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*Liveness in Free-Choice Petri Nets
An Algebraic Approach*

François Baccelli et Bruno Gaujal

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———— THÈME 1 ————



*Rapport
de recherche*

Liveness in Free-Choice Petri Nets An Algebraic Approach

François Baccelli et Bruno Gaujal

Thème 1 — Réseaux et systèmes
Projet mistral, sloop

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Abstract: In this paper, we show that the evolution equations for timed free choice Petri net can be used to prove logical properties of the underlying un-timed net. For example, this approach leads to new methods and algorithms to check liveness and several other basic properties, such as absence or presence of deadlocks. The main tools for proving these properties are graph theory and idempotent algebras.

Key-words: Petri nets, $(\min,+)$ algebra, liveness.

(Résumé : tsvp)

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Vivacité des réseaux à choix libres approche algébrique

Résumé : Dans cet article, nous montrons que les équations d'évolution pour un réseau de Petri à choix libres temporisé permettent d'établir des propriétés logiques du réseau non temporisé sous-jacent. Par exemple, cette approche amène de nouvelles méthodes et de nouveaux algorithmes pour tester la vivacité du réseau et d'autres propriétés comme la présence ou l'absence de blocages dans le réseau. Les principaux outils utilisés sont la théorie des graphes et les algèbres idempotentes.

Mots-clé : Réseaux de Petri, algèbre $(\min,+)$, vivacité.

1 Introduction

Timed event graphs can be seen as linear dynamic systems in the so-called $(\max, +)$ -algebra [2]. This algebraic viewpoint provides a rich variety of results on stability, control and computational problems arising in both deterministic and stochastic event graphs. The aim of the present paper is to show that a similar algebraic approach can be generalized to Free Choice nets (FCNets). The analysis, which works both in the bounded and the unbounded case, is based on a decomposition of the net into its maximal $(\max, +)$ -linear (i.e. event graph) subnetworks and a non-linear “routing” subnet; this decomposition is introduced together with basic definitions on Petri nets in Section 2.

In Section 3, this decomposition is then used to derive the evolution equations for the timed case. Up to a minor transformation, the dynamics of a FCNet admit a representation in terms of a ‘coupling’ of two linear systems. The first one pertains to the event graph components, and satisfies a non-autonomous $(\min, +)$ -linear evolution equation with an input from the second system, whereas the second one is “essentially” a non-autonomous $(+, \times)$ -linear system with an input from the first system. A few basic properties of this class of evolution equations are established, and in particular an equation for the total firing, which plays a central role in the analysis of both logical and temporal properties.

In Section 4, this decomposition is used in order to analyze logical properties of FCNets. We analyze various such properties including starvation, liveness and deadlocks as properties of the evolution equations. We give in particular a simple proof of Commoner’s liveness condition (i.e. the network is live if and only if each siphon contains a trap with a non-empty marking) which is purely based on the total firing equation (in Appendix 3). In the case of single input-FCNets (where all event-graph subnets arising in the decomposition are single-input $(\max, +)$ -linear systems) we show that our characterization of liveness leads to a polynomial time algorithm (in the size of the net).

2 Timed Petri net

2.1 Dynamics of Petri Nets

A Petri net is a 4-tuple $(\mathcal{P}, \mathcal{T}, \mathcal{C}, \mathcal{M}_0)$ where \mathcal{P} is the set of places, \mathcal{T} is the set of transitions, \mathcal{C} the set of directed arcs between places and transitions or between transitions and places (\mathcal{C} is a subset of $\mathcal{P} \times \mathcal{T} \cup \mathcal{T} \times \mathcal{P}$). \mathcal{M}_0 is the initial marking in the places. We denote by $\bullet t$ the set $\{p \in \mathcal{P} : (p, t) \in \mathcal{C}\}$ (i.e. the set of all input places of t). We define similarly the sets t^\bullet , $\bullet p$, p^\bullet as the set of output places of t , the set of input transitions of p and the set of output transitions of p , respectively.

All the nets which we considered in this paper are simple, in that arcs are not weighted; in a non-simple net, each arc has an integer value which gives the number of tokens produced (resp. consumed) by the transition at the origin (resp. end) of the arc. Simple nets are nets where all arcs have value 1.

A timed Petri net is a Petri net with temporal data attached to transitions: $\sigma^t(n)$ is a data which gives the duration of the n -th firing of transition t . This means that if transition t begins to fire for the n -th time at epoch e , this firing will end at epoch $e + \sigma^t(n)$; tokens are then taken out of input places and put into output places of t according to the firing rule of the untimed Petri net.

For more on the matter, and in particular for the definitions of *deadlocks*, *liveness*, *structural liveness* etc. which are used in the paper, the reader is advised to consult the survey paper by Murata [15].

2.2 Free Choice Nets

Free choice nets (FCNet) are Petri nets verifying the following conditions: $|p^\bullet| > 1$ implies that for all $t \in p^\bullet$, $|\bullet t| = 1$, or equivalently $\bullet t = \{p\}$. In words, whenever two transitions share an input place, they have no other input place.

Free choice nets have been extensively studied in the 70's [9] and have regained interest recently [11, 17, 10] because they constitute a nice compromise between power of description and tractability of problems.

Several 'semantics' can be used for the resolution of conflicts. The most common is called the *race policy*. Another one, called the *routing policy*, was introduced in [1].

Let p be a place with several output transitions.

- In the case of the race policy, the resolution of conflicts is purely based on the temporal data. For FCNets, this policy boils down to the following: as soon as the n -th token enters place p , the transitions of p^\bullet immediately start their n -th firing (this is possible since they have no other pre-conditions by hypothesis), which takes $\sigma^t(n)$ for transition t . The transition which completes first wins the race for the n -th token, and consumes this token.
- In the case of the routing policy, *routing data* are attached with each place with several output transitions. Place p has a routing sequence $\nu^p : \mathbb{N} \rightarrow p^\bullet$, where $\nu^p(n)$ gives the transition $t \in p^\bullet$ to which the n -th token to enter place p is routed. The routing sequences can for instance be periodic or random. If this token is the k -th to be routed to transition $t \in p^\bullet$, then this token is immediately consumed by the transition (due to the FC property) where it experiences a *firing time* of $\sigma^t(k)$.

In fact, *in the free choice case*, the race policy is a special case of routing policy. Assume that the net evolves according to the race policy. Then the n -th token to enter place p is routed to transition

$$\nu^p(n) = \operatorname{argmin}_{t \in p^\bullet} \sigma^t(n),$$

(at least whenever this argument is unique). This function can also be seen as predefined routing data which does not depend on the firing times of the transitions of the net, but for those of p^\bullet of course. So, up to an adequate renumbering of the firing times of p^\bullet , one can view race as a special case of routing. The converse construction is also easy to make.

Finally, FCNets with a routing policy have the power of description of Turing machines. This can be shown by a reduction of the Boolean Dataflow Model (BDF) to a FCNet with routing and using the fact that the BDF model has the expressive power of a Turing Machine (see [7]).

In what follows, we shall adopt the routing semantics.

2.3 Decomposition into Marked Graph Components

Let \mathcal{T}' be the set of transitions with at least one input place. A place p in a FCNet F is *serial* if $|\bullet p \cap \mathcal{T}'| = |p^\bullet| = 1$. Let $\mathcal{A} \subset \mathcal{T}'$ be the subset of transitions such that all their upstream places are serial, and let $\mathcal{B} = \mathcal{T}'/\mathcal{A}$ be the set of transitions with at least one non-serial input place.

We define a binary relation \mathcal{L} by: $t, t' \in \mathcal{T}'$, $t\mathcal{L}t'$ if there is a serial place p verifying $\bullet p = \{t\}$ and $p^\bullet = \{t'\}$. Let \mathcal{K} be the transitive closure of \mathcal{L} . The binary relation \mathcal{K} is a parallelism relation on \mathcal{T}' . We partition the set \mathcal{T}' into its maximal \mathcal{K} -classes, $\mathcal{T}_1, \dots, \mathcal{T}_n$. We construct a decomposition of F in the following way:

$$\mathcal{P}_i = \{p \in \mathcal{P} \mid p \text{ serial and } \bullet p, p^\bullet \in \mathcal{T}_i\}, \quad \text{for all } i.$$

The marked graph component (MGC) \mathcal{G}_i of F is the sub-Petri net $(\mathcal{P}_i, \mathcal{T}_i, \mathcal{C} \cap (\mathcal{P}_i \times \mathcal{T}_i \cup \mathcal{T}_i \times \mathcal{P}_i))$ of F . One can easily check that \mathcal{G}_i is a marked graph and is maximal in the sense that no marked graph included in F contains \mathcal{G}_i , except \mathcal{G}_i itself. A marked graph component \mathcal{G}_i is *degenerated* if \mathcal{G}_i is reduced to a single transition (with no places).

A few definitions are in order

- A network is *autonomous* if each transition has at least one input place, and *non-autonomous* otherwise.
- In a non-autonomous network, the transitions which have no input place are called *entry* transitions; the places which follow them are called *entry places*.
- The places which do not belong to any component \mathcal{G}_i are either places with several input transitions and/or several output transitions, or entry places. This set of places will be called the set of *routing* places in the following, and it will be denoted \mathcal{R} .
- A MGC \mathcal{G}_i is *input-connected* if for each transition t in \mathcal{G}_i , there is a directed path from a routing place to t .

An example of decomposition is given in Figure 1.

2.4 Classification of Free Choice Nets

The following classification of the marked graph components of a FCNet is based upon their links with the routing places.

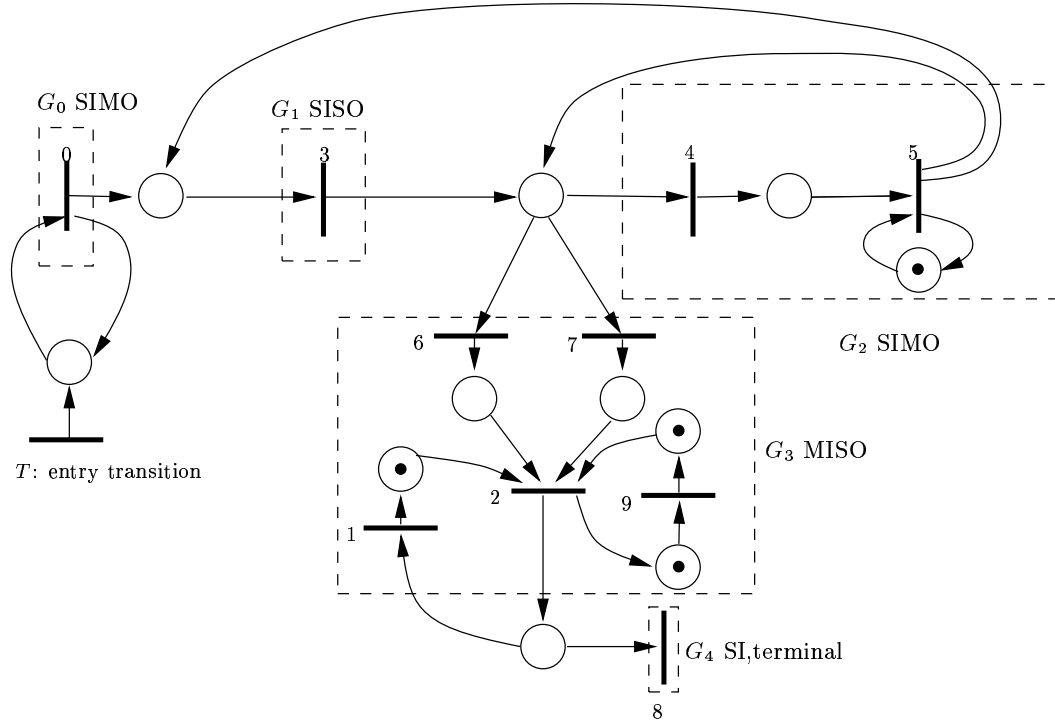


Figure 1: Decomposition of a FCNet into MGC's – T denotes an exogenous input.

