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Silvia C. Di Marco, Roberto L.V. Gonzalez

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*A numerical procedure for  
minimizing the maximum cost*

Silvia C. DI MARCO - Roberto L.V. GONZALEZ

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A NUMERICAL PROCEDURE FOR  
MINIMIZING THE MAXIMUM COST

UNE PROCÉDURE NUMÉRIQUE POUR LA  
MINIMISATION DU COÛT MAXIMUM

Silvia C. Di Marco \*    Roberto L.V. González \*

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\*CONICET - Dpto. Matemática, Fac. Cs. Ex., Ing. y Agr., Universidad Nacional de Rosario - Avda. Pellegrini 250 - 2000 Rosario - Argentine. This paper is included in the activities developed in the frame of the Cooperation Project INRIA - Instituto de Matemática Beppo Levi, Coordinators of the project: E. Rofman - R. González

## Abstract

In this paper we consider the numerical solution of the Bellman quasi-variational inequality (QVI) associated to the problem that consists in finding the minimum of the maximum of a scalar functional on a trajectory. We present an approximation method for the numerical solution of the QVI which employs both discretization on time and on spatial variables. In this way, we obtain a fully discrete problem that has an unique solution. We give an optimal estimate for the error between the approximated solution and the optimal cost function of the original problem. Also, numerical examples are presented.

## Résumé

Nous étudions ici la solution numérique d'une inéquation quasi-variationnelle associée à la minimisation du maximum d'une fonctionnelle définie sur la trajectoire d'un système dynamique gouverné par une équation différentiel ordinaire. Nous faisons la présentation d'une méthode d'approximation qui emploie des discretisations en espace et en temps. Nous obtenons des estimations optimales pour la vitesse de convergence des solutions approchées vers le coût optimal du problème originel.

**Keywords:** *minimax problems, quasi-variational inequalities, Bellman equation, viscosity solution, numerical solution.*

**AMS Classification:** Primary: 49D25, 49A40      Secondary: 49C05, 49C20

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# 1 Introduction and description of the problem

## 1.1 Description of the problem

We consider in the interval  $[0, T]$  a dynamic system which evolves according to the ordinary differential equation

$$\frac{dy}{ds}(s) = g(y(s), \alpha(s)) \quad 0 \leq t \leq s \leq T, \quad (1)$$

with initial condition

$$y(t) = x \in \Omega \subseteq \mathfrak{R}^n, \Omega \text{ an open domain.} \quad (2)$$

The optimal control problem consists in minimizing the functional  $J$

$$\begin{aligned} J : [t, T] \times \Omega \times \mathcal{U}(t, T) &\mapsto \mathfrak{R} \\ (t, x, \alpha(\cdot)) &\mapsto J(t, x, \alpha(\cdot)) = \text{ess sup} \{f(y(s), \alpha(s)) : s \in [t, T]\}. \end{aligned} \quad (3)$$

The set of controls is denoted by  $\mathcal{U}(t, s)$ ,

$$\mathcal{U}(t, s) = \{\alpha : [t, s] \mapsto A \subset \mathfrak{R}^m : \alpha(\cdot) \text{ measurable}\}.$$

This problem arises, for example, when we want to minimize the maximum deviation of the controlled trajectories with respect to a given special trajectory. This differs from those problems usually considered in the optimal control literature, where an accumulated cost is minimized. As considering an accumulated cost is not always the best method to qualify a controlled system with an unique scalar parameter, problems of this type have received considerable interest in recent publications (see e.g. [2]).

The objective of this work is to approximate in a numerical way the value function  $u$

$$\begin{aligned} u : [0, T] \times \Omega &\mapsto \mathfrak{R} \\ (t, x) &\mapsto u(t, x) = \inf \{J(t, x, \alpha(\cdot)) : \alpha \in \mathcal{U}(t, T)\}. \end{aligned} \quad (4)$$

The principal results are the following ones:

In §2, we obtain a discrete time approximation using a finite differences scheme and we give an estimate of the error of this approximation.

In §3, by using linear finite elements, we obtain a fully discrete approximation that converges to the solution of the original problem with rate  $\sqrt{k}$ .

In §4, we generalize the previous results, for the case where the cost functional includes an integral cost.

In §5, we show the optimality of the estimation  $\sqrt{k}$ .

In §6, we present some numerical applications with computational results.

## 1.2 The value function

### 1.2.1 Technical assumptions

Let  $BUC(\Omega \times A)$  be the set of the bounded and uniformly continuous functions on  $\Omega \times A$ .

We assume that  $f$  and  $g$  satisfy the following hypotheses:

A<sub>1</sub>)  $g : \Omega \times A \mapsto \mathfrak{R}^r$ ,  $g \in BUC(\Omega \times A)$

$$\|g(x, a)\| \leq M_g, \|g(x, a) - g(\hat{x}, a)\| \leq L_g \|x - \hat{x}\|, \forall x, \hat{x} \in \Omega, \forall a \in A \quad (5)$$

A<sub>2</sub>)  $f : \Omega \times A \mapsto \mathfrak{R}$ ,  $f \in BUC(\Omega \times A)$

$$0 \leq f(x, a) \leq M_f, |f(x, a) - f(\hat{x}, a)| \leq L_f \|x - \hat{x}\|, \forall x, \hat{x} \in \Omega, \forall a \in A \quad (6)$$

A<sub>3</sub>)  $A$  is compact

A<sub>4</sub>) The trajectory  $y(\cdot)$  remains in  $\Omega$ , for any control that belongs to  $\mathcal{U}(t, T)$ .

### 1.2.2 Properties of the value function

The following properties have been established by Barron-Ishii in [2]:

- $u \in BUC([0, T] \times \Omega)$ , moreover, the function  $u$  is Lipschitz continuous in its second variable with Lipschitz constant  $L_u$

$$L_u = L_f \exp(L_g T). \quad (7)$$

- The value function  $u$  satisfies the following dynamic programming principle  $\forall t \in [0, T)$ ,  $x \in \Omega$

$$u(t, x) = \inf \left\{ \max \left\{ \text{ess sup}_{\tau \in [t, s]} f(y(\tau), \alpha(\tau)), u(s, y(s)) \right\} : \alpha \in \mathcal{U}(t, s) \right\}, \quad (8)$$

with final condition

$$u(T, x) = \min \{ f(x, a) : a \in A \}. \quad (9)$$

- The function  $u$  is the unique viscosity solution of the following quasi-variational inequality,  $t \in [0, T)$ ,  $x \in \Omega$ ,

$$\max \{ u_t(t, x) + H(t, x, u(t, x), \nabla_x u(t, x)), \min_{a \in A} f(x, a) - u(t, x) \} = 0, \quad (10)$$

where

$$H(t, x, r, q) = \min \{ q \cdot g(x, z) : z \in A / f(x, z) \leq r \}.$$



## 2 A discrete time scheme of approximation

### 2.1 The discrete time problem

Here we introduce an auxiliary problem that is a natural discretization of the optimal cost  $u$  defined in (4). We divide the interval  $[0, T]$  into  $\mu$  subintervals with common length  $h = T/\mu$ . Let us define, for every  $n = 0, \dots, \mu$ ,

$$\mathcal{U}^h(n) = \{\alpha(\cdot) \in \mathcal{U}(nh, T) : \alpha \text{ is constant in } [kh, (k+1)h), k = n, \dots, \mu\} \quad (11)$$

$$J^h(n, x, \alpha^h(\cdot)) = \max_{k=n, \dots, \mu} f(y^h(k), \alpha^h(kh)) \quad (12)$$

$$u^h(n, x) = \min_{\alpha^h \in \mathcal{U}^h(n)} J^h(n, x, \alpha^h(\cdot)) \quad (13)$$

where  $y^h$  is a discrete time process. It is recursively defined,  $\forall k = n, \dots, \mu - 1$ , by

$$\begin{cases} y^h(k+1) = y^h(k) + hg(y^h(k), \alpha^h(kh)) \\ y^h(n) = x \end{cases} \quad (14)$$

**Remark 2.1** The process  $y^h$  is the Euler discretization of the continuous system (1). In (13) the minimum exists because  $\mathcal{U}^h(n)$  is compact in  $A^{\mu-n}$  and  $J^h$  is a continuous mapping from  $\mathcal{U}(n)$  to  $\mathfrak{R}$  by virtue of hypotheses  $A_1$ - $A_3$ .

### 2.2 Properties of the function $u^h$

In the following lemma we will establish the dynamic programming principle verified by the discrete time cost. It gives a recursive way to compute the function  $u^h$ . The proof follows classical lines and we omit it for the sake of brevity.

**Lemma 2.1** For every  $n = 0, \dots, \mu - 1$ ,  $u^h$  verifies the recursive relation

$$u^h(n, x) = \min_{a \in A} \{ \max\{ f(x, a), u^h(n+1, x + hg(x, a)) \} \}, \quad (15)$$

and the final condition

$$u^h(\mu, x) = \min_{a \in A} f(x, a). \quad (16)$$

It is easy to prove –using well known techniques as those that can be seen in [11]– that the function  $u^h$  is bounded and Lipschitz continuous in its second variable with Lipschitz constant  $L_u$ . These results are announced in the following lemma.

**Lemma 2.2** For every  $n = 0, \dots, \mu$ ,  $x$  and  $\hat{x} \in \Omega$ , it is valid that

$$0 \leq u^h(n, x) \leq M_f, \quad (17)$$

$$|u^h(n, x) - u^h(n, \hat{x})| \leq L_u \|x - \hat{x}\|. \quad (18)$$

### 2.3 Approximation of controls with step functions

To compute the discrete time optimal cost function, in (13), we optimize the functional  $J^h$  on the set  $\mathcal{U}^h(n)$ , whose elements are step functions. To prove the convergence of  $u^h$  to  $u$ , we need to establish relations between controls of  $\mathcal{U}(nh, T)$  and  $\mathcal{U}^h(n)$ .

Since the relation  $\mathcal{U}^h(n) \subseteq \mathcal{U}(nh, T)$ , is obvious, we have to prove that  $\mathcal{U}^h(n)$  approximates the set  $\mathcal{U}(nh, T)$  in the following sense

$$\overline{\bigcup_{\mu=1}^{\infty} \mathcal{U}^h(0)} = \mathcal{U}(0, T),$$

where the closure is taken in the weak\* topology of  $L^\infty(0, T)$ .

For any  $\mu_1$ , we take  $h_1 = T/\mu_1$  and we divide the interval  $[0, T]$ , into  $\mu_1$  subintervals  $I_i$ , where  $I_i = [t_i, t_{i+1})$ ,  $t_i = ih_1$ , and  $i = 0, \dots, \mu_1 - 1$ .

Given  $\alpha(\cdot) \in \mathcal{U}(0, T)$ , we transform it by eliminating (in zero measure sets) those values of  $\alpha(\cdot)$  such that  $f(y(t_i), \alpha(s))$  is greater than the  $\text{ess sup } \{f(y(t_i), \alpha(t)) : t \in (0, T)\}$ ; in other words, we consider the operator

$$Q : \mathcal{U}(0, T) \mapsto \mathcal{U}(0, T)$$

with the following properties:

$$\begin{cases} Q(\alpha(\cdot))(s) = \alpha(s) & \text{in a.e. } s \in I_i \\ \text{ess sup}_{s \in I_i} f(y(t_i), \alpha(s)) = \sup_{s \in I_i} f(y(t_i), Q(\alpha(\cdot))(s)). \end{cases} \quad (19)$$

**Remark 2.2** The operator  $Q$  depends on the chosen partition of  $[0, T]$ , but we omit that in the writing to simplify the notation.

