices.

*Output:* The smallest circle containing  $P$  and disjoint with  $Int(Q)$ , if one exists.

1. Compute  $FSArcs(P)$ .
2. **if** no root of  $FSArcs(P)$  cuts  $Q$ 
  - then** OUTPUT(the smallest circle enclosing  $P$ ); STOP.
  - else**  $A \leftarrow$  a root of  $FSArcs(P)$  which cuts  $Q$ .
3. **while**  $Q$  is not empty **do**
  - 3.1.  $E \leftarrow \text{next}(Q)$
  - 3.2. **while**  $A$  does not cut  $E$  **do**
    - if**  $Q$  is empty, go to 4
    - else**  $E \leftarrow \text{next}(Q)$ .
  - 3.3. **while** there exists a  $child_c(A)$  which cuts  $E$  **do**
    - $A \leftarrow child_c(A)$ .
  - 3.4 **if**  $A$  is a terminal arc of  $FSArcs(P)$ 
    - then** OUTPUT('CH( $P$ ) and  $Q$  intersect'); STOP.
4.  $Arc \leftarrow$  the arc externally tangent to  $Q$  and passing through the endpoints of  $A$ .
5. **if** the complementary arc of  $Arc$  cuts the polygon  $Q$ 
  - then** OUTPUT('CH( $P$ ) and  $Q$  are not separable').
  - else** OUTPUT(circle of  $Arc$ ).

**End of the Algorithm**

### 3.2 The Correctness of the Algorithm.

We prove here that the algorithm outputs the smallest circle enclosing  $P$  and external to  $Q$  if such a circle exists.

First, we observe that if the algorithm terminates in step 2, it outputs the smallest circle enclosing  $P$  which is clearly the smallest circle enclosing  $P$  and external to  $Q$  if this circle does not intersect  $Q$ .

Then notice that if algorithm stops with a terminal arc in step 3.4, the current edge  $E$  of  $Q$  intersects that terminal arc which is an edge of  $CH(P)$ , thus  $CH(P)$  and  $Q$  intersect and there is no separating circle.

If the algorithm does not stop in step 3.4, the while loop of step 3 terminates when the list  $Q$  is empty. Then, the current arc  $A$  is not a terminal arc and it cuts  $Q$  but its children don't. Indeed, every edge of  $Q$  scanned while the current arc is  $A$  don't cut  $A$ , hence these edge don't cut the children of  $A$  because the convex hull of any arc contains the convex hull of any of its descendant in  $FSArcs(P)$ . As for the edge scanned before arc  $A$  is the current arc, they don't cut the children of an ancestor of  $A$  that is an ancestor of the children of  $A$ .

Let us show that the arc  $Arc$  computed in step 4 is uniquely defined. Let  $s_p$  and  $s_q$  be the endpoints of the current arc  $A$  at the end of step 3. The segment joining the center of the circle including arc  $A$  and the center of the circle including its children is an edge of the furthest site Voronoi diagram of  $P$ ; this edge is the locus of the centers of circles which enclose  $P$  and pass through  $s_p$  and  $s_q$ . The arc  $s_p s_q$  of the circle including  $A$  cuts  $Q$  while the arc  $s_p s_q$  of the circle including the children of  $A$  does not. By continuity, there is a point on this furthest site Voronoi edge which is the center of a circle through  $s_p$  and  $s_q$ , enclosing  $P$  and whose arc  $s_p s_q$  is tangent to  $Q$ . This circle is the extension of  $Arc$ .

In step 5, when the complementary arc  $s_q s_p$  of  $Arc$  cuts the polygon  $Q$ , there exists a small disk  $d$  internal to  $Q$  and centered in some point  $x$  on  $s_p s_q$ . Remind that  $Arc$  is not greater than  $\pi$  and that it is tangent in some point  $y$  to an edge of  $Q$ . Thus it is possible to modify slightly the circle of  $Arc$ , so that it encloses point  $y$  but neither of  $s_p$  and  $s_q$ , and still intersects a part of disk  $d$ . Then, the condition of lemma 2 are verified and there exists no circle separating  $P$  and  $Q$ . Note that, as step 2 does not compute all the roots cutting  $Q$ , and step 4.2 does not test all the children of  $A$  for cutting  $Q$  the non-separability of  $P$  and  $Q$  is not detected earlier.