A MGC \mathcal{G}_i is said *Single Input* (SI) if $\#\{t \in \mathcal{T}_i, \bullet t \notin \mathcal{P}_i\} = 1$. In this case, we will denote $t^{(i)}$ and call *input transition* of \mathcal{G}_i this unique transition. \mathcal{G}_i is said *Multiple Input* (MI) if $\#\{t \in \mathcal{T}_i, \bullet t \notin \mathcal{P}_i\} > 1$. If $\#\{t \in \mathcal{T}_i, \bullet t \notin \mathcal{P}_i\} = 0$, the MGC is *autonomous*. Similarly, a MGC \mathcal{G}_i is said *Single Output* (SO), *Multiple Output* (MO) or *terminal* depending on the value of $\#\{t \in \mathcal{T}_i, t^\bullet \notin \mathcal{P}_i\}$.

Thus all non-autonomous and non-terminal MGC's of a FCNet can be put in one of the four classes, SISO, SIMO, MISO, MIMO. A FCNet is said SI (resp. SO) if all its MGC's are SI (resp. SO) and MI (resp. MO) otherwise.

3 Algebraic Representation of Timed Free Choice Nets

In the following, we will first focus on timed FCNets; we will return later to properties of untimed nets in §4. We give an evolution equation for timed FCNets, which was first established in [5], and deduce from it an equation for the total number of events in the net.

3.1 Transformation of a FCNet

The FC condition states that no transition with more than one input place is preceded by a place with more than one output transition. In what follows, we will consider nets satisfying the following reinforcement of the FC condition:

A transition $t \in \mathcal{B}$ with more than one input place is never preceded by a place with more than one input transition.

We show in Appendix 1 that this assumption is not restrictive in the sense that a FCNet can always be transformed into an equivalent FCnet where this property is satisfied.

Note that under the above assumption, each transition of \mathcal{B} has *exactly one* input place, and this place is necessarily non-serial; it may either be the input of several transitions of \mathcal{B} or the output of several transitions, or both.

In what follows, all FCNets will be assumed to satisfy the above property.

In the following, it will also sometimes be useful to assume that all MGC's are *input connected*. By this, we mean that for all transitions t in a MGC \mathcal{G} , there is a path from an input transition of \mathcal{G} to t . We show in Appendix 1 that this assumption introduces no loss of generality either.

3.2 Evolution Equations for Timed Free Choice Nets

Counters For all transition $t \in \mathcal{T}'$, let $X^t(u)$ denote the *counter* associated with t , namely, the number of firings initiated by t before time u . We will consider the version of this function which is continuous to the right. Let $Y(u)$ be the vector $\{X^t(u), t \in \mathcal{A}\}$, $u \in \mathbb{R}^+$, where the transitions are arranged in some order, and let $Z(u)$ be the vector $\{X^t(u), t \in \mathcal{B}\}$, $u \in \mathbb{R}^+$.

We will denote $\nu^p(m)$ the m -th *routing decision* of place p ($\nu^p(m) \in p^\bullet$) and $\mathcal{H}^t(m)$ the sum

$$\mathcal{H}^t(m) = \sum_{l=1}^m 1_{\nu^{\bullet t}(l)=t}, \quad t \in \mathcal{B}. \quad (1)$$

The routing functions are said to be *fair*, if for all $t \in \mathcal{B}$, $\mathcal{H}^t(m)$ goes to infinity when m goes to infinity. Note that all routing functions defined as above are *conservative* in the sense that for all t ,

$$\sum_{s \in (\bullet t)^\bullet} \mathcal{H}^s(m) = m.$$

Throughout the paper, the default option is that all routing functions are fair.

Exogenous Arrivals In the case of non-autonomous nets, the entry transitions trigger *exogenous arrivals*. We will characterize these arrivals through *entry counters* defined as follows: for all $t \in \mathcal{B}$, we denote $R^t(u)$ the cumulated exogenous arrivals in the place $\bullet t$ up to time u . We will have $R^t(u) = \text{Constant}$ if $\bullet t$ is not an entry place. Let $R(u)$ be the

$\mathbb{N}^{|\mathcal{B}|}$ -vector $\{R^t(u), t \in \mathcal{B}\}$. Note that this vector may carry redundant information, like for instance when two \mathcal{B} transitions admit the same non-serial place as input place.

The initial condition of the entry vector $R(0-) = R_o$, is the $\mathbb{N}^{|\mathcal{B}|}$ -vector of initial markings in the routing places of $\bullet\mathcal{B}$: $R_o^t = c$ if $t \in \mathcal{B}$ is such that $\bullet t$ has an \mathcal{M}_0 -marking (prior to any exogenous arrival) of c . In the autonomous case, $R(u) = R_o$, for all $u \geq 0$.

Equations for Constant Integer Firing Times We shall first consider the case when firing times (and exogenous arrival times when appropriate) are all constant, positive, and integer multiples of a common number, which will be taken equal to 1 without loss of generality.

We will show in Section 6 how to address the case with varying (and in particular stochastic) firing times, which can be handled with a similar method. We will denote M the (integer-valued) upper-bound on the firing times.

Theorem 1 *Under the above assumptions, for all $k \in \mathbb{Z}$, the counting vectors $\{Y(k), Z(k)\}$ satisfy the following evolution equation:*

$$Y(k) = 0, \quad Z(k) = 0, \quad \forall k < 0, \quad (2)$$

and, for $k \geq 0$,

$$Y(k) = \bigoplus_{l=1}^M (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l)) \quad (3)$$

$$Z(k) = \mathcal{H} \left(\sum_{l=1}^M (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k) \right). \quad (4)$$

In this evolution equation, (\oplus, \otimes) respectively denote matrix products and additions in the $(\min, +)$ semi-ring (see [2]), whereas $(+, \times)$ denote the same operations but in the conventional algebra. The matrices used in the recurrence equations are defined from the net structure as follows:

- The $|\mathcal{A}| \times |\mathcal{A}|$ matrix A_l is defined by $A_l(t, t') = c$, if the firing time of $t' \in \mathcal{A}$ is l , and there is a serial place between $t' \in \mathcal{A}$ and t , with \mathcal{M}_0 -marking equal to c ; $\varepsilon (= \infty)$ otherwise. If there are more than one serial places between t' and t , we take c equal to the minimum of their \mathcal{M}_0 -markings.
- The $|\mathcal{A}| \times |\mathcal{B}|$ matrix B_l is defined by $B_l(t, t') = c$, if the firing time of $t' \in \mathcal{B}$ is l , and if there is a serial place between $t' \in \mathcal{B}$ and $t \in \mathcal{A}$, with \mathcal{M}_0 -marking equal to c ; ε otherwise.
- The $|\mathcal{B}| \times |\mathcal{B}|$ matrix P_l is defined by $P_l(t, t') = 1$, if the firing time of $t' \in \mathcal{B}$ is l , and there is a routing place connecting t' to t ; 0 otherwise.

- The $|\mathcal{B}| \times |\mathcal{A}|$ matrix Q_l is defined by $Q_l(t, t') = 1$, if the firing time of $t' \in \mathcal{A}$ is l , and there is a place connecting t' to $t \in \mathcal{B}$; 0 otherwise.
- For all vectors of integers $Z = (Z^1, \dots, Z^q)$, where $q = |\mathcal{B}|$, $\mathcal{H}(Z)$ is the vector of integers:

$$\mathcal{H}(Z) = (\mathcal{H}^1(Z^1), \dots, \mathcal{H}^q(Z^q)).$$

Proof: Equation (3) is obtained in a way which is similar to that used for establishing the evolution equation for event graphs in [2]. For instance, the number of firings initiated by transition $t \in \mathcal{A}$ at time k cannot exceed the minimum of the number of tokens arrived in the places of $\bullet t$ by time k , which is exactly

$$\bigoplus_{l=1}^M (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l))^t.$$

Furthermore, $Y^t(k)$ is equal to this quantity because transitions are assumed to fire as soon as they are enabled. For obtaining Equation (4), the key observation is that, due to our preliminary assumption, a transition t which belongs to \mathcal{B} has at most one input arc, which allows us to write (4), and so the number of firings it initiates by time k is simply the ' \mathcal{H}^t -filtering' of the total number of arrivals $N^p(k)$ into place $p = \bullet t$, up to time k , that is

$$N^p(k) = \left(\sum_{l=1}^M (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k) \right)^t.$$

■

Example: For the example of Figure 1, assuming that all firing times are equal to 1, we have $\mathcal{A} = \{2, 5, 9\}$, $\mathcal{B} = \{0, 1, 3, 4, 6, 7, 8\}$ and $M = 1$. Note that the entry transition does not belong to \mathcal{T}' and appears only in vector R . Up to the renumbering $(2, 5, 9), (0, 1, 3, 4, 6, 7, 8)$, we have

$$A_1 = \begin{pmatrix} \varepsilon & \varepsilon & 1 \\ \varepsilon & 1 & \varepsilon \\ 1 & \varepsilon & \varepsilon \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & \varepsilon & \varepsilon & 0 & 0 & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } R(k) = \begin{pmatrix} R^0(k) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Remark: *Linearity.* The \mathcal{H} function satisfies the relation

$$\mathcal{H}^t(m+n) = \mathcal{H}^t(m) + \mathcal{H}^t(n) \circ \tau_m, \quad (5)$$

where $\mathcal{H}^t(\cdot) \circ \tau_m$ is the \mathcal{H}^t function associated with the shifted sequence $\tau_m \nu^p(\cdot)$, namely the sequence

$$\tau_m \nu^p(n) = \nu^p(n+m). \quad (6)$$

For instance, whenever the sequence $\nu^p(\cdot)$ is random and stationary (for instance i.i.d., i.e. made of random variables which are independent and identically distributed), then $\mathbb{E}[\mathcal{H}^t(n) \circ \tau_m] = \mathbb{E}[\mathcal{H}^t(n)]$, so that

$$\mathbb{E}[\mathcal{H}^t(m+n)] = \mathbb{E}[\mathcal{H}^t(m)] + \mathbb{E}[\mathcal{H}^t(n)]. \quad (7)$$

So in this case, \mathcal{H} is ‘linear in expectation’. ■

Remark: *FCNet Realizing an Evolution Equation.* We have shown above how to establish evolution equations from the structure of the net. The construction of a net satisfying a given evolution equation is possible as well, as shown below. Consider the following data:

- an integer $M \geq 1$;
- $\mathbb{N} \cup \{\varepsilon\}$ -valued matrices A_l ($n \times n$) and B_l ($n \times m$), $l = 1, \dots, M$;
- $\{0, 1\}$ -valued matrices P_l ($m \times m$) and Q_l ($m \times n$), $l = 1, \dots, M$;
- a collection of non-decreasing functions: $\mathcal{H}^t : \mathbb{N} \rightarrow \mathbb{N}$, $t = 1, \dots, m$;
- a collection of \mathbb{N}^m -valued functions $R(k)$, non-decreasing in $k \geq -1$.