**Remark 2.3** It is clear that  $\alpha$  and  $Q(\alpha)$  have the same action on the system (1), in the sense that

$$\int_{t_i}^{t_{i+1}} g(y(t_i), Q(\alpha(\cdot))(s)) ds = \int_{t_i}^{t_{i+1}} g(y(t_i), \alpha(s)) ds. \quad (20)$$

**Definition 2.1** Let  $A(\alpha, i) := \{Q(\alpha(\cdot))(s) : s \in I_i\}$  and  $\Gamma(\alpha, i) := \overline{A(\alpha, i)}$  its closure. Clearly, by hypothesis  $A_3$ ,  $\Gamma(\alpha, i)$  is compact.

**Lemma 2.3** For every  $i = 0, \dots, \mu_1 - 1$ , there exists a step function  $\alpha_\omega$  that takes at most  $r + 1$  constant values in  $\Gamma(\alpha, i)$ , on  $I_i$ , such that

$$\int_{t_i}^{t_{i+1}} g(y(t_i), \alpha(s)) ds = \int_{t_i}^{t_{i+1}} g(y(t_i), \alpha_\omega(s)) ds, \quad (21)$$

$$\max_{s \in I_i} f(y(t_i), \alpha_\omega(s)) \leq \sup_{s \in I_i} f(y(t_i), \alpha(s)). \quad (22)$$

**Proof:** As  $\Gamma(\alpha, i)$  is compact, using the results of Proposition 4.1 in [13], it is possible to construct a function  $\alpha_\omega$  such that,

$$\int_{t_i}^{t_{i+1}} g(y(t_i), Q(\alpha(\cdot))(s)) ds = \int_{t_i}^{t_{i+1}} g(y(t_i), \alpha_\omega(s)) ds. \quad (23)$$

By definition of  $Q(\alpha)$ , we have

$$\int_{t_i}^{t_{i+1}} g(y(t_i), Q(\alpha(\cdot))(s)) ds = \int_{t_i}^{t_{i+1}} g(y(t_i), \alpha(s)) ds. \quad (24)$$

From (23) and (24), we get (21).

To prove (22), let us note that if  $t \in I_i$ , then  $Q(\alpha(\cdot))(t) \in \Gamma(\alpha, i)$ . Therefore,

$$\max_{s \in I_i} f(y(t_i), Q(\alpha(\cdot))(s)) \leq \sup_{a \in \Gamma(\alpha, i)} f(y(t_i), a).$$

In addition, since  $\Gamma(\alpha, i) = \overline{A(\alpha, i)}$ , we have

$$\max_{\Gamma(\alpha, i)} f(y(t_i), \cdot) = \sup\{f(y(t_i), a) : a \in A(\alpha, i)\}.$$

From definition of  $A(\alpha, i)$ , it results

$$\sup_{A(\alpha, i)} f(y(t_i), \cdot) = \sup_{s \in I_i} f(y(t_i), Q(\alpha(\cdot))(s)).$$

Taking into account (19), it follows that

$$\max_{s \in I_i} f(y(t_i), Q(\alpha(\cdot))(s)) \leq \sup_{s \in I_i} f(y(t_i), Q(\alpha(\cdot))(s)) = \text{ess sup}_{s \in I_i} f(y(t_i), \alpha(s)).$$

□

**Remark 2.4** By virtue of Lemma 2.3, we can define an operator  $\mathcal{E}$

$$\mathcal{E} : \mathcal{U}(0, T) \mapsto \mathcal{U}(0, T)$$

such that  $\mathcal{E}(\alpha)$  is a step function that verifies (21) and (22). To simplify the writing  $\alpha_w$  denotes the image of such mapping.

**Note 2.1** To simplify the notation and the exposition,  $C$ ,  $M$  and  $K$  denote any generic constant whose value depends on the context where it appears. Such constants only depend on the functions  $f$  and  $g$  of the problem but they are independent of the parameters  $h$ ,  $k$ ,  $\rho$  of discretization.

The following lemma gives an estimate for the difference between the original trajectory of the system and the system's response corresponding to the step control  $\alpha_w(\cdot)$ .

**Lemma 2.4** Let  $y(\cdot)$  be the response to the control  $\alpha(\cdot)$  and  $y_w(\cdot)$ , the response to the control  $\alpha_w(\cdot)$ , then

$$\|y(t) - y_w(t)\| \leq M h_1, \quad (25)$$

where

$$M = M_g(2 + \exp(L_g t)). \quad (26)$$

**Proof:** We define,  $\forall i = 0, \dots, \mu - 1$ ,

$$E_i = \|y(t_i) - y_w(t_i)\|.$$

In consequence, it is valid that

$$E_{i+1} \leq E_i + \left\| \int_{t_i}^{t_{i+1}} (g(y(s), \alpha(s)) - g(y_w(s), \alpha_w(s))) ds \right\|. \quad (27)$$

To estimate the second term of (27), we write

$$\begin{aligned} & \left\| \int_{t_i}^{t_{i+1}} (g(y(s), \alpha(s)) - g(y_w(s), \alpha_w(s))) ds \right\| \\ & \leq \left\| \int_{t_i}^{t_{i+1}} (g(y(s), \alpha(s)) - g(y(t_i), \alpha(s))) ds \right\| \\ & \quad + \left\| \int_{t_i}^{t_{i+1}} (g(y(t_i), \alpha(s)) - g(y(t_i), \alpha_w(s))) ds \right\| \end{aligned}$$

$$\begin{aligned}
 & + \left\| \int_{t_i}^{t_{i+1}} (g(y(t_i), \alpha_w(s)) - g(y_w(t_i), \alpha_w(s))) ds \right\| \\
 & + \left\| \int_{t_i}^{t_{i+1}} (g(y_w(t_i), \alpha_w(s)) - g(y_w(s), \alpha_w(s))) ds \right\|.
 \end{aligned} \tag{28}$$

From (1) and (5), we have

$$\|y(s) - y(t_i)\| = \left\| \int_{t_i}^s g(y(\theta), \alpha(\theta)) d\theta \right\| \leq M_g(s - t_i). \tag{29}$$

Then, using (5) and (29), we estimate the first term of (28) in the following way:

$$\begin{aligned}
 & \left\| \int_{t_i}^{t_{i+1}} (g(y(s), \alpha(s)) - g(y(t_i), \alpha(s))) ds \right\| \\
 & \leq \int_{t_i}^{t_{i+1}} \|g(y(s), \alpha(s)) - g(y(t_i), \alpha(s))\| ds \\
 & \leq L_g \int_{t_i}^{t_{i+1}} \|y(s) - y(t_i)\| ds \leq L_g M_g \int_{t_i}^{t_{i+1}} (s - t_i) ds \leq L_g M_g \frac{h_1^2}{2}.
 \end{aligned} \tag{30}$$

We can estimate in a similar way the fourth term of (28):

$$\left\| \int_{t_i}^{t_{i+1}} (g(y_w(t_i), \alpha_w(s)) - g(y_w(s), \alpha_w(s))) ds \right\| \leq L_g M_g \frac{h_1^2}{2}. \tag{31}$$

From (21), the second term of (28) is zero, i.e.

$$\left\| \int_{t_i}^{t_{i+1}} (g(y(t_i), \alpha(s)) - g(y(t_i), \alpha_w(s))) ds \right\| = 0. \tag{32}$$

From (5), the third term of (28) can be estimated as follows,

$$\left\| \int_{t_i}^{t_{i+1}} (g(y(t_i), \alpha_w(s)) - g(y_w(t_i), \alpha_w(s))) ds \right\|$$

$$\begin{aligned}
&\leq \int_{t_i}^{t_{i+1}} \|g(y(t_i), \alpha_w(s)) - g(y_w(t_i), \alpha_w(s))\| ds \\
&\leq L_g \int_{t_i}^{t_{i+1}} \|y(t_i) - y_w(t_i)\| ds \leq E_i L_g h_1.
\end{aligned} \tag{33}$$

From (27), (28), (30)–(33), we get

$$E_{i+1} \leq E_i(1 + L_g h_1) + L_g M_g h_1^2.$$

Taking into account that  $E_0 = 0$ , by induction we can prove that

$$E_i \leq L_g M_g h_1^2 \sum_{j=0}^{i-1} (1 + L_g h_1)^j.$$

Since

$$\sum_{j=0}^{i-1} (1 + L_g h_1)^j = \frac{(1 + L_g h_1)^i - 1}{(1 + L_g h_1) - 1},$$

we obtain

$$E_i \leq M_g h_1 (1 + L_g h_1)^i.$$

Let  $t \in I_i$ , for some  $i = 0, \dots, \mu_1 - 1$ , we have

$$\|y(t) - y_w(t)\| \leq \|y(t_i) - y_w(t_i)\| + \left\| \int_{t_i}^t (g(y(s), \alpha(s)) - g(y_w(s), \alpha_w(s))) ds \right\|. \tag{34}$$

To estimate (34), we take into account that

$$\left\| \int_{t_i}^t (g(y(s), \alpha(s)) - g(y_w(s), \alpha_w(s))) ds \right\| \leq 2M_g h_1,$$

and thus, we obtain

$$\|y(t) - y_w(t)\| \leq 2M_g h_1 + M_g h_1 (1 + L_g h_1)^i \leq M_g h_1 (2 + \exp(L_g t)).$$

□

## 2.4 Approximation of controls with uniform-step functions

The control  $\alpha_w$  is a step function which has, at most,  $r + 1$  steps in each interval of length  $h_1$ . We are interested in approximating the set  $\mathcal{U}(0, T)$  with uniform-step functions, so we need a suitable element of  $\mathcal{U}^h(0)$ , that approximates  $\alpha_w$  in the sense described below.

### 2.4.1 Definition of the control $\alpha_w^h$

We introduce a parameter  $\nu$  and define

$$h = \frac{h_1}{[\nu]},$$

where  $[s]$  is the integer part of  $s$ .

For any interval  $I_i = [t_i, t_{i+1})$ ,  $t_i = ih_1$ , we know that  $\alpha_w$  takes, at most,  $r + 1$  different values denoted by  $\alpha_{ij}$ ,  $i = 0, \dots, \mu - 1$ ,  $j = 0, \dots, r$ . Accordingly,  $\lambda_{ij}$  denotes the length of the subinterval where  $\alpha_w = \alpha_{ij}$ .