Finally, when the complementary arc of  $Arc$  does not cut polygon  $Q$ , no edge of  $Q$  cuts the interior of the circle of  $Arc$ , and the circle encloses  $P$ . Hence, it is a separating circle. On the other hand, by the construction of  $Arc$ , it follows from lemma 3, that this circle is the smallest separating circle enclosing  $P$ .

### 3.3 The Complexity of the Algorithm.

The first step relies on well known optimal algorithms. By [Lee83], the convex hull of  $P$  is computed in  $O(m)$  time. Within the same complexity, [AGSS89] computes the furthest site Voronoi diagram of a convex polygon, which results in the construction of  $FSArcs(P)$  and the smallest circle enclosing  $P$ .

Step 2 may be computed easily within  $O(m + n)$  time in the following way. In  $O(n)$  time, all edges of  $Q$  are tested for intersection with the interior of the smallest circle enclosing  $P$ . Then any of the edges found to intersect this disk is tested for cutting by the roots of  $FSArcs(P)$ . As there is  $m$  roots at most, this is done within  $O(m)$  time.

Steps 3.1 and 3.2 are executed at most  $O(n)$  times overall, as each execution results in skipping an element of  $\mathcal{Q}$ . Step 3.3 is executed  $O(m)$  times at most, as  $FSArcs(P)$  has  $O(m)$  complexity. Step 3.4 is executed at most once. Hence, the overall complexity of step 3 is  $O(m + n)$ .

Step 4 is executed in constant time and Step 5 in  $O(m)$  time, thus we can conclude by the following result.

**Theorem 4** *In  $O(m + n)$  time and space it is possible to determine if two given polygons, one with  $m$  and the other one with  $n$  vertices, are separable by a circle. The smallest separating circle may be found within the same bounds.*

Observe that, although it makes no sense to ask for a circle separating two polygons with not-disjoint interiors, Algorithm Smallest Separating Circle still works in this case. The algorithm will either detect the intersection of the two polygons in step 2 or stops with a terminal arc in step 3.3. The algorithm also works in the case when polygon  $Q$  is not necessarily simple. Moreover, the algorithm extends on the case when the first polygonal curve contains the second one, i.e. when we want to separate the unbounded region lying outside the external curve, from the region bounded by the internal curve. It is easy to observe that the algorithm generalizes also to the case of separation of connected planar straight line graphs. We say that two graphs are separated by a circle if no edge of the first graph intersects the interior of the circle while no edge of the second graph intersects the exterior of the circle. Indeed, in linear time each graph may be transformed to a polygon, obtained by the traversal of the external face of the graph. As some edges may be traversed twice, the polygon is not simple in general. However, the algorithm still works in this case.

Furthermore it is to be noticed that our method can be extended to answer separability queries when the allowed separating curves are the homothets of a given convex curve. Indeed, the algorithm relies on Lemma 3 which still holds if the circles



are replaced by the homothets of a given convex curve because two homothets convex curve intersect in at most two points. In that case, the algorithm computes the furthest site Voronoi diagram of polygon  $P$  for the convex distance associated with the given convex curve.

## 4 Largest Circles Inscribed in Convex Polygons.

In this section we study another version of the problem of circular separability. Suppose that we want to separate a convex polygon  $P$  from a set of points lying inside the polygon. Suppose as well, that the polygon  $P$  may be preprocessed, so that for each set  $S$  of points given as a query, separation of  $P$  from  $S$  may be decided efficiently. We also address the question when a part of the query is the line, delimiting a halfplane in which the separating circle must lie.

### 4.1 Point Set Queries.

We start by the case of single point queries.

**Theorem 5** *It is possible to preprocess a convex  $n$ -gon  $P$  in  $O(n)$  time and space, so that given a query point  $x$ , the largest circle enclosing  $x$  and internal to  $P$  may be found in  $O(\log n)$  time.*

**Proof :** Compute  $SkT(P)$  and a planar partition of  $P$  induced by  $SkT(P)$  in the following way. Each vertex of  $SkT(P)$  is the center of a circle internal to  $P$  which has at least three tangent points with  $P$  and is called a Voronoi circle. For each Voronoi circle, we consider the arcs extending between two consecutive tangent points. Each such arc which is not greater than  $\pi$  is included in the planar map see Fig. 6. In this way we obtain a partition of the interior of  $P$ . One region is the interior of the largest circle  $C$  inscribed in  $P$ . Other regions are bounded by two circular arcs and two parts of edges of  $P$ . Regions adjacent to vertices of  $P$  may be considered of the same type, with one of the arcs degenerated to a single point. As  $SkT(P)$  is computed in  $O(n)$  time and space using [AGSS89], the planar map may be computed within the same bounds. Observe that if the query point  $x$  lies inside  $C$ , the largest separating circle is  $C$  itself. If point  $x$  lies outside  $C$ , the largest separating circle passes through  $x$  and is tangent to the two portions of edges of  $P$ , bounding the region of the map which contains point  $x$ . Thus, the largest separating circle may be found in constant time, once point  $x$  has been located in the planar map. By well known methods, following the idea of [Kir83], our planar map may be

preprocessed in  $O(n)$  time and space, so that the point location would be possible in  $O(\log n)$  time.  $\square$

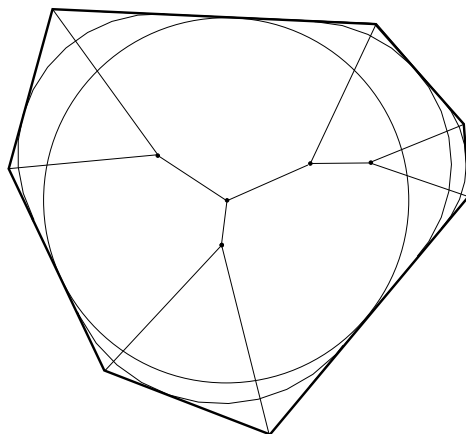


Figure 6: Planar map induced by the arcs of Voronoi circles

**Theorem 6** *It is possible to preprocess a convex  $n$ -gon  $P$  in  $O(n)$  time and space, so that given as a query a set  $S$  of  $k$  points, the largest circle enclosing  $S$  and internal to  $P$  may be computed in  $O(k \log n)$  time and  $O(n + k)$  space.*

**Proof:** Construct the planar map, as in Fig. 6 in the preprocessing step.  $SkT(P)$  is the dual graph of the map. Let  $p$  be a point of  $S$ . We observe that all maximal disks included in  $Int(P)$  and containing  $p$  are centered on a subtree of  $SkT(P)$  rooted at the center of the largest internal circle passing through  $p$ . Thus, if two points  $p$  and  $q$  of  $S$  belong to two different cells of the planar map which correspond to the unrelated vertices of  $SkT(P)$ , i.e. such that none of this two vertices is an ancestor of the other one, no circle internal to  $P$  contains both points  $p$  and  $q$ . Hence, if  $S$  is enclosed in a circle internal to  $P$ , all points of  $S$  must belong to cells, whose duals belong to a descending path of  $SkT(P)$ . To answer the query, we perform first the point location in the map of each element of  $S$ . We check next if the cells of the query points correspond to a descending path in  $SkT(P)$ . For each query point  $q$  we compute the largest circle inscribed in  $P$  and containing  $q$ . The smallest among all

these circles is the candidate for the circle containing  $S$ . It is sufficient if all points of  $S$  belong to the candidate circle. The complexity of the algorithm is dominated by the point location step, taking  $O(k \log n)$  time.  $\square$

Remark, that the smallest circle internal to  $P$ , and containing a set of  $k$  points, may be computed using the technique from the previous section. The set of  $k$  points must first be connected to form the set of vertices of a polygon. We can conclude by the following alternative result

**Corollary 7** *Given a convex  $n$ -gon  $P$  and a set  $S$  of  $k$  points, the largest circle containing  $S$  and internal to  $P$  may be found in  $O(k \log k + n)$  time and  $O(n + k)$  space.*

## 4.2 Queries Involving Lines.

We consider first the case when the query consists of a single line, determining a halfplane which must contain the resulting circle.

**Theorem 8** *It is possible to preprocess a convex  $n$ -gon  $P$  in  $O(n \log n)$  time and space, so that given a query line  $l$ , the largest circle internal to  $P$  and lying in a closed halfplane  $H_l^+$ , determined by  $l$ , may be found in  $O(\log n)$  time.*

**Proof :** Let  $v_f \in H_l^+$  be the vertex of  $P$  which lies at the largest distance from  $l$ . The part of the boundary of  $P$  lying in  $H_l^+$  is split by  $v_f$  into two chains of edges. The largest circle  $C$  inscribed in  $P \in H_l^+$  must be tangent to each of these two chains.  $C$  is then centered on the path of  $SkT(P)$  joining its root with vertex  $v_f$ . See Fig. 7. Observe that the center of  $C$  splits this path into two parts, one subpath contains the centers of Voronoi circles intersecting  $l$ , while the other subpath contains the centers of circles contained in  $H_l^+$ .

To answer the query, we first find in  $O(\log n)$  time vertex  $v_f$ . Then we perform a binary search on the path joining the root of  $SkT(P)$  with vertex  $v_f$ , to find an edge of  $SkT(P)$  containing the center of  $C$ . Now we can find  $C$  in constant time.

In order to perform above algorithm, an appropriate search structure must be build in the preprocessing time. It is sufficient to add to each vertex of  $SkT(P)$  the pointers to its ancestors at distance  $2^i$ , for  $i = 1, 2, \dots, \lfloor \log n \rfloor$ . It is possible to construct such structure in  $O(n \log n)$  time and space, during a standard tree-traversal of  $SkT(P)$ .  $\square$

Our next result considers the case when the query is given as a pair of lines, determining a wedge in which the solution circle must be contained.

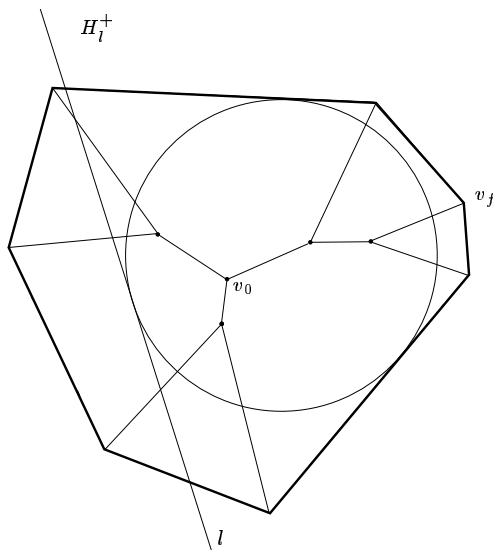


Figure 7: The largest circle contained in  $P \cap H_l^+$  is centered on the path joining  $v_f$  and  $v_0$

**Theorem 9** *It is possible to preprocess a convex  $n$ -gon  $P$  in  $O(n \log n)$  time and space, so that given as a query two lines  $l_1$  and  $l_2$ , the largest circle  $C$  internal to  $P$ , and lying in the closed wedge determined by  $l_1$  and  $l_2$  may be found in  $O(\log n)$  time.*

**Proof :** Three cases are possible. The resulting circle  $C$  is tangent to both lines  $l_1$  and  $l_2$ , it is tangent to one of them, or  $C$  does not meet any of the two lines. Suppose that  $C$  is tangent to  $l_1$  and  $l_2$ . Consider the space of circles introduced in the Preliminaries section. Take a halfplane  $\mathcal{H}_{l_1}$ , originating at line  $l_1$  of  $x$ - $y$  plane, having 45 degrees angle with the vertical axis. When  $C$  is tangent to  $l_1$ ,  $\phi(C)$  must belong to  $\mathcal{H}_{l_1}$ . In our case  $\phi(C)$  is the intersection of the line  $\delta = \mathcal{H}_{l_1} \cap \mathcal{H}_{l_2}$  with  $Skel(P)$ . Hence, the problem reduces to finding an intersection of a line with a convex polyhedron, which may be answered in  $O(\log n)$  time, supposing  $O(n \log n)$  computation of the hierarchical representation of  $Skel(P)$  in the preprocessing time.

The algorithm takes four cases into consideration. In the first case, the largest circle inscribed in  $P$  is output as the solution as long as it does not intersect  $l_1$  nor

$l_2$ . In the second case, the largest circle contained in  $P \cap H_{l_1}^+$  is computed. This circle is the solution of our problem if it does not intersect  $l_2$ . Similarly, in the third case, the largest circle contained in  $P \cap H_{l_2}^+$  is computed and then checked for the intersection with  $l_1$ . Finally, the largest circle contained in  $P \cap H_{l_1}^+ \cap H_{l_2}^+$  is found using the above method. Obviously, the solution exists only when  $P \cap H_{l_1}^+ \cap H_{l_2}^+ \neq \emptyset$ . Except for the first case, our algorithm uses  $O(\log n)$  time, supposing  $O(n \log n)$  time preprocessing.  $\square$

Similar technique is used to solve the mixed query problem, when the resulting circle must contain a given point, and it must lie on one side of a given line.

**Theorem 10** *It is possible to preprocess a convex  $n$ -gon  $P$  in  $O(n \log n)$  time and space, so that given as a query a line  $l$  and a point  $x \in H_l^+$ , the largest circle  $C$  internal to  $P$ , enclosing  $x$  and lying in the closed halfplane  $H_l^+$ , may be found in  $O(\log n)$  time.*

**Proof :** Suppose that  $C$  is tangent to  $l$  and contains  $x$  on its boundary.  $\phi(C)$  lies then on a parabola  $\wp$ , being the intersection of  $H_{l_1}^+$  with the vertical cone originating at  $x$ , having 45 degrees apex. It is possible to adapt the algorithm of line intersection queries to the case of the intersections between  $Skel(T)$  and parabola  $\wp$ . Indeed, the parabola  $\wp$  intersect  $Skel(T)$  and each polyhedron of the hierarchical decomposition of  $Skel(T)$  in at most two points.

The algorithm checks if the largest circle inscribed in  $P$  contains  $x$  and lies in  $H_l^+$ . If this is not the case we find, like in Theorem 5, the largest circle containing  $x$ , and we output this circle if it lies in  $H_l^+$ . Otherwise, we continue, like in Theorem 8, computing the largest circle inscribed in  $P$ , which lies in  $H_l^+$ . We output this circle when it contains  $x$ . Finally, if no circle was output yet, we find circle  $C$  tangent to  $l$  and containing  $x$  on its boundary using the above method. The solution does not exist when the parabola  $\wp$  does not intersect  $Skel(P)$ . The complexity of the query algorithm is  $O(\log n)$ . The preprocessing is dominated by the  $O(n \log n)$  hierarchical decomposition and the construction of the search structure needed in Theorem 8.  $\square$

Observe that Theorems 5, 6, 7 and 8 can be easily generalized to query concerning the homothets of a given convex curve. In the same way, theorem 9 and 10 can be generalized : the mapping  $\phi$  from the homothet convex curves to points in the three dimensional space is defined analogously than in the case of circles by choosing a reference point internal to the convex curve and a particular point on the convex curve whose (euclidian) distance to the reference point will be consider as the radius of the convex curve. Then the locus of points which are the images of curve internal to  $P$  and tangent to  $P$  is still a polyedron  $Skel(P)$  and the locus of points which are

images of curves tangent to a line  $l$  is still an hyperplan  $\mathcal{H}_l$ . The cone which is the image of the convex curves passing through a point  $x$  is no longer a circular cone but a cone whose sections perpendicular to the vertical axis are the homothets of a convex curve dual to the given convex curve.

## 5 Conclusion and Open Problems.

The paper studied two types of problems concerning circular separability. In Section 3, the problem of separability of two simple polygons is solved. Section 4 concerns the problem of the largest circle inscribed in a convex polygon, given some query point and/or line constraints. The natural way to approach the circular separability problems is to employ some mixture of the furthest point and the closest point Voronoi diagrams. However, in many cases the naive way of making use of this method leads to a quadratic algorithm. Consider, for example, the case of the largest circle separating two simple polygons. Such circle is of one of the two possible types: it is either tangent to three edges of the external polygon, or it is tangent to two edges of the external polygon and one vertex of the internal one. The circle of the first type may be found in  $O(n \log m)$  time, considering Voronoi circles centered at vertices of  $Vor(Q)$ , the closest point Voronoi diagram of the external polygon, and localising their centers in  $FSVor(P)$ , the furthest site Voronoi diagram of the internal polygon. To find the separating circle of the second type, we may superpose  $Vor(Q)$  and  $FSVor(P)$ . Taking into consideration, one by one, each portion of an edge of  $Vor(Q)$ , lying in some face of  $FSVor(P)$ , leads to the investigation of all the candidate circles of the second type. However, such structure needs  $O(mn)$  space. We strongly believe, that the largest circle separating two simple polygons may be found in better than quadratic time.

Using a convex distance to compute the Voronoi diagram, our method can be tuned to answer separation queries for separating curves which are the homothets of a given convex curve.

It is natural to try to extend our approach to higher dimensions. The method from [OKM86], detecting the spherical separability of two sets of points, is based on linear programming and it gives  $O(n)$  solution in any dimension. However, the paraboloid transformation method, used in [OKM86], seems not applicable in the case of simple polygons. Our algorithm achieves the linear bound scanning two structures: (1) the list of edges of one polygon, and (2) a path in the furthest point Voronoi diagram of the vertices of another polygon. The solution circle is always centered on this path. In the three-dimensional space, the center of the separating

sphere may not belong to a Voronoi edge of any of the two polyhedra. Our "edge-marching" approach is then not directly applicable to the higher dimensional case.

It is also tempting to ask for the solutions of the higher-dimensional version of the problems from section 4. The single point queries may be solved in  $O(\log n)$  time by the similar, point location approach. The separating spheres are tangent to two or three polyhedral faces. The cells are separated by parts of disks, orthogonal to polyhedral edges, as well as spherical and conical surfaces. However, it is not clear how to answer queries involving two or more points.

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## References

- [ABO<sup>+</sup>89] A. Aggarwal, H. Booth, J. O'Rourke, S. Suri, and C. K. Yap. Finding minimal convex nested polygons. *Inform. Comput.*, 83(1):98–110, October 1989.
- [AGSS89] A. Aggarwal, L. J. Guibas, J. Saxe, and P. W. Shor. A linear-time algorithm for computing the Voronoi diagram of a convex polygon. *Discrete Comput. Geom.*, 4:591–604, 1989.
- [BCD<sup>+</sup>94] J.-D. Boissonnat, J. Czyzowicz, O. Devillers, J.-M. Robert, and M. Yvinec. Convex tours of bounded curvature. In *European Symp. on Algorithms*, pages 254–265, 1994. to appear.
- [BG94] H. Brönnimann and M. T. Goodrich. Almost optimal set covers in finite vc-dimension. In *Proc. 10th Annu. ACM Sympos. Comput. Geom.*, 1994.
- [Bha88] B. K. Bhattacharya. Circular separability of planar point sets. In G. T. Toussaint, editor, *Computational Morphology*, pages 25–39. North-Holland, Amsterdam, Netherlands, 1988.
- [DJ90] G. Das and D. Joseph. The complexity of minimum convex nested polyhedra. In *Proc. 2nd Canad. Conf. Comput. Geom.*, pages 296–301, 1990.

- 
- [DK85] D. P. Dobkin and D. G. Kirkpatrick. A linear algorithm for determining the separation of convex polyhedra. *J. Algorithms*, 6:381–392, 1985.
- [EP88] H. Edelsbrunner and F. P. Preparata. Minimum polygonal separation. *Inform. Comput.*, 77:218–232, 1988.
- [Fis86] S. Fisk. Separating point sets by circles and the recognition of digital disks. *IEEE Trans. Pattern Anal. Mach. Intell.*, 8:554–556, 1986.
- [KA84] C. E. Kim and T. Anderson. Digital disks and a digital compactness measure. In *Proc. 16th Annu. ACM Sympos. Theory Comput.*, pages 117–124, 1984.
- [Kir83] D. G. Kirkpatrick. Optimal search in planar subdivisions. *SIAM J. Comput.*, 12:28–35, 1983.
- [Lay71] S.R. Lay. On separation by spherical surfaces. *Amer. Math. Monthly*, 78:1112–1113, 1971.
- [Lee83] D. T. Lee. On finding the convex hull of a simple polygon. *Internat. J. Comput. Inform. Sci.*, 12:87–98, 1983.
- [Meg84] N. Megiddo. Linear programming in linear time when the dimension is fixed. *J. ACM*, 31:114–127, 1984.
- [Mou92] D. M. Mount. Intersection detection and separators for simple polygons. In *Proc. 8th Annu. ACM Sympos. Comput. Geom.*, pages 303–311, 1992.
- [MS92] J. S. B. Mitchell and S. Suri. Separation and approximation of polyhedral surfaces. In *Proc. 3rd ACM-SIAM Sympos. Discrete Algorithms*, pages 296–306, 1992.
- [OKM86] J. O’Rourke, S. R. Kosaraju, and N. Megiddo. Computing circular separability. *Discrete Comput. Geom.*, 1:105–113, 1986.





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