Assume that this set of data is *realizable*, which consists in the following assumptions:

- for all $j = 1, \dots, n$, there is a unique $1 \leq l(j) \leq M$, such that for all $l \neq l(j)$, the columns $A_l(\cdot, j)$ and $Q_l(\cdot, j)$ are empty (which translates the fact that transition j has a uniquely defined firing time $l(j)$).
- For all $j = 1, \dots, m$, there is a unique $l(j)$ such that for all $l \neq l(j)$, the columns $B_l(\cdot, j)$ and $P_l(\cdot, j)$ are empty.
- There exists a partition $\mathcal{E}^1, \dots, \mathcal{E}^r$ of $\{1, \dots, m\}$, such that

$$\sum_{s \in \mathcal{E}^q} \mathcal{H}^s(n) = n, \quad \forall n \in \mathbb{N}, \quad q \in \{1, \dots, r\}, \quad (8)$$

and such that for all $l = 1, \dots, M$, and $i \in \mathcal{E}^q$,

$$P_l(i, j) = P_l^{(q)}(j), \quad Q_l(i, j) = Q_l^{(q)}(j), \quad R(i)(k) = R^{(q)}(k), \quad \forall k$$

(the set \mathcal{E}^q will consist of all transitions of \mathcal{B} which share a common input place).

- For all $j = 1, \dots, m$, either $j \in \mathcal{E}^q$ with $|\mathcal{E}^q| > 1$, or $|\{i \text{ s.t. } P_{l(i)}(j, i)\}| + |\{i \text{ s.t. } Q_{l(i)}(j, i)\}| > 1$ or $R^j(\cdot)$ is non-constant (the places of \mathcal{B} are non-serial).

Associated with realizable data, one can construct a FCNet with

- $m + n$ non-entry transitions, (n in \mathcal{A} , m in \mathcal{B}), with respective firing time $l(j)$ for transition j ;
- As many serial places as there are non- ε entries in the matrices A_l and B_l : if $A_l(i, j) = c$, there is a serial place p with an initial marking of c and arcs from j to p and from p to i (and the same for B_l);
- r routing places: the set of input transitions of place q is

$$\{j \text{ s.t. } \exists l \text{ with } P_l^q(j) = 1\} \cup \{i \text{ s.t. } \exists l \text{ with } Q_l^q(i) = 1\}$$

Its set of output transitions is \mathcal{E}^q . Its initial marking is R^q .

- As many entry transitions t^q as there are non-constant $R^q(\cdot)$ functions, with an arc from t^q to q .

The net constructed above admits an evolution equation which is that of Theorem 1. In that sense, there exists a FCNet *realizing* an evolution equation of the form (3), (4). ■

Remark: *Constructiveness.* Even if the initial net is such that all its firing times are positive, the transformation of FC nets which was made at the beginning of § 3 may make it necessary to consider the case with some zero firing time transitions; this may translate into an extra term of the form $B_0 \otimes Z(k)$ in (3). Such an additional term preserves the ‘constructiveness’ of the evolution equation. By constructiveness of the generalized equation

$$Y(k) = \bigoplus_{l=0}^M (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l)) \quad (9)$$

$$Z(k) = \mathcal{H} \left(\sum_{l=0}^M (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R(k) \right) \quad (10)$$

(it generalizes the evolution equations of Theorem 1 because l now ranges from 0 to M), we mean that there exists an ordering of the coordinates of $X(k) = (Y(k), Z(k))$, say $X^{i_1}(k), \dots, X^{i_{|\mathcal{T}|}}(k)$, such that, for all $j = 1, \dots, |\mathcal{T}|$, the line corresponding to $X^{i_j}(k)$ in (9)-(10) is such that no term of the form $X^{i_l}(k)$, $l \geq j$, can be found in the right hand side. This property is of course essential for simulation purposes.

There is an easy algebraic characterization of this property: let S be the $\mathcal{T} \times \mathcal{T}$ matrix defined by

$$S = \begin{pmatrix} \text{supp}(A_0) & \text{supp}(B_0) \\ \text{supp}(P_0) & \text{supp}(Q_0) \end{pmatrix}, \quad (11)$$

where the support, $\text{supp}(A)$, of a (\oplus, \otimes) -matrix A is a matrix of the same size and such that $\text{supp}(A)_{i,j} = 1$ if $A_{i,j} \neq \varepsilon$, and 0 otherwise; similarly the support, $\text{supp}(P)$, of a $(+, \times)$ -matrix P is a matrix of the same size and such that $\text{supp}(P)_{i,j} = 1$ if $P_{i,j} \neq 0$, and 0 otherwise. The equations are constructive iff there exists a permutation U such that $U^{-1} \times S \times U$ is strictly lower triangular.

All the results that we prove in this paper can be extended to nets with constructive evolution equations. ■

Remark: *Localization.* From the above equations, it is easily checked by induction that the state variables $X(k)$ satisfy the following ‘localization’ property: if $R(k)$ and $R'(k)$, $k \geq 0$ are two functions which coincide up to time K , then two nets which would only differ in their input counters R and R' are such that their state variables $X(k)$ and $X'(k)$ coincide up to time K . ■

Remark: *Parallel Simulation.* The equations (3)-(4) capture the evolution of the net in terms of matrix-vector multiplications. They can be used to simulate the system using efficient parallel algorithms (see [6]). ■

Remark: *Fluid Version.* A fluid version of these equations, which consists in replacing the \mathcal{H} operator by a (conventional algebra) matrix, was recently proposed and investigated in [8]. ■

3.3 Total Firing

3.3.1 Total Firing Equation

Let $Y = Y(\infty)$ and $Z = Z(\infty)$ denote the vectors counting the total number of firings of the transitions.

Lemma 1 *The integer-valued vectors Z and Y satisfy the system of equations*

$$Y = A \otimes Y \oplus B \otimes Z \quad (12)$$

$$Z = \mathcal{H}(P \times Z + Q \times Y + R), \quad (13)$$

where $A = \bigoplus_{l=1}^M A_l$, $B = \bigoplus_{l=1}^M B_l$, $P = \sum_{l=1}^M P_l$, $Q = \sum_{l=1}^M Q_l$ and $R = \lim_{k \rightarrow \infty} R(k)$.

We first prove a general lemma on this type of fixed point equations.

Lemma 2 *Let A, B, P, Q and R be any non-negative integer valued matrices with respective dimensions $(n \times n)$, $(n \times m)$, $(m \times m)$, $(m \times n)$ and $(n \times 1)$. Let $\mathcal{H} : \mathbb{N}^m \rightarrow \mathbb{N}^m$ be a non-decreasing map. Then the system of equations*

$$\begin{aligned} Y &= A \otimes Y \oplus B \otimes Z \\ Z &= \mathcal{H}(P \times Z + Q \times Y + R), \end{aligned}$$

admits a minimum¹ non-negative solution (for the coordinate-wise partial order).

Proof: The operator

$$\begin{aligned} \Phi : \mathbb{N}^{|A|} \times \mathbb{N}^{|B|} &\rightarrow \mathbb{N}^{|A|} \times \mathbb{N}^{|B|} \\ (Y, Z) &\rightarrow (A \otimes Y \oplus B \otimes Z, \mathcal{H}(P \times Z + Q \times Y + R)) \end{aligned}$$

is component-wise non-decreasing and integer-valued. Any finite non-negative solution (S_1, S_2) of (12)-(13) is a finite fixed point of Φ : $(S_1, S_2) = \Phi(S_1, S_2)$.

Let us now define the sequence of vectors $(\alpha(k), \beta(k))_{k \in \mathbb{N}}$ in $\mathbb{N}^{|A|} \times \mathbb{N}^{|B|}$ by $(\alpha(0), \beta(0)) = (0, 0)$ and

$$(\alpha(k+1), \beta(k+1)) = \Phi(\alpha(k), \beta(k)).$$

This sequence is component-wise non-decreasing and so, it has a limit when k goes to infinity denoted (α, β) . The vector (α, β) is a fixed point of Φ (actually, whenever (α, β) is finite, the limit vector is reached in a finite number of steps).

A straightforward induction on n shows that for all other non-negative fixed points (S_1, S_2) of Φ , $(\alpha, \beta) \leq (S_1, S_2)$, where the order relation is component-wise. ■

Lemma 3 *The system of equations (12)-(13) admits a minimum non-negative solution (for the coordinate-wise partial order). It admits a finite non-negative solution if and only if the net reaches a deadlock. Furthermore, the total firing vector is the minimum non-negative solution of this system.*

Proof: The first assertion is a direct consequence of the previous lemma. If the net reaches a deadlock, the total firing vector $X = (Y, Z)$ is finite and non-negative and Lemma 1 states that this vector is a solution of (12)-(13).

Conversely, let us assume that this system admits a finite solution. Then the variables (α, β) in the proof of the previous lemma are necessarily finite. Consider the sequences $(Y'(k), Z'(k))$ defined by $(Y'(k), Z'(k)) = (0, 0)$, for $k < 0$ and

$$\begin{aligned} Y'(k) &= \bigoplus_{l=1}^M (A_l \otimes Y'(k-l) \oplus B_l \otimes Z'(k-l)) \\ Z'(k) &= \mathcal{H} \left(\sum_{l=1}^M (P_l \times Z'(k-l) + Q_l \times Y'(k-l)) + R \right) \end{aligned}$$

¹The minimum element over any partially ordered set E is the smallest element in E (it may not always exist) while a minimal element does not admit a smaller element in E

for $k \geq 0$. Using the fact that $R(k) \leq R$, we prove by induction that $(Y(k), Z(k)) \leq (Y'(k), Z'(k))$ for all k . Similarly, an induction shows that for all $k \in \mathbb{N}$, $(Y'(k-1), Z'(k-1)) \leq (\alpha(k), \beta(k))$ component-wise. Therefore, $(Y(\infty), Z(\infty)) \leq (\alpha, \beta)$ which is finite, Therefore $(Y(\infty), Z(\infty)) = (\alpha, \beta)$. ■

Remark: This proof also provides another way of computing the total firing vector when the system reaches a deadlock. Indeed, we can iterate the operator Φ on a null vector until convergence to a fixed point which will be the total firing vector. ■

Remark: *Invariance of Total Firing.* A striking property is that this system and its minimum solution does not depend on the variables σ^t anymore: in other words, all properties based on total firing (like starvation, liveness, deadlock etc.) are associated with the routing functions, the topology, the initial marking and the R vector only, and *not* with timing variables (neither firing times nor arrival epochs of exogenous tokens).

This invariance property also holds true when the firing times are non-constant but finite (see §6). Similar observations with different levels of formalization can be found in [1, 16]. ■

3.3.2 Simplification for Live MGC's

A MGC will be said to be *live* if each of its cycles contains at least one token. In case all the MGC's are live, the system of equations (12)-(13) can be simplified in that one of its unknowns can be eliminated. Let $f_k = \min_{s,t} A^k(t, s)$. From the super-additive inequality $f_{k+l} \geq f_k + f_l$, it follows that the limit

$$\frac{\min_{s,t} A^k(t, s)}{k} \rightarrow \gamma,$$

holds as k tends to ∞ . The constant γ (possibly infinite) is the $(\min, +)$ Perron-Frobenius eigenvalue of A , and is given by the following formula [2]:

$$\gamma = \inf_{s \in \mathcal{S}} \frac{n(s)}{|s|},$$

where \mathcal{S} is the set of all circuits in the MGC's, $n(s)$ is the total number of tokens in circuit s , $|s|$ is the length (number of places) of s , and with the convention that the infimum over an empty set is infinite.

The liveness of all MGC's implies that $\gamma > 0$, so that f_k tends to ∞ . Let

$$A^* = \bigoplus_{k=0}^{\infty} A^k, \tag{14}$$

where A^k is the k -th power of A in $(\min, +)$ (this series is well defined since the sequence $H_k \stackrel{\text{def}}{=} A^0 \oplus A^1 \oplus \dots \oplus A^k$ is monotone). Let C be the matrix

$$C = A^* \otimes B.$$

For all k ,

$$Y = H_k \otimes B \otimes Z \oplus A^{k+1}Y,$$

and using the fact that f_k tends to ∞ , we get the simplified system (E):

$$Y = C \otimes Z \tag{15}$$

$$Z = \mathcal{H}(P \times Z + Q \times (C \otimes Z) + R). \tag{16}$$

Note that the equation for total firing only depends on A and B via matrix C ; in other words, in this case, all properties pertaining to total firing only depend on C, \mathcal{H}, P, Q and R . ■

Remark: *Input Connectedness of a MGC.* The MGC's are all input-connected if and only if the matrix $C = A^* \otimes B$ has no empty line (i.e. no line composed only of ε 's, where $\varepsilon = \infty$). ■

3.3.3 Reduced Net

To a FCNet with evolution equation as in Theorem 1 and with live MGC's, we associate the evolution equation

$$Y(k+1) = C \otimes Z(k) \tag{17}$$

$$Z(k+1) = \mathcal{H}(P \times Z(k) + Q \times Y(k) + R(k)). \tag{18}$$

The corresponding set of data is realizable, with $M = 1$, and so there exists a FCNet with all its firing times equal to 1, which is a realization of this evolution equation, and which we call the *reduced net* associated with the original net. The reduced net of the FCNet depicted in Figure 1 with total number of entries by T equal to 1, is given in Figure 2.

The total number of firings of the transitions in this net is the same as that of the corresponding transitions in the original net. In that sense, it is possible to test all properties pertaining to total firing on this reduced net. ■

Remark: *Total Firing—the SI case.* In the SI case, the equations for total firing can be further simplified. In this case, C has at most one finite element per line (exactly one in the input-connected case), and $C \otimes Z$, is just a permutation U of Z plus a constant L (which

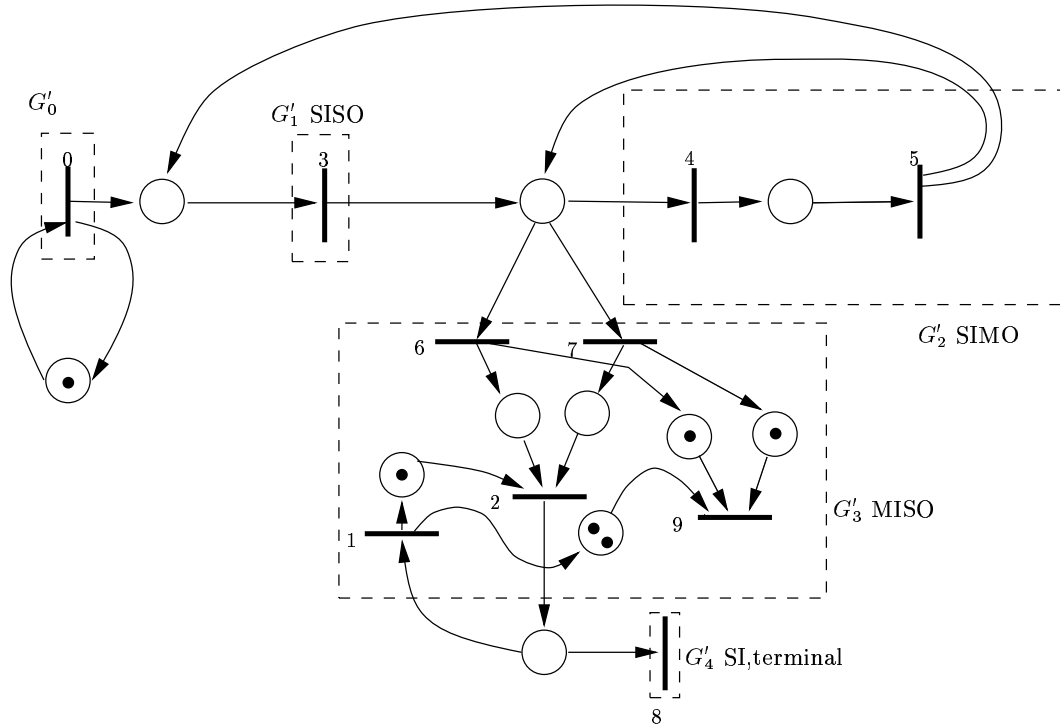


Figure 2: Reduced net associated with the FCNet depicted in Figure 1.

we may have to take equal to ∞ in the non input-connected case): $C \otimes Z = U \times Z + L$. Let $Q' = Q \times U$,

$$P' = P + Q', \quad R' = R + Q \times L. \quad (19)$$

By elimination of Y , Equation (15) becomes

$$Z = \mathcal{H}(P' \times Z + R') \quad (20)$$

$$Y = C \otimes Z. \quad (21)$$

■

3.4 SI-Subnets of a FCNet

Associated with a FCNet with live input-connected MGC's, and its matrix C , we define the set of matrices $C_{\langle s \rangle}$, $s \in \{1, \dots, K\}$; each matrix is obtained by picking only one non- ε element in each line of C : if for some i and j , $C(i, j) = c < \infty$ and s is such that $C_{\langle s \rangle}(i, j) = c$, then $C_{\langle s \rangle}(i, k) = \varepsilon$, for all $k \neq j$. Here K denotes the number of possible combinations. Note that if the original net is SI, then $K = 1$ and $C_{\langle 1 \rangle} = C$.

When the original net is MI, for each $s \in \{1, \dots, K\}$, the evolution equation

$$Y(k+1) = C_{\langle s \rangle} \otimes Z(k) \quad (22)$$

$$Z(k+1) = \mathcal{H}(P \times Z(k) + Q \times Y(k) + R(k)). \quad (23)$$

has realizable data, and so there exists a SI-FCNet $F_{\langle s \rangle}$ realizing this equation. Each of the $F_{\langle s \rangle}$ nets derived in such a way from the original net F , will be called a *SI-subnet* of F ; $F_{\langle s \rangle}$ can be constructed directly from the net F' realizing Equation (17) as follows: for each transition t_i of \mathcal{G}_l , one of its F' -predecessors is selected (necessarily in the set $\mathcal{B} \cap G_l$), say t_j , and in $F_{\langle s \rangle}$, t_i has t_j for unique predecessor. All other transitions t_k of $\mathcal{B} \cap G_l$ are disconnected from t_i by removing the place between them. Note that a SI-subnet of a FCNet is a SI-FCNet.

Figure 3 illustrates this transformation, together with the fact that it may create transitions with no output place. It may also split certain MGC's into disconnected components.

Another example associated with the net of Figure 1 is considered in Figure 6. There are 6 different SI-subnets of \mathcal{G}'_3 (and therefore, of the whole net) all displayed in Figure 6.

The equations for total firing of this new system ($E_{\langle s \rangle}$) reads:

$$Y_{\langle s \rangle} = C_{\langle s \rangle} \otimes Z_{\langle s \rangle} \quad (24)$$

$$Z_{\langle s \rangle} = \mathcal{H}(P \times Z_{\langle s \rangle} + Q \times (C_{\langle s \rangle} \otimes Z_{\langle s \rangle}) + R). \quad (25)$$

Since $C_{\langle s \rangle}$ has only one finite element per line, one can use the simplifications proposed in (20): we can write $C_{\langle s \rangle} \otimes Z_{\langle s \rangle} = U_{\langle s \rangle} \times Z_{\langle s \rangle} + L_{\langle s \rangle}$, where $U_{\langle s \rangle}$ is a permutation. So if we define $Q_{\langle s \rangle} = Q \times U_{\langle s \rangle}$, $P_{\langle s \rangle} = P + Q_{\langle s \rangle}$ and $R_{\langle s \rangle} = R + Q \times L_{\langle s \rangle}$, then Equation (25) becomes

$$Z_{\langle s \rangle} = \mathcal{H}(P_{\langle s \rangle} \times Z_{\langle s \rangle} + R_{\langle s \rangle}).$$

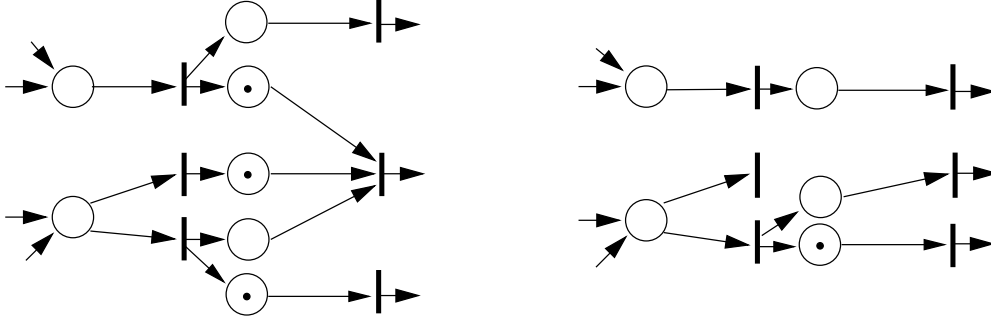


Figure 3: Construction of a SI-Subnet from a reduced net by keeping only one connection for each output transition.

The following lemma will be crucial for the analysis of logical properties in §4.

Lemma 4 *If (Y, Z) is the minimum solution of (E) then, we can find $s \in \{1, \dots, K\}$, such that (Y, Z) is the minimum solution of $(E_{<s>})$.*

Proof: We choose s_0 , or equivalently the non- ε element of line i of C , for all i , as follows:

$$C_{<s_0>}(i, j) = \begin{cases} C(i, j) \geq 0 & \text{if } C(i, j) + Z^j \leq C(i, k) + Z^k \forall k \\ \varepsilon & \text{otherwise} \end{cases}$$

(if there are more than one j achieving the minimum, one of them is chosen in an arbitrary way). So $C_{<s_0>} \otimes Z = C \otimes Z$. This and the equations

$$Y = C \otimes Z \tag{26}$$

$$Z = \mathcal{H}(P \times Z + Q \times (C \otimes Z) + R), \tag{27}$$

show that

$$Y = C_{<s_0>} \otimes Z \tag{28}$$

$$Z = \mathcal{H}(P \times Z + Q \times (C_{<s_0>} \otimes Z) + R), \tag{29}$$

and so, (Y, Z) is a solution of $(E_{<s_0>})$ which implies that $(Y, Z) \geq (Y_{<s_0>}, Z_{<s_0>})$. Since $(Y, Z) \leq (Y_{<s>}, Z_{<s>})$ for all s , then necessarily $(Y, Z) = (Y_{<s_0>}, Z_{<s_0>})$. ■

4 Logical Properties

All nets considered in this section will be assumed to be autonomous.

The aim of the present section is to address logical properties of the net like liveness, starvation and deadlocks via the MGC decomposition and the associated equations, particularly the total firing equations (Lemma 1).

Two classes of such properties will be considered: \mathcal{H} -bound and \mathcal{H} -free properties. By \mathcal{H} -free, we mean here properties which hold for all fair \mathcal{H} , whereas properties true for a specific \mathcal{H} function (or a specific class of \mathcal{H} functions) will be called \mathcal{H} -bound.

4.1 \mathcal{H} -Bound Properties

4.1.1 Starvation

A transition is in \mathcal{H} -starvation for some initial marking if during the evolution of the net, it fires only a finite number of times. If all transitions are in \mathcal{H} -starvation, we will say that the net is in \mathcal{H} -starvation. \mathcal{H} -starvation can naturally be formulated in terms of finiteness properties of the minimum solution of Equations (12)-(13). Since the total firing vector does not depend on the timing variables σ^t , \mathcal{H} -starvation is actually an assumption on \mathcal{H} (and on the topology of the net of course).

In the SI case, whenever the routing decisions are probabilistic and independent, one can use Equation (20) to assess this property by reduction to simple linear algebra tests:

Lemma 5 *Let F be a SI-FCNet. Assume that the routing sequences are realizations of the following random scheme: the sequences $\{\nu^p(\cdot)\}$ are realizations of mutually independent random sequences, and for each p the sequence $\{\nu^p(\cdot)\}$ is a sequence of realizations of independent and identically distributed random variables. Assume in addition that R' (defined in (19)) is finite. Let Ψ be the $|\mathcal{B}| \times |\mathcal{B}|$ -diagonal (conventional algebra) matrix with t -th diagonal term equal to $\mathbb{P}[\nu^{\bullet t}(1) = t]$. Then, whenever the Perron-Frobenius eigenvalue of the positive matrix $\Psi P'$ (where P' was defined in (19)) is strictly less than 1, then for almost all realizations of the routing function \mathcal{H} in the above class, the net is in starvation.*

Proof: Taking expectations on both sides of (20), we obtain (on line t)

$$\begin{aligned} \mathbb{E}[Z^t] &= \mathbb{E}\left[\sum_{i=1}^{(P'Z+R')^t} 1_{\nu^{\bullet t}(i)=t}\right] = \sum_{i=1}^{\infty} \mathbb{E}[1_{(P'Z+R')^t \geq i} 1_{\nu^{\bullet t}(i)=t}] \\ &= \sum_{i=1}^{\infty} \mathbb{E}[1_{(P'Z+R')^t > i-1} 1_{\nu^{\bullet t}(i)=t}]. \end{aligned}$$

But it is clear that the events $\{(P'Z + R')^t \leq i - 1\}$ and $\{\nu^{\bullet t}(i) = t\}$ are independent, and so

$$\begin{aligned} \mathbb{E}[Z^t] &= \sum_{i=1}^{\infty} \mathbb{P}[(P'Z + R')^t > i - 1] \mathbb{P}[\nu^{\bullet t}(1) = t] = \mathbb{E}[(P'Z + R')^t] \mathbb{P}[\nu^{\bullet t}(1) = t] \\ &= (P' \mathbb{E}[Z] + R')^t \mathbb{P}[\nu^{\bullet t}(1) = t]. \end{aligned}$$

Finally

$$\mathbb{E}[Z] = \Psi P' \mathbb{E}[Z] + \Psi R'. \quad (30)$$

The fact that the Perron-Frobenius eigenvalue of $\Psi P'$ is strictly less than one implies that (30) admits a unique finite solution \tilde{Z} . But this in turn implies that the expectation of the minimum solution Z of (20)-(21) is finite, and coincides in fact with the unique finite solution of (30). In order to see this, consider sequence $Z(k)$ of Lemma 6. It is easy to check that

1. $Z(k)$ converges monotonically to Z ;
2. $\mathbb{E}(Z(k)) < \infty$, for all k ;
3. $\mathbb{E}(Z(k)) \leq \Psi P' \mathbb{E}[Z(k)] + \Psi R'$.

But (2) and (3) imply that $\mathbb{E}(Z(k)) \leq \tilde{Z}$, and so, when using (1) and the monotone convergence theorem, we obtain that

$$\mathbb{E}(Z) = \mathbb{E}(\lim_k Z(k)) = \lim_k \mathbb{E}(Z(k)) \leq \tilde{Z}.$$

Therefore $\mathbb{E}(Z)$ is finite (in fact since $\mathbb{E}(Z)$ is a solution of (30), then necessarily $\mathbb{E}(Z) = \tilde{Z}$). ■

Remark: *On Lemma 5.* This lemma covers both the autonomous case (in this case, the condition $R' < \infty$ will be satisfied whenever the MGC's are all input connected and the initial marking in the routing places is finite) and the non-autonomous case with exogenous arrivals (there, one has to add a third condition that the total number of exogenous arrivals is finite as well, or equivalently that the entry transitions are in starvation).

Extensions of this to non-i.i.d. sequences is possible using ergodic theory, and was actually done in the SISO case (see [3]). ■

In the MI case, we give below a sufficient condition for starvation, which reduces to checking this property for the SI-subnets of the net.

Lemma 6 *The net F , with live MGC's, \mathcal{H} -starves iff one of its SI-subnet $F_{\langle s \rangle}$ \mathcal{H} -starves.*

Proof If $F_{\langle s \rangle}$ \mathcal{H} -starves for some s , then $(Y_{\langle s \rangle}, Z_{\langle s \rangle}) < \infty$. But since $C \leq C_{\langle s \rangle}$, $(Y, Z) \leq (Y_{\langle s \rangle}, Z_{\langle s \rangle})$, and so F \mathcal{H} -starves as well. Conversely, if F \mathcal{H} -starves, in view of Lemma 4, one of its SI-subnets \mathcal{H} -starves. ■

We conclude this subsection with a lemma which will be used later on.

Lemma 7 *If the net starves, then the deadlock which is reached is such that each routing place is empty.*

Proof Let G be the $|\mathcal{B}| \times |\mathcal{B}|$ -matrix defined by $K(t, t') = 1$ if $t \in \mathcal{B}$ and $t' \in (\bullet t)^\bullet$ (or equivalently if t and t' belong to the same \mathcal{E}^l). Since for all finite vectors Z

$$K \times \mathcal{H}(Z) = Z,$$

then the solution of (12)-(13) satisfies

$$K \times Z = P \times Z + Q \times Y + R, \quad (31)$$

which implies that the total number of tokens that ever leave place p is indeed equal to the sum of the initial number of tokens in p plus the total number that ever enter p . ■

4.2 \mathcal{H} -Free Properties

Observe that a \mathcal{H} -free property bearing on the total firing vector is actually an *untimed* property since it does neither depend on the firing times (from Lemma 1 and the following remark) nor on the routing decisions, by definition).

4.2.1 Deadlocks

For a FCNet, a deadlock is a marking D under which no transition is enabled (i.e $\forall t \in T, \exists p \in \bullet t$, with $D(p) = 0$). In a FCNet this is a \mathcal{H} -free property. The most basic question concerning deadlocks is certainly whether the initial marking is a deadlock.

Lemma 8 *The initial marking is a deadlock if and only if $R_o = 0$ and A and B satisfy the following property: for each i , either the line $A(i, \cdot)$ or the line $B(i, \cdot)$ have at least one element equal to 0. When all MGC's are live, the initial marking is a deadlock iff $R_o = 0$ and the matrix C has at least one zero per line (which implies that all MGC's are input connected).*

Proof: From Lemma 3, the initial marking is a deadlock if and only if $(0, 0)$ is a solution of the equations

$$Y = A \otimes Y \oplus B \otimes Z \quad (32)$$

$$Z = \mathcal{H}(P \times Z + Q \times Y + R_o). \quad (33)$$

Assume that $(0, 0)$ is a solution. Then from (33), $0 = \mathcal{H}(R_o)$, and necessarily $R_o = 0$ since \mathcal{H} is conservative. Similarly, (32) implies that for each i , either the line $A(i, \cdot)$ or the line $B(i, \cdot)$ have at least one element equal to 0.

Conversely, $R_o = 0$ and (A, B) satisfies the above property, using the fact that $\mathcal{H}(0) = 0$, we see that $(0, 0)$ is a solution of (E). The second result is obtained in a similar way from (15). ■

Remark: *Case of Live MGC's.* The last characterization of the lemma is equivalent to the following property of the net: all routing places are empty and for any transition of the net, there is a token free path from some routing place to this transition. ■

4.2.2 Starvation

The \mathcal{H} -free version of starvation reads: for all possible routings, the net eventually enters a deadlock.

In the SI-FCNet case, it is easily checked that this condition boils down to the absence of circuits containing routing places in the net.

By the same type of arguments as in the \mathcal{H} -bound case, whenever there exists a SI-subnet of the net which is in \mathcal{H} -free starvation, so is the original net.

4.2.3 Liveness

A FCNet is live if for all reachable marking \mathcal{M} , for all transition t , there exists a sequence of firings leading from \mathcal{M} to some marking where t is enabled. Note that this is an untimed notion.

A FCNet with some of its MGC's which are not live cannot be live. In this section, we will therefore concentrate on the case with live MGC's.

Our first step consists in showing that liveness can be checked on the total firing equations.

Lemma 9 *The untimed net is live if and only if for all fair routing functions, the minimum solution of (12),(13) is $(Y, Z) = (\infty, \infty)$.*

Proof: Suppose that some fair \mathcal{H} , the minimum solution $X = (Y, Z)$ of (12), (13) is not infinite. Let us assume that X_1, X_2, \dots, X_h are the finite elements of X . So, for this evolution of the net, the transitions of $\tau = \{t_1, \dots, t_h\}$ fire a finite number of times, for all timed evolution of the net. Let us chose some positive values for the firing times of the transitions, and let d_0 be the epoch when all the transitions of τ stop firing in the corresponding timed evolution of the net. So we have $X_i(u) = X_i$, for all $0 \leq i \leq h$ and $u \geq d_0$. Let $t_i \in \tau$.

- If $t_i \in \mathcal{B}$, then by fairness of \mathcal{H} , the place $p_i = \bullet t_i$ receives a finite number of tokens during the evolution of the net. Let $d_1(i)$ be the epoch when the last token enters place p_i . From time $d_1(i) + 1$ on, place p_i is empty: if one denotes $\mathcal{M}(u)$ the marking at time u , then $\mathcal{M}(u)(p_i) = 0$ for all $u \geq d_1(i) + 1$.
- If $t_i \in \mathcal{A}$, then all its input places are serial. There exists a place $p_i \in \bullet t_i$, such that $\mathcal{M}(u)(p_i) = 0$, for all $u \geq d_0$ (otherwise we would have $X_i(u + 1) > X_i(d_0)$).

Choose $d_2 > d_0 \vee (\max_{i=1, \dots, h} (d_1(i) + 1))$; so for all $t_i \in \tau$, there exists at least one place $p_i \in \bullet t_i$ which is empty under the marking $\mathcal{M}(u)$, for all $u \geq d_2$.

Now, we show that $S = \{p_1, \dots, p_h\}$ forms a *siphon* (a siphon is a set of places σ verifying the set inclusion $\bullet\sigma \subset \sigma\bullet$ – see § 5). Let $p_i \in S$. Since $\mathcal{M}(u)(p_i) = 0$, for all $u \geq d_2$, then p_i receives a finite number of tokens during the evolution of the net. This number is $\sum_{j \in \bullet p_i} X_j$. Therefore, all input transitions of p_i are in τ and have an input place in S . So S is a siphon indeed.

Therefore, there exists a reachable marking ($\mathcal{M}(d_2)$) of the untimed net under which siphon S is *empty* (each place of S has no tokens in $\mathcal{M}(d_2)$). But an empty siphon remains empty for all further evolutions of the net as from this marking on, there exists no sequence of firings of the untimed net capable of enabling a transition of τ . Therefore, the net is not live. This concludes the proof of the fact that the liveness property implies that $(Y, Z) = (\infty, \infty)$ for all fair \mathcal{H} .

For the converse, we will use the fact that the invariance of total firing holds true for non-constant firing times as well (see §6). For each reachable marking \mathcal{M} (in the untimed sense), there exists a choice of firing durations and routing decisions which leads to \mathcal{M} in the timed version of the net as well. Actually, the choice concerns a finite initial subsequence of the routing and the firing sequences. If the minimum solution of (12),(13) is $(Y, Z) = (\infty, \infty)$, for all fair \mathcal{H} , then it is in particular infinite for any fair continuation of this specific initial subsequence of routing decisions. This implies that for all fair continuations of the \mathcal{H} sequence and for all t , transition t becomes eventually enabled from \mathcal{M} , This in turn implies that a marking which enables t is reachable from \mathcal{M} , which proves liveness. ■

Lemma 10 *The minimum solution of Equations (12),(13) is $(Y, Z) = (\infty, \infty)$ iff for all SI-subnets $F_{\langle s \rangle}$ of F , $(Y_{\langle s \rangle}, Z_{\langle s \rangle}) = (\infty, \infty)$.*

Proof: The condition is clearly necessary as $(Y, Z) \leq (Y_{\langle s \rangle}, Z_{\langle s \rangle})$, for all $s = 1, \dots, K$. It is also sufficient as there exists a SI-subnet $F_{\langle s_0 \rangle}$ such that $(Y, Z) = (Y_{\langle s_0 \rangle}, Z_{\langle s_0 \rangle})$ (see Lemma 4). ■

By Lemmas 9 and 10, we can now concentrate on the infinite behavior of SI-FCNets with fair routings.

4.2.4 Liveness of SI-FCNets

In the following, F will be a SI-FCNet with a fair routing and live input-connected MGC's. As mentioned earlier, the assumption that all MGC's are input-connected can be made with no loss of generality.

First, we recall that if F is a SI-FCNet with live input-connected MGC's, then the total number of firings is given by equations of the form:

$$Y = C \otimes Z \tag{34}$$

$$Z = \mathcal{H}(PZ + R), \tag{35}$$

where R is finite.

We will focus on variable Z which characterizes completely the infinite behavior of the net and forget about $Y = C \otimes Z$. Indeed $(Y, Z) = \infty \Leftrightarrow Z = \infty$.

We decompose P into its strongly connected components, which admit a partial order, referred to as the *reduced order*.

We can choose a total order of the strongly connected components such that

- P has a triangular block form;
- For all strongly connected components π , the degenerated components (of size one) which are direct successors of π in the reduced partial order are put as early as possible in the total order, namely arranged in (some) sequence just after this component. We will say that these degenerated components are *associated* with π .

An *initial block* of P is now constructed by aggregating into a single block matrix a strongly connected component without predecessors in the reduced order, and all the degenerated components which are associated with it. Thus matrix P has the following block form:

$$P = \begin{pmatrix} P_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & P_h & 0 & \cdots & 0 \\ L_{h+1,1} & \cdots & \cdots & L_{h+1,h} & P_{h+1} & \cdots & 0 \\ \vdots & & & & \ddots & \ddots & \vdots \\ L_{c,1} & \cdots & & & & L_{c,c-1} & P_c \end{pmatrix},$$

where P_1, \dots, P_h are the sub-matrices associated with the initial blocks (these matrices have no null lines) and P_{h+1}, \dots, P_c are strongly connected components (possibly degenerated). Each initial block P_l is such that the restriction of the routing function on the corresponding set of coordinates is fair and conservative; the first property would not hold if the associated components had not been added.

The liveness of a SI-FCNet can be checked on its initial components as shown by the following lemma:

Lemma 11 *The minimum solution $Z = (Z_1, \dots, Z_c)$ of Equation (35) is infinite if and only if it is infinite on all initial blocks, i.e. $Z_i = \infty$ for all $i = 1, \dots, h$.*

Proof: The “only if” direction is immediate. For the converse suppose that $Z_1 = \infty, \dots, Z_h = \infty$. Then for all $k > h$, $Z_k = \mathcal{H}_k(L_{k,1}Z_1 + \cdots + L_{k,k-1}Z_{k-1} + P_k Z_k + R_k)$, where at least one of the matrices $L_{k,i}$ is non-null, so that at least one of the elements in Z_k is infinite. By strong connectedness of P_k and fairness of \mathcal{H}_k , all the elements of Z_k are infinite. ■

In the following, we will focus on initial blocks.

Lemma 12 *The minimum solution of the restriction of the equation $Z = \mathcal{H}(PZ + R)$ to the coordinates of an initial block, is infinite if and only if it is non-finite. This property is true when the routing is fair, and does not require that it be conservative.*

Proof: If $\exists i$ s.t. $Z^i = \infty$ and i is some coordinate in the strongly connected component, then all coordinates in this component are ∞ as well, and all associated coordinates also in view of fairness. If i is an associated coordinate, then necessarily at least one of the coordinates of the strongly connected component is infinite, and the rest of the proof is as above. ■

The Projection Operator We define $\Pi_{[j]}$ a *projection operator* (in the linear algebra sense) along dimension j . By definition, if P is some square matrix, $\Pi_{[j]}(P)$ is the matrix obtained when deleting line j and column j ; similarly, for all functions: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form $\mathcal{H}(X) = (\mathcal{H}^1(X^1), \dots, \mathcal{H}^n(X^n))$, we denote $\Pi_{[j]}(\mathcal{H})$ the function: $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ such that

$$\Pi_{[j]}(\mathcal{H})(\Pi_{[j]}(X)) = (\mathcal{H}^1(X^1), \dots, \mathcal{H}^{j-1}(X^{j-1}), \mathcal{H}^{j+1}(X^{j+1}), \dots, \mathcal{H}^n(X^n)).$$

Note that the projection of a conservative routing function is not necessarily conservative.

Lemma 13 *Let P be any matrix with non-negative entries and \mathcal{H} be a non-decreasing integer-valued function. If j is a null column of P , then Z_0 , the minimum solution of $Z = \mathcal{H}(PZ + R)$, is finite if and only if $\Pi_{[j]}(Z_0)$ is finite. Let U_0 be the minimum solution of:*

$$U = \Pi_{[j]}(\mathcal{H})(\Pi_{[j]}(P)U + \Pi_{[j]}(R)). \quad (36)$$

Then $U_0 = \Pi_{[j]}(Z_0)$.

Proof: If j is a null column of P , then for all $i \neq j$ $(PZ_0 + R)^i = (\Pi_{[j]}(P)\Pi_{[j]}(Z_0) + \Pi_{[j]}(R))^i$. Therefore

$$(\Pi_{[j]}(Z_0))^i = [\Pi_{[j]}(\mathcal{H})(\Pi_{[j]}(P)\Pi_{[j]}(Z_0) + \Pi_{[j]}(R))]^i,$$

so that $\Pi_{[j]}(Z_0) \geq U_0$. But conversely, if \tilde{U} denotes the vector $(U_1, \dots, U_{j-1}, \mathcal{H}^j(R^j), U_{j+1}, \dots, U_m)$, then one gets from (36) that \tilde{U} is a solution of $Z = \mathcal{H}(PZ + R)$, and so $\Pi_{[j]}(Z_0) \leq U_0$. This proves the second assertion. For the first property, it is enough to show that $U < \infty$ implies $Z^j < \infty$. But this follows from the fact that $(PZ)^j = \sum_{i \neq j} P(j, i)U^i$. ■

The Selection Operator Let \mathcal{H} and P be data satisfying the conditions of realizability. For all coordinates j , let $\mathcal{S}_{[j]}(\mathcal{H})$ be the routing function which always selects coordinate j :

$$\mathcal{S}_{[j]}(\mathcal{H})^i(n) = \begin{cases} \mathcal{H}^j(n) & \text{if } j \notin \mathcal{E}^l, \\ 0 & \text{if } j \in \mathcal{E}^l \text{ and } i \neq j, \\ n & \text{if } j = i, \end{cases} \quad (37)$$

where l is the index such that $j \in \mathcal{E}^l$ (see Equation (8)).

Similarly, let $\mathcal{S}_{[j]}(P)$ and $\mathcal{S}_{[j]}(R)$ be the matrix and vector given by:

$$\mathcal{S}_{[j]}(P)(i, \cdot) = \begin{cases} P(i, \cdot) & \text{if } i \notin \mathcal{E}^l, \\ 0 & \text{if } i \in \mathcal{E}^l \text{ and } i \neq j, \\ P(j, \cdot) & \text{if } i = j, \end{cases} \quad \mathcal{S}_{[j]}(R)(i) = \begin{cases} R(i) & \text{if } i \notin \mathcal{E}^l, \\ 0 & \text{if } i \in \mathcal{E}^l \text{ and } i \neq j, \\ R(j) & \text{if } i = j. \end{cases} \quad (38)$$

Note that $\mathcal{S}_{[j]}(\mathcal{H})$ is not a fair routing for the partition $\{\mathcal{E}^l\}_l$. However, if \mathcal{H} is fair and conservative then $\Pi_{[j]} \circ \mathcal{S}_{[j]}(\mathcal{H})$ is fair and conservative for the following modified partition:

$$\{\mathcal{E}^l\}_{l \neq l(j)} \cup \bigcup_{i \in \mathcal{E}^{l(j)}, i \neq j} \{i\},$$

where $l(j)$ is the index of the set which contains coordinate j .

Lemma 14 *Let P, R and \mathcal{H} be realizable matrices and routing function. Let j be a null column of P . The minimum solution Z_0 of $Z = \mathcal{H}(PZ + R)$ is finite for all fair \mathcal{H} if and only if the minimum solution V_0 of*

$$V = \Pi_{[j]} \circ \mathcal{S}_{[j]}(\mathcal{H}) \left(\Pi_{[j]} \circ \mathcal{S}_{[j]}(P)V + \Pi_{[j]} \circ \mathcal{S}_{[j]}(R) \right) \quad (39)$$

is finite, for all fair \mathcal{H} .

Proof: Assume that V_0 is non-finite. Since $\Pi_{[j]}(\mathcal{H}) \geq \Pi_{[j]} \circ \mathcal{S}_{[j]}(\mathcal{H})$, and $\Pi_{[j]}(P) \geq \Pi_{[j]} \circ \mathcal{S}_{[j]}(P)$, then using Lemma 13, we obtain that $\Pi_{[j]}(Z_0) \geq V_0$ and $\Pi_{[j]}(Z_0)$ is non-finite. So Z_0 is non-finite.

In order to prove the converse, it is enough to prove that if V_0 is finite for some fair \mathcal{H} , then there exists a fair \mathcal{H}_c such that Z_0 is finite. From $\Pi_{[j]} \circ \mathcal{S}_{[j]}(\mathcal{H})$, which is a routing function on the set of coordinates $\{1, 2, \dots, j-1, j+1, \dots, m\}$, we construct a routing function \mathcal{H}_c on $\{1, \dots, j, \dots, m\}$, which is fair and conservative for the initial partition \mathcal{E}^l , $l = 1, \dots, r$ (that associated with the initial data P, R and \mathcal{H}), in the following way: pick \mathcal{H}_f , any fair routing on $\{1, \dots, j, \dots, m\}$, and define $T = \Pi_{[j]}(P)V_0 + \Pi_{[j]}(R)$. We first define \mathcal{H}_c^i for $i \notin \mathcal{E}^{l(j)}$ by $\mathcal{H}_c^i(n) = \mathcal{H}_f^i(n)$. For $i \in \mathcal{E}^{l(j)}$, $i \neq j$, we take

$$\mathcal{H}_c^i(n) = \begin{cases} \Pi_{[j]} \circ \mathcal{S}_{[j]}(\mathcal{H})^i(n) & \text{if } n \leq T^i; \\ \mathcal{H}_f^i(n - T^i) & \text{if } n \geq T^i \end{cases}$$

and $\mathcal{H}_c^j(n)$ is deduced from this and the assumption that \mathcal{H}_c is conservative. We have

$$\begin{aligned} V_0 &= \Pi_{[j]} \circ \mathcal{S}_{[j]}(\mathcal{H}) \left(\Pi_{[j]} \circ \mathcal{S}_{[j]}(P)V_0 + \Pi_{[j]} \circ \mathcal{S}_{[j]}(R) \right) = \Pi_{[j]}(\mathcal{H}_c) \left(\Pi_{[j]} \circ \mathcal{S}_{[j]}(P)V_0 + \Pi_{[j]} \circ \mathcal{S}_{[j]}(R) \right) \\ &= \Pi_{[j]}(\mathcal{H}_c) \left(\Pi_{[j]}(P)V_0 + \Pi_{[j]}(R) \right). \end{aligned}$$

The second equality follows from the fact that for all $i \neq j$, $\Pi_{[j]}(\mathcal{H}_c)^i(\cdot) = \Pi_{[j]} \circ \mathcal{S}_{[j]}(\mathcal{H})^i(\cdot)$, up to T^i , and from the property that $T^i \geq (\Pi_{[j]} \circ \mathcal{S}_{[j]}(P)V_0 + \Pi_{[j]} \circ \mathcal{S}_{[j]}(R))^i$. The third one follows from the fact that for all $i \in \mathcal{E}^l$, $i \neq j$, $\Pi_{[j]}(\mathcal{H}_c)^i(T^i) = 0$, which implies that $\Pi_{[j]} \circ \mathcal{S}_{[j]}(P)V_0$ can be replaced by $\Pi_{[j]}(P)V_0$ and $\Pi_{[j]} \circ \mathcal{S}_{[j]}(R)$ by $\Pi_{[j]}(R)$. So V_0 is a solution of

$$U = \Pi_{[j]}(\mathcal{H}_c)(\Pi_{[j]}(P)U + \Pi_{[j]}(R)) \quad (40)$$

and if we denotes U_0^c the minimum solution of this equation, then $U_0^c \leq V_0$.

But in view of Lemma 13, the projection along j of the minimum solution Z_0^c of $Z = \mathcal{H}_c(PZ + R)$ coincides with the minimum solution U_0^c of (40). Therefore $\Pi_{[j]}Z_0^c = U_0^c \leq V_0$. Since V_0 is finite, then $\Pi_{[j]}Z_0^c$ is finite, which in turn implies that Z_0^c is finite in view of Lemma 13. \blacksquare

Lemma 15 *If the matrix P does not have any null column and if \mathcal{H} is conservative, then the minimum solution of $Z = \mathcal{H}(PZ + R)$ is non-finite if and only if $R \neq 0$.*

Proof: If $R = 0$, then $Z = 0$ is a solution of Equation (35).

Conversely, let us suppose that Equation (35) has a finite solution, Z . Then we have

$$\sum_i Z^i = \sum_i \mathcal{H}^i(\sum_j (P(i, j)Z^j) + R^i) = \sum_i h(i)(\sum_j (P(i, j)Z^j) + R^i),$$

where h_i is the real number

$$h(i) \stackrel{\text{def}}{=} \frac{\mathcal{H}^i(\sum_j (P(i, j)Z^j) + R^i)}{\sum_j (P(i, j)Z^j) + R^i}.$$

Now, we have,

$$\sum_i Z^i = \sum_i \sum_j (h(i)P(i, j)Z^j) + \sum_i h(i)R^i = \sum_j Z^j (\sum_i h(i)P(i, j)) + \sum_i h(i)R^i.$$

From the assumption on P , for all j , there exists a i such that $P(i, j) \geq 1$. Assume that $i \in \mathcal{E}^l$. Then for all $i' \in \mathcal{E}^l$, $P(i', j) \geq 1$, and so

$$\sum_i h(i)P(i, j) = \sum_{i \in \mathcal{E}^l} h(i)P(i, j) \geq \sum_{i \in \mathcal{E}^l} h(i) \geq 1,$$

where we have used the conservation assumption. Therefore we have

$$\sum_i Z^i \geq \sum_j Z^j + \sum_i h(i)R^i. \quad (41)$$

By the same arguments as above, if $i \in \mathcal{E}^l$,

$$\sum_i h(i)R^i = \sum_l \sum_{i \in \mathcal{E}^l} h(i)R^{(l)} = \sum_l R^{(l)},$$

where $R^{(l)}$ denotes the (constant) value of R^i for $i \in \mathcal{E}^l$. So (41) is possible if and only if $R = 0$. \blacksquare

Consider the following algorithm, where \mathcal{H} , P and R are the routing function, the matrix and the vector associated with the FCNet. Let \mathcal{H}_k , P_k and R_k , $k = 1, \dots, h$, be the projections of these objects on the set of coordinates corresponding to the initial blocks. For each k , the algorithm constructs a sequence $\widehat{\mathcal{H}}_k$, \widehat{P}_k and \widehat{R}_k which is initially equal to \mathcal{H}_k , P_k and R_k , and which is transformed during the execution of the program into objects of smaller sizes by certain projections and selections.

```

CheckLiveness( $\mathcal{H}, P, R$ )
LIVE := TRUE
for each initial component ( $\mathcal{H}_k, P_k, R_k$ ) do
  ( $\widehat{\mathcal{H}}_k, \widehat{P}_k, \widehat{R}_k$ ) := ( $\mathcal{H}_k, P_k, R_k$ )
  (loop 1) While  $\widehat{P}_k \neq 0$ 2 and  $\widehat{P}_k$  has empty columns do
    for each  $j$  such that  $\widehat{P}_k(\cdot, j) = 0$  do ( $\widehat{\mathcal{H}}_k, \widehat{P}_k, \widehat{R}_k$ ) :=  $\Pi_{[j]} \circ \mathcal{S}_{[j]}(\widehat{\mathcal{H}}_k, \widehat{P}_k, \widehat{R}_k)$  od
  od
  if  $\widehat{P}_k = 0$  then LIVE := FALSE
  if  $\widehat{R}_k = 0$  then LIVE := FALSE
od
return(LIVE)

```

Theorem 2 *Let F be a SI-FCNet, with total firing equation $Z = \mathcal{H}(PZ + R)$. Then the minimum solution of this equation is infinite (or equivalently F is live) if and only if $\text{CheckLiveness}(\mathcal{H}, P, R) = \text{TRUE}$.*

Proof: By Lemmas 11 and 12, the minimum solution of this equation is infinite if and only, for each initial block the equation $Z = \mathcal{H}_k(P_k Z + R_k)$ has a non-finite minimum solution.

Now, for a given k , we define the following boolean functions:

- $N(\widehat{\mathcal{H}}_k, \widehat{P}_k, \widehat{R}_k)$ is true iff the minimum solution of the equation $Z = \widehat{\mathcal{H}}_k(\widehat{P}_k Z + \widehat{R}_k)$ is non-finite,
- $C(\widehat{\mathcal{H}}_k, \widehat{P}_k, \widehat{R}_k)$ is true iff the routing function $\widehat{\mathcal{H}}_k$ is conservative and fair.

The property $C(\widehat{\mathcal{H}}_k, \widehat{P}_k, \widehat{R}_k)$ is inherited by (loop 1); by this we mean that since it holds when the loop starts, then it also holds at any step and in particular when the loop is complete.

From Lemma 14, the property $N(\widehat{\mathcal{H}}_k, \widehat{P}_k, \widehat{R}_k)$ is preserved by (loop 1); by this we mean that it holds true when the loop starts if and only if it holds true when it is complete;

²By $\widehat{P}_k = 0$, we mean that all the columns of this matrix are empty.

In order to conclude, note that the final matrix \widehat{P}_k does not have any null column when not null. We can then use Lemma 15 to conclude on this case.

If $\widehat{P}_k = 0$, then the equation $Z_k = \widehat{H}_k(\widehat{P}_k Z + \widehat{R}_k)$ admits a finite solution (equal to $\widehat{H}_k(\widehat{R}_k)$) and it is not live. ■

Note that this characterization of liveness is solely based on the total firing equation. In that, it differs from previous characterizations (see §5 for a complete discussion).

Corollary 1 *Checking liveness of a SI-FCNet is polynomial in the size of the net.*

Remark: *Acyclic Routing.* The previous algorithm also has an interpretation at the net level. Checking liveness of a SI-FCNet boils down to finding a choice of one of the output arcs of each routing place, such that if all tokens are sent to the chosen output arc, then the net is acyclic. When such a choice exists, the net is non-live, and one of the possible such choices is given by the algorithm `CheckLiveness`, through the sequence of selection operators $S_{[i]}$ used in the execution of the algorithm.

Remark: *Live Transitions.* In case the net is not live, the algorithm `CheckLiveness` also provides a way to characterize the set of live transitions. Let U and V be $\{0, 1\}$ -valued vectors of size n and m respectively, with $U^i = 1$ (resp. $V^j = 1$) if and only if transition i (resp. j) is live. Then (U, V) is the minimum positive fixed point of the following equations, where the first line is in the (\min, \times) (denoted here $(\underline{\oplus}, \underline{\otimes})$), and the second in the (\max, \times) algebra (denoted here $(\overline{\oplus}, \overline{\otimes})$)

$$\begin{aligned} U &= \text{supp}(C) \underline{\otimes} V \\ V &= (\text{supp}(P) \overline{\otimes} V) \overline{\oplus} (\text{supp}(Q) \overline{\otimes} U) \overline{\oplus} W, \end{aligned}$$

where $\text{supp}(M)$ denotes the support matrix of M , and W is the $\{0, 1\}$ -valued vector defined by: $W^i = 1$ if and only if i belongs to an initial block and the projection $\Pi_{[i]}$ is never used in `CheckLiveness`.

An example of the use of this algorithm to check liveness is given in Appendix 2 (§ 6) on the FCNet given in Figure 1.

5 The Commoner Condition

In this section, we show that the characterization of liveness for FCNet given in the section 4.2.3 which only uses the total firing equation also implicitly contains a proof of Commoner's theorem [9], [10].

5.1 Siphons and Traps of a FCNet

The following definition is a general Petri net definition:

Definition 1 A siphon is a set of places S verifying $\bullet S \subset S^\bullet$. A trap is a set of places T verifying $T^\bullet \subset \bullet T$.

In [9], the following necessary and sufficient condition of liveness for FCNets is given.

Theorem 3 A FCNet is live if and only if every siphon contains a marked trap.

The structural liveness of a FCNet can be characterized by checking whether every siphon contains a trap.³

A siphon is minimal if it does not contain a proper smaller siphon. Since the union of two traps is a trap, we can talk about the maximal trap included in a set of places. Theorem 3 can easily be reduced to the following equivalent version:

A FCNet is live if and only if the maximal trap included in every minimal siphon is marked.

Once a siphon S is given, one can test in quadratic time (in the number of places of the siphon) whether this siphon is structurally live (i.e. whether the maximal trap included in S is the empty set or not), by running the following procedure on S :

MaxTrap

1. Tag all the places which are not in S .
2. Tag all the places of S which have at least one output transition with only tagged output places. (i.e. all the places $p \in S$ s.t. there exists one transition t in p^\bullet with t^\bullet made of tagged places only).
3. Repeat step 2 until no new place is tagged.

Lemma 16 The places which remain untagged when the run of MaxTrap is complete form the maximal trap U included in siphon S .

Proof: First we prove that U is a trap. If U is empty, then U is a trap. If U is not empty, let $p \in U$. By construction of U , for each $t \in p^\bullet$, $t^\bullet \cap U \neq \emptyset$. This is the definition of a trap.

Then, we prove that this trap is maximal. For this we show that if p is tagged, then no trap containing p is included in S . The proof is by induction on i , the step at which p is tagged.

If p is tagged at step 1, p does not belong to S , and therefore, a trap containing p cannot be included in S .

³Unfortunately, checking this condition can be exponential in the size of the net. In [11, 10], polynomial conditions of liveness and boundedness of FCNets are derived. However, checking liveness for general unbounded FCNets is co-NP-complete, see [10].

Suppose that no place tagged at step j , $j \leq i$ belongs to a trap included in S . Let p be a place tagged at step $i + 1$. Then by construction, there exists a transition t in p^\bullet such that some places of t^\bullet have been tagged at previous steps. But all trap containing p must contain at least one place in t^\bullet , and from the induction assumption, this trap is not included in S . ■

This lemma also allows one to test the liveness of a minimal siphon in quadratic time (in the number of places included in the siphon). Indeed, this can be checked by testing whether the maximal trap included in the siphon is non-empty and marked.

5.2 A Proof of Commoner's Theorem

In this section, we will only consider FCNets with live MGC's. From previous sections, we know that the liveness of such a FCNet can be checked on its reduced net. For a SI-subnet F' of F we denote P_k an initial block of F' . Let W_k be the set of transitions of P_k and \widehat{W}_k be the set of transitions left in matrix \widehat{P}_k (resulting from the application of `CheckLiveness`).

Lemma 17 *A minimal siphon in a reduced FCNet F is of the form $\bullet W_k$, for some SI-subnet of F .*

Proof: Let S be a minimal siphon in the reduced net F . Then for any transition t in $\bullet S$, the minimality of S implies that S contains a unique place in $\bullet t$. This means that the net made with the transitions in $\bullet S$ and the places in S is SI. Therefore it is included in a SI-subnet of the original net.

Let F' be a SI-subnet of F and P_k be an initial block of F' . Then the set $\bullet W_k$ of places in P_k is a siphon for if $p \in \bullet W_k$ then all transitions in $\bullet p$ are in W_k . But due to the strong connectedness property of $\bullet W_k$, this siphon is minimal. ■

Lemma 18 *The maximal trap included in an initial block of a SI-FCNet is made of all the places in ${}^*\widehat{W}_k$, where *W denotes the input places of transitions in W , under the topology of a the net realization of \widehat{P}_k .*

Proof: We show below that the construction of the maximal trap by algorithm `MaxTrap` can be seen as one construction of \widehat{W}_k by algorithm `CheckLiveness`.

Let $U(i)$ be the set of places still untagged at step i in `MaxTrap` and let $\widehat{W}_k(i)$ be the set of transitions corresponding to the matrix \widehat{P}_k obtained after i applications of the Projection/Selection transformation in `CheckLiveness`. We show by induction on i that ${}^*\widehat{W}_k(i) = U(i)$.

At the beginning, we have ${}^*\widehat{W}_k(0) = \bullet W_k = U(0)$. If at step i a new place is tagged, that is $U(i + 1) = U(i) \setminus p$, this means that all the transitions in p^\bullet are removed from $\widehat{W}_k(i)$ at step $i + 1$ in `CheckLiveness` by successive projections along each of them. ■

The combination of these two lemmas gives a new proof of Commoner's Theorem based on Theorem 2. Indeed, the algorithm `CheckLiveness` tests if every minimal siphon $(\bullet W_k)$ contains a maximal trap $(\bullet \widehat{W}_k)$ containing tokens.

6 Generalization to Variable Firing Times

We now consider the case when firing times are still integer-valued and bounded, but variable with time. Let $\sigma^t(m)$ be the firing time of the m -th firing of transition t . Let $\zeta^t(k)$ be the minimum of M and the time which elapsed since the last time t has started firing before time k . If we consider the variables to be left-continuous, we have:

Lemma 19

$$Y(k) = \bigoplus_{l=1}^M (A_l(k) \otimes Y(k-l) \oplus B_l(k) \otimes Z(k-l)) \quad (42)$$

$$Z(k) = \mathcal{H} \left(\sum_{l=1}^M (P_l(k) \times Z(k-l) + Q_l(k) \times Y(k-l)) + R(k) \right), \quad (43)$$

with $A_l(k)(t, t') = c$, the number of tokens in the initial marking of the place between t' and t if $l = \zeta^t(k) \wedge \sigma^{t'}(X^{t'}(k-1))$, ∞ otherwise (with a similar definition for B) and $P_l(k)(t, t') = 1$, if there is a place between t' and t and $l = \zeta^t(k) \wedge \sigma^{t'}(X^{t'}(k-1))$, 0 otherwise (with a similar definition for Q).

Bounded Firing Times A system with bounded stochastic firing times with integer values can be transformed into a system with constant firing times by replacing every timed transition with a random firing time by a subnetwork as in Figure 4.

Indeed, let $\sigma(n)$ be an integer-valued random process bounded by M . Then, the distribution of $\sigma(n)$ is determined by the sequence $P_1(n), \dots, P_M(n)$, with $\mathbb{P}(\phi(n) = i) = P_i(n)$.

Note that if the original net was a SI-FCNet, the net obtained after transformation is a MI-FCNet.

Unbounded Firing Times A system with an geometric distributed firing time can also be transformed into an equivalent system with constant firing times. Each transition with a geometric firing time is replaced by the subnetwork depicted in Figure 5.

If α is the parameter of the distribution, then the \mathcal{H} function routes the token back into a unit firing time transition with probability α .

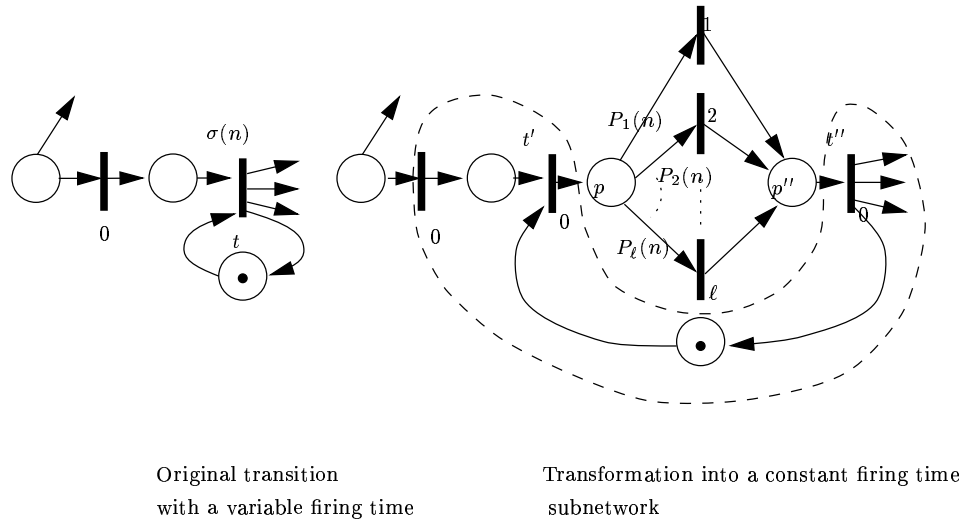


Figure 4: Transformation of a random firing time transition into a subnetwork composed with constant firing time transitions

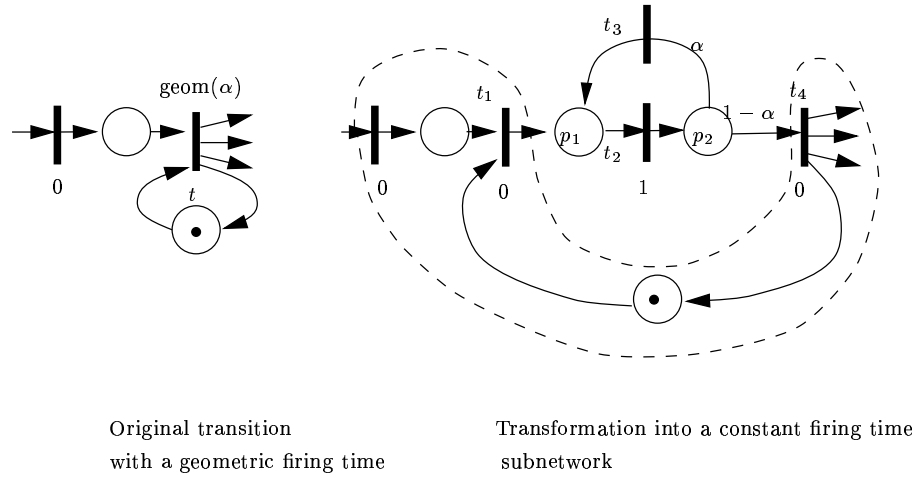


Figure 5: Transformation of a geometric firing time transition into a subnetwork composed with constant firing time transitions

Appendix

Appendix 1: Transformation of a FCNet

Conformization Any FCNet can be transformed into an *equivalent* net which satisfies this property: for all $p \in \bullet t$ which violates this property, p and the arcs involving p are

deleted and replaced by a triple p', t', p'' , with the following arcs (the arcs in the new net are denoted by \star , whereas we use \bullet for those of the initial net): $\star p' = \bullet p$, $p''\star = p\bullet = t$, $p'\star = t'$, $\star t' = p'$, $t'\star = p''$, $\star p'' = t'$, and all other arcs are unchanged. Whenever t' is given a firing time equal to zero, for all given sequences of tokens arrival times into p of the original net, if one takes the same sequence of arrival times into p' of the transformed net, then the enabling times of t are the same in both nets. In this sense, the two FCNets are equivalent.

Input Connectedness Any FCNet F can be transformed into an *equivalent* net F' with input connected MGC's. Let \mathcal{G} be a MGC of F which is not input connected. In each initial strongly connected component S of \mathcal{G} , choose an arbitrary place p . In F' , the place p is considered as an entry place by adding an entry transition into p which never fires ($R^{p\bullet}(k) = 0 \forall k$). In F' , $p\bullet$ is in \mathcal{B} and becomes an input transition of the MGC $\mathcal{G}\{p\bullet\}$ of F' . Now all the transitions in S are input connected (thanks to $p\bullet$) and the net F' is equivalent to F . Finally, note that one may have to conformize F' (if $p\bullet$ has several input places).

Appendix 2: An example of Liveness Analysis

We consider the FCNet of Figure 1 and we apply the algorithm described in §4.2.3, to check its liveness. First, note that all the MGC's are live (they do not contain any empty cycle), therefore, we can reduce the net as in Figure 2. We will illustrate the behavior of the algorithm `CheckLiveness` by showing, after each transformation of the matrices, a FCNet realization of the new system.

We start with the net in Figure 2. $\mathcal{A} = \{2, 5, 9\}$ and $\mathcal{B} = \{0, 1, 3, