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***Structural, temporal and stochastic properties
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PROGRAMME 1

Architectures parallèles,
bases de données,
réseaux et systèmes distribués



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Structural, temporal and stochastic properties of unbounded free-choice Petri nets

François Baccelli *, Sergei Foss ** et Bruno Gaujal *

Programme 1 — Architectures parallèles, bases de données, réseaux et systèmes distribués
Projet Mistral

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Abstract: In this paper, we show how a decomposition of a free choice Petri net into a “routing” network and marked graph subnetworks (i.e. linear subnetworks in the $[\max, +]$ setting) leads to new methods and algorithms to test structural as well as temporal properties of the net. Although several results hold for general free choice nets, the paper primarily focuses on the class of single input-free choice nets, defined here. We show how this decomposition in linear subnets allows one to:

- (in the untimed case) check liveness in polynomial time;
- (in the timed case) establish evolution equations which allow to represent the system as a coupling of two linear systems, a $(\min, +)$ -linear system, and a quasi $(+, \times)$ -linear one;
- (in the stochastic case) check stability, i.e. the fact that the marking remains bounded in probability.

The main tools for proving these properties are graph theory, idempotent algebras and ergodic theory.

Key-words: Petri nets, $(\min, +)$ algebra, stability, ergodic theory

(Résumé : tsvp)

*{Francois.Baccelli}{Bruno.Gaujal}@sophia.inria.fr

**Novosibirsk State University, 630090 Novosibirsk, Russia

Propriétés structurelles, temporelles et stochastiques des réseaux de Petri à choix libres

Résumé : Dans cet article, nous montrons comment une décomposition d'un réseau de Petri à choix libres en un graphe de routage et des sous-réseaux graphes marqués (c'est à dire des sous-réseaux linéaires dans le contexte $[\max, +]$) conduit à de nouvelles méthodes pour tester des propriétés structurelles et temporelles du réseau. Bien que certains résultats soient vrais pour des réseaux généraux, cet article s'intéresse surtout à la classe des réseaux à entrée unique, qui est définie dans l'article. Cette décomposition nous permet

- (dans le cas non-temporisé) de vérifier la vivacité du réseau en temps polynomial;
- (dans le cas temporisé) d'établir les équations d'évolution qui décrivent le système comme le couplage d'un système $[\min, +]$ -linéaire et d'un système quasi $[+, \times]$ -linéaire;
- (dans le cas stochastique) de tester la stabilité du système, c'est à dire le fait que le marquage des places reste borné en probabilité.

Les outils utilisés pour montrer ces résultats sont la théorie des graphes, l'algèbre idempotente et la théorie ergodique

Mots-clé : Réseaux de Petri, algèbre $(\min, +)$, stabilité, théorie ergodique

1 Introduction

Timed event graphs can be seen as linear dynamic systems in the so-called $(\max, +)$ -algebra [2]. The algebraic formalism used in the above approach provides a rich variety of results on stability, control and computational problems arising in both deterministic and stochastic event graphs. The aim of the present paper is to show that a similar algebraic approach can partially be generalized to free choice nets. The whole analysis is based on a decomposition of the net into its maximal $(\max, +)$ -linear (i.e. event graph) subnetworks and a non-linear “routing” subnet; this decomposition is introduced together with basic definitions on Petri nets in Section 2. Several results obtained in the paper bear on general free choice nets, including the decomposition and evolution equations for the timed case. However, the main focus of the paper is the class of single-input free choice nets (SI-FCNets), where all event-graph subnets arising in the decomposition are single-input $(\max, +)$ -linear systems.

This decomposition is first used in Section 3, in order to analyze structural properties of *unbounded* free choice nets. In the case of SI-FCNets, we show that Commoner’s characterization of liveness in terms of siphons and traps [5] (i.e. the network is live iff each siphon contains a trap with a non-empty marking), when added to the above decomposition, leads to polynomial time algorithms (in the size of the net) for checking liveness. The analysis is mainly based on graph theory.

The decomposition is then used in Section 4 to derive the evolution equations for the timed case. Up to a minor transformation, the dynamics of a free choice net admit a canonical representation in terms of a ‘coupling’ of two linear systems. The first one pertains to the event graph components, and satisfies a non-autonomous $(\min, +)$ -linear evolution equation with an input from the second system, whereas the second one is “essentially” a nonautonomous $(+, \times)$ -linear system with an input from the first system. A few basic properties of this class of evolution equations are established, and in particular monotonicity and conservation properties which play a key role in the analysis of stability.

The last problem which is analyzed in the paper is stability (Sections 5 and 6). Stability is the property that a timed (resp. stochastic) unbounded Petri net may have, and which implies that its marking remains bounded (resp. bounded in probability). This is not a structural property in the sense that it strongly depends on the initial marking and on the timing variables used in the net. The main result is a set of quasi necessary and sufficient conditions for stability for non-autonomous SI-FCNets. The conditions are expressed in terms of inequalities between two types of linear characteristics of the net: the $(\max, +)$ -Perron Frobenius maximal eigenvalues (resp. the Lyapunov exponents) of the $(\max, +)$ -linear subsystems on one side, and certain fixed points pertaining to the $(+, \times)$ subsystem on the other side. Whenever this condition is fulfilled, we also show that the marking process converges to a periodic (resp. stationary) integer-valued process (resp. stochastic process). The main tools used in this part are the results already known for $(\max, +)$ -linear systems and an ergodic theory construction which relies on the monotonicity properties of the evolution equations. A few special cases of this stability result are already known in the literature: for Jackson queueing networks, which happen to be SISO FCNets (in fact, all linear subsystems are here single input, single output, one dimensional systems), the stability condition boils

down to comparing the mean service rate of each queue with the solution of the so called traffic equation (see [3]). So, the set of inequalities which characterize stability in the general SI-FCNet case can also be seen as a generalization of the well known conditions for stability in the Jackson network case. First order ergodic theorems (showing the existence of rates under general conditions) were also obtained in the SISO case in [6].

2 Timed Petri net

2.1 Dynamics of Petri Nets

A Petri net is a t-uple $(\mathcal{P}, \mathcal{T}, \mathcal{C}, \mathcal{M}_0)$ where \mathcal{P} is the set of places, \mathcal{T} is the set of transitions, \mathcal{C} the set of arcs between places and transitions or between transitions and places (\mathcal{C} is a subset of $\mathcal{P} \times \mathcal{T} \cup \mathcal{T} \times \mathcal{P}$). \mathcal{M}_0 is the initial marking in the places. We denote by $\bullet t$ the set $\{p \in \mathcal{P} : (p, t) \in \mathcal{C}\}$ (i.e. the set of all input places of t). We define similarly the sets t^\bullet , $\bullet p$, p^\bullet as the set of output places of t , the set of input transitions of p and the set of output transitions of p , respectively.

A timed Petri net is a Petri net with temporal data attached to transition: $\sigma^t(n)$ is a data which gives the duration of the n -th firing of transition t . This means that if transition t begins to fire for the n -th time at epoch e , this firing will end at epoch $e + \sigma^t(n)$; tokens are then taken out of input places and put into output places of t according to the firing rule of the untimed Petri net.

For more on the matter, and in particular for the definitions of *deadlocks*, *liveness*, *structural liveness* etc. which are used in the paper, the reader is advised to consult the survey paper by Murata [10].

2.2 Free Choice Nets

Free choice nets (FCNet) are Petri nets verifying the following conditions: $\forall p \in \mathcal{P}, t_1, t_2 \in p^\bullet, t_1 \neq t_2, \bullet t_1 = \bullet t_2 = \{p\}$. In other words, whenever two transitions share an input place, they have no other input place.

Free choice nets have been extensively studied in the 70's [5] and have regained interest recently [7], [11] because they constitute a nice compromise between power of description and tractability of problems.

Several 'semantics' can be used for the resolution of conflicts. The most common is called the *race policy*. Another one called the *routing policy* was introduced in [1].

Let p be a place with several output transitions.

In the case of the race policy, the resolution of conflicts is purely based on the temporal data. For free choice nets, this policy boils down to the following: as soon as the n -th token enters place p , the transitions of p^\bullet immediately start their n -th firing (this is possible since they have no other pre-conditions by hypothesis), which takes $\sigma^t(n)$ for transition t . The transition which completes first wins the race for the n -th token, and consumes this token.

In the case of the routing policy, *routing data* are attached with each place with several output transitions. Place p has a routing sequence $\nu^p : \mathbb{N} \rightarrow p^\bullet$, where $\nu^p(n)$ gives the transition $t \in p^\bullet$ to which the n -th token to enter place p is routed. The routing sequences can be periodic or random. If this token is the k -th to be routed to transition $t \in p^\bullet$, then this token is immediately consumed by the transition (due to the FC property) where it experiences a firing time of $\sigma^t(k)$.

In fact, *in the free choice case*, the race policy is a special case of routing policy. Assume that the net evolves according to the race policy. Then the n -th token to enter place p is routed to transition

$$\nu^p(n) = \operatorname{argmin}_{t \in p^\bullet} \sigma^t(n),$$

(at least whenever this argument is unique). This function can also be seen as predefined routing data which does not depend on the firing times of the transitions of the net, but for those of p^\bullet of course. So, up to an adequate renumbering of the firing times of p^\bullet , one can view race as a special case of routing. The converse construction is also easy to make.

In what follows, we shall adopt the routing semantics.

2.3 Decomposition into Marked Graph Components

A place p in a FCNet F is *serial* if $|\bullet p| = |p^\bullet| = 1$. Let \mathcal{A} be the set of transitions such that all their upstream places are serial, and let \mathcal{B} be the set of transitions which do not belong to \mathcal{A} .

First we define a relation \mathcal{L} by: $t, t' \in T$, $t\mathcal{L}t'$ if there is a serial place p verifying $\{\bullet p, p^\bullet\} = \{t, t'\}$. Let \mathcal{K} be the transitive closure of \mathcal{L} . \mathcal{K} is a parallelism relation. We partition the set of transitions T into its maximal \mathcal{K} -classes, $\mathcal{T}_1, \dots, \mathcal{T}_n$. We construct a decomposition of F in the following way: $\mathcal{P}_i = \{p \in \mathcal{P} \mid p \text{ serial and } \bullet p, p^\bullet \in \mathcal{T}_i\}$, for all i .

The marked graph component (MGC) G_i of F is the sub-Petri net of F ($\mathcal{P}_i, \mathcal{T}_i, \mathcal{C} \cap (\mathcal{P}_i \times \mathcal{T}_i \cup \mathcal{T}_i \times \mathcal{P}_i)$). One can easily check that G_i is a marked graph and is maximal in the sense that no marked graph included in F contains G_i , except G_i itself. A marked graph component G_i is degenerated if G_i is reduced to a single transition (with no places).

The places which do not belong to any component G_i are the places with several input transitions and/or several output transitions. These places will be called *routing* places in the following. The set of routing places is denoted \mathcal{R} .

In a first step, we will only consider networks with an exogeneous input into some places (although the case when input can also take place in transitions, will also be used and analyzed in the paper). Any place with such an input will be put into the set of routing places, by definition.

Some examples of decompositions are given in figures 1 and 2.

2.4 The Routing Net

If $F = (\mathcal{P}, \mathcal{T}, \mathcal{C})$ be a free choice net, we define its *routing graph* $H = (\mathcal{R}, \mathcal{G}, \mathcal{C}', w)$ as follows: \mathcal{R} is the set of routing places, \mathcal{G} is the set of MGC's and \mathcal{C}' is the set of arcs in \mathcal{C} between

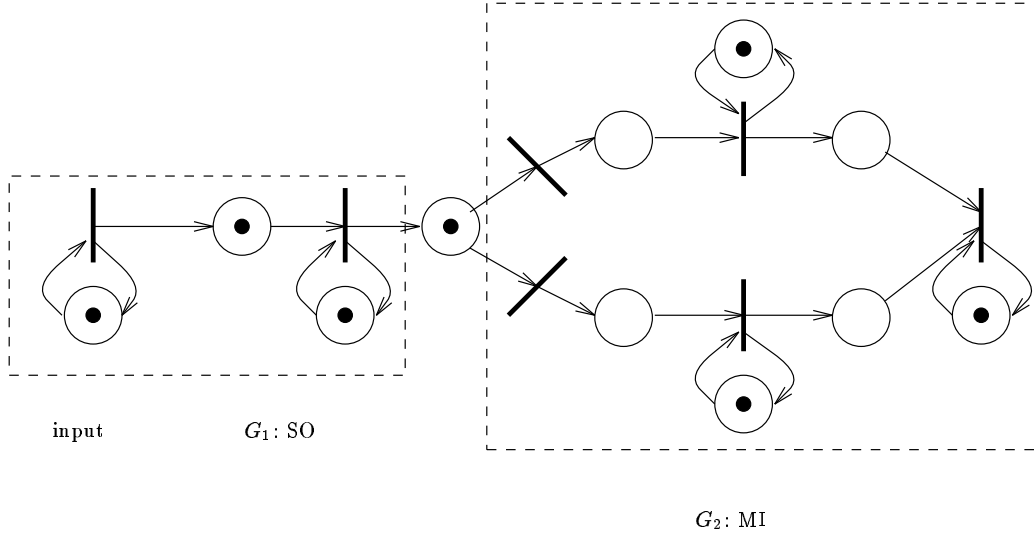


Figure 1: This FCNet is decomposed into two MGC's.

routing places and MGC's. The routing net is a weighted Petri net which is obtained by replacing each MGC by a transition and keeping only the routing places. If there is an arc from the routing place p to MGC G (resp. from G to p) in the original net, we put an arc from p to transition G in the routing net (resp. from G to p). The weight $w_{p,G}$ (resp. $w_{G,p}$) of this arc equals the number of arcs from p to G , i.e. $\#(p^\bullet \cap G)$ (resp. from G to p) in the original FCNet.

Note that G denotes a transition in the routing net as well as a MGC in the original net. Similarly, p denotes a place in the routing net as well as in the original one. We also use the notation $G \in p^\star$ (resp. $G \in \star p$), if there is an arc from p to G (resp. from G to p) in the routing net.

Finally, if E is a set of places in the original FCNet, E_r denotes its projection on the routing graph, i.e. $E_r = E \cap R$.

The routing net of the FCNet depicted in Figure 2 is shown in Figure 3.

Note that the routing net of a FCNet is a FCNet.

2.5 Classification of Free Choice Nets

We propose a classification of the marked graph components of a FCNet based upon its links with the routing places.

A MGC G_i is said *Single Input* (SI) if $\#\{t \in \mathcal{T}_i, \bullet t \notin \mathcal{P}_i\} = 1$. In this case, we will denote $t^{(i)}$ and call *input transition* this unique transition of G_i . G_i is said *Multiple Input* (MI) if $\#\{t \in \mathcal{T}_i, \bullet t \notin \mathcal{P}_i\} > 1$. A MGC G_i is said *Single Output* (SO) if $\#\{t \in \mathcal{T}_i, t^\bullet \notin \mathcal{P}_i\} = 1$. A MGC G_i is said *Multiple Output* (MO) if $\#\{t \in \mathcal{T}_i, t^\bullet \notin \mathcal{P}_i\} > 1$.

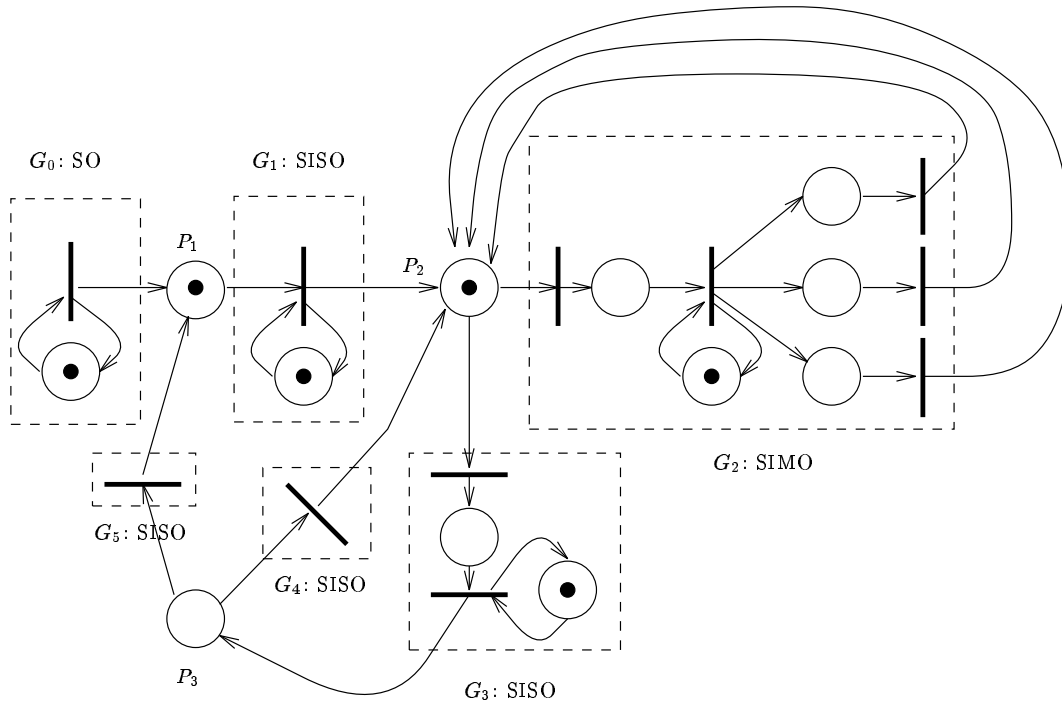


Figure 2: This is a SI-FCNet. Indeed all its MGC's are SI.

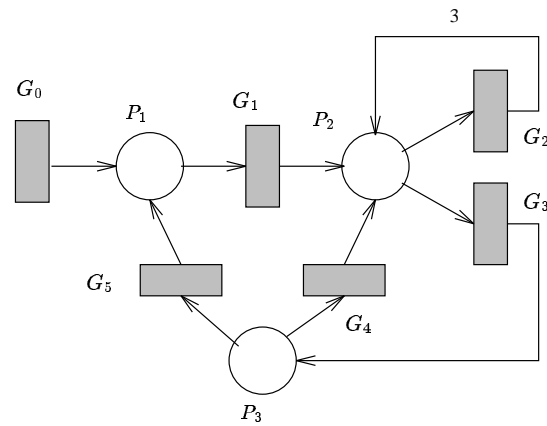


Figure 3: Since the graph F is a SI-FCNet, its routing graph can be interpreted as a weighted free choice net. The default weight on an arc is one and only the weights different than 1 are explicitly written.

Thus all MGC's of a FCNet can be put in one of the four classes, SISO, SIMO, MISO, MIMO. A FCNet is said SI (resp. SO) if all its MGC's are SI (resp. SO) and MI (resp. MO) otherwise.

Note that if F is a SI-FCNet, then its routing net H is a SI-FCNet.

3 Structural Properties of SI-FCNets

In this section we show that certain structural properties of a FCNet can be checked on its routing net. We will mainly focus on structural liveness (see [10]), but other properties like boundedness can also be tested at the routing net level only.

Note that if the FCNet is not in the SI class, the routing net does not help to decide on the liveness of the original FCNet. We illustrate this property the example in Figure 4, which depicts a FCNet and its routing net. The FCNet is not live whereas the routing net is live.

3.1 Traps, Siphons and Commoner's Liveness Condition

In [5], a necessary and sufficient condition of liveness for (simple¹) FCNets is given. This condition uses the notions of siphons and traps.

Definition 1 *A siphon is a set of places S verifying $\bullet S \subset S \bullet$. A trap is a set of places T verifying $T \bullet \subset \bullet T$.*

Commoner [5] established the following theorem:

Theorem 1 *A (non-weighed) FCNet is live if and only if every siphon contains a marked trap.*

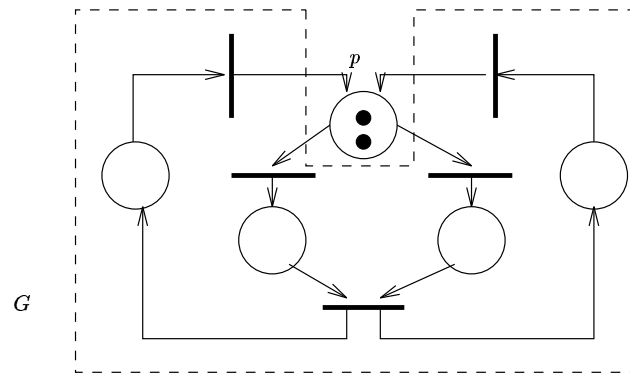
The structural liveness of a FCNet can be characterized by checking whether *every siphon contains a trap*.²

A siphon is minimal if it does not contain a smaller siphon. Since the union of two traps is a trap, we can talk about the maximal trap included in a set of places. The structural version of Theorem 1 can easily be reduced to the following equivalent version:

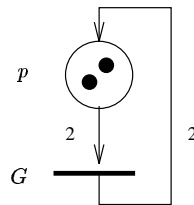
A simple FCNet is structurally live if and only if the maximal trap included in every minimal siphon is not the empty set.

¹All the nets which we considered in this paper are simple, in that arcs are not weighed; in a non-simple net, each arc has an integer value which gives the number of tokens produced (resp. consumed) by the transition at the origin (resp. end) of the arc. Simple nets are nets where all arcs have value 1.

²Unfortunately, checking this condition can be exponential in the size of the net. In [7], polynomial conditions of liveness and boundedness of FCNets are derived under the form of an integer linear program. However, to the best of our knowledge, nothing is known in the case of unbounded FCNets.



A non live MIMO-FCN



Its routing graph is live

Figure 4: This FCNet is composed by only one routing place p and one MGC G . The routing net is live and the original net is not.

Once a siphon S is given, one can test in *quadratic* time (in the number of places of the siphon) whether this siphon is structurally live (i.e. whether the maximal trap included in S is the empty set or not), by running the following (at most quadratic) procedure on siphon S :

- 1 Tag all the places which are not in S .
- 2 Tag all the places of S which have one downstream transition leading only to already tagged places. (i.e. all the places $p \in S$ s.t. there exists one transition t in p^\bullet with t^\bullet made of tagged places only.
- 3 Repeat step 2 until no new place is tagged.

Lemma 1 *The places which remain untagged when the run is complete form the maximal trap U included in siphon S .*

Proof: First we prove that U is a trap. If U is empty, then U is a trap. If U is not empty, let $p \in U$. By construction of U , for each $t \in p^\bullet$, $t^\bullet \cap U \neq \emptyset$. This is the definition of a trap.

Then, we prove that this trap is maximal. We prove that if p is tagged then no trap containing p is included in S . The proof is by induction on i , the step at which p has been tagged.

If p was tagged at step 1, p does not belong to S , therefore, a trap containing p cannot be included in S .

Suppose that a place that has been tagged at step j , $j \leq i$ does not belong to a trap included in S . Let p be a place tagged at step $i + 1$. Then by construction, there exists a transition in p^\bullet such that the whole set t^\bullet has been tagged at previous steps. But all traps containing p must contain at least one place in t^\bullet and from the induction assumption, this trap is not included in S . ■

This lemma also allows one to test the liveness of a minimal siphon in quadratic time (in the number of places included in the siphon). Indeed, for this, it is necessary and sufficient to check whether the maximal trap included in the siphon is marked.

In the following section, we show that in the class of SI-FCNets, we can check (structural) liveness by looking at the (structural) liveness of the routing net.

3.2 Liveness of SI-FCNets

In the following, we will focus on SI-FCNets. Let $F = (\mathcal{P}, \mathcal{T}, \mathcal{C})$ be a SI-FCNet and $H = (\mathcal{R}, \mathcal{G}, \mathcal{C}')$ its routing net. We distinguish two classes of minimal siphons in F .

- (K_1) A minimal siphon S belongs to class K_1 if it contains a cycle included in a MGC. In this case, the siphon is reduced to this cycle since a cycle in a marked graph is a siphon.

- (K_2) A minimal siphon S belongs to K_2 if it contains routing places. Note that a K_2 -siphon cannot contain a cycle included in a MGC, because it would not be minimal. So, the set of places of such a siphon which belong to MGC G_i can only consists of places in acyclic paths of G_i leading from the input-transition of G_i to the output-transitions in *S .

Lemma 2 *If S is a minimal siphon in K_2 , then S_r , its projection on the routing net ($S_r = S \cap \mathcal{R}$) is a siphon.*

Proof: Let p be a place in $\mathcal{R} \cap S$ and G a transition in *p . We have to prove that there is a place q such that

$$q \text{ belongs to } \mathcal{R} \cap S \text{ and } G \in q^*. \quad (1)$$

Since G belongs to *p , there is a transition t which belongs to $G \cap {}^\bullet p$. Since S is a siphon, there is a place q_0 in S such that q_0^\bullet contains t . If q_0 is not in G , then G belongs to q_0^* and condition (1) is satisfied.

If q_0 is in G then it is serial, and there exists a place q_1 in S such that $q_0 \in q_1^{\bullet\bullet}$. More generally, consider the sequence q_2, \dots, q_n, \dots , constructed in the same way (namely q_k is a place of S , and $q_{k-1} \in q_k^{\bullet\bullet}$), and stopped at the first place which does not belong to G . Since S cannot contain any cycle in G (S is a K_2 -siphon), the last sequence stops after a finite number of steps, and so it ends with a place q_m in S , which does not belong to G . So G belongs to q_m^* , and place q_m satisfies condition (1).

This is true for all MGC in ${}^\bullet p$ and for all places p in S_r . So S_r is a siphon. \blacksquare

Lemma 3 *Let S be a minimal siphon in the SI-FCNet F of type K_2 and S_r its restriction to the routing net H . If U is a trap included in S for F , then $U \cap \mathcal{R}$ is a trap included in S_r for H . If V is trap included in S_r for H , then $U = V \cup \{p \in V^* : p \text{ belongs to a path in } S \text{ from a place in } V \text{ to a place in } V\}^3$ is a trap included in S for F .*

Proof: First we prove that if U is a trap included in S , then $U_r = U \cap \mathcal{R}$ is a trap included in S_r . The proof is similar to that of the previous lemma.

We have to prove that if p is a place in U_r , then for all transitions G in H such that $G \in p^*$, there is a place q in U_r , s.t. G^* contains q .

Let G be such that $G \in p^*$. Then there exists $t \in G$ s.t. t belongs to p^\bullet . Since U is a trap, there is a place in ${}^\bullet t \cap U$, say q_0 . If q_0 is not in G , then $G \in {}^*q_0$ and so we can take $q = q_0$. If q_0 is in G , then it is serial, and there exists a place q_1 in U , such that $q_0 \in {}^{\bullet\bullet}q_1$. We construct similarly a finite sequence q_1, q_2, \dots, q_n , of places belonging to U , with $q_{k-1} \in {}^{\bullet\bullet}q_k$, and such that all places of the sequence but the last, q_n belong to G . (since S contains no cycle in G , neither does U). We then take $q = q_n$.

³By $p \in V^*$ we mean that p belongs to a MGC G such that $G \in V^*$.

We now prove the second part of the lemma. Let p be a place in U and t be any transition in p^\bullet . In order to prove that $U = V \cup \{p \in S : p \in V^*\}$ is a trap, we have to show that there is a place p' in U s.t. t belongs to $\bullet p'$.

First consider the case when p belongs to \mathcal{R} . In this case, there is a G in p^* with $t \in G$. Now, since V is a trap, there is a place q in V s.t. $G \in {}^*q$. From the definition of U , all the places in $G \cap S$ are in U . Since S is a siphon, for all transitions in $G \cap \bullet q$, there is a place q_0 in S with $q \in q_0^{\bullet\bullet}$. If q_0 is not in G , then by the single input property of G , $q_0 = p$ and $p' = q$ answers the question. If q_0 is in G , construct the sequence q_1, \dots, q_{n-1} of places in $G \cap S$ with $q_{k-1} \in q_k^{\bullet\bullet}$. As noted previously, all these places are distinct because S does not contain any cycle in G . So if n is such that q_{n-1} is the last place to be in G , then $\bullet\bullet q_{n-1}$ is reduced to a single place: p and we can choose $p' = q_{n-1}$.

If p does not belong to \mathcal{R} , then p belongs to a MGC (say G) and to S . Note that since V is a trap, ${}^*G \in V$ and $G^* \cap V \neq \emptyset$. The places in $G \cap S$ form paths of G , each path leading from the input transition of G to a transition in $G \cap {}^*S$. Indeed let $q \in G \cap S$. If $q^{\bullet\bullet} \notin S$, then $S - \{q\}$ is a siphon and S would not be minimal.

Now, p belongs to one of those paths, leading to a place in V . Let p' be the successor of p on this path. If $p' \in G$, then it verifies all the conditions of places of U . If $p' \notin G$, then p' is a routing place and is in $V \subset U$. Therefore, p has a successor in U , so U is a trap. ■

Note from the definition that $V = U_r$.

The previous lemmas also hold for general FCNets if we replace each MGC by a complete bipartite graph between its entry-transitions and its exit-transitions instead of replacing it by a single transition. Unfortunately, in this case, the following lemma does not hold which makes this new kind of reduction quite useless to test liveness.

Lemma 4 *The number of minimal siphons in H is polynomial in $|\mathcal{R}|$.*

Proof: Let S be a minimal siphon containing place p . Then for all input transitions of p , S must contain one of their input places. Since H is SI , there is only one input place per transition. Thus one has no choice and there is at most one siphon containing place p . Therefore the number of siphons in the routing net is linear in the number of places and we can find all the minimal siphons in $O(|\mathcal{R}|^2)$. ■

Theorem 2 *Checking (structural) liveness of a SI-FCNet is polynomial.*

Proof: For any minimal siphon S in F , S_r is a siphon in H (Lemma 2) and S contain a trap U is equivalent to S_r contains a trap U_r (Lemma 3). Therefore, the structural liveness of F and H are equivalent, since all the MGC which are marked graphs are always structurally live.

The following algorithm checks the structural liveness of a SI-FCNet:

- 1 Construct the routing net. $[O(|\mathcal{T}|^2)]$

- 2 For each siphon in the routing net check whether the maximal included trap is not the empty set. [$O(|\mathcal{R}|^3)$ using Lemmas 1 and 4]

To check liveness, one has to go one more step:

- 3 Make sure that for each siphon its maximal trap is marked. If the trap in H is empty, go back to the original graph, delete all places with tokens in each MGC and check whether there is still a path from the entry-transition to all exit-transitions with at least one place in the trap as output. [$O(|\mathcal{T}|^3)$]

4 Evolution Equations for Timed Free Choice Nets

In the following, we will focus on timed FCNets. A transition is in *starvation* if during the evolution of the net, it fires only a finite number of times. A *deadlock* is a situation where all the transitions are starving. In particular, a deadlock is a state of the system under which no firing will ever take place. For a FCNet, a deadlock is a marking D under which no transition is enabled (i.e. $\forall t \in T, \exists p \in \bullet t$, with $\text{Marking}(p) = 0$).

4.1 Transformation of a FCNet

The FC nets we will consider will be assumed to satisfy the following assumption:

A transition $t \in \mathcal{B}$ with more than one incoming arc (i.e. with an and-convergence) is never preceded by a place p with more than one incoming arc (i.e. with an or-convergence)

This restriction introduces no loss of generality as any FC net can be transformed into an *equivalent* net which satisfies this assumption: because of the FC constraint, a transition t as above cannot be preceded by a place with multiple outgoing arcs; so, each place p as above can be replaced by a triple p', t', p'' , where $\bullet p' = \bullet p$, $p'' \bullet = p \bullet = t$, and where $p' \bullet = t'$, $\bullet t' = p'$, $t' \bullet = p''$, $\bullet p'' = t'$, without altering the time evolution of the net (provided t' is given a firing time equal to zero).

Note that under the above assumption, each transition of \mathcal{B} has exactly one upstream place, and that this place is necessarily non-serial. This place may either have an or-divergence (in which case it precedes several transitions of \mathcal{B}) or an or-convergence, or both.

4.2 State Vectors

Counters Let $X^t(u)$ denote the counter associated with t , namely, the number of firings initiated by transition t by time u . We will consider the version of this process which is continuous to the right. Let $Y(u)$ be the vector $\{X^t(u), t \in \mathcal{A}, u \in \mathbb{R}\}$, where the transitions are arranged in some order, and let $Z(u)$ be the vector $\{X^t(u), t \in \mathcal{B}, u \in \mathbb{R}\}$.

We shall first consider the case when firing times are constant, positive, and all multiple of a common number, which will be taken equal to 1 without loss of generality. We will show in Section 7 how to address the case with varying (and in particular stochastic) firing times, which can be treated with a similar method. We will denote M the (integer-valued) upper-bound on the firing times. We will denote $\nu^p(m)$ the m -th routing decision from place p ($\nu^p(m) \in p^\bullet$) and $\Pi^t(m)$ the sum

$$\Pi^t(m) = \sum_{l=1}^m 1_{\nu^{\bullet t}(l)=t}, \quad t \in \mathcal{B}, \quad (2)$$

where $\nu^p(\cdot)$ is the routing function associated with place p .

Exogeneous Arrival Process As mentioned in the introduction, our main focus concerning stability will be the non-autonomous case, namely the case of nets with an exogeneous input process. We will denote $\{T_n\}_{n \in \mathbb{N}}$ this input process:

$$0 = T_0 < T_1 < \dots < T_n < T_{n+1} < \dots$$

We will consider the case when the arrival times T_n (or equivalently the jump times of the associated counter) are integer-valued. The associated counters are defined as follows: for all $t \in \mathcal{B}$, we denote $R^t(k)$ the cumulated exogeneous input in place $\bullet t$ (this place is uniquely defined as observed in the last subsection) up to time k . So these counters satisfy the evolution equation

$$R^t(k) = R^t(k-1) + I^t(k), \quad k \geq 0, \quad (3)$$

where $I^t(k) = j$ iff $k \in \{T_n\}_n$ and the exogeneous arrival at k brings j tokens to place $\bullet t$. We will denote $R(k)$ the $\mathbb{N}^{|\mathcal{B}|}$ -vector $R^t(k)$, $t \in \mathcal{B}$. Note that this vector may carry redundant information (for instance whenever two \mathcal{B} transitions admit the same non-serial place as input place). The initial condition of this recursion, $R(-1)$, is the $\mathbb{N}^{|\mathcal{B}|}$ -vector of initial markings in the routing places of $\bullet \mathcal{B}$: $R(-1)(t) = c$ if $t \in \mathcal{B}$ is such that $\bullet t$ has an \mathcal{M}_0 -marking (prior to any exogeneous arrival) of c .

Theorem 3 *Under the above assumptions, for all $k \in \mathbb{Z}$, the counting vectors $\{Y(k), Z(k)\}$ satisfy the following evolution equation:*

$$Y(k) = 0, \quad Z(k) = 0, \quad \forall k < 0, \quad (4)$$

and, for $k \geq 0$,

$$Y(k) = \bigoplus_{l=1}^M (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l)) \quad (5)$$

$$Z(k) = \Pi \left(\sum_{l=1}^M (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k) \right). \quad (6)$$

In this evolution equation, (\oplus, \otimes) respectively denote matrix products and additions in the $(\min, +)$ semi-ring (see [2]), whereas $(+, \times)$ denote the same operations but in the conventional algebra. The matrices used in the recurrence equations are defined from the net structure as follows:

- The matrix A_l on $\mathcal{A} \times \mathcal{A}$ is defined by $A_l(t, t') = c$ if the firing time of $t \in \mathcal{A}$ is l and there is a serial place between $t' \in \mathcal{A}$ and t , with \mathcal{M}_0 -marking equal to c ; ∞ otherwise. If there are more than one serial places between t' and t , we take c equal to the minimum of their \mathcal{M}_0 -markings.
- The matrix B_l on $\mathcal{A} \times \mathcal{B}$ is defined by $B_l(t, t') = c$ if the firing time of $t \in \mathcal{A}$ is l and there is a serial place between $t' \in \mathcal{B}$ and t , with \mathcal{M}_0 -marking equal to c ; ∞ otherwise.
- The matrix P_l on $\mathcal{B} \times \mathcal{B}$ is defined by $P_l(t, t') = 1$ if the firing time of $t \in \mathcal{B}$ is l and there is a place connecting t' to t ; 0 otherwise.
- The matrix Q_l on $\mathcal{B} \times \mathcal{A}$ is defined by $Q_l(t, t') = 1$ if the firing time of $t \in \mathcal{B}$ is l and there is a place connecting t' to t ; 0 otherwise.
- For all vectors of integers $Z = (Z^1, \dots, Z^q)$, where $q = |\mathcal{B}|$, $\Pi(Z)$ is the vector of integers:

$$\Pi(Z) = (\Pi^1(Z^1), \dots, \Pi^q(Z^q)).$$

Proof: Equation (5) is obtained in a way which is similar to that used for establishing the evolution equation for event graphs in [2]. For instance, the number of firings initiated by transition $t \in \mathcal{A}$ at time k cannot exceed the minimum of the number of tokens arrived in the places of $\bullet t$ by time k , which is exactly

$$\bigoplus_{l=1}^M (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l))^t.$$

Furthermore, $Y^t(k)$ is equal to this quantity because transitions are assumed to fire as soon as they are enabled. For obtaining Equation (6), the key observation is that, due to our preliminary assumption, a transition t which belongs to \mathcal{B} has at most one input arc, which allows us to write (6), and so the number of firings it initiates by time k is simply the ' Π^t -filtering' of the total number of arrivals into place $\bullet t$ up to time k , that is

$$\left(\sum_{l=1}^M (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k) \right)^t.$$

■

Remark: *Constructiveness.* Even if the initial net is such that all its firing times are positive, the transformation of FC nets which was made at the beginning of § 4 may make it necessary to consider the case with some zero firing time transitions; this may translate into an extra term of the form $B_0 \otimes Z(k)$ in (5). Such an additional term preserves the ‘constructiveness’ of the evolution equation. By constructiveness of the generalized equation:

$$Y(k) = \bigoplus_{l=0}^M (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l)) \quad (7)$$

$$Z(k) = \Pi \left(\sum_{l=0}^M (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k) \right) \quad (8)$$

(it generalizes the evolution equations of the theorem because l now ranges from 0 to M), we mean that there exists an ordering of the coordinates of $X(k) = (Y(k), Z(k))$, say $X^{i_1}(k), \dots, X^{i_{|\mathcal{T}|}}(k)$, such that, for all $j = 1, \dots, |\mathcal{T}|$, the line corresponding to $X^{i_j}(k)$ in (7)-(8) is such that no term of the form $X^{i_l}(k)$, $l \geq j$ can be found in the right hand side.

There is an easy algebraic characterization of this property: let S be the $\mathcal{T} \times \mathcal{T}$ matrix defined by

$$S = \begin{pmatrix} \text{supp}(A_0) & \text{supp}(B_0) \\ \text{supp}(P_0) & \text{supp}(Q_0) \end{pmatrix}, \quad (9)$$

where the support, $\text{supp}(A)$, of a (\oplus, \otimes) -matrix A is a matrix of the same size and such that $(\text{supp}(A))_{i,j} = 1$ if $A_{i,j} \neq \varepsilon$, and 0 otherwise; similarly the support, $\text{supp}(P)$, of a $(+, \times)$ -matrix P is a matrix of the same size and such that $(\text{supp}(P))_{i,j} = 1$ if $P_{i,j} \neq 0$, and 0 otherwise. The equations are constructive iff there exists a permutation U such that $U^{-1} \times S \times U$ is strictly lower triangular.

All the results that we prove in this paper can be extended to nets with constructive evolution equations.

Remark: *Localization.* From the above equations, it is easily checked by induction that the state variables $X(k)$ satisfy the following ‘localization’ property: if $R(k)$ and $R'(k)$, $k \geq 0$ are two functions which coincide up to time K , then two nets which would only differ in their input counters R and R' are such that their state variables $X(k)$ and $X'(k)$ coincide up to time K .

Remark: *Linearity.* The Π function satisfies the relation

$$\Pi^t(m+n) = \Pi^t(m) + \Pi^t(n) \circ \tau_m, \quad (10)$$

where $\Pi^t(\cdot) \circ \tau_m$ is the Π^t function associated with the shifted sequence $\tau_m \nu^p(\cdot)$, namely the sequence

$$\tau_m \nu^p(n) = \nu^p(n+m). \quad (11)$$

For instance, whenever the sequence $\nu^p(\cdot)$ is random and i.i.d. (i.e. made of random variables which are independent and identically distributed), then $E[\Pi^t(n) \circ \tau_m] = E[\Pi^t(n)]$, so that

$$E[\Pi^t(m+n)] = E[\Pi^t(m)] + E[\Pi^t(n)]. \quad (12)$$

So in this case, Π is ‘linear in expectation’.

4.3 Total Number of Firings

Let $Y = Y(\infty)$ and $Z = Z(\infty)$ denote the vectors counting the total number of firings of the transitions. One can characterize the presence or the absence of deadlocks (see the beginning of Section 4 for the definition of deadlocks) and related properties, like starvation, directly from Y and Z : for instance, given an initial marking \mathcal{M}_0 such that $R(-1) = 0$, the system without exogeneous input (i.e. $R(k) = R(-1) = 0, \forall k$) is deadlocked if and only if the Z and Y vectors associated with this $R(k)$ function are both equal to 0.

Lemma 5 *The integer-valued vectors Z and Y satisfy the system of equations*

$$Y = A \otimes Y \oplus B \otimes Z \quad (13)$$

$$Z = \Pi(P \times Z + Q \times Y + R), \quad (14)$$

where $A = \bigoplus_{l=1}^M A_l$, $B = \bigoplus_{l=1}^M B_l$, $P = \sum_{l=1}^M P_l$, $Q = \sum_{l=1}^M Q_l$ and $R = \lim_{k \rightarrow \infty} R(k)$.

Remark: *Independence on timing* A striking property is that this system does not depend on the variables σ^t anymore: in other words, all properties like starvation, liveness, deadlock and intermediates are associated with the switching functions, the topology, the initial marking and the R vector only, and *not* with timing variables (neither firing times nor arrival epochs of exogeneous tokens).

This property holds true in the case when the firing times are non-constant but finite (see §7).

This property allows us to establish relations between the liveness of the underlying untimed Petri net and the absence of deadlock or starvation in the timed version of the net.

In particular, if the untimed FCNet is live, then the timed version never reaches a deadlock (this is not true for general Petri nets). Furthermore, if the untimed net is live and if all the functions Π^t are equitable⁴ then, there will be no starvation in the timed version, that is, all the transitions will fire infinitely often. For more on this see [8]. ■

Lemma 6 *The system of equations (13)-(14) admits a minimal non-negative solution (for the coordinatewise partial order). It admits a finite non-negative solution if and only if the net reaches a deadlock. Furthermore, the vector counting the total numbers of events in the deadlock is the minimal non-negative solution of this system.*

⁴The function Π^t , $t \in \mathcal{T}$ is equitable if $\Pi^t(m)$ goes to infinity when m goes to infinity

Proof: The operator

$$\begin{aligned} \Phi : \mathbb{N}^{|\mathcal{A}|} \times \mathbb{N}^{|\mathcal{B}|} &\rightarrow \mathbb{N}^{|\mathcal{A}|} \times \mathbb{N}^{|\mathcal{B}|} \\ (Y, Z) &\rightarrow (A \otimes Y \oplus B \otimes Z, \Pi(P \times Z + Q \times Y + R)) \end{aligned}$$

is componentwise non-decreasing and integer-valued. Any finite non-negative solution (S_1, S_2) of (13)-(14) is a finite fixed point of Φ : $(S_1, S_2) = \Phi(S_1, S_2)$.

Let us now define the sequence of vectors $(\alpha(k), \beta(k))_{k \in \mathbb{N}}$ in $\mathbb{N}^{|\mathcal{A}|} \times \mathbb{N}^{|\mathcal{B}|}$ by $(\alpha(0), \beta(0)) = (0, 0)$ and

$$(\alpha(k+1), \beta(k+1)) = \Phi(\alpha(k), \beta(k)).$$

This sequence is componentwise non-decreasing and so, it has a limit when k goes to infinity denoted (α, β) . The vector (α, β) is a fixed point of Φ (actually, whenever (α, β) is finite, the limit vector is reached in a finite number of steps).

A straightforward induction on n shows that for all other non-negative fixed point (S_1, S_2) of Φ , $(\alpha, \beta) \leq (S_1, S_2)$, where the order relation is componentwise, which proves the first assertion of the lemma.

If the net reaches a deadlock, the total firing vector $X = (Y, Z)$ is finite and non-negative and Lemma 5 states that this vector is a solution of (13)-(14).

Conversely, let us assume that this system admits a finite solution. Then (α, β) is necessarily finite. Consider the sequences $(Y'(k), Z'(k))$ defined by $(Y'(k), Z'(k)) = (0, 0)$, for $k < 0$ and

$$\begin{aligned} Y'(k) &= \bigoplus_{l=1}^M (A_l \otimes Y'(k-l) \oplus B_l \otimes Z'(k-l)) \\ Z'(k) &= \Pi \left(\sum_{l=1}^M (P_l \times Z'(k-l) + Q_l \times Y'(k-l)) + R \right) \end{aligned}$$

for $k \geq 0$. We get $(Y(k), Z(k)) \leq (Y'(k), Z'(k))$ by induction. Similarly, we verify by induction that for all $k \in \mathbb{N}$, $(Y'(k-1), Z'(k-1)) \leq (\alpha(k), \beta(k))$ componentwise. Therefore, $(Y(\infty), Z(\infty)) \leq (\alpha, \beta)$ which is finite, Therefore $(Y(\infty), Z(\infty)) = (\alpha, \beta)$. ■

Remark: *Total number of firings.* This proof also provides another way of computing the total number of firings in the case when the system reaches a deadlock. Indeed, we can iterate the operator Φ on a null vector until convergence to a fixed point which will be the total number of firings. ■

The system of equations (13)-(14) can be separated. Let

$$A^* = \bigoplus_{k=0}^{\infty} A^k \tag{15}$$

(the series always converges) and let C be the *transfer* matrix

$$C = A^* \otimes B. \quad (16)$$

We get the system (E):

$$Y = C \otimes Z \quad (17)$$

$$Z = \Pi(P \times Z + Q \times (C \otimes Z) + R). \quad (18)$$

Note that the equation for the total number of events only depends on A and B via the transfer matrix C ; in other words, all properties pertaining to the total number of events (e.g. deadlocks, liveness etc.) only depend on C , Π , P and Q .

5 Stability Analysis of FCNets

This section gives results on general FC nets. Some of the general results obtained here translate into more practical results in the SI case, studied in the next section.

5.1 Reference Deadlock

In this section, we will consider the class of networks which admit a *reference deadlock*. Roughly speaking, the initial marking of the network should be this reference deadlock, and for all finite input processes, the network should eventually reach the reference deadlock. More precisely, the following three assumptions are made:

Assumption 1 (A_1) *The initial marking is a deadlock, i.e. $R = 0$ implies that $Y = 0$ and $Z = 0$.*

Assumption 2 (A_2) *For all finite R , the net enters a deadlock, i.e. R finite implies that Z and Y are finite.*

Assumption 3 (A_3) *For all R such that the network reaches a deadlock, then this deadlock is the reference deadlock.*

This last assumption can be formulated in terms of T-invariants. A T-invariant of a Petri net is a firing vector $V = (V_1, \dots, V_{|\mathcal{T}|})$ such that when transition i is fired V_i times, for all $i \in \mathcal{T}$, then the marking is equal to the initial marking. Let us view for a moment the total input R^t into place $p = \bullet t$ as the total number of firings of an (extra) input transitions which feeds p . Then, (A_3) states that (X, R) is a T-invariant of this extended net.

5.2 Conditions for (A_1) – (A_3); the non-Structural Case

This section focuses on sufficient (and sometimes necessary) conditions for (A_1) – (A_3) to hold for a given Π function. We call them non-structural in that they depend on the chosen Π function.

5.2.1 Assumption (A_1)

Assumption (A_1) is easy to check via (E). Indeed, with $R = 0$, these equations become

$$Y = C \otimes Z \quad (19)$$

$$Z = \Pi(P \times Z + Q \times Y). \quad (20)$$

Lemma 7 *Assumption (A_1) is satisfied if and only if the matrix C has at least one zero element per line.*

Proof: Using Lemma 6, $Y = 0$ and $Z = 0$ if and only if $(0, 0)$ is a solution of these equations. Assume that $(0, 0)$ is a solution. This implies that the matrix C has at least one zero element on each line. Conversely, if C has at least one zero element, using the fact that $\Pi(0) = 0$, we see that $(0, 0)$ is solution of (E). ■

This characterization of (A_1) is equivalent to the following property of the net: for any transition of the net, there is an empty path from some routing place to this transition.

5.2.2 Assumption (A_2)

Since the total firing vector does not depend on the timing variables σ^t , (A_2) is actually an assumption on Π (and the topology of the net of course).

In §6.2, we will give a sufficient and quasi necessary condition for (A_2) to hold in the SI-FCNet case, whenever the routing sequences are i.i.d. This condition is based on the computation of the Perron-Frobenius eigenvalue of a certain matrix.

In the MI case, we can give a sufficient condition for (A_2) to hold that reduces to checking (A_2) in a SI-network associated to the original network, called a SI-projection of the net.

Let \underline{C} be the *support matrix* of the transfer matrix $C = A^* \otimes B$:

$$\underline{C}_{i,j} = \begin{cases} \infty & \text{if } C_{i,j} = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

We also define the *SI-projections* of the net. We define the set of matrices $C_{\langle s \rangle}$, $s \in \{1, \dots, K\}$ by picking only one zero element in each line of \underline{C} , K being the total number of possible combinations. Note that if the original net is SI and verifies (A_1) , then $K = 1$ and $C_{\langle 1 \rangle} = \underline{C} = C$.

In the case when the original net is MI, for each $s \in \{1, \dots, K\}$, the matrix $C_{\langle s \rangle}$ corresponds to a SI-FCNet derived from the original net which we call a *SI-projection*. Consider row $C[i]$. Let us assume that the element $C[i][j] = 0$ and s is such that $C_{\langle s \rangle}[i][j] = 0$ and that $C_{\langle s \rangle}[i][k] = \infty$, for $k \neq j$. This transfer matrix corresponds to that of a SI-FCNet where for each transition t_i in \mathcal{A} , only transition t_j in \mathcal{B} remains connected to t_i . All other transitions t_k are disconnected from t_i by removing the place between the two transitions. Figure 5 illustrates this transformation. Note that this transformation may create transitions with no output place, like in Figure 5. Their firings remove tokens from the system.

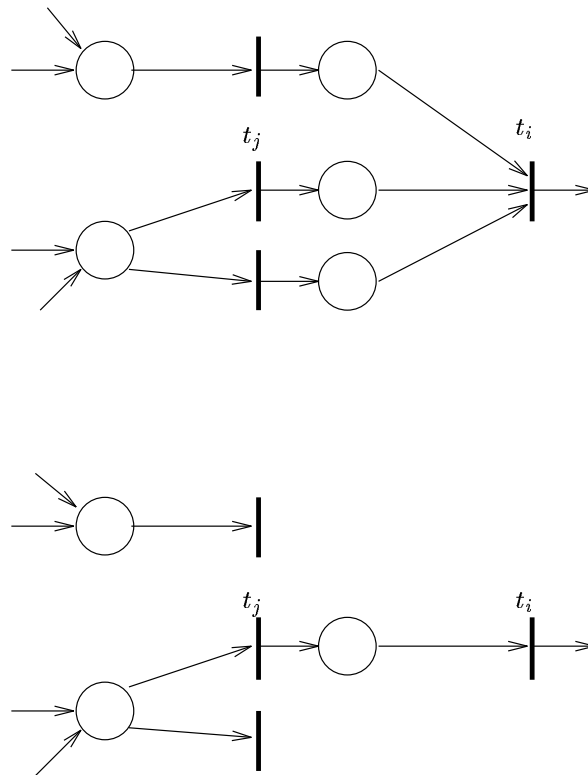


Figure 5: Transformation of the transfer matrix of the original net into that of a SI-FCNet, by keeping only one connection for transition t_i .

The total firing equations of this new system ($E_{\langle s \rangle}$) is:

$$Y_{\langle s \rangle} = C_{\langle s \rangle} \otimes Z_{\langle s \rangle} \quad (21)$$

$$Z_{\langle s \rangle} = \Pi(P \times Z_{\langle s \rangle} + Q \times (C_{\langle s \rangle} \otimes Z_{\langle s \rangle}) + R). \quad (22)$$

Since $C_{\langle s \rangle}$ has only one finite element per line (which is zero), $C_{\langle s \rangle} \otimes Z_{\langle s \rangle}$, is just a permutation $U_{\langle s \rangle}$ of $Z_{\langle s \rangle}$: $C_{\langle s \rangle} \otimes Z_{\langle s \rangle} = U_{\langle s \rangle} \times Z_{\langle s \rangle}$. Let $Q_{\langle s \rangle} = Q \times U_{\langle s \rangle}$ and $P_{\langle s \rangle} = P + Q_{\langle s \rangle}$. Equation (22) becomes

$$Z_{\langle s \rangle} = \Pi(P_{\langle s \rangle} \times Z_{\langle s \rangle} + R) = \Phi_{\langle s \rangle}(Z_{\langle s \rangle}).$$

Lemma 8 *Suppose that $C = \underline{C}$. If there exists a SI-projection which verifies assumption (A_2) , then so does the original net.*

Proof Let $C_{\langle s \rangle}$ be the transfer matrix of the SI-projection verifying assumption (A_2) and $(Y_{\langle s \rangle}, Z_{\langle s \rangle})$ be the total firing vector of this system which is finite. If $C = \underline{C}$ then, $C \leq C_{\langle s \rangle}$ which implies that $(Y, Z) \leq (Y_{\langle s \rangle}, Z_{\langle s \rangle})$. Therefore (Y, Z) is finite and the original net verifies assumption (A_2) . ■

5.2.3 Assumption (A_3)

In §6.1, we will also show that for SI-FCNets, assumption (A_3) is satisfied whenever (A_2) holds.

As we show below, in the MI case, checking (A_3) can be reduced to testing properties of the set of the SI-projections of the original net. This is the consequence of three following lemmas, where we assume that (A_1) and (A_2) hold.

Lemma 9 *Assumption (A_3) is satisfied if and only if the $X = (Y, Z)$ vector is constant on each MGC, G :*

$$\forall t_1, t_2 \in G, X^{t_1} = X^{t_2} \quad (23)$$

Proof: Since the network is assumed to reach a deadlock, then X is finite. If this is the reference deadlock, then in any MGC G , all the transitions fire the same number of times and (23) holds. Conversely if for all MGC G , $\forall t_1, t_2 \in G$, $X^{t_1} = X^{t_2}$, then all the MGC's reach the reference deadlock and consequently, the whole network reaches the reference deadlock. In order to see this last point, it is enough to show that for any deadlock, the number of tokens in each routing place is necessarily zero. Let F be the $|\mathcal{B}| \times |\mathcal{B}|$ -matrix defined by $F(t, t') = 1$ if $t \in \mathcal{B}$ and $t' \in (\bullet t)^\bullet$. Since for all finite vectors Z

$$F \times \Pi(Z) = Z,$$

then the solution of (13)-(14) satisfies

$$F \times Z = P \times Z + Q \times Y + R, \quad (24)$$

which implies that the total number of tokens that ever leave place p is indeed equal to the sum of the initial number of tokens in p plus the total number that ever enter p . ■

This characterization is interesting because it can be checked on a set of SI-FCNets derived from the original net using a projection presented below.

Support Matrix Condition We recall that \underline{C} is the support matrix of the transfer matrix C . Let $(\underline{X}, \underline{Y})$ be the smallest solution of Equations (E) for the total number of events in the system with transfer matrix \underline{C} in place of C :

$$\underline{Y} = \underline{C} \otimes \underline{Z} \quad (25)$$

$$\underline{Z} = \Pi(P \times \underline{Z} + Q \times \underline{Y} + R). \quad (26)$$

Equivalently, $(\underline{X}, \underline{Y})$ is the total firing vector in the net where the initial marking corresponds to matrix \underline{C} .

Lemma 10 *The system with transfer matrix C verifies (A_3) if and only if the system with transfer matrix \underline{C} verifies (A_3) .*

Proof: First, we show that if $(\underline{Y}, \underline{Z})$ verifies (A_3) (or equivalently (23)), so does (Y, Z) . If \underline{Z} satisfies (23), then $\underline{C} \otimes \underline{Z} = C \otimes \underline{Z}$ (because (A_1) is assumed to hold), so that $\underline{Z} = \Pi(P \times \underline{Z} + Q \times (C \otimes \underline{Z}) + R)$. Therefore, $\underline{Z} \geq Z$. But since $\underline{C} \leq C$, $\underline{Z} \leq Z$. Therefore, $\underline{Z} = Z$ and Z satisfies (23). This implies that (Y, Z) also satisfies (23).

Now, we assume that (Y, Z) satisfies (23) for all input vectors R . Let (Y', Z') be the minimal solution of

$$Y' = (C \otimes Z') \oplus \underline{Y} \quad (27)$$

$$Z' = \Pi(P \times Z' + Q \times (C \otimes Z') + R) \oplus \underline{Z}. \quad (28)$$

Obviously, $(Y', Z') \leq (\underline{Y}, \underline{Z})$. On the other hand, $(\underline{Y}, \underline{Z})$ is the minimal solution of (25)-(26) which can be rewritten as:

$$\begin{aligned} \underline{Y} &= (\underline{C} \otimes \underline{Z}) \oplus \underline{Y} \\ \underline{Z} &= \Pi(P \times \underline{Z} + Q \times \underline{Y} + R) \oplus \underline{Z}. \end{aligned}$$

Now, the fact that $\underline{C} \leq C$ implies that $(Y', Z') \geq (\underline{Y}, \underline{Z})$ and so $(\underline{Y}, \underline{Z}) = (Y', Z')$.

This means that under a proper choice of the timing of the transitions (see the definition of delayed networks in §6.3), the counters of the system with matrix C can reach the value $(\underline{Y}, \underline{Z})$ before the total firing vector (Y, Z) is reached. That is, $(\underline{Y}, \underline{Z})$ is *firable* under C . Let \mathcal{M} be the marking reached after having fired $(\underline{Y}, \underline{Z})$ under C . Note that since $(\underline{Y}, \underline{Z})$ is firable under \underline{C} , $\mathcal{M} \geq \mathcal{M}_0$ (this is true because $(\underline{Y}, \underline{Z})$ can be fired without using the initial tokens in the C -net). Let R' and D be the input vector and transfer matrix associated with \mathcal{M} . We know that $R' \geq 0$ and $D \geq C$. The fact that $(\underline{Y}, \underline{Z})$ is firable under C implies that $(Y - \underline{Y}, Z - \underline{Z})$ is the smallest solution of

$$Y - \underline{Y} = (D \otimes (Y - \underline{Z})) \quad (29)$$

$$Z - \underline{Z} = \Pi(P \times (Z - \underline{Z}) + Q \times (D \otimes (Z - \underline{Z})) + R'). \quad (30)$$

But since $D \geq C$, $D \otimes (Z - \underline{Z}) + R' = C \otimes (Z - \underline{Z}) + R''$ with $R'' = R' + Q(D \otimes (Z - \underline{Z}) - C \otimes (Z - \underline{Z})) \geq R'$. We get

$$Z - \underline{Z} = \Pi(P \times (Z - \underline{Z}) + Q \times (C \otimes (Z - \underline{Z}) + R'')).$$

Now, this equation is the original equation with a different input vector. By assumption, $(Z - \underline{Z})$ verifies (23). This combined with the fact that Z verifies (23), implies that $\underline{Z} = Z - (Z - \underline{Z})$ also verifies (23) and so $(\underline{Y}, \underline{Z})$ verifies (23). \blacksquare

SI-projections Condition We recall that the vector $(Y_{\langle s \rangle}, Z_{\langle s \rangle})$ is the minimal solution of the system $E_{\langle s \rangle}$ associated with the SI-projection of the net with transfer matrix $C_{\langle s \rangle}$.

Lemma 11 *Under $(A_1), (A_2)$, the variables $(\underline{Y}, \underline{Z})$ verify Condition (23) if and only if, for all s , the variables $(Y_{\langle s \rangle}, Z_{\langle s \rangle})$ verify Condition (23).*⁵

Proof: First note that according to Lemma 7, an arbitrary vector $(\underline{Y}, \underline{Z})$ verifies (23) if and only if \underline{Z} verifies (23). Therefore in the following we will focus exclusively on vector \underline{Z} .

First, we prove that if $Z_{\langle s \rangle}$ verifies (23) for all s , then \underline{Z} verifies (23). If \underline{Z} is a solution of (\underline{E}) then, we can find $s \in \{1, \dots, K\}$ such that \underline{Z} is a solution of $(E_{\langle s \rangle})$. For each line of C , we choose the element corresponding to the minimal component of \underline{Z} :

$$C_{\langle s \rangle}[i][j] = \begin{cases} 0 & \text{if } \underline{Z}_j = \min_{k: C[i][k]=0} \underline{Z}_k \\ \infty & \text{otherwise,} \end{cases}$$

so that $C_{\langle s \rangle} \otimes Z_{\langle s \rangle} = C \otimes Z_{\langle s \rangle}$. This shows that $\underline{Z} \geq Z_{\langle s \rangle}$, since $Z_{\langle s \rangle}$ is the minimal solution of $(E_{\langle s \rangle})$. But conversely, if $Z_{\langle s \rangle}$ verifies (23), then $Z_{\langle s \rangle}$ is also a solution of (\underline{E}) and therefore, $Z_{\langle s \rangle} \geq \underline{Z}$, which implies $\underline{Z} = Z_{\langle s \rangle}$ and so, \underline{Z} verifies (23).

We now prove that if \underline{Z} verifies (23), then so does $Z_{\langle s \rangle}$, for any s . We must introduce a few notations. For each row i of matrix C (which corresponds to a transition, t_i in \mathcal{A}), we denote $t_{\langle s \rangle}^i$ the only transition in \mathcal{B} which remains connected to transition t_i in the system associated with matrix $C_{\langle s \rangle}$. In order to keep notation simple, we also note $Z_{\langle s \rangle}^{t_{\langle s \rangle}^i} = Z_{\langle s \rangle}^{(i)}$.

We suppose that there exists s such that $Z_{\langle s \rangle}$ does not verify (23). We must distinguish two cases:

1. For all $t_i \in \mathcal{A}$, for all $t_j \in \mathcal{B}$ such that $\underline{C}[i][j] = 0$, $Z_{\langle s \rangle}^{(i)} \leq Z_{\langle s \rangle}^{t_j}$. In this case, $Z_{\langle s \rangle}$ is a solution of (\underline{E}) , so $\underline{Z} \leq Z_{\langle s \rangle}$. Since \underline{Z} verifies (23), then $\underline{C} \otimes \underline{Z} = C_{\langle s \rangle} \otimes \underline{Z}$ and so, \underline{Z} is also a solution of $E_{\langle s \rangle}$. Thus, $\underline{Z} = Z_{\langle s \rangle}$, which contradicts the fact that $Z_{\langle s \rangle}$ does not verify (23). This ends this case.
2. There exists $t_i \in \mathcal{A}$ and $t_j \in \mathcal{B}$ with $\underline{C}[i][j] = 0$ such that $Z_{\langle s \rangle}^{(i)} > Z_{\langle s \rangle}^{t_j}$. Then we construct $u \in \{1, \dots, K\}$ by picking t_j instead of $t_{\langle s \rangle}^i$. We have $t_{\langle u \rangle}^k = t_{\langle u \rangle}^i$, $\forall k \neq i$

⁵By this, we mean that $X_{\langle s \rangle}^t = X_{\langle s \rangle}^u$ for all transitions t and u which belong to the same MGC of the original net.

and $t_{\langle u \rangle}^i = t_j$. We have:

$$\begin{aligned} Z_{\langle s \rangle} &= \Pi(PZ_{\langle s \rangle} + Q(C_{\langle s \rangle} \otimes Z_{\langle s \rangle} + R)) \\ &> \Pi(PZ_{\langle u \rangle} + Q(C_{\langle u \rangle} \otimes Z_{\langle s \rangle} + R)). \end{aligned}$$

This implies $Z_{\langle s \rangle} > Z_{\langle u \rangle}$ using the fixed point theorem of Knapster-Tarsky for increasing functions in the $|\mathcal{B}|$ dimensional cube $[0, Z_{\langle s \rangle}]$.

Now, we consider $Z_{\langle u \rangle}$. Note that $Z_{\langle u \rangle}$ does not verify (23), otherwise we would have $Z_{\langle u \rangle} \geq Z_{\langle s \rangle}$.

- If $Z_{\langle u \rangle}$ falls in case 1, then $Z_{\langle u \rangle} \geq \underline{Z}$ and if \underline{Z} verifies (23), then $Z_{\langle u \rangle} = \underline{Z}$ and cannot verify (23). Therefore, in this case, \underline{Z} does not verify (23).
- If $Z_{\langle u \rangle}$ falls in case 2, then we repeat the previous construction but for $Z_{\langle u \rangle}$ this time. We get $Z_{\langle v \rangle}$, $Z_{\langle u \rangle} > Z_{\langle v \rangle}$ and the proof continues with $Z_{\langle v \rangle}$. The sequence $Z_{\langle s \rangle} > Z_{\langle u \rangle} > Z_{\langle v \rangle} > \dots$ is necessarily finite and the last term of the sequence necessarily falls in case 1. This implies that \underline{Z} does not verify (23).

This ends the proof of the lemma and gives a characterization of (A_3) using the SI-projections of the net. ■

5.3 Conditions for $(A_1) - (A_3)$; the Structural Case

This section focuses on conditions ensuring $(A_1) - (A_3)$ for all Π .

5.3.1 (A_2)

Assumption (A_2) has a structural counterpart which reads:

For all finite R , the net enters a deadlock for all possible routing.

In Petri net theory, this property is also called *promptness* with respect to the input. In the SI-FCNet case, it is easily checked that this promptness condition boils down to the absence of circuits containing routing places in the net.

5.3.2 (A_3)

The structural counterpart of (A_3) reads:

If the network reaches a deadlock, then this deadlock is the reference deadlock, for all possible routing Π , i.e. the net admits a unique deadlock (if any).

All the SI-FCNets satisfy this condition (see §6.2). Some MI-FCNets also satisfy this condition as exemplified in §7.

Lemma 12 *In a SI-FCNet, the set of all minimal solutions of the equations*

$$Z = \Pi(PZ + Q(C \otimes Z) + R), \tag{31}$$

for all the functions Π coincide with the minimal solutions of the equation

$$FZ = PZ + Q(C \otimes Z) + R. \quad (32)$$

Proof: First note that in a SI-FCNet, the expression $PZ + Q(C \otimes Z) + R$ can be rewritten $P'Z + R$, for some P' ; this is true because C has only one zero element per line. If $FZ = P'Z + R$ and Z is minimal for (32), then one can construct Π_Z such that $Z = \Pi_Z(P, Z + R)$. Z is also a minimal solution of this equation, because for any other solution Z' we have $Z' = \Pi_Z(P'Z' + R)$ and therefore $FZ' = P'Z' + R$ which implies $Z' \geq Z$.

Conversely, let Z be a minimal solution of (31). Then Z verifies (32). We show that Z is minimal for (32) as well. Let us suppose that there exists $Z' \leq Z$ and verifying $FZ' = P'Z' + R$. We have, $F(Z - Z') = P'(Z - Z')$, which means that $Z - Z'$ is a T-invariant of the routing net associated with the FCNet. Suppose that $\Pi(P'Z' + R)_i < Z'_i$. This means that $Z_i < Z'_i$ which is impossible. Therefore, $\Pi(P'Z' + R) \geq Z'$. But this means that Z' is firable in the system with routing Π and this implies $Z' = Z$ since the markings after having fired Z or Z' are the same, and this marking is a deadlock. ■

Theorem 4 Under (A_1) and (A_2) a FCNet verifies the structural version of (A_3) if and only if the minimal solutions of all the systems:

$$FZ_{\langle s \rangle} = P_{\langle s \rangle}Z_{\langle s \rangle} + R, \quad s \in \{1, \dots, K\}$$

verify Condition (23).

Proof: This result is a direct consequence of Lemmas 10, 11 and 12. ■

Remark: *Complexity.* The system $FZ_{\langle s \rangle} = P_{\langle s \rangle}Z_{\langle s \rangle} + R$ is linear, and one can find its solutions in polynomial time. Then, testing (23) is done in linear time. The only problem with this characterization is that K , the total number of SI-projections, can be exponential in the size of the original net. ■

Remark: *Non minimal solutions.* We have seen in the proof of Lemma 12 that the equation $FZ = P'Z + R$ has non-minimal solutions. Let U be a non zero T-invariant of the net. Then $F(Z + U) = P'(Z + U) + R$. Conversely, if $Z' \geq Z$ is a solution of the equation, then, $Z' - Z$ is a T-invariant of the net. ■

5.4 Restriction of the Arrival Process

Let m and n be two integers such that $0 \leq m \leq n$. Let $N_{[m,n]}$ be the $[m, n]$ -restriction of the point process N , namely the point process $\{T_l\}_{m \leq l \leq n}$. For instance, $N_{[0,0]}$ has only one

point at $T_0 = 0$. Let $R_{[m,n]}^t$ be the counter of the exogeneous arrivals to place $\bullet t$, for $N_{[m,n]}$:

$$R_{[m,n]}^t = \sum_{k \in \mathbb{Z}} I_{[m,n]}^t(k), \quad (33)$$

where $I_{[m,n]}^t(k) = I^t(k)$ if $k \in \{T_l\}_{m \leq l \leq n}$ and 0 otherwise.

Starting from 0 Let us first discuss the processes associated with $N_{[0,n]}$. Assumption (A_2) allows us to say that the network with the input process $(R_{[0,n]}(k))_{k \in \mathbb{N}}$ reaches a deadlock. We denote by $Z_{[0,n]}$ and $Y_{[0,n]}$ the total firing vectors for this system. With $\Pi_{[0,\infty]} \equiv \Pi$, they verify the equations

$$\begin{aligned} Y_{[0,n]} &= A \otimes Y_{[0,n]} \oplus B \otimes Z_{[0,n]} \\ Z_{[0,n]} &= \Pi_{[0,\infty]} (P \times Z_{[0,n]} + Q \times Y_{[0,n]} + R_{[0,n]}) . \end{aligned}$$

If $t \in \mathcal{B}$, we denote by $S_{[0,n]}^t$ the total number of tokens that enter place $\bullet t$. Finally, let $S_{[0,n]} = \{S_{[0,n]}^t, t \in \mathcal{B}\}$. By definition, we have

$$S_{[0,n]} = P \times Z_{[0,n]} + Q \times Y_{[0,n]} + R_{[0,n]} = F \times Z_{[0,n]}. \quad (34)$$

Since $Z_{[0,n]}$ is finite, so is $S_{[0,n]}$.

Starting from m Now, we introduce the system generated by the restricted input process $(R_{[m,n]}(k))_{k \in \mathbb{N}}$, with $0 \leq m \leq n$. We assume that this system is in its reference deadlock at time T_m^- . We connect this system with the original one by taking the following routing sequences:

$$\nu_{[m,\infty]}^p(k) = \nu^p(k + S_{[0,m-1]}^t) = \tau_{S_{[0,m-1]}^t} \nu^p(k), \quad (35)$$

for all $p = \bullet t$, where the last equality makes use of the notation introduced in Equation (11). This definition states that routing sequences are ‘marks’ of the arrival point process.

With this definition, it is easy to check that the function $\Pi_{[m,\infty]}$ is actually defined from Π by the relation

$$\Pi_{[m,\infty]}^t(k) \equiv \sum_{l=1}^k \mathbf{1}_{\nu_{[m,\infty]}^{\bullet t}(l)=t} = \Pi^t(k + S_{[0,m-1]}^t) - \Pi^t(S_{[0,m-1]}^t). \quad (36)$$

Finally we define the vectors $X_{[m,n]}(k)$, $Y_{[m,n]}(k)$ and $Z_{[m,n]}(k)$, by the equations

$$\begin{aligned} Y_{[m,n]}(k) &= \bigoplus_{l=1}^M (A_l \otimes Y_{[m,n]}(k-l) \oplus B_l \otimes Z_{[m,n]}(k-l)) \\ Z_{[m,n]}(k) &= \Pi_{[m,\infty]} \left(\sum_{l=1}^M (P_l Z_{[m,n]}(k-l) + Q_l Y_{[m,n]}(k-l)) + R_{[m,n]}(k) \right), \end{aligned}$$

with the initial conditions : $\forall k < T_m, Y_{[m,n]}(k+1) = 0$ and $Z_{[m,n]}(k) = 0$.

5.5 The Separability and Monotonicity Properties

In this section, we prove that under the three assumptions (A_1) - (A_3) , the variables $X_{[m,n]}(k)$ satisfy a set of monotonicity and conservation properties (which are known as the ‘monotone-separable properties’ - see Appendix 1) which form the basis for analyzing stability. Let $\mathcal{X}_{[m,n]}$ denote the *maximal dater* of the net with input $N_{[m,n]}$, i.e. the first time larger than T_n at which this net reaches the reference deadlock again (see Appendix 1). We need a preliminary lemma.

Lemma 13 *Let $0 < m < n$. If $T_m > \mathcal{X}_{[0,m-1]} + M$, then for all $k \geq T_m$, $X_{[0,n]}(k) = X_{[0,m-1]} + X_{[m,n]}(k)$.*

Proof: The proof is by induction on k . For $k \geq T_m$, the assumption $T_m > \mathcal{X}_{[0,m-1]} + M$ implies that

$$X_{[0,m-1]}(k-l) = X_{[0,m-1]} \quad \forall 1 \leq l \leq M.$$

In addition, from the localization property, for $k = T_m$ and any $l \geq 1$

$$X_{[0,m-1]}(k-l) = X_{[0,n]}(k-l).$$

So for $k = T_m$ and $1 \leq l \leq M$,

$$X_{[0,n]}(k-l) = X_{[0,m-1]}.$$

Therefore, we obtain from Equation (6) that for $k = T_m$

$$\begin{aligned} Y_{[0,n]}(k) &= Y_{[0,m-1]} \\ Z_{[0,n]}(k) &= \Pi \left((P Z_{[0,m-1]} + Q Y_{[0,m-1]} + R_{[0,m-1]} + I(T_m)) \right). \end{aligned}$$

So, using the fact that $Y_{[m,n]}(T_m) = 0$ and the definition of $\Pi_{[m,\infty]}$ (Equation (36)), we obtain the desired relations for $k = T_m$:

$$\begin{aligned} Y_{[0,n]}(k) &= Y_{[0,m-1]} + Y_{[m,n]}(k) \\ Z_{[0,n]}(k) &= Z_{[0,m-1]} + Z_{[m,n]}(k). \end{aligned}$$

For $k > T_m$, the induction assumption, which we assume to hold up to rank $k-1$, gives

$$\begin{aligned} Y_{[0,n]}(k) &= \bigoplus_{l=1}^M (A_l \otimes (Y_{[0,m-1]} + Y_{[m,n]}(k-l)) \oplus B_l \otimes (Z_{[0,m-1]} + Z_{[m,n]}(k-l))) \\ Z_{[0,n]}(k) &= \Pi \left(\sum_{l=1}^M (P_l (Z_{[0,m-1]} + Z_{[m,n]}(k-l)) \right. \\ &\quad \left. + Q_l (Y_{[0,m-1]} + Y_{[m,n]}(k-l))) + R_{[0,m-1]} + R_{[m,n]}(k) \right). \end{aligned}$$

Using the result of Lemma 9 and Relation (36), we get

$$\begin{aligned} Y_{[0,n]}(k) &= Y_{[0,m-1]} + \bigoplus_{l=1}^M (A_l \otimes Y_{[m,n]}(k-l) \oplus B_l \otimes Z_{[m,n]}(k-l)) \\ Z_{[0,n]}(k) &= \Pi \left(\sum_{l=1}^M (P_l Z_{[m,n]}(k-l) + Q_l Y_{[m,n]}(k-l)) + S_{[0,m-1]} + R_{[m,n]}(k) \right). \end{aligned}$$

This shows that

$$\begin{aligned} Y_{[0,n]}(k) &= Y_{[0,m-1]} + Y_{[m,n]}(k) \\ Z_{[0,n]}(k) &= Z_{[0,m-1]} + Z_{[m,n]}(k). \end{aligned}$$

■

Corollary 1 $X_{[0,n]} = X_{[0,m-1]} + X_{[m,n]}$.

Proof: This is an immediate corollary of the previous lemma, considering the fact that $X_{[0,n]}$ does not depend on T_m . ■

Lemma 14 (External Monotonicity) *If the two arrival counting measures R and R' satisfy the inequality $R'_{[m,n]}(k) \geq R_{[m,n]}(k)$ for all k , then $X'_{[m,n]}(k) \geq X_{[m,n]}(k)$ for all k .*

Proof: The vector $X_{[m,n]}(k)$ is an increasing function of $(X_{[m,n]}(k-l))_{l=1 \dots M}$ and of $R_{[m,n]}(k)$. The proof follows by a straightforward induction. ■

Lemma 15 (Conservation) $X_{[m,n]}$ is finite and independent of the arrival times.

Proof: Corollary 1 says that $X_{[m,n]} = X_{[0,n]} - X_{[0,m-1]}$. Therefore, $X_{[m,n]}$ is finite and independent of the arrival times. ■

Lemma 16 (Separability) *Let $m < r < n$. Suppose that $T_r > \mathcal{X}_{[m,r-1]} + M$. Then*

$$\begin{aligned} \text{if } k < T_r, \text{ then } X_{[m,n]}(k) &= X_{[m,r-1]}(k), \\ \text{if } k \geq T_r, \text{ then } X_{[m,n]}(k) &= X_{[m,r-1]} + X_{[r,n]}(k). \end{aligned}$$

Proof: The case $k < T_r$ follows from the localization property. For $k \geq T_r$, the proof holds by induction on k . It is very similar to the proof of Lemma 13 (in fact, for $m = 0$, it is exactly the same). ■

Lemma 17 (Homogeneity) *Let $R'_{[m,n]}$ be the arrival process shifted by a constant C , $R'_{[m,n]}(k) = R_{[m,n]}(k + C)$. Then, $X'_{[m,n]}(k) = X_{[m,n]}(k + C)$.*

Proof: By induction on k . ■

So, under assumptions (A_1) - (A_3) , the counters $X^t(k)$ of a FCNet satisfy the conditions of the monotone-separable framework given in the Appendix 1.

Let

$$\begin{aligned} \mathcal{W}_{[m,n]} &\equiv \mathcal{X}_{[m,n]} - T_n, \\ W_{[m,n]}^t(k) &\equiv X_{[m,n]}^t - X_{[m,n]}^t(T_n + k), \quad k \geq 0. \end{aligned}$$

The variable $\mathcal{W}_{[m,n]}$ gives the residual time to deadlock, whereas $W_{[m,n]}^t(k)$ gives the residual number of firings of transition t after time k ; both definitions assume that the arrival process is stopped at time T_n .

Since the counters $X^t(k)$ satisfy the conditions of the monotone-separable framework, the following theorem holds (see [4]):

Theorem 5 For all n , $\mathcal{W}_{[m-1,n]} \geq \mathcal{W}_{[m,n]}$; for all $k \geq 0$, t and n , $W_{[m-1,n]}^t(k) \geq W_{[m,n]}^t(k)$, so that

$$\exists \lim_{m \rightarrow -\infty} \uparrow \mathcal{W}_{[m,n]} \equiv \mathcal{W}_{[-\infty,n]}, \quad \exists \lim_{m \rightarrow -\infty} \uparrow W_{[m,n]}^i(t) \equiv W_{[-\infty,n]}^i(t), \quad (37)$$

where the notation $\lim_{m \rightarrow -\infty} \uparrow x(m)$ indicates that $x(m)$ is a non-decreasing function of m .

5.6 Stochastic Assumptions

All the random variables defined in what follows are assumed to be carried by some probability space $(\Omega, \mathcal{F}, P, \theta)$, where θ is a shift on Ω which is P -ergodic and such that $P[f \circ \theta] = P[f]$ for all measurable functions $f : \Omega \rightarrow \mathbb{R}^+$ (see [2] for more details on this formalism).

We assume that the point process associated with the counting measure $R_{[-\infty, +\infty]}(k)$ is θ -stationary and ergodic, and that it has a finite intensity. When taking $\{T_0 = 0\}$, this θ -stationarity assumption here means that

$$R_{[n, \infty]}(T_n + k) = R_{[0, \infty]}(k) \circ \theta^n \quad (38)$$

for all $k \in \mathbb{R}$ and $n \in \mathbb{N}$. Actually, this relation also allows us to continue the point process to the left, i.e. to define a stationary ergodic point process $\{N_n\}_{n \in \mathbb{Z}}$.

Consider the \mathcal{T} -valued sequences $\nu_{[0, \infty]} = \{\nu_{[0, \infty]}^p(k)\}$, $p \in \mathcal{R}, k \in \mathbb{N}$ describing the routing decisions; we also assume that the following compatibility relation holds for all n .

$$\nu_{[n, \infty]} = \nu_{[0, \infty]} \circ \theta^n. \quad (39)$$

This equation also allows to continue the routing sequences to negative indices.

Let $\tau_S \nu$ denote the sequences $(\tau_{S^t} \nu^{\bullet t}, t \in \mathcal{B})$. Equation (35) and the above relation imply the compatibility relation:

$$\nu_{[0, \infty]} \circ \theta^n = \tau_{S_{[0, n-1]}} \nu_{[0, \infty]} \quad (40)$$

where $S_{[0,n]}^t$ is the function defined in Equation (34).

It should be clear that under the above assumptions, the functions $\Pi_{[n,\infty]}^t$ satisfy the compatibility property

$$\Pi_{[n,\infty]} = \Pi_{[0,\infty]} \circ \theta^n, \quad (41)$$

so that the following compatibility relations also hold:

$$\begin{aligned} \mathcal{W}_{[n,n+k]} &= \mathcal{W}_{[0,k]} \circ \theta^n, \\ W_{[n,n+l]}^t(k) &= W_{[0,l]}^t(k) \circ \theta^n, \quad k \geq 0. \end{aligned}$$

Since all the firing times are positive, one can easily check that condition (42) below

$$\{\mathcal{W}_{[-n,0]} \rightarrow_{n \rightarrow \infty} \infty\} \stackrel{\text{a.s.}}{=} \left\{ \exists i / W_{[-n,0]}^i \rightarrow_{n \rightarrow \infty} \infty \right\} \quad (42)$$

is satisfied.

Therefore, the following theorem holds (see [4]):

Theorem 6 *Under the above statistical assumptions, the following a.s. limit takes place for all c -dilation⁶ $c.N$ of N , with $c \geq 0$:*

$$\exists \lim_{n \rightarrow \infty} \frac{\mathcal{W}_{[0,n]}(c.N)}{n} = \Gamma(c) \quad (43)$$

where $\Gamma(c)$ is a constant. If the intensity λ of the input point process is such that $\lambda\Gamma(0) < 1$, then the limits $\mathcal{W}_{[-\infty,n]}(1.N) \equiv \mathcal{W}_{[-\infty,n]}$ and $W_{[-\infty,n]}^t(k)$ are a.s. finite for all n , k and t .

So, under the condition $\lambda\Gamma(0) < 1$, the residual processes $W_{[-\infty,n]}^t(\cdot)$ admit a finite stationary regime which is given by the relation

$$W_{[-\infty,n]}^t(\cdot) = W_{[-\infty,0]}^t(\cdot) \circ \theta^n. \quad (44)$$

More precisely

Corollary 2 *Whenever $\lambda\Gamma(0) < 1$, if the initial condition is the reference deadlock, the stochastic process $W_{[0,n]}^t(\cdot)$ converges in law to the stochastic process $W_{[-\infty,0]}^t(\cdot)$, for all $t \in \mathcal{T}$.*

Proof: This follows from the relation

$$W_{[0,n]}^t(\cdot) = W_{[-n,0]}^t(\cdot) \circ \theta^n$$

and from the fact that

$$W_{[-n,0]}^t(\cdot) \uparrow = W_{[-\infty,0]}^t(\cdot) < \infty$$

⁶See Appendix 1 for the definition of dilations. Of course we have to limit ourselves to c rational in order to stay within the framework of this paper, namely, up to a change of scale, all timing variables are integer-valued.

which imply that

$$P[W_{[0,n]}^t(k) > x] = P[W_{[-n,0]}^t(k) > x] \leq P[W_{[-\infty,0]}^t(k) > x],$$

for all $x \in \mathbb{R}^+$ and $k \in \mathbb{N}$. ■

In other words, the residual processes $W_{[-n,0]}^t(\cdot)$ are bounded in probability indeed (in fact, we proved more since we showed that they converge in law to finite stationary processes).

It is easy to construct the stationary marking process out of the stationary residual processes. The number of tokens $M_{[m,n]}^p(k)$ in place p at time $k + T_n$, for the input process $N_{[m,n]}$, is given by the relation

$$N_{[m,n]}^p(k) = M^p + \sum_{t \in p^\bullet} W_{[m,n]}^t(k) - \sum_{t' \in \bullet p} W_{[m,n]}^{t'}(k), \quad (45)$$

where M^p denotes the marking of p in the reference deadlock. So, if $\lambda\Gamma(0) < 1$, we have constructed a stationary (θ -compatible) version of the marking process in place p , say $M^p(\cdot)$, by taking

$$M^p(T_n + k) = M^p + \sum_{t \in p^\bullet} W_{[-\infty,n]}^t(k) - \sum_{t' \in \bullet p} W_{[-\infty,n]}^{t'}(k), \quad (46)$$

for $k \in [0, T_{n+1} - T_n)$.

Remark: *Continuation of the routing sequences.* Let $t \in \mathcal{B}$ and $p = \bullet t$. Note that since $\nu_{[0,\infty]}^p(k) = \nu_{[n,n]}^p(l)$ for $l = k - S_{[0,n-1]}^t \leq S_{[n,n]}^t$, the infinite sequences $\nu_{[0,\infty]}$ are fully determined by the finite sub-sequences

$$(\nu_{[n,n]}^p(k), p \in \mathcal{R}, 0 \leq k \leq S_{[n,n]}^t)_{n \geq 0}$$

Thus, with our framework, for all nodes p , the whole routing sequence $\nu_{[0,\infty]}^p(k)$ is simply the concatenation of the routing sequences $\nu_{[n,n]}^p(l)$, $1 \leq l \leq S_{[n,n]}^t$.

6 More on the Stability of SI-FCNets

Throughout this section, we will assume that the free choice net under consideration is SI, and we will denote $t^{(i)}$ the unique transition of \mathcal{B} which is an input of G_i .

6.1 Total Number of Events

If F is a SI-FCNet, the equations for the total number of events can be put in a simplified form.

We recall that the total number of events verify

$$Y = C \otimes Z \quad (47)$$

$$Z = \Pi(P \times Z + Q \times (C \otimes Z) + R). \quad (48)$$

Since the network is SI, the matrix C has at most one finite element per line. Therefore, we can define C', R' by $C \otimes Z + R = C' \times Z + R'$ and P' by $P' = P + Q \times C'$. In this case, the total number of firings verify the equations:

$$Z = \Pi(P' \times Z + R') \quad (49)$$

$$Y = C \otimes Z. \quad (50)$$

A MGC G_i is *input-connected* if for each transition in G_i , there is a path from $t^{(i)}$ to t .

This translates into the following algebraic criterion: all MGC's are input-connected if and only if the matrix $C = A^* \otimes B$ has no line composed only of ε 's, where $\varepsilon = \infty$. Since this matrix has at most one non- ε element per line (from the SI assumption), all MGC's are input-connected if and only if C has exactly one non- ε element per line.

6.2 Reference Deadlock for SI-FCNets

Lemma 18 *Let F be a SI-FCNet with all its MGC's input-connected. If F can reach a deadlock, then this deadlock is unique.*

Proof: If a routing place p contains a token, then one of the transitions in p^\bullet is enabled, thus this marking is not a deadlock. Let t be a transition in G_i , let us follow the longest path in G_i without tokens. This path leads to a transition which is enabled except if it is the input transition. Now, a marking verifying these conditions is necessarily unique. ■

Therefore, for SI-FCNets with input-connected components, assumption (A_3) is redundant for a reference deadlock to exist.

Lemma 19 *Assume that the routing sequences $\{\nu^p(i)\}$ are i.i.d. in i and mutually independent in p . Let Ψ be the $|\mathcal{B}| \times |\mathcal{B}|$ -diagonal (conventional algebra) matrix with t -th diagonal term equal to $P[\nu^t(1) = t]$. Then condition (A_2) is satisfied whenever the Perron-Frobenius eigenvalue of the positive matrix $\Psi P'$ is strictly less than 1.*

Proof: Taking expectations on both sides of (49), we obtain (on line t)

$$\begin{aligned} E[Z^t] &= E\left[\sum_{i=1}^{(P'Z+R')^t} 1_{\nu^{\bullet t(i)}=t}\right] \\ &= \sum_{i=1}^{\infty} E[1_{(P'Z+R')^t \geq i} 1_{\nu^{\bullet t(i)}=t}] \\ &= \sum_{i=1}^{\infty} E[1_{(P'Z+R')^t > i-1} 1_{\nu^{\bullet t(i)}=t}]. \end{aligned}$$

But it is clear that the events $\{(P'Z + R')^t \leq i - 1\}$ and $\{\nu^{\bullet t}(i) = t\}$ are independent, and so

$$\begin{aligned} E[Z^t] &= \sum_{i=1}^{\infty} P[(P'Z + R')^t > i - 1] P[\nu^{\bullet t}(1) = t] \\ &= E[(P'Z + R')^t] P[\nu^{\bullet t}(1) = t] \\ &= (P'E[Z] + R')^t P[\nu^{\bullet t}(1) = t]. \end{aligned}$$

Finally

$$E[Z] = \Psi P'E[Z] + \Psi R'. \quad (51)$$

The fact that the Perron-Frobenius eigenvalue of $\Psi P'$ is strictly less than one implies that (51) admits a unique finite solution \tilde{Z} . But this in turn implies that the expectation of the minimal solution Z of (49)-(50) is finite, and coincides in fact with the unique finite solution of (51). In order to see this, consider sequence $Z(k)$ of Lemma 6. It is easy to check that

1. $Z(k)$ converges monotonically to Z ;
2. $E(Z(k)) < \infty$ for all k ;
3. $E(Z(k)) \leq \Psi P'E[Z(k)] + \Psi R'$.

But (2) and (3) imply that $E(Z(k)) \leq \tilde{Z}$, and so, when using (1) and the monotone convergence theorem, we obtain that

$$E(Z) = E(\lim_k Z(k)) = \lim_k E(Z(k)) \leq \tilde{Z}.$$

Therefore $E(Z)$ is finite (in fact since $E(Z)$ is a solution of (51), then necessarily $E(Z) = \tilde{Z}$). ■

A branching process interpretation of this result is given in Appendix 2.

6.3 Stability Region

The aim of this section is the computation of the constant $\Gamma(0)$ defined in Theorem 6, which allows us to characterize the stability region of the network, as we know from this theorem.

Delayed Networks We now consider a more general framework with inputs both on routing places as above, but also on transitions. Inputs on transitions mean *delays*. Our assumption will be that each transition is possibly delayed by an exogeneous integer-valued input process, say $U^j(k)$ for transition j , with the interpretation that this transition cannot

initiate more than $U^j(k)$ firings by time k . The evolution equations for such a network are then given by

$$Y(k) = \left(\bigoplus_{l=1}^M (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l)) \right) \oplus V(k) \quad (52)$$

$$Z(k) = \Pi \left(\sum_{l=1}^M (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k) \right) \oplus W(k), \quad (53)$$

where $V(k)$ denotes the vector $U^j(k)$, $j \in \mathcal{A}$. and $W(k)$ the vector $U^j(k)$, $j \in \mathcal{B}$.

The case with no inputs on transitions is a particular case obtained when taking $U^j(k) = \infty$ for all j and k .

All the monotonicity properties extend to this more general framework. In addition, we obtain by an immediate induction that the solution of the above system satisfies the following property:

Lemma 20 *If the two delay processes U and U' satisfy the inequality $U(k) \geq U'(k)$ for all k , then $X(k) \geq X'(k)$ for all k .*

Total Number of Firings We have the following generalization for the equations of Lemma 5

$$Y = A \otimes Y \oplus B \otimes Z \oplus V \quad (54)$$

$$Z = \Pi(P \times Z + Q \times Y + R) \oplus W, \quad (55)$$

with $V = V(\infty)$, $W = W(\infty)$. Note that the total number of events does only depend on the delay process through V and W . In particular, whenever $V = \infty$ and $W = \infty$, the total numbers of events in the delayed network and the non-delayed one are the same.

6.3.1 Computation of $\Gamma(0)$

Consider a SI network, where all MGC's are input-connected. Assumption (A_1) and (A_1) are assumed to hold (so that (A_2) also holds, and the net is separable). The constant $\Gamma(0)$ is that associated with the network with arrival point process $0.N$. Thus, in the following, all exogeneous arrivals take place at time 0.

For all $m \leq n$, let

$$X_{[m,n]}^{(i)} = X_{[m,n]}^j, \quad j \in G_i, \quad (56)$$

denote the total number of firings of (any transition in) G_i , whenever the input process is $0.N_{[m,n]}$ (see the remark following Assumption (A_2)). Let

$$b^i = \gamma^i E[X_{[0,0]}^{(i)}], \quad (57)$$

where γ^i denotes the maximal $(\max, +)$ -Lyapunov exponent of the event-graph G_i (see below and [2], Chapter 7), and finally let

$$b = \max_{i=1, \dots, I} b^i. \quad (58)$$

The parameter b^i will be referred to as the *load factor* of MGC G_i .

Lemma 21 *Under the foregoing assumptions, $\Gamma(0) \geq b$.*

Proof: For all $i = 1, \dots, I$, the total number of tokens to arrive in $\bullet t^{(i)}$ and dedicated to G_i is $X_{[1,n]}^{(i)}$ (whenever the input is $0.N_{[1,n]}$). This and external monotonicity (applied to G_i) imply that

$$\Gamma(0) \geq \lim_n \frac{X_{[1,n]}^{(i)}}{n} \frac{\tilde{\mathcal{X}}_{[1,n]}^{(i)}}{X_{[1,n]}^{(i)}},$$

where $\tilde{\mathcal{X}}_{[1,n]}^{(i)}$ denotes the maximal dater of the MGC G_i in isolation, with an initial number of tokens in its input place equal to $X_{[1,n]}^{(i)}$.

Using the additive property for the total number of events (see Corollary 1) and the strong law of large numbers (SLLN), we obtain that

$$\lim_n \frac{X_{[1,n]}^{(i)}}{n} = E[X_{[0,0]}^{(i)}] \quad a.s.$$

From the first order ergodic theorem on marked graphs (see [2], Chapter 7)

$$\lim_n \frac{\tilde{\mathcal{X}}_{[1,n]}^{(i)}}{X_{[1,n]}^{(i)}} = \gamma^i \quad a.s.$$

Note that if the MGC G^i is acyclic, then $\gamma^i = 0$.

So $\Gamma(0) \geq b^i$, for all i . ■

The main result of this section is the following theorem:

Theorem 7 *Under the foregoing assumptions, $\Gamma(0) = b$.*

The proof of this theorem is based on an induction on the number I of MGC's, which we will assume to be positive. So we start with:

Lemma 22 *Theorem 7 holds for $I = 1$.*

Proof: Since we are in the SI case, whenever $I = 1$, there is exactly one routing place, that is $p = \bullet t^{(1)}$, and the exogeneous input process is necessarily directed to this place. This place necessarily has a unique successor, and so $\Pi^{t^{(1)}}$ is degenerate and equal to identity. This place has no input arc either: if there were an input arc, it would originate from a transition of G_1 , and the net would not be separable. So the case $I = 1$ reduces to the case of an event graph. ■

The induction requires the construction of two upper-bound networks.

First Upper-Bound Network Assume that the expectation of the number of exogeneous arrivals to MGC G_I is positive, that is

$$E[\Pi^{t^{(I)}}(R_{[0,0]}^{t^{(I)}})] > 0.$$

This is not a restriction, up to a change of numbering of MGC's. The following lemma is an immediate consequence of the first order ergodic theorems in [2], Chapter 7:

Lemma 23 *When multiplying all firing times in MGC G_i by a common positive constant s , its Lyapunov exponent becomes $s\gamma^i$.*

There are two kinds of MGC's: those which are acyclic (for instance single non recycled transitions), and those which contain a cycle. The maximal (max, +)-Lyapunov exponent of any MGC of the first kind is 0, and due to our assumption of positiveness on firing times, that of any MGC of the second kind is positive.

In order to define the first upper bound network, we transform the MGC's as follows:

- For each MGC G_i of the first kind, we replace its input transition $t^{(i)}$, with original successor set $P^{(i)}$, by the triple $t^{(i)}, p^{(i)}, t'^{(i)}, q^{(i)}$, where $\bullet t^{(i)}$ is unchanged, $t^{(i)\bullet} = p^{(i)}$, $\bullet t'^{(i)} = \{p^{(i)}, q^{(i)}\}$, and $t'^{(i)\bullet} = P^{(i)} \cup q^{(i)}$; this transformation, which essentially consists in adding a recycled transition just after the input transition of G_i , leaves the net in the class of SI input-connected FC nets and does not alter the time evolution of the net provided the additional transition $t'^{(i)}$ has firing time equal to zero. It transforms the net into an equivalent net with MGC's of the second kind only. This net is constructive if the initial one is.
- For each MGC G_i , $i \neq I$, which is originally of the second kind, we multiply all firing times by the constant $c_i = b / (\gamma^i E[X_{[0,0]}^{(i)}])$, which is larger than or equal to 1. If G_I is originally of the second kind, all firing times of transitions in G_I multiplied by $c_I = (b + d) / (\gamma^I E[X_{[0,0]}^{(I)}])$, where d is some positive real number, so that $s_I \geq 1$.
- For each MGC G_i , $i \neq I$, which is originally of the first kind, we increase the firing times of the additional transition $t'^{(i)}$ to the value $s_i = b / (E[X_{[0,0]}^{(i)}])$, and if G_I is originally of the first kind, we increase the firing times of $t'^{(I)}$ to the value $s_I = (b + d) / (E[X_{[0,0]}^{(I)}])$, where d is as above.⁷

Lemma 23 and monotonicity with respect to firing times (see [1]), imply that this is an upper bound network for the original one. Therefore, the constant $\Gamma(0)$ of the original net is bounded from above by that of this first upper-bound network, where all MGC's have a load factor equal to b , but for G_I , the load factor of which is $b + d$.

⁷In order to keep the network in the class of nets with integer-valued firing times (up to a scaling), we need that all c_i 's and s_i 's be rational, which we will assume from now on. If it is not the case, we have take arbitrarily close rational parameters, and adapt slightly the rest of the proof.

Thus, it is enough to prove that the constant associated with this upper-bound network is bounded from above by $b + d$ to conclude the proof of the theorem. Indeed, we will then have $\Gamma(0) \leq b + d$ for all $d > 0$, that is $\Gamma(0) \leq b$.

In order to keep light notations, we will assume from now on that our initial network satisfies the properties $b^I = b + d$, $b^i = b$, $i \neq I$ (or equivalently, we will use the same notations for the initial and the first upper-bound network).

Second Upper-Bound Network Let M and n be positive integers, and c be a positive real number such that

$$Mc < \frac{E[\Pi^{t^{(I)}}(R_{[0,0]}^{t^{(I)}})]}{E[X_{[0,0]}^{(I)}]} \leq 1. \quad (59)$$

Let \mathcal{E}_n be the event

$$\mathcal{E}_n = \left\{ \Pi^{t^{(I)}}(R_{[0,n]}^{t^{(I)}}) \geq X_{[0,Mc n]}^{(I)} \right\}.$$

In view of (59), we obtain from the SLLN that

$$\lim_n \mathcal{E}_n = \Omega \quad a.s. \quad (60)$$

We now define a sequence $U_m(k)$ of delay processes, all defined on the event \mathcal{E}_n . The second upper-bound network is obtained by delaying the first upper-bound network with the delay function $U_{M+1}(k)$. This sequence is defined by induction on $m = 1, \dots, M + 1$ as follows:

Definition of U_1 .

For all transitions $j \in G_I$, we take

$$U_1^j(k) = X_{[0, cn]}^j, \quad \forall k. \quad (61)$$

On \mathcal{E}_n , the number of events of $j \in G_I$, in the U_1 -delayed network, is exactly $X_{[0, cn]}^{(I)}$, and the associated daters, which will be denoted $\tilde{\mathcal{X}}_{1, [0, n]}^j$, $j \in G_I$, are not affected by the behavior of the rest of the network (since, on \mathcal{E}_n , transition $t^{(I)}$ is never blocked due to a lack of tokens in place $\bullet t^{(I)}$ before its last firing).

Let \mathbf{V}_1 be the maximal dater of G_I in this delayed subnetwork:

$$\mathbf{V}_1 = \max_{j \in G_I} \tilde{\mathcal{X}}_{1, [0, n]}^j(X_{[0, cn]}^j) + \sigma^j. \quad (62)$$

For all transitions j which do not belong to G_I , we define the delays $U_1(k)$ by

$$U_1^j(k) = \begin{cases} 0 & \text{for } 0 \leq k < \mathbf{V}_1; \\ X_{[0, cn]}^j & \text{for } \mathbf{V}_1 \leq k. \end{cases} \quad (63)$$

We denote $\tilde{\mathcal{X}}_{1, [0, n]}^j$ the daters of this delayed network and $\tilde{X}_{1, [0, n]}^j$ the corresponding counters for all j . In particular, $\tilde{\mathcal{X}}_{1, [0, n]}^j$ is its maximal dater.

Lemma 24 For all transitions j , the total number of events $\tilde{X}_{1,[0,n]}^j$ on j is such that

$$\tilde{X}_{1,[0,n]}^j = X_{[0,cn]}^j. \quad (64)$$

Proof: With these delays, at most $X_{[0,cn]}^j$ firings are allowed on transition j . We have to prove that the total number of events on transition j is *exactly* $X_{[0,cn]}^j$. For this, we compare the solutions $\tilde{Y}_{1,[0,n]}$ and $\tilde{Z}_{1,[0,n]}$ of Equations (54)-(55) for the delayed system, which read

$$\begin{aligned} \tilde{Y}_{1,[0,n]} &= A \otimes \tilde{Y}_{1,[0,n]} \oplus B \otimes \tilde{Z}_{1,[0,n]} \oplus Y_{[0,cn]} \\ \tilde{Z}_{1,[0,n]} &= \Pi \left(P \times \tilde{Z}_{1,[0,n]} + Q \times \tilde{Y}_{1,[0,n]} + R_{[0,n]} \right) \oplus Z_{[0,cn]}. \end{aligned}$$

and of the non-delayed system with input process $0.N_{[0,cn]}$, which can be rewritten as follows:

$$\begin{aligned} Y_{[0,cn]} &= A \otimes Y_{[0,cn]} \oplus B \otimes Z_{[0,cn]} \oplus Y_{[0,cn]} \\ Z_{[0,cn]} &= \Pi \left(P \times Z_{[0,cn]} + Q \times Y_{[0,cn]} + R_{[0,cn]} \right) \oplus Z_{[0,cn]} \end{aligned}$$

(the \oplus -addition of the extra terms $Y_{[0,cn]}$ and $Z_{[0,cn]}$ leaves the solution of the system unchanged).⁸

Since $R_{[0,n]} \geq R_{[0,cn]}$, we obtain from the monotonicity property of the solution Y, Z of the system

$$\begin{aligned} Y &= A \otimes Y \oplus B \otimes Z \oplus Y_{[0,cn]} \\ Z &= \Pi \left(P \times Z + Q \times Y + R \right) \oplus Z_{[0,cn]}. \end{aligned}$$

seen as a function of R , that $\tilde{Y}_{1,[0,n]} \geq Y_{[0,cn]}$ and $\tilde{Z}_{1,[0,n]} \geq Z_{[0,cn]}$. But from the very equation satisfied by $\tilde{Y}_{1,[0,n]}$ and $\tilde{Z}_{1,[0,n]}$, $\tilde{Y}_{1,[0,n]} \leq Y_{[0,cn]}$ and $\tilde{Z}_{1,[0,n]} \leq Z_{[0,cn]}$, and so $\tilde{Y}_{1,[0,n]} = Y_{[0,cn]}$ and $\tilde{Z}_{1,[0,n]} = Z_{[0,cn]}$. ■

So at time \mathbf{V}_1 , for all transitions j which do not belong to G_I , no firing has yet taken place, and exactly $X_{[0,cn]}^j$ firings still have to take place.

Lemma 25

$$\lim_n \frac{\mathcal{V}_1}{n} = c\gamma^I E[X_{[0,0]}^{(I)}] = c(b+d) \quad a.s. \quad (65)$$

Proof: Immediate in view of the first-order ergodic theorem on marked graphs, the additive property for total number, of events and the SLLN. ■

⁸Note that the two systems are quite different. For instance, if $\Pi^{t^{(I)}}(R_{[0,n]}^{t^{(I)}}) \geq Z_{[0,cn]}^{t^{(I)}}$, then the delayed system is such that transition $t^{(I)}$ is never blocked because of a lack of tokens in place $\bullet t^{(I)}$ before the last firing of transition $t^{(I)}$, which is not necessarily the case in the non-delayed system with input process $0.N_{[0,cn]}$.

Lemma 26

$$\lim_n \frac{\tilde{\mathcal{X}}_{1,[0,n]}}{n} = \max_{i \neq I} c \gamma^i E[X_{[0,0]}^{(i)}] = c(2b + d) \quad a.s. \quad (66)$$

Proof: Consider the subnetwork F' of the initial FC net F , which is composed of the set of transitions $\mathcal{T} - G_I$ and of the set of places $\mathcal{P} - \mathcal{P}_I$. Network F' is a SI-FC net with less than I input connected MGC's. When considering F with the delays $U_1(k)$, its subnetwork F' has all its exogeneous arrivals taking place at time \mathbf{V}_1 , and we know from Lemma 24 that the total number of events on transition j for this input is equal to $X_{[0,cn]}^j$. Since this is true for all c and n , F' necessarily satisfies (A_1) and (A_2) , and so we are in a position to assume that Theorem 7 holds for this network (induction assumption). Thus

$$\lim_n \frac{\tilde{\mathcal{X}}_{1,[0,n]}}{n} = \lim_n \frac{\mathcal{V}_1}{n} + \max_{i \neq I} c \gamma^i E[X_{[0,0]}^{(i)}] = c(2b + d) \quad a.s.$$

■

Definition of \mathbf{U}_m , $1 < m \leq M$.

Assume that the function $U_p(k)$, $p \leq m$, and the parameters \mathbf{V}_p , and $\tilde{\mathcal{X}}_{p,[0,n]}$, $p \leq m$, are defined for some $1 \leq m < M$, and are such that

$$\lim_n \frac{V_p}{n} = cp(b + d) \quad a.s. \quad (67)$$

and

$$\lim_n \frac{\tilde{\mathcal{X}}_{p,[0,n]}}{n} = c(p(b + d) + b) \quad a.s. \quad (68)$$

for all $p = 1, \dots, m$, so that the event

$$\mathcal{F}_{m,n} = \mathcal{E}_n \cap \left\{ \tilde{\mathcal{X}}_{p-1,[0,n]} < \mathbf{V}_p, \forall p \leq m \right\} \quad (69)$$

tends to Ω a.s. when n goes to ∞ . This holds for $m = 1$ when taking $\tilde{\mathcal{X}}_{0,[0,n]} = cb$.

We define the same quantities for $m + 1$, on the event $\mathcal{F}_{m,n}$, as follows:

For $j \in G_I$, we take

$$U_{m+1}^j(k) = X_{[0,(m+1)cn]}^j, \quad \forall k. \quad (70)$$

Since we are on \mathcal{E}_n , the total number of events of $j \in G_I$ in this delayed network is exactly $X_{[0,(m+1)cn]}^{(I)}$, and the associated daters $\tilde{\mathcal{X}}_{m+1,[0,n]}^j$, $j \in G_I$ are not affected by the behavior of the rest of the network. So we can define

$$\mathbf{V}_{m+1} = \max_{j \in G_I} \tilde{\mathcal{X}}_{m+1,[0,n]}^j (X_{[0,(m+1)cn]}^j) + \sigma^j. \quad (71)$$

By the same arguments as in the case $m = 1$, we prove that:

$$\lim_n \frac{\mathcal{V}_{m+1}}{n} = (m+1)c\gamma^I E[X_{[0,0]}^{(I)}] = (m+1)c(b+d) \quad a.s.$$

Since $d > 0$, this, Equation (68) and the SLLN allow us to prove that $\lim_n \mathcal{F}_{m+1,n} = \Omega$ a.s.

For $j \notin G_I$, we take

$$U_{m+1}^j(k) = \begin{cases} U_m^j(k) & \text{for } 0 \leq k < \mathbf{V}_{m+1}; \\ X_{[0,(m+1)cn]}^j & \text{for } \mathbf{V}_{m+1} \leq k. \end{cases} \quad (72)$$

Let $\tilde{\mathcal{X}}_{m+1,[0,n]}^j$ and $\tilde{\mathcal{X}}_{m+1,[0,n]}^j$ respectively denote the daters and counters in this delayed network, and let $\tilde{\mathcal{X}}_{m+1,[0,n]}^j$ be its maximal dater.

By the same arguments as those in the proof of Lemma 24, we show that on $\mathcal{F}_{m,n}$, the total number of events to take place on transition j from time \mathbf{V}_{m+1} on is exactly $X_{[mcn+1,(m+1)cn]}^j$. This, plus arguments similar to those in the proof of Lemma 26 allow us to show that

$$\lim_n \frac{\tilde{\mathcal{X}}_{m+1,[0,n]}^j}{nc} = (m+1)(b+d) + b \quad a.s. \quad (73)$$

Definition of \mathbf{U}_{M+1} .

The last delay functions are defined on the event $\mathcal{F}_n = \mathcal{F}_{M,n}$. Let

$$\mathbf{W}_n = \tilde{\mathcal{X}}_{M,[0,n]} = \max_{j \notin G_I} \tilde{\mathcal{X}}_{M,[0,n]}^j(X_{[0,Mcn]}^j) + \sigma^j. \quad (74)$$

For all j , we take

$$U_{M+1}^j(k) = \begin{cases} U_M^j(k) & \text{for } 0 \leq k < \mathbf{W}_n; \\ X_{[0,n]}^j & \text{for } \mathbf{W}_n \leq k. \end{cases} \quad (75)$$

By the same arguments as above, on the event $\mathcal{F}_{M,n}$

$$\lim_n \frac{\mathbf{W}_n}{nc} = M(b+d) + b \quad a.s. \quad (76)$$

Proof of Theorem 7 From Lemma 20, adding the delay process $U_{M+1}(k)$ to the first upper-bound network leads to a second upper-bound network, the variables of which will be denoted \bar{X}_n^j and $\bar{\mathcal{X}}_n^j$. From Lemma 20, we have

$$\mathcal{X}_{[0,n]} \leq [0,n]. \quad (77)$$

By the same argument as in Lemma 22, at time \mathbf{W}_n , the serial places all have the marking of the reference deadlock, whereas for all $t \in \mathcal{B}$, the routing place $p = \bullet t$ has a marking

equal to $R_{[Mc_{n+1}, n]}^t$. So, from time \mathbf{W}_n on, the U_{M+1} -delayed network coincides with the network with arrival point process $\mathbf{W}_n + 0.N_{[c_{n+1}, n]}$. So in view of homogeneity

$${}_{[0, n]} = \mathbf{W}_n + \mathcal{X}_{[c_{n+1}, n]}. \quad (78)$$

From (77) and (78), we finally obtain the relation:

$$\mathcal{X}_{[0, n]} 1_{\mathcal{F}_n} \leq (\mathbf{W}_n + \mathcal{X}_{[Mc_{n+1}, n]}) 1_{\mathcal{F}_n}. \quad (79)$$

Since the sequence $1_{\mathcal{F}_n}$ tends to 1 with coupling (i.e. from some finite random number on, it is constant and equal to 1), by dividing both sides by n and letting n go to ∞ , we obtain from Equation (76) that

$$\Gamma(0) \leq Mc(b+d) + bc + (1-Mc)\Gamma(0).$$

This in turn implies

$$\Gamma(0) \leq \frac{M(b+d)+b}{M}.$$

Since this holds for all d and since M can be taken arbitrarily large, we finally obtain that $\Gamma(0) \leq b$. \blacksquare

6.4 Multiple Stationary Regimes

The marking process may admit several stationary regimes. Here is a simple example. Consider the FCnet of Figure 6. The routing in place p_2 is the periodic sequence: $(t_2, t_3, t_2, t_3, \dots)$. In the separated version, each external token is first consumed by t_1 , then by t_2 , then once more by t_1 , and after that it leaves the network via t_3 . The arrival epochs are at time $0, 1, 2, \dots$, and each arrival brings a single token to place p_1 . The firing times $\sigma_1, \sigma_2, \sigma_3$ are rational, with $\sigma_3 = 0$ and $2\sigma_1 + \sigma_2 = 1$.

This is clearly a separable model which satisfies all the assumptions of the previous section. Let us call *workload* at time u in place p_1 or p_2 the time for the downstream transitions to clear the tokens present in this place at time u . Then $\forall d$, such that $0 \leq d < \sigma_1$, we have the following stationary regime: if the initial workload at time $0-$ is $(d, 0)$ (i.e. transition t_1 can only start serving the first token at time d), then we will have the same workload at time $1-$.

7 Generalization to Variable Firing Times

Variable Firing Times We now consider the case when firing times are still integer-valued and bounded, but variable with time. Let $\sigma^t(m)$ be the firing time of the m -th firing of transition t . Let $\zeta^t(k)$ be the minimum of M and the time which elapsed since the last time t has started firing before time k . If we consider the variables to be left-continuous, we have:

Lemma 27

$$Y(k) = \bigoplus_{l=1}^M (A_l(k) \otimes Y(k-l) \oplus B_l(k) \otimes Z(k-l)) \quad (80)$$

$$Z(k) = \Pi \left(\sum_{l=1}^M (P_l(k) \times Z(k-l) + Q_l(k) \times Y(k-l)) + R(k) \right), \quad (81)$$

with $A_l(k)(t, t') = c$, the number of tokens in the initial marking of the place between t' and t if $l = \zeta^t(k) \wedge \sigma^{t'}(X^{t'}(k-1))$, ∞ otherwise (with a similar definition for B) and $P_l(k)(t, t') = 1$, if there is a place between t' and t and $l = \zeta^t(k) \wedge \sigma^{t'}(X^{t'}(k-1))$, 0 otherwise (with a similar definition for Q).

Finally, one can notice that a system with bounded stochastic firing times with integer values can be transformed into a system with constant firing times by replacing every timed transition with a random firing time by a subnetwork as in figure 7.

Indeed, let $\sigma(n)$ be an integer-valued random process bounded by M . Then, the distribution of $\sigma(n)$ is determined by the sequence $P_1(n), \dots, P_M(n)$, with $\mathbf{P}(\phi(n) = i) = P_i(n)$.

Note that if the original net was a SI-FCNet, the net obtained after transformation is a MI-FCNet. However this particular MI-FCNet falls within the separable and monotone framework because it verifies assumptions (A_1) , (A_2) and (A_3) if a similar net with constant firing times does so. For see this, just apply Lemma 9 to transitions t_1 , t_2 and t_3 which form a MI-MGC.

However, one should note that the computation of γ_0 has to be different for *MI-FCNets*.

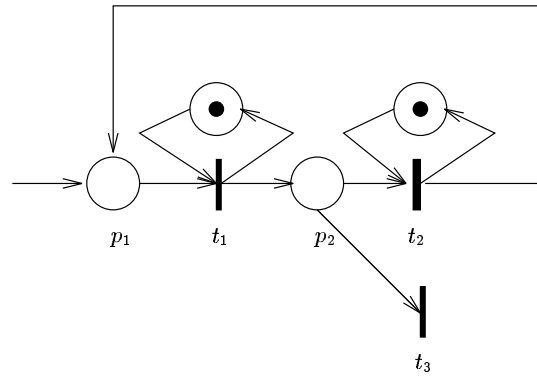


Figure 6: An example of a SI separable network with multiple stationary regimes.

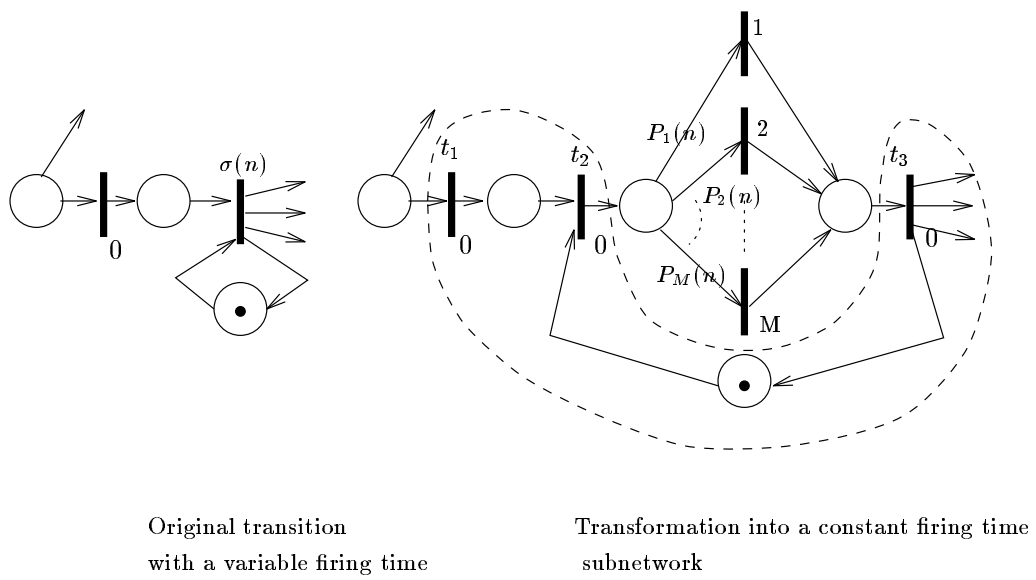


Figure 7: Transformation of a random firing time transition into a subnetwork composed with constant firing time transitions

Appendix

Appendix 1: Separable-Monotone Framework for Counters and Daters

The framework summarized in this section is that of [4]. We consider a discrete event system composed of a set of nodes, and submitted to an input point process $N = \{T_n\}_{n \in \mathbb{Z}}$. Let $c.N$ be the c -dilation of N , namely the point process with arrivals $c.N = \{cT_n\}_{n \in \mathbb{Z}}$. Let $N_{[m,n]}$ be the $[m, n]$ -restriction of the point process N , namely the point process $\{T_l\}_{m \leq l \leq n}$. We will say that $N_{[m,n]} \leq N'_{[m,n]}$ if $T_l \leq T'_l$ for all $m \leq l \leq n$. In what follows, this point process will also be characterized through its counting measure $R_{[m,n]} : \mathbb{R} \rightarrow \mathbb{N}$, where $R_{[m,n]}(t)$ counts the number of points of $N_{[m,n]}$ which are less than t .

The discrete event system is characterized by two equivalent sets of variables:

- The daters: $\mathcal{X}_{[m,n]}^i(k) \in \mathbb{R}$ will denote the epoch of the k -th event on node i , when the system is submitted to $N_{[m,n]}$ (here, we take $k \in \mathbb{N}$ and $\mathcal{X}_{[m,n]}^i(k) = \infty$ if there are less than k events on node i).
- The counters: $X_{[m,n]}^i(t) \in \mathbb{N}$ will denote the number of events which take place on node i before time t (we will take this function left-continuous).

Note that counters and daters are related by

$$X_{[m,n]}^i(t) = \sum_{k \in \mathbb{N}} 1_{\{\mathcal{X}_{[m,n]}^i(k) \leq t\}}. \quad (82)$$

The separable-monotone framework consists of the following set of assumptions:

External Monotonicity If $N_{[m,n]} \leq N'_{[m,n]}$, then for all k and i (with obvious notations), $\mathcal{X}_{[m,n]}^i(k) \leq \{\mathcal{X}'\}_{[m,n]}^i(k)$, which is equivalent to the property that for all t and i $X_{[m,n]}^i(t) \geq \{X'\}_{[m,n]}^i(t)$.

Conservation Let

$$X_{[m,n]}^i \equiv \lim_{t \rightarrow \infty} X_{[m,n]}^i(t). \quad (83)$$

This limit exists since the function is non-decreasing. In words, $X_{[m,n]}^i$ counts the *total* number of events on node i for $N_{[m,n]}$. We assume that $X_{[m,n]}^i$ is *finite* and *independent* of the values taken by the variables T_l , $n \leq l \leq m$ (provided m, n and $\{T_l\}$ are finite of course). Of particular interest to us will be the *maximal dater* defined by:

$$\mathcal{X}_{[m,n]} = \max_i \mathcal{X}_{[m,n]}^i(X_{[m,n]}^i). \quad (84)$$

Separability The separability assumption states that if $T_{l+1} \geq \mathcal{X}_{[m,l]} + M$, for some non-negative M , then

$$\begin{aligned} \mathcal{X}_{[m,n]}^i(k) &= \mathcal{X}_{[m,l]}^i(k), & k \leq X^i[m,l] \\ \mathcal{X}_{[m,n]}^i(k + X^i[m,l]) &= \mathcal{X}_{[l+1,m]}^i(k), & k \geq 1 \end{aligned} \quad (85)$$

or equivalently

$$\begin{aligned} X_{[m,n]}^i(t) &= X_{[m,l]}^i(t), & t < T_{l+1} \\ X_{[m,n]}^i(t) &= X_{[m,l]}^i + X_{[l+1,m]}^i(t), & t \geq T_{l+1}. \end{aligned} \quad (86)$$

It is easy to check that the separation and the conservation properties imply that for all $m \leq l < n$, $X_{[m,n]}^i = X_{[m,l]}^i + X_{[l+1,n]}^i$ regardless of $\{T_l\}$.

Homogeneity The homogeneity assumption states that if $T'_l = T_l + c$, then $\{\mathcal{X}'\}_{[m,n]}^i(k) = \mathcal{X}_{[m,n]}^i(k) + c$ for all k and i or equivalently that $\{X'\}_{[m,n]}^i(t+c) = X_{[m,n]}^i(t)$ for all t and i .

Appendix 2: A Branching Process Interpretation

This appendix focuses on a multitype branching process interpretation of the equations of Lemma 19. The assumptions are that the net is SI. For all input transition t of a MGC, let O^t be the set

$$O^t = \{q \in \mathcal{R} \mid \exists t' \in G_i \text{ s.t. } q = t'^{\bullet}\},$$

where q is counted with multiplicity n if there are n arcs going from G_i to q . We will then say that q is an *offspring* of t with multiplicity n .

We now describe the dynamics of a pseudo marking process (this marking process is different from the one in the real system) on the set of places of \mathcal{R} , which is driven by the routing functions only. Fix an arbitrary priority order on the places of \mathcal{R} : p_j has priority over p_{j+1} etc. Assume that the jump of R at T_0 brings $m^{\bullet t}$ tokens to place $\bullet t$, for all $t \in \mathcal{B}$. If $m^{p_1} > 0$, one token of p_1 is *moved* following the routing decision $t = \nu_{[0,0]}^{p_1}(1)$. Let n^{p_i} be the multiplicity of the offspring p_i of t (this multiplicity is zero if p_i does not belong to O^t). By definition, such a move leads to the new marking defined by $m^{p_i} + n^{p_i}$ for all $i > 1$ and $m^{p_1} + n^{p_1} - 1$ for p_1 . If the new marking of p_1 is still positive, we move one token of p_1 as above, but according to the routing decision $\nu_{[0,0]}^{p_1}(2)$, which leads to a new marking; the procedure is repeated up to the time when no tokens are left in p_1 (this may never happen in which case this first step of the procedure never stops). We then move one token of type p_2 according to the routing decision $\nu_{[0,0]}^{p_2}(1)$, provided there is at least one token in place p_2 in the last obtained marking. This may possibly create new tokens of type p_1 . The general rule is actually to move a token with highest priority at each step, according to the residual routing decisions. The procedure stops whenever there are no tokens left in the routing places.

Lemma 28 *Under the foregoing assumptions, (A_1) (and therefore (A_2)) is satisfied if and only if the above procedure stops after an almost surely finite number of steps.*

The proof is omitted. It is based on a generalization of the Euler property for directed graphs called the Euler-Ordered property, which is introduced in [3]. Note that if the above stopping property holds for this specific ordering of the moves, it will hold for any other ordering.

In the particular case of i.i.d. routing decisions, independent on different nodes, one can naturally associate a multitype branching process with the set \mathcal{R} by saying that an individual of type p has a set of offspring O^t with probability $P = P(\nu^p = t)$. Properties (A_1) (and (A_2)) will then a.s. hold whenever this multitype branching process is subcritical (namely whenever its population dies out a.s. for all finite initial conditions). This property boils down to checking that the maximal eigenvalue of the branching matrix is strictly less than 1 ([9]).

Future Research The approach presented here can be extended in various ways. Here is the list of the most obvious extensions that we intend to investigate in the future:

- Extension of the results to non-integer valued, random, unbounded firing times;
- Computation of $\Gamma(0)$ in the MI-separable case;
- Extension to the autonomous case (without input process);
- Extension to weighed nets.

References

- [1] Baccelli F., Cohen G., Gaujal B., *Recursive Equations and Basic Properties of Timed Petri Nets*, DEDS: Theory and Applications, 1(4)415-439, 1992.
- [2] Baccelli F., Cohen G., Olsder G.J, Quadrat J.P. *Synchronization and Linearity*, Wiley 1992.
- [3] Baccelli F., Foss S., *Stability of Jackson-type Queueing Networks, I*, INRIA Report n. 1845 (1993), To appear in QUESTA, 1994.
- [4] —————, *On the Saturation Rule for the Stability of Queues*, INRIA Report n. 2015, (1993), to appear in J.A.P., 1995.
- [5] Commoner F., *Deadlocks in Petri Nets*, Applied Data Research, CA-7606-2311, 1972.
- [6] Dumas V., *Stabilité des réseaux de Jackson* Mémoire de stage de DEA, Département de mathématiques appliquées, Ecole Polytechnique, Palaiseau, France, 1992.
- [7] Esparza L., Silva M., *On the Analysis and Synthesis of Free Choice Systems*, G. Rozenberg editor, Advances in Petri Nets, Vol. 483 of LNCS, 243-286, 1990.

- [8] Gaujal B., *Parallélisme et simulation des systèmes à événements discrets*, Thèse de Doctorat, Université de Nice Sophia-Antipolis, 1994.
- [9] Harris T.E., *The Theory of Branching Processes*, Springer, Berlin, 1969.
- [10] Murata, T., *Petri nets: Properties, analysis and applications*, Proceedings of the IEEE, 77:541-580, 1989.
- [11] Thiagarajan P.S., Voss K., *A Fresh Look at Free Choice Nets*, Information and Control, (62)85-113, 1984.



Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irista, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

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