We define

$$\begin{aligned} t_{i0} &= t_i & i &= 0, \dots, \mu_1 - 1 \\ t_{ij} &= t_i + \sum_{k=0}^{j-1} \lambda_{ik} & j &= 1, \dots, r + 1 \\ \tilde{t}_{ij} &= h[t_{ij}/h] & i &= 0, \dots, \mu_1 - 1, j = 0, \dots, r + 1. \\ \alpha_w^h(s) &= \alpha_{ij} & \forall s &\in [\tilde{t}_{ij}, \tilde{t}_{i,j+1}) \end{aligned} \tag{35}$$

Therefore,  $\alpha_w^h = \alpha_w$  in  $I_i$  except at most in  $r + 1$  intervals of length  $\eta_{ij}$ , with  $\eta_{ij} = t_{ij} - \tilde{t}_{ij} \leq h$ . Then,  $\alpha_w^h = \alpha_w$  in  $I_i$  except in a set which measure is smaller than

$$\sum_{j=1}^{r+1} \eta_{ij} \leq (r + 1)h. \tag{36}$$

### 2.4.2 Properties of the control $\alpha_w^h$

The procedure of construction of  $\alpha_w^h$ , implies

$$\alpha_w^h \in \mathcal{U}^h(0),$$

$$\{\alpha_w^h(s), s \in I_i\} = \{\alpha_w(s), s \in I_i\}.$$

By the previous assertion, it is valid that

$$\max_{s \in I_i} f(y(t_i), \alpha_w^h(s)) = \max_{s \in I_i} f(y(t_i), \alpha_w(s)). \tag{37}$$

From (22) and (37), we have

$$\max_{s \in I_i} f(y(t_i), \alpha_w^h(s)) \leq \text{ess sup}_{s \in I_i} f(y(t_i), \alpha(s)). \quad (38)$$

The following lemma gives an estimate for the difference between the original trajectory of the system and the response corresponding to the uniform-step control  $\alpha_w^h(\cdot)$ .

**Lemma 2.5** *Let  $y(\cdot)$  be the response to the control  $\alpha(\cdot)$  and  $y_w^h(\cdot)$  the response to the control  $\alpha_w^h(\cdot)$ , then*

$$\|y(t) - y_w^h(t)\| \leq K\sqrt{h}, \quad (39)$$

where

$$K = M_g \sqrt{\frac{2(r+1)}{L_g}} (4 + 3 \exp(L_g t)). \quad (40)$$

**Proof:** To estimate the difference between  $y(\cdot)$  and  $y_w^h(\cdot)$ , we write

$$\|y(t) - y_w^h(t)\| \leq \|y(t) - y_w(t)\| + \|y_w(t) - y_w^h(t)\|. \quad (41)$$

The first term of (41) was analyzed in lemma 2.4 and has the estimate (25).

To estimate the second term of (41), let  $t \in I_i$ , for some  $i = 0, \dots, \mu_1 - 1$

$$\|y_w(t) - y_w^h(t)\| \leq \|y_w(t_i) - y_w^h(t_i)\| + \left\| \int_{t_i}^t (g(y_w(s), \alpha_w(s)) - g(y_w^h(s), \alpha_w^h(s))) ds \right\|. \quad (42)$$

The second term of (42) is bounded by

$$\left\| \int_{t_i}^t (g(y_w(s), \alpha_w(s)) - g(y_w^h(s), \alpha_w^h(s))) ds \right\| \leq 2M_g h_1. \quad (43)$$

To estimate the first one, we define  $\forall i = 0, \dots, \mu_1 - 1$

$$E_i = \|y_w(t_i) - y_w^h(t_i)\|, \quad (44)$$

then

$$E_{i+1} \leq E_i + \left\| \int_{t_i}^{t_{i+1}} (g(y_w(t_i), \alpha_w(s)) - g(y_w^h(t_i), \alpha_w^h(s))) ds \right\|. \quad (45)$$



To estimate the second term of the right side of (45), we write

$$\begin{aligned} & g(y_w(s), \alpha_w(s)) - g(y_w^h(s), \alpha_w^h(s)) = \\ &= (g(y_w(s), \alpha_w(s)) - g(y_w(t_i), \alpha_w(s))) + (g(y_w(t_i), \alpha_w(s)) - g(y_w^h(t_i), \alpha_w(s))) \\ &+ (g(y_w^h(t_i), \alpha_w(s)) - g(y_w^h(t_i), \alpha_w^h(s))) + (g(y_w^h(t_i), \alpha_w^h(s)) - g(y_w^h(s), \alpha_w^h(s))). \end{aligned}$$

It is clear that the following estimates hold:

$$\left\| \int_{t_i}^{t_{i+1}} (g(y_w(s), \alpha_w(s)) - g(y_w(t_i), \alpha_w(s))) ds \right\| \leq L_g M_g \frac{h_1^2}{2}, \quad (46)$$

$$\left\| \int_{t_i}^{t_{i+1}} (g(y_w^h(t_i), \alpha_w^h(s)) - g(y_w^h(s), \alpha_w^h(s))) ds \right\| \leq L_g M_g \frac{h_1^2}{2}, \quad (47)$$

$$\left\| \int_{t_i}^{t_{i+1}} (g(y_w(t_i), \alpha_w(s)) - g(y_w^h(t_i), \alpha_w(s))) ds \right\| \leq L_g E_i h_1. \quad (48)$$

By (36), the following result holds

$$\left\| \int_{t_i}^{t_{i+1}} (g(y_w^h(t_i), \alpha_w(s)) - g(y_w^h(t_i), \alpha_w^h(s))) ds \right\| \leq 2M_g(r+1)h. \quad (49)$$

In consequence, by substituting (46)–(49) into (45), we have

$$E_{i+1} \leq E_i(1 + L_g h_1) + 2M_g(r+1)h + L_g M_g h_1^2.$$

By induction, and taking into account (44), we obtain

$$E_i \leq (2M_g(r+1)h + L_g M_g h_1^2) \sum_{j=0}^{i-1} (1 + L_g h_1)^j.$$

Since

$$\sum_{j=0}^{i-1} (1 + L_g h_1)^j = \frac{(1 + L_g h_1)^i - 1}{L_g h_1},$$

and

$$(1 + L_g h_1)^i \leq \exp(L_g t),$$

we obtain the estimate

$$E_i \leq (2M_g(r+1)h + L_g M_g h_1^2) \frac{\exp(L_g t)}{L_g h_1}. \quad (50)$$

The minimum corresponding to the right side of (50) is achieved at

$$\nu = \sqrt{\frac{2(r+1)}{h L_g}}. \quad (51)$$

We suppose, for the sake of simplicity, that  $\nu$  is an integer. The general case can be proved similarly with no difficulty.

Consequently, from (50) and (51), it results

$$E_i \leq \exp(L_g t) 2M_g \sqrt{\frac{2(r+1)h}{L_g}}.$$

By using (43), we have

$$\|y_w(t) - y_w^h(t)\| \leq 2M_g \sqrt{\frac{2(r+1)}{L_g}} \exp(L_g t) \sqrt{h} + 2M_g h_1. \quad (52)$$

Finally, from (25), (26) and (52) we have the estimate

$$\|y(t) - y_w^h(t)\| \leq K \sqrt{h},$$

where  $K = M_g \sqrt{\frac{2(r+1)}{L_g}} (4 + 3 \exp(L_g t))$

□

## 2.5 Rate of convergence

In the previous section, for each policy  $\alpha \in \mathcal{U}(0, T)$  and its associated response  $y(\cdot)$ , we have defined an approximating control  $\alpha_w^h \in \mathcal{U}^h(0)$  and we have obtained an estimate for the difference between the corresponding trajectories. In this section, we will give an estimate of the difference between  $u^h$  and  $u$ .

We define, for every  $n = 0, \dots, \mu$ , the following auxiliary function

$$u^e(n, x) = \min_{\alpha^h \in \mathcal{U}^h(n)} J(nh, x, \alpha^h(\cdot)). \quad (53)$$

**Theorem 2.1** Let  $u(0, x)$  be the optimal cost of original problem and  $u^h(0, x)$  the discrete time cost defined in (13), then

$$|u(0, x) - u^h(0, x)| \leq M\sqrt{h}. \quad (54)$$

**Proof:** To estimate the difference between  $u(0, x)$  and  $u^h(0, x)$ , we write

$$|u(0, x) - u^h(0, x)| \leq |u(0, x) - u^e(0, x)| + |u^e(0, x) - u^h(0, x)|. \quad (55)$$

Taking into account that the error associated to Euler's integration method applied to the system (1) is of order  $h$  and following the methodology used in [5, 6], it is possible to prove that

$$|u^e(0, x) - u^h(0, x)| \leq Ch. \quad (56)$$

The proof follows well known techniques and it is here omitted for the sake of brevity.

Let us now estimate the first term of the right side of (55):

1. As  $U^h(0) \subset U(0, T)$ , we have by definition (53)

$$u(0, x) \leq u^e(0, x). \quad (57)$$

2. Conversely, we consider  $\alpha \in U(0, T)$  and  $\alpha_w^h \in U^h(0)$  the control defined in (35), that is associated to  $\alpha$ . Then we have

$$J(0, x, \alpha_w^h) = \text{ess sup}_{s \in [0, T]} f(y_w^h(s), \alpha_w^h(s)). \quad (58)$$

Let  $s \in I_i$ ,

$$\begin{aligned} f(y_w^h(s), \alpha_w^h(s)) &= (f(y_w^h(s), \alpha_w^h(s)) - f(y(s), \alpha_w^h(s))) + \\ &+ (f(y(s), \alpha_w^h(s)) - f(y(t_i), \alpha_w^h(s))) + f(y(t_i), \alpha_w^h(s)), \end{aligned} \quad (59)$$

by (6) and Lemma 2.5, we have

$$|f(y_w^h(s), \alpha_w^h(s)) - f(y(s), \alpha_w^h(s))| \leq C\sqrt{h}, \quad (60)$$

$$|f(y(s), \alpha_w^h(s)) - f(y(t_i), \alpha_w^h(s))| \leq Ch. \quad (61)$$

From (38), we obtain

$$\begin{aligned} \text{ess sup}_{s \in [0, T]} f(y(t_i), \alpha_w^h(s)) &= \max_{i=0, \dots, \mu_1-1} \{ \max_{s \in I_i} f(y(t_i), \alpha_w^h(s)) \} \\ &\leq \max_{i=0, \dots, \mu_1-1} \{ \text{ess sup}_{s \in I_i} f(y(t_i), \alpha(s)) \} \leq J(0, x, \alpha(\cdot)) + L_f M_g h. \end{aligned} \quad (62)$$

Consequently, from (3), (59)–(62), we get

$$J(0, x, \alpha_w^h(\cdot)) \leq J(0, x, \alpha(\cdot)) + M\sqrt{h},$$

therefore

$$u^e(0, x) \leq J(0, x, \alpha_w^h(\cdot)) \leq J(0, x, \alpha(\cdot)) + M\sqrt{h}. \quad (63)$$

Taking the infimum over  $\mathcal{U}$  in the right side of (63), we have

$$u^e(0, x) \leq u(0, x) + M\sqrt{h}. \quad (64)$$

From (57) and (64), we get

$$|u^e(0, x) - u(0, x)| \leq M\sqrt{h}. \quad (65)$$

Finally, from (56) and (65), we obtain

$$|u(0, x) - u^h(0, x)| \leq M\sqrt{h}.$$

□

**Remark 2.5** In the previous proof, we suppose, without losing of generality, that

$$h \leq 1.$$

**Remark 2.6** In a similar way as it was done in Theorem 2.1, we can also prove that,  $\forall n = 0, \dots, \mu$ , the following inequality holds

$$|u(n, x) - u^h(n, x)| \leq M\sqrt{h}.$$

### 3 Fully discrete solutions

In the previous section we have approximated the function  $u$  with one obtained by discretizing the original problem in its time variable. This approximation scheme is not directly implementable to be computed numerically. To obtain a fully discrete approximation with this property, we discretize the space  $\Omega$ , using the methodology described in [14, 15].

#### 3.1 Elements of the discrete problem

We identify the discretization of the spatial variables with the parameter  $k$ , which also indicates the size of the discretization.

##### 3.1.1 Approximation of the domain $\Omega$

We consider a family of quasi-uniform triangulations of  $\Omega$ , which is denoted by  $\{S^k\}_k$  and verify,

- For all  $k$ ,  $S^k$  is a finite collection of closed simplices  $\{S_j^k\}_j$ , contained in  $\overline{\Omega}$ .
- $\bigcup_j S_j^k = \overline{\Omega}_k$  where  $\overline{\Omega}_k$  is a polyhedron such that, as  $k$  goes to 0,  $\Omega_k$  converges to  $\Omega$  in the following sense
  - $\Omega_k \subset \Omega$
  - $\forall Z$  compact,  $Z \subset \Omega$ ,  $\exists \bar{k}(Z)$  such that  $Z \subset \Omega_k \forall k < \bar{k}(Z)$ .
- If  $S_j^k \in S^k$ ,  $S_p^k \in S^k$ ,  $S_j^k \neq S_p^k$ , we have
  - $\text{int}(S_j^k) \cap \text{int}(S_p^k) = \emptyset$ .
  - Either  $S_j^k \cap S_p^k = \emptyset$  or  $S_j^k$  and  $S_p^k$  have in common a whole  $(r - m)$  - edge,  $m = 1, \dots, r$
- $\max_j (\text{diam}(S_j^k)) = k$
- $\exists h_0 > 0$  for which  $x + hg(x, a) \in \Omega_k$ ,  $\forall x \in \Omega_k$ ,  $\forall a \in A$ ,  $\forall h < h_0$ .
- If  $d_j$  is the diameter of the simplex  $S_j^k$ , then  $\exists \chi_1 > 0$ ,  $\exists \chi_2 > 0$  independent on the discretization, such that
  - the simplex  $S_j^k$  has a sphere of radius  $r_j$  in its interior and it results

$$r_j \geq \chi_1 d_j, \quad (66)$$

- for any simplex  $S_j^k$ ,

$$k \leq \chi_2 d_j. \quad (67)$$

Let  $V_k = \{x^i, i = 1, \dots, N\}$  be the vertices of  $\Omega_k$ , arbitrarily arranged and  $N$  its cardinal. Every  $x \in \Omega_k$  is a convex combination of the vertices  $x^i$  of the simplex to which  $x$  belongs. Hence,  $\forall a \in A$  there exists a matrix  $N \times N$ , with components  $\gamma_j(x^i, a)$ , such that:

$$\gamma_j(x^i, a) \geq 0, \quad \forall i, j = 1, \dots, N. \quad (68)$$

$$\sum_{j=1}^N \gamma_j(x^i, a) = 1, \quad \forall i = 1, \dots, N. \quad (69)$$

$$x^i + hg(x^i, a) = \sum_{j=1}^N \gamma_j(x^i, a)x^j, \quad \forall i = 1, \dots, N. \quad (70)$$

### 3.1.2 Definition of the approximate space $F_k$

We consider the set  $W_k$  of functions  $w : \Omega_k \mapsto \mathfrak{R}$ ,  $w \in W^{1,\infty}(\Omega_k)$ , such that  $\partial w / \partial x$  is constant in the interior of each simplex of  $\Omega_k$ , i.e., the functions  $w$  are linear finite elements and they are characterized by their values on  $V_k$ . We denote

$$F_k = (W_k)^{\mu+1}.$$

**Note 3.1** The elements of  $F_k$  will be denoted by  $w_k^h(n, x)$ ,  $n = 0, \dots, \mu$ ,  $x \in V_k$ .

### 3.1.3 Definition of the fully discrete solution

Taking in mind the equation (15)–(16), we define the fully discrete solution to be the function  $u_k^h \in F_k$  such that,  $\forall i = 1, \dots, N$  and  $\forall n = 0, \dots, \mu - 1$ , verifies

$$u_k^h(n, x^i) = \min_{a \in A} \left\{ \max \left\{ f(x^i, a), \sum_{j=1}^N \gamma_j(x^i, a) u_k^h(n+1, x^j) \right\} \right\}, \quad (71)$$

$$u_k^h(\mu, x^i) = \min_{a \in A} f(x^i, a).$$

**Remark 3.1** Obviously, the solution of (71) is unique and can be computed recursively. This allows us to implement the computational procedure.

**Remark 3.2** If  $A$  is an infinite set, then (71) is not a fully discrete scheme in a strict sense. To obtain it, the final step consists in approximating this compact set by a suitable finite one. We have omitted such approximation in this paper to simplify the exposition. Because of this, from now on, we consider that  $A$  is a finite set. To simplify the proof of the following theorems we also suppose that  $\Omega = \mathfrak{R}^r$ . In the general case,  $\Omega \neq \mathfrak{R}^r$ , it is also possible to prove them under suitable hypothesis of regularity in the boundary and by an external approximation of the domain. The corresponding techniques can be seen in [12].

## 3.2 Central result

### 3.2.1 Rate of convergence

To obtain an estimate of the rate of convergence of the fully discrete solution, we establish an auxiliary result given by the following theorem, which brings an estimate for the difference between the discrete time cost and the fully discrete cost (defined in (15) and (71), respectively). The proof is based on regularization techniques and the obtained estimate depends on the regularization parameter  $\rho$ .

**Theorem 3.1** *Let  $x \in V_k$  and  $n = 0, \dots, \mu$ , then it results*

$$|u^h(n, x) - u_k^h(n, x)| \leq M\rho + T \left( L_u L_g \rho + L_u \frac{k^2}{\rho h} \right). \quad (72)$$

To prove this theorem, we apply properties established in lemmas introduced in the following subsection. The proof consists in obtaining estimates for the differences between subsolutions and supersolutions of problems introduced ad-hoc.

The central result of convergence is given by the following theorem which establishes an estimate of the difference between the optimal cost and the fully discrete solution.

**Theorem 3.2** *If there exist constants  $c_1$  and  $c_2$  such that*

$$c_1 k \leq h \leq c_2 k, \quad (73)$$

*then there exists  $C$  such that  $\forall x \in V_k$  and  $\forall n = 0, \dots, \mu$ , it results*

$$|u(nh, x) - u_k^h(n, x)| \leq C\sqrt{k}. \quad (74)$$

**Proof:** By using both the results of theorems 2.1 and 3.1, it results

$$|u(nh, x) - u_k^h(n, x)| \leq C\sqrt{h} + M\rho + T \left( L_u L_g \rho + L_u \frac{k^2}{\rho h} \right). \quad (75)$$

The estimate obtained in (75) attains its minimum value at  $\rho = Ck/\sqrt{h}$ , in consequence

$$|u(nh, x) - u_k^h(n, x)| \leq C \left( \sqrt{h} + \frac{k}{\sqrt{h}} \right).$$

Taking into account (73), we obtain

$$|u(nh, x) - u_k^h(n, x)| \leq C\sqrt{k}.$$

□

### 3.2.2 Definitions and auxiliary properties

Let  $\beta(\cdot) \in C^\infty(\mathbb{R}^r)$ , with the following properties

$$\begin{cases} \beta(x) \geq 0, & \forall x \in \mathbb{R}^r, \\ \text{supp}(\beta) \subseteq B(0, 1), \\ \int_{\mathbb{R}^r} \beta(x) dx = 1. \end{cases} \quad (76)$$

We define

$$\beta_\rho(x) = \rho^{-r} \beta(x/\rho), \quad \forall \rho \in \mathbb{R}^+. \quad (77)$$

We define  $\forall n = 0, \dots, \mu$ , the function  $u_\rho^h$ , regularization of the function  $u^h$ , as follows

$$u_\rho^h(n, x) = \int_{\mathbb{R}^r} u^h(n, x - y) \beta_\rho(y) dy. \quad (78)$$

By virtue of the convolution properties,  $\forall x, \hat{x} \in \Omega$ , the function  $u_\rho^h$  verifies

$$|u^h(n, x) - u^h(n, \hat{x})| \leq L_u \|x - \hat{x}\|, \quad (79)$$

$$|u_\rho^h(n, x) - u^h(n, x)| \leq L_u \rho. \quad (80)$$

**Remark 3.3** The proofs of the following lemmas are presented in the appendix in order to clarify the exposition, arriving directly to our central result of convergence.

**Lemma 3.1** For every  $x \in V_k$ ,  $n = 0, \dots, \mu - 1$ , it results

$$u_\rho^h(n, x) \leq \min_{a \in A} \{ \max \{ f_\rho(x, a), u_\rho^h(n+1, x + hg(x, a)) + L_u L_g h \rho \} \}, \quad (81)$$

where  $f_\rho(x, a) = f(x, a) + K \rho$  and  $K = L_f + L_u(2 + L_g h)$ .

From the previous lemma, we get the following corollary which establishes that the function  $u_\rho^h$  is a subsolution of the perturbed problem (82), or in other words, that the inequality (83) holds.

**Corollary 3.1** Let  $x \in V_k$ ,  $n = 0, \dots, \mu - 1$ , and  $v_\rho \in C(\Omega_k)^{\mu+1}$  be recursively defined by

$$v_\rho(n, x) = \min_{a \in A} \{ \max \{ f_\rho(x, a), v_\rho(n+1, x + hg(x, a)) + h L_u L_g \rho \} \}, \quad (82)$$

$$v_\rho(\mu, x) = \min_{a \in A} f_\rho(x, a).$$



Then,

$$u_\rho^h \leq v_\rho. \quad (83)$$

**Definition 3.1**  $u_{\rho,h}^h$  denotes the linear interpolation of  $u_\rho^h$ . In other words, the element of  $F_k$ , such that,

$\forall n = 0, \dots, \mu$ , and  $\forall x \in V_k$ , verifies

$$u_{\rho,h}^h(n, x) = u_\rho^h(n, x). \quad (84)$$

In [19], it is proved that the difference between  $u_\rho^h$  and its linear interpolation can be estimated in the following way:

$\forall x \in \Omega_k$

$$|u_{\rho,h}^h(n, x) - u_\rho^h(n, x)| \leq L_u \frac{k^2}{\rho}. \quad (85)$$

The following lemma establishes that the function  $u_{\rho,h}^h$  is a subsolution of the problem (86).

**Lemma 3.2** Let  $x \in V_k$ ,  $n \in \{0, \dots, \mu-1\}$ , and  $\epsilon_{\rho,h}$  be the element of  $F_k$  recursively defined by

$$\epsilon_{\rho,h}(n, x) = \min_{a \in A} \left\{ \max \left\{ f_\rho(x, a), \epsilon_{\rho,h}(n+1, x + hg(x, a)) + hL_u L_g \rho + L_u \frac{k^2}{\rho} \right\} \right\}, \quad (86)$$

$$\epsilon_{\rho,h}(\mu, x) = \min_{a \in A} f_\rho(x, a),$$

then,

$$u_{\rho,h}^h \leq \epsilon_{\rho,h}. \quad (87)$$

The following lemma establishes a relation between the element  $\epsilon_{\rho,h}$  of  $F_k$  defined in (86) and the solution of the fully discrete problem.

**Lemma 3.3** Let  $x \in V_k$  and  $n \in \{0, \dots, \mu\}$ , then

$$\epsilon_{\rho,h}(n, x) \leq u_k^h(n, x) + K\rho + T \left( L_u L_g \rho + L_u \frac{k^2}{h\rho} \right). \quad (88)$$

**Definition 3.2** Let  $\{a_n : n = 0, \dots, \mu\}$  be a sequence of elements of  $A$  and  $z$  the function recursively given by:

$$z(n, x) = \max\{f(x, a_n), z(n+1, x + hg(x, a_n))\}, \quad (89)$$

$$z(\mu, x) = f(x, a_\mu),$$

**Remark 3.4** The function  $z(\cdot, x)$  is the cost corresponding to the functional (3), where, although the control policy  $\{a_n\}$  can be non-stationary, it does not depend on the initial position of the system. Although the function  $z$  depends on  $\{a_n\}$ , we do not explicitly write this dependence in order to simplify the notation.

The function  $z$  has properties similar to those of  $u$ , i.e.

$$|z(n, x)| \leq M_f,$$

$$|z(n, x) - z(n, \hat{x})| \leq L_u \|x - \hat{x}\|.$$

**Definition 3.3** We denote  $z_\rho$  the regularized function of  $z$ , i.e.,  $\forall n = 0, \dots, \mu$

$$z_\rho(n, x) = \int_{\mathbb{R}^r} z(n, x - y) \beta_\rho(y) dy, \quad (90)$$

where  $\beta_\rho$  is the function defined in (77).

By the properties of convolution, it is valid that

$$|z_\rho(n, x) - z(n, x)| \leq L_u \rho. \quad (91)$$

**Definition 3.4** Let  $z_{\rho,k}$  be the linear interpolation of  $z_\rho$ . In other words, the function of  $F_k$  such that,  $\forall x \in V_k, \forall n = 0, \dots, \mu$ , verifies

$$z_{\rho,k}(n, x) = z_\rho(n, x). \quad (92)$$

We use again the result obtained in [19], which establishes that the difference between  $z_\rho$  and its linear interpolation can be estimated,  $\forall x \in \Omega_k$ , in the following way

$$|z_{\rho,k}(n, x) - z_\rho(n, x)| \leq L_u \frac{k^2}{\rho}. \quad (93)$$

**Lemma 3.4** Let  $x \in V_k, n \in \{0, \dots, \mu\}$  and  $\tilde{a} = \{\tilde{a}_m : m = 0, \dots, \mu\}$  be a policy whose restriction for the values  $n, n+1, \dots, \mu$  is an optimal policy from the initial conditions  $(n, x)$ ; i.e.

$$J^h(n, x, \tilde{a}) = u^h(n, x), \quad (94)$$

then,  $\forall m = n, \dots, \mu$

$$z(m, y_x^h(m)) = u^h(m, y_x^h(m)), \quad (95)$$

where  $y_x^h$  is defined by the recurrence

$$\begin{aligned} y_x^h(m+1) &= y_x^h(m) + hg(y_x^h(m), \tilde{a}_m) \\ y_x^h(n) &= x \end{aligned} \tag{96}$$

and  $z$  is the function defined by (89), where  $\{a_n\}$  is replaced by the special sequence  $\{\tilde{a}_n\}$ .

The following lemma and its corollary establish that the function  $z_\rho$  is a supersolution of the problem (98) defined below.

**Lemma 3.5** For every  $x \in V_k$  and  $n = 0, \dots, \mu - 1$ , it results

$$z_\rho(n, x) \geq \max \left\{ \tilde{f}_\rho(x, \tilde{a}_n), z_\rho(n+1, x + hg(x, \tilde{a}_n)) - hL_u L_g \rho \right\}, \tag{97}$$

where  $\tilde{f}_\rho(x, \tilde{a}_n) = f(x, \tilde{a}_n) - K\rho$ , and  $K = L_f + L_u(2 + L_g h)$ .

**Corollary 3.2** Let  $x \in V_k$ ,  $n \in \{0, \dots, \mu - 1\}$ , and  $\phi_\rho \in C(\Omega_k)^{\mu+1}$  be recursively defined by

$$\phi_\rho(n, x) = \max \left\{ \tilde{f}_\rho(x, \tilde{a}_n), \phi_\rho(n+1, x + hg(x, \tilde{a}_n)) - hL_u L_g \rho \right\}, \tag{98}$$

$$\phi_\rho(\mu, x) = \tilde{f}_\rho(x, \tilde{a}_\mu).$$

Then,

$$z_\rho \geq \phi_\rho. \tag{99}$$

The following lemma establishes that the function  $z_{\rho,h}$  given by definition 3.4, is a supersolution of the problem (100) introduced below.

**Lemma 3.6** Let  $x \in V_k$ ,  $n = 0, \dots, \mu - 1$ , and  $\sigma_{\rho,h}$  be the element of  $F_k$  recursively defined by

$$\sigma_{\rho,h}(n, x) = \max \left\{ \tilde{f}_\rho(x, \tilde{a}_n), \sigma_{\rho,h}(n+1, x + hg(x, \tilde{a}_n)) - hL_u L_g \rho - L_u \frac{k^2}{\rho} \right\}, \tag{100}$$

$$\sigma_{\rho,h}(\mu, x) = \tilde{f}_\rho(x, \tilde{a}_\mu),$$

then,

$$z_{\rho,h} \geq \sigma_{\rho,h}. \tag{101}$$

The following lemma establishes that the solution of the fully discrete problem defined in (71) is a subsolution of the maximizing stopping time problem (102), defined as follows.

**Lemma 3.7** *Let  $x \in V_k$ ,  $n \in \{0, \dots, \mu - 1\}$ , and  $\xi_k$  be the element of  $F_k$  recursively defined by*

$$\begin{aligned}\xi_k(n, x) &= \max \left\{ \tilde{f}(x, \tilde{a}_n), \xi_k(n+1, x + hg(x, \tilde{a}_n)) \right\}, \\ \xi_k(\mu, x) &= f(x, \tilde{a}_\mu),\end{aligned}\tag{102}$$

then, for every  $n = 0, \dots, \mu$  we have

$$\xi_k \geq u_k^h.\tag{103}$$

The following lemma establishes a relation between the solution of the problem (102) and the solution of the problem (100).

**Lemma 3.8** *For every  $x \in V_k$ ,  $n = 0, \dots, \mu$ , it results*

$$\sigma_{\rho, k}(n, x) \geq \xi_k(n, x) - K\rho - T \left( L_u L_g \rho + L_u \frac{k^2}{h\rho} \right).\tag{104}$$

### 3.2.3 Proof of a preliminary estimate

Now we can prove Theorem 3.1. To do that, we write

$$\begin{aligned}u^h(n, x) - u_k^h(n, x) &= (u^h(n, x) - u_\rho^h(n, x)) + (u_\rho^h(n, x) - u_{\rho, k}^h(n, x)) \\ &\quad + (u_{\rho, k}^h(n, x) - \varepsilon_{\rho, k}(n, x)) + (\varepsilon_{\rho, k}(n, x) - u_k^h(n, x)).\end{aligned}\tag{105}$$

The terms of (105) can be estimated in the following way:

By virtue of (80), we have

$$u^h(n, x) - u_\rho^h(n, x) \leq L_u \rho,$$

also, by (84)

$$u_\rho^h(n, x) - u_{\rho, k}^h(n, x) = 0,$$

by Lemma 3.2

$$u_{\rho,h}^h(n, x) - \varepsilon_{\rho,h}(n, x) \leq 0,$$

and by Lemma 3.3

$$\varepsilon_{\rho,h}(n, x) - u_k^h(n, x) \leq K\rho + T \left( L_u L_g \rho + L_u \frac{k^2}{h\rho} \right).$$

Therefore

$$u^h(n, x) - u_k^h(n, x) \leq M\rho + T \left( L_u L_g \rho + L_u \frac{k^2}{h\rho} \right). \quad (106)$$

Let now  $n$  and  $x$  be arbitrary elements and  $\tilde{a}$  a control policy such that, taking into account Lemma 3.4, has the following property: its restriction to the values  $m = n, \dots, \mu$ , is a discrete optimal policy for the functional  $J^h$  and the initial values  $(n, x)$ . The function  $z$  used in the remains of the proof is chosen according to this  $\tilde{a}$ . To obtain an inequality similar to (106) but where the sign  $\geq$  holds, we write

$$\begin{aligned} u^h(n, x) - u_k^h(n, x) &= (u^h(n, x) - z(n, x)) + (z(n, x) - z_\rho(n, x)) \\ &\quad + (z_\rho(n, x) - z_{\rho,h}(n, x)) + (z_{\rho,h}(n, x) - \sigma_{\rho,h}(n, x)) \\ &\quad + (\sigma_{\rho,h}(n, x) - \xi_k(n, x)) + (\xi_k(n, x) - u_k^h(n, x)). \end{aligned} \quad (107)$$

These terms are estimated in the following form.

By Lemma 3.4

$$u^h(n, x) - z(n, x) = 0,$$

by (91)

$$z(n, x) - z_\rho(n, x) \geq -L_u \rho,$$

by (92)

$$z_\rho(n, x) - z_{\rho,h}(n, x) = 0,$$

by Lemma 3.6

$$z_{\rho,h}(n, x) - \sigma_{\rho,h}(n, x) \geq 0,$$

by Lemma 3.8

$$\sigma_{\rho,h}(n, x) - \xi_k(n, x) \geq -K\rho - T \left( L_u L_g \rho + L_u \frac{k^2}{h\rho} \right)$$

and, by Lemma 3.7

$$\xi_k(n, x) - u_k^h(n, x) \geq 0.$$

Consequently,

$$u^h(n, x) - u_k^h(n, x) \geq -M\rho - T \left( L_u L_g \rho + L_u \frac{k^2}{h\rho} \right). \quad (108)$$

From (106) and (108), we obtain the inequality (72), i. e.

$$|u^h(n, x) - u_k^h(n, x)| \leq M\rho + T \left( L_u L_g \rho + L_u \frac{k^2}{h\rho} \right).$$

□

## 4 Generalization to accumulated costs

Similarly as we have done for function (3), we can deal with the general problem of minimizing a functional  $\hat{J}$  that includes an integral cost. Specifically, we consider the functional

$$\hat{J}(t, x, \alpha(\cdot)) = \text{ess sup} \left\{ \int_t^s b(y(\theta), \alpha(\theta)) d\theta + f(y(s), \alpha(s)) : s \in [t, T] \right\},$$

where  $y(\cdot)$  is the solution of the dynamical system (1) and the function  $b$  is bounded and Lipschitz continuous in  $x$ . Here, the optimal cost function  $u$  takes the form

$$u : (t, x) \in [0, T] \times \Omega \mapsto \inf \left\{ \hat{J}(t, x, \alpha(\cdot)) : \alpha \in \mathcal{U}(t, T) \right\}. \quad (109)$$

Similarly to (71), we define recursively the fully discrete approximation of  $u$  as follows,

$$\forall x^i \in V_k, u_k^h(\mu, x^i) = \min_{a \in A} f(x^i, a) \text{ and, } \forall n = 0, \dots, \mu - 1$$

$$u_k^h(n, x^i) = \min_{a \in A} \left\{ \max \left\{ f(x^i, a), hb(x^i, a) + \sum_{j=1}^{\mu} \gamma_j(x^i, a) u_k^h(n+1, x^j) \right\} \right\}. \quad (110)$$

where  $\gamma_j(x^i, a)$  are the functions defined in (70).

Also for this problem, it holds the following convergence result:

**Theorem 4.1** *If there exist constants  $c_1$  and  $c_2$  such that  $c_1 k \leq h \leq c_2 k$ , then there exists a constant  $C$  such that  $\forall x \in V_k, \forall n = 0, \dots, \mu$*

$$|u(nh, x) - u_k^h(n, x)| \leq C\sqrt{k}. \quad (111)$$

Hence, for the general problem we can get fully discrete approximations and an estimate of convergence of the same order as that obtained for the original problem.

## 5 Optimality of the estimate

In this problem, even though the data  $f$  and  $g$  are semiconcave in  $x$ , it is not possible to improve the estimate  $\sqrt{h}$  which appears in (54), as it was done in the problem studied in [13, 15]. There, under semiconcavity hypothesis on  $f$  and  $g$ , it was shown that the optimal cost function  $u$  also results semiconcave. In that case, the estimate for  $\|u - u^h\|$  can be improved to order  $h$ , improvement that was critical to prove an estimate of type  $k^{2/3}$  for the fully discrete approximation. The following example shows that, for the minimax problem, an improvement of this type is not possible.

Let the dynamic of the system be

$$\begin{cases} \frac{dy}{dt}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y(t), & \forall t \in [0, 10], \\ y(0) = x \in \mathbb{R}^2. \end{cases} \quad (112)$$

We define the instantaneous cost function as follows

$$f(y(t)) = (1 - (y_1^2(t) + y_2^2(t))) y_1(t), \quad (113)$$

where  $y_i(\cdot)$  is the  $i$ -th component of  $y(\cdot)$ .

The system moves freely in  $\mathbb{R}^2$  and the functions  $f$  and  $g$  verify the hypotheses  $A_1$ - $A_4$ . Clearly, they are semiconcave.

Let  $r = \sqrt{x_1^2 + x_2^2}$ . It is easy to check, after elementary calculus that the optimal cost function is

$$u(0, x) = \max_{t \in [0, 10]} f(y(t)) = |1 - r^2| r, \quad (114)$$

which is not semiconcave.

The discretization procedure introduced here coincides with the methodology studied in [14]. In that work, the authors proved that the estimate  $\sqrt{k}$  is critical. In fact, here we can prove – using the triangulation shown in Fig. 1 and calculations entirely similar to those employed in [14, 15] – that the error  $|u(0, x) - u_k^h(0, x)|$  verifies

$$|u(0, x) - u_k^h(0, x)| \geq C\sqrt{k}. \quad (115)$$

In consequence, we have that for approximations of this type the better error that can be expected, is of order  $\sqrt{k}$ .



## 6 Numerical example

The problem consists in the calculus of the optimal trajectory from a given initial position, in a rectangular domain  $\Omega = [-2, 2] \times [-2, 2]$ . The optimal trajectory minimizes the maximum value of  $f$  along its way, i.e. the cost functional is given by

$$J(t, x, \alpha(\cdot)) = \text{ess sup}_{s \in [t, T]} f(y(s), s). \quad (116)$$

The function  $f$  is independent on the control and has the shape shown in the Fig. 2 (this figure corresponds to the function  $f(x, 0)$ ).

We consider a regular discretization  $\Omega_k$ , whose set of nodes is  $V_k = \{x^i, i = 1, \dots, N\}$ . The nodes of  $\Omega_k$  have the general form  $(ik, jk), i = -N_1, \dots, N_1, j = -N_1, \dots, N_1, k = 2/N_1$  and  $N = (2N_1 + 1) \times (2N_1 + 1)$ .

The system evolves up to  $T = 5$ . We divide the interval  $[0, 5]$  into  $\mu$  sub-intervals with common length  $h = T/\mu$ .

In this problem,  $A = \{0, 1, 2, 3, 4\}$ , and the admissible controls are five admissible directions of movement given by

$$g(x_1, x_2, \alpha) = (1 - (x_1 - 1)^+ - (x_1 + 1)^-)^+ \times (1 - (x_2 - 1)^+ - (x_2 + 1)^-)^+ \times g_o(\alpha), \quad (117)$$

where

$$g_o(\alpha) = \begin{cases} (0, 0) & \text{if } \alpha = 0 \\ (1, 0) & \text{if } \alpha = 1 \\ (0, 1) & \text{if } \alpha = 2 \\ (-1, 0) & \text{if } \alpha = 3 \\ (0, -1) & \text{if } \alpha = 4. \end{cases}$$

Taking into account the description above, the fully discrete problem becomes

$$\forall i = 1, \dots, N \text{ and } \forall n = 0, \dots, \mu - 1,$$

$$u_k^h(n, x^i) = \max \left\{ f(x^i, nh), \min_{a \in A} u_k^h(n+1, x^i + hg(x^i, a)) \right\}, \quad (118)$$

$$u_k^h(\mu, x^i) = f(x^i, \mu h).$$

The system (118) gives us a recursive procedure to compute the optimal cost at each time and at each node. At each iteration, we store the direction that produces the optimum, operation which enables us to construct the optimal trajectories, like that one shown in Fig. 3.

## 7 Conclusions

Here, we have developed a discretization procedure to obtain the numerical solution of the problem of minimizing the maximum cost, analyzed from the continuous point of view by Barron-Ishii in [2].

The numerical procedure obtained is easily implementable and it converges to the solution of the original problem, with an error estimate of the form

$$|u - u_k^h| \leq C\sqrt{k}. \quad (119)$$

This estimate was shown to be optimal.

The optimality of the estimate (119) stems from the fact that this minimax problem is a disguised differential game problem, where the controller is trying to minimize the cost

$$J(t, x, \alpha(\cdot), \tau) = f(y(\tau), \alpha(\tau)), \quad (120)$$

while an unidentified opponent – using fully information of actions of the other player – chooses at any instant the stopping time  $\tau$  of the process. As a result of the complete game, the pay-off (120) is given.

This action of the second privileged player has as an effect that the first player must – in a strict way – minimize a functional that is not semiconcave with respect to the spatial variable  $y$ .

In this way, once a fully discretization – using finite differences or finite elements – is applied, the resultant fully discrete optimal control problem reflects this property in the validity of the estimate of type  $\sqrt{k}$ .

Except for very special trajectories, where some carefully chosen triangulations may be used, it seems not possible in general to get approximations with better convergence properties.

In fact, the error worsens as time grows and for the case  $T = \infty$ , the phenomenon of non-convergence arises. This undesirable effect is studied more detailed in our work [10].

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## 9 Appendix

**Proof of Lemma 3.1:** Let  $x \in V_k$  and  $n = 0, \dots, \mu - 1$ ; as we have proved in Lemma 2.1,

$$u^h(n, x) = \min_{a \in A} \{ \max \{ f(x, a), u^h(n+1, x + hg(x, a)) \} \},$$

$$u^h(\mu, x) = \min_{a \in A} f(x, a),$$

then,  $\forall a \in A$ ,  $u^h$  satisfies

$$u^h(n, x) \leq \max \{ f(x, a), u^h(n+1, x + hg(x, a)) \}. \quad (121)$$

Let  $R = L_f + L_u(1 + L_g h)$ .

We analyze two cases concerning  $\max \{ f(x, a), u^h(n+1, x + hg(x, a)) \}$

- $\max \{ f(x, a), u^h(n+1, x + hg(x, a)) \} \leq f(x, a) + R\rho$

It results from (121)

$$u^h(n, x) \leq f(x, a) + R\rho. \quad (122)$$

By using (80), we have

$$u_\rho^h(n, x) \leq u^h(n, x) + L_u \rho; \quad (123)$$

consequently, from (122), it follows that

$$u^h(n, x) + L_u \rho \leq f(x, a) + R\rho + L_u \rho = f(x, a) + K\rho = f_\rho(x, a). \quad (124)$$

From (123) and (124), we obtain

$$u_\rho^h(n, x) \leq f_\rho(x, a). \quad (125)$$

- $\max \{ f(x, a), u^h(n+1, x + hg(x, a)) \} > f(x, a) + R\rho$

This condition implies

$$\max \{ f(x, a), u^h(n+1, x + hg(x, a)) \} = u^h(n+1, x + hg(x, a)). \quad (126)$$

Let us see that  $\forall y \in B_\rho(x)$  (where  $B_\rho(x)$  denotes a ball centered on  $x$  of radius  $\rho$ ), the following inequality holds,

$$u^h(n, y) \leq u^h(n+1, y + hg(y, a)). \quad (127)$$

Let us suppose that (127) is false, i.e.,  $\exists \tilde{y} \in B_\rho(x)$ , such that

$$u^h(n, \tilde{y}) > u^h(n+1, \tilde{y} + hg(\tilde{y}, a)). \quad (128)$$

As

$$u^h(n, \tilde{y}) \leq \max\{f(\tilde{y}, a), u^h(n+1, \tilde{y} + hg(\tilde{y}, a))\},$$

the inequality (128) implies

$$f(\tilde{y}, a) > u^h(n+1, \tilde{y} + hg(\tilde{y}, a)). \quad (129)$$

By virtue of the fact that  $f$  is Lipschitz continuous and (18), we obtain

$$\begin{aligned} f(x, a) + L_f \rho &\geq f(\tilde{y}, a) > u^h(n+1, \tilde{y} + hg(\tilde{y}, a)) \\ &\geq u^h(n+1, x + hg(x, a)) - L_u(1 + L_g h) \rho. \end{aligned}$$

Hence,

$$u^h(n+1, x + hg(x, a)) - f(x, a) < L_f \rho + L_u(1 + L_g h) \rho = R \rho,$$

which contradicts (126).

By convolution of  $u^h$  with  $\beta_\rho$  we have, by virtue of (127),

$$\begin{aligned} u_\rho^h(n, x) &= \int_{B_\rho(x)} u^h(n, x-y) \beta_\rho(y) dy \\ &\leq \int_{B_\rho(x)} u^h(n+1, x-y + hg(x-y, a)) \beta_\rho(y) dy \\ &\leq \int_{B_\rho(x)} u^h(n+1, x-y + hg(x, a)) \beta_\rho(y) dy \\ &\quad + \int_{B_\rho(x)} L_u \|x-y + hg(x-y, a) - (x-y + hg(x, a))\| \beta_\rho(y) dy \\ &\leq u_\rho^h(n+1, x + hg(x, a)) + L_u L_g h \int_{B_\rho(x)} \beta_\rho(y) dy \\ &= u_\rho^h(n+1, x + hg(x, a)) + L_u L_g h \rho. \end{aligned}$$

Consequently,

$$u_\rho^h(n, x) \leq u_\rho^h(n+1, x + hg(x, a)) + L_u L_g h \rho. \quad (130)$$

From (125) and (130),  $\forall a \in A$ ,  $u_\rho^h$  verifies

$$u_\rho^h(n, x) \leq \max \{ f_\rho(x, a), u_\rho^h(n+1, x + hg(x, a)) + L_u L_g h \rho \},$$

then

$$u_\rho^h(n, x) \leq \min_{a \in A} \{ \max \{ f_\rho(x, a), u_\rho^h(n+1, x + hg(x, a)) + L_u L_g h \rho \} \}. \quad (131)$$

□

**Proof of Corollary 3.1.** For  $n = \mu$ , the following relation holds

$$u_\rho^h(\mu, x) \leq u^h(\mu, x) + L_u \rho = \min_{a \in A} f(x, a) + L_u \rho \leq \min_{a \in A} f_\rho(x, a) = v_\rho(\mu, x). \quad (132)$$

From (131), (132) and (82), we can prove by induction the following inequality

$$u_\rho^h \leq v_\rho. \quad (133)$$

From (132), the relation (133) holds for  $n = \mu$ . Let us assume that it is true for  $n = m+1$  and we will prove that the same happens for  $n = m$ . From (131), the induction assumptions and (82), we have

$$\begin{aligned} u_\rho^h(m, x) &\leq \min_{a \in A} \{ \max \{ f_\rho(x, a), u_\rho^h(m+1, x + hg(x, a)) + L_u L_g h \rho \} \} \\ &\leq \min_{a \in A} \{ \max \{ f_\rho(x, a), v_\rho(m+1, x + hg(x, a)) + L_u L_g h \rho \} \} = v_\rho(m, x). \end{aligned}$$

□

**Proof of Lemma 3.2:** From the definition of  $u_\rho^h$  and (84), we have

$$u_{\rho, h}^h(\mu, x) = u_\rho^h(\mu, x) \leq \varepsilon_{\rho, h}(\mu, x). \quad (134)$$

Let  $x \in V_h$  and  $n = 0, \dots, \mu - 1$ ; from Lemma 3.1, it follows that,  $\forall a \in A$ ,

$$u_\rho^h(n, x) \leq \max \{ f_\rho(x, a), u_\rho^h(n+1, x + hg(x, a)) + h L_u L_g \rho \}.$$

- If  $\max \{f_\rho(x, a), u_\rho^h(n+1, x + hg(x, a)) + hL_u L_g \rho\} = f_\rho(x, a)$ , (84) implies

$$u_{\rho, k}^h(n, x) = u_\rho^h(n, x) \leq f_\rho(x, a). \quad (135)$$

- If  $\max \{f_\rho(x, a), u_\rho^h(n+1, x + hg(x, a)) + hL_u L_g \rho\} = u_\rho^h(n+1, x + hg(x, a)) + hL_u L_g \rho$ , from (84), it results

$$u_{\rho, k}^h(n, x) = u_\rho^h(n, x) \leq u_\rho^h(n+1, x + hg(x, a)) + hL_u L_g \rho.$$

Since  $x + hg(x, a) \in \Omega_k$ , using (85), we obtain

$$u_{\rho, k}^h(n, x) \leq u_{\rho, k}^h(n+1, x + hg(x, a)) + hL_u L_g \rho + L_u \frac{k^2}{\rho}. \quad (136)$$

For each  $a \in A$ , at least one of the inequalities (135) or (136), must be true. Consequently,

$$u_{\rho, k}^h(n, x) \leq \max \left\{ f_\rho(x, a), u_{\rho, k}^h(n+1, x + hg(x, a)) + hL_u L_g \rho + L_u \frac{k^2}{\rho} \right\},$$

then, since  $a$  is an arbitrary element of  $A$ , we have

$$u_{\rho, k}^h(n, x) \leq \min_{a \in A} \left\{ \max \left\{ f_\rho(x, a), u_{\rho, k}^h(n+1, x + hg(x, a)) + hL_u L_g \rho + L_u \frac{k^2}{\rho} \right\} \right\}. \quad (137)$$

From (134) and (137), also by induction, we can prove as we have done in Corollary 3.1 that

$$u_{\rho, k}^h \leq \varepsilon_{\rho, k}.$$

□

### Proof of Lemma 3.3:

We use backward induction for  $n = 0, \dots, \mu$ .

From (71), we have

$$u_k^h(\mu, x) = \min_{a \in A} f(x, a). \quad (138)$$

By definition of  $\varepsilon_{\rho, k}$ , we obtain

$$\varepsilon_{\rho, k}(\mu, x) = \min_{a \in A} f_\rho(x, a) = \min_{a \in A} f(x, a) + K\rho = u_k^h(\mu, x) + K\rho.$$

Let us suppose that for a generic  $n$ ,

$$\varepsilon_{\rho, k}(n, x) \leq u_k^h(n, x) + K\rho + (\mu - n) \left( hL_u L_g \rho + L_u \frac{k^2}{\rho} \right).$$

**Note 9.1** The previous inequality also holds in points of the form  $x + hg(x, a)$  that do not necessarily belong to  $V_k$ . This is because the functions  $u_k^h$  and  $\varepsilon_{\rho, k}$  are in  $F_k$ , so they verify the property of linearity and the assumption (70).

We must prove the inequality for  $n - 1$

$$\begin{aligned}
\varepsilon_{\rho, k}(n-1, x) &= \min_{a \in A} \left\{ \max \left\{ f_\rho(x, a), \varepsilon_{\rho, k}(n, x + hg(x, a)) + hL_u L_g \rho + L_u \frac{k^2}{\rho} \right\} \right\} \\
&\leq \min_{a \in A} \left\{ \max \left\{ f_\rho(x, a), u_k^h(n, x + hg(x, a)) + K\rho + (\mu - n + 1) \left( hL_u L_g \rho + L_u \frac{k^2}{\rho} \right) \right\} \right\} \\
&\leq \min_{a \in A} \left\{ \max \left\{ f_\rho(x, a), u_k^h(n, x + hg(x, a)) \right\} \right\} + K\rho + (\mu - n + 1) \left( hL_u L_g \rho + L_u \frac{k^2}{\rho} \right) \\
&= u_k^h(n-1, x) + K\rho + (\mu - n + 1) \left( hL_u L_g \rho + L_u \frac{k^2}{\rho} \right).
\end{aligned}$$

Consequently, by induction,

$$\begin{aligned}
\varepsilon_{\rho, k}(0, x) &\leq u_k^h(0, x) + K\rho + \mu \left( hL_u L_g \rho + L_u \frac{k^2}{\rho} \right) \\
&\leq u_k^h(0, x) + K\rho + T \left( L_u L_g \rho + L_u \frac{k^2}{\rho h} \right).
\end{aligned}$$

□

**Proof of Lemma 3.4:** By definition of  $\{\tilde{a}_n : n = 0, \dots, \mu\}$ , if  $n = 0, \dots, \mu - 1$ , we have

$$u^h(n, y_{x_0}^h(n)) = J(n, y_{x_0}^h(n), \tilde{a}_n) = \max \{ f(y_{x_0}^h(n), \tilde{a}_n), u^h(n+1, y_{x_0}^h(n) + hg(y_{x_0}^h(n), \tilde{a}_n)) \}$$

$$u^h(\mu, y_{x_0}^h(\mu)) = f(y_{x_0}^h(\mu), \tilde{a}_\mu),$$

then,  $\forall n = 0, \dots, \mu$ , it follows that

$$z(n, y_{x_0}^h(n)) = u^h(n, y_{x_0}^h(n)). \quad (139)$$

□

**Proof of Lemma 3.5 and Corollary 3.2:** They are similar to the proofs of Lemma 3.1 and Corollary 3.1 respectively, and they are here omitted.

**Proof of Lemma 3.6:** Let  $x \in V_k$  and  $n = 0, \dots, \mu - 1$ . According to the definition of  $z_{\rho,k}$  and the result of Lemma 3.5, the following relations are valid:

$$z_{\rho,k}(n, x) = z_{\rho}(n, x) \geq \tilde{f}_{\rho}(x, \tilde{a}_n), \quad (140)$$

$$z_{\rho,k}(n, x) = z_{\rho}(n, x) \geq z_{\rho}(n+1, x + hg(x, \tilde{a}_n)) - hL_u L_g \rho. \quad (141)$$

As  $x + hg(x, \tilde{a}_n) \in \Omega_k$ , using (93), we obtain

$$z_{\rho}(n+1, x + hg(x, \tilde{a}_n)) \geq z_{\rho,k}(n+1, x + hg(x, \tilde{a}_n)) - L_u \frac{k^2}{\rho}. \quad (142)$$

From (141) and (142) we arrive to

$$z_{\rho,k}(n, x) \geq z_{\rho,k}(n+1, x + hg(x, \tilde{a}_n)) - hL_u L_g \rho - L_u \frac{k^2}{\rho}. \quad (143)$$

Therefore, from (140) and (143),  $z_{\rho,k}$  verifies

$$z_{\rho,k}(n, x) \geq \max \left\{ \tilde{f}_{\rho}(x, \tilde{a}_n), z_{\rho,k}(n+1, x + hg(x, \tilde{a}_n)) - hL_u L_g \rho - L_u \frac{k^2}{\rho} \right\}. \quad (144)$$

The inequality (101) can be proved by induction as we have done in Lemma 3.1 because the relation for the final condition is obviously true. Rather than use this, we do the proof using a discrete maximum principle verified by the operator associated to (100). (cf [3]).

Let us suppose that (101) does not hold. Then, there exists an index  $m$  such that it is the maximum element of the set  $\{0, 1, \dots, \mu - 1\}$  for which

$$\min_{x \in V_k} (z_{\rho,k}(m, x) - \sigma_{\rho,k}(m, x)) < 0. \quad (145)$$

Since  $z_{\rho,k}$  and  $\sigma_{\rho,k}$  belong to  $F_k$  and, for every  $x \in V_k$ ,  $x + hg(x, a)$  can be written as a convex sum of elements in  $V_k$ , it follows from (145) that

$$z_{\rho,k}(m+1, x + hg(x, a)) \geq \sigma_{\rho,k}(m+1, x + hg(x, a)). \quad (146)$$

From (144), (145), (100) and (146) we have

$$\begin{aligned} z_{\rho,k}(m, x) &\geq \max \left\{ \tilde{f}_{\rho}(x, \tilde{a}_m), z_{\rho,k}(m+1, x + hg(x, \tilde{a}_m)) - hL_u L_g \rho - L_u \frac{k^2}{\rho} \right\} \\ &> \max \left\{ \tilde{f}_{\rho}(x, \tilde{a}_m), \sigma_{\rho,k}(m+1, x + hg(x, \tilde{a}_m)) - hL_u L_g \rho - L_u \frac{k^2}{\rho} \right\} \\ &= \sigma_{\rho,k}(m, x). \end{aligned}$$



This contradicts (145). Thus,

$$z_{\rho,h} \geq \sigma_{\rho,h}.$$

□

**Proof of Lemma 3.7:** Taking in mind the definition of  $\xi_k$ , we deduce

$$\xi_k(\mu, x) = f(x, \tilde{a}_\mu) \geq \min_{a \in A} f(x, a) = u_k^h(\mu, x). \quad (147)$$

Given a  $n = 0, \dots, \mu - 1$ , let us suppose that,  $\forall x \in V_k$ ,

$$\xi_k(\mu - n + 1, x) \geq u_k^h(\mu - n + 1, x). \quad (148)$$

As we have seen in Note 9.1, this inequality also holds at any point of the form  $x + hg(x, a)$  belonging to  $\Omega_k$ .

Let us see that the inequality also holds in  $\mu - n$ .

$$\begin{aligned} \xi_k(\mu - n, x) &= \max\{f(x, \tilde{a}_{\mu-n}), \xi_k(\mu - n + 1, x + hg(x, \tilde{a}_{\mu-n}))\} \\ &\geq \max\{f(x, \tilde{a}_{\mu-n}), u_k^h(\mu - n + 1, x + hg(x, \tilde{a}_{\mu-n}))\} \\ &\geq \min_{a \in A} \{\max\{f(x, a), u_k^h(\mu - n + 1, x + hg(x, a))\}\} = u_k^h(\mu - n, x). \end{aligned}$$

Consequently, by induction we prove,  $\forall n = 0, \dots, \mu$ ,

$$\xi_k(n, x) \geq u_k^h(n, x).$$

□

**Proof of Lemma 3.8:** It is similar to the proof of Lemma 3.3 and it is here omitted.

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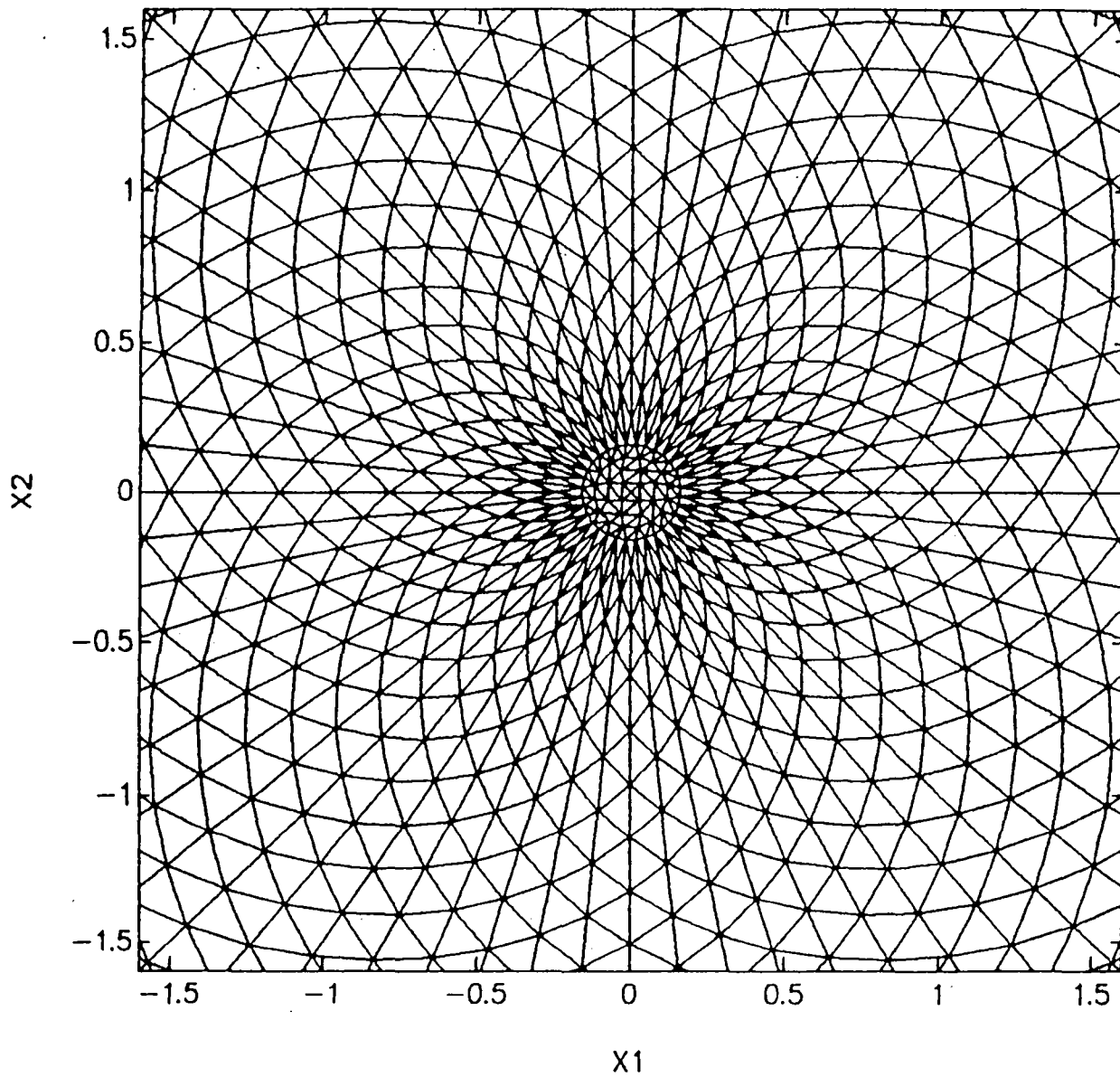


Figure 1: An Usual Mesh

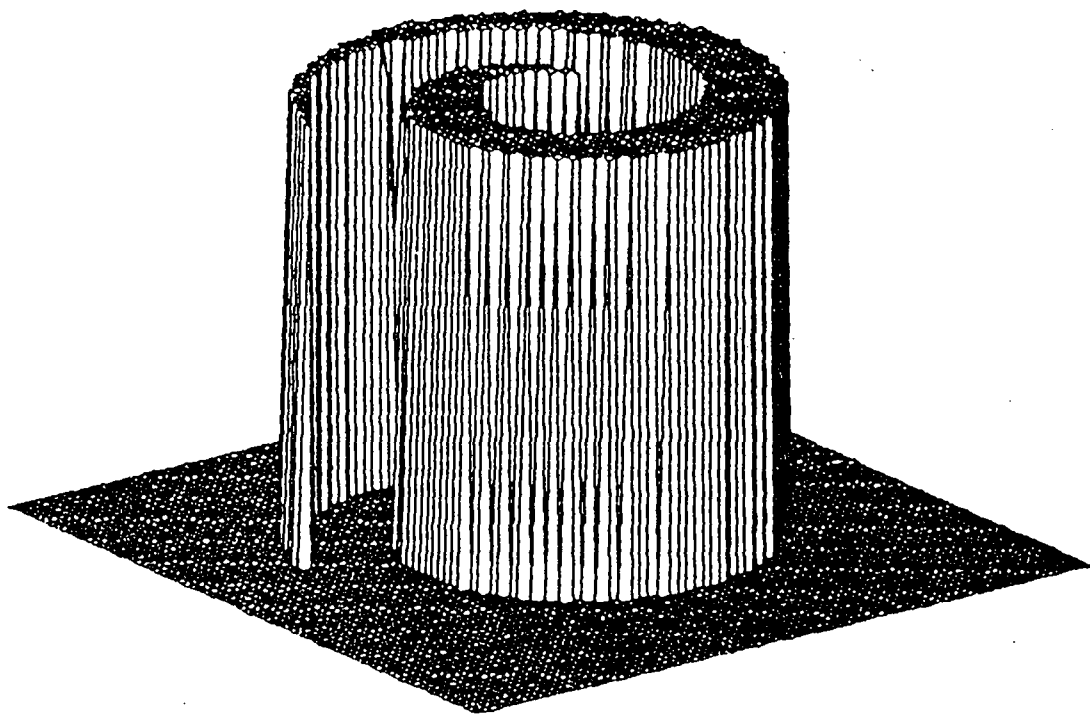


Figure 2: Function  $f(0, \cdot)$

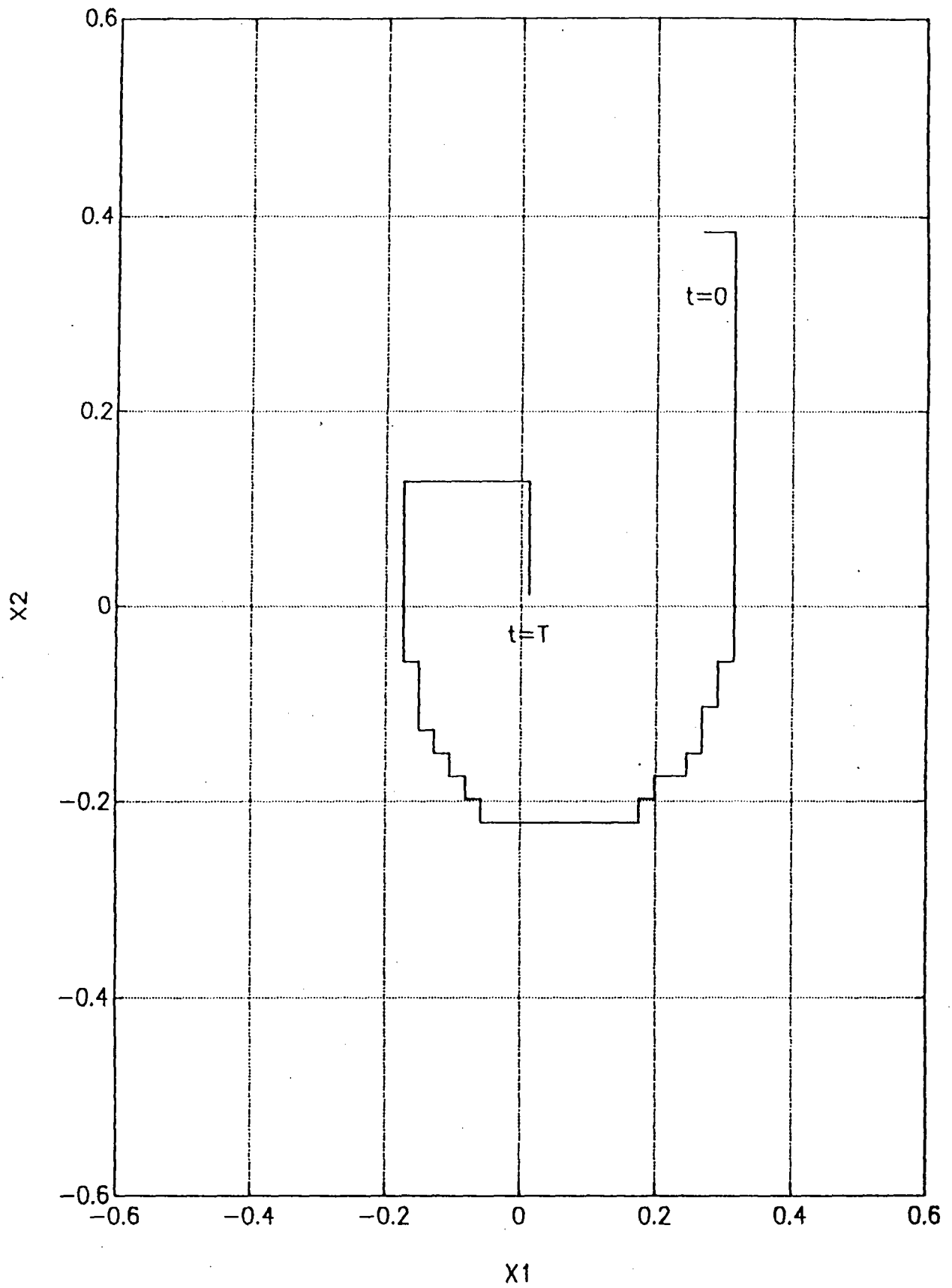


Figure 3: An Optimal Trajectory



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Unité de recherche INRIA Rocquencourt  
Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 Le Chesnay Cedex (France)  
Unité de recherche INRIA Lorraine - Technopôle de Nancy-Brabois - Campus scientifique  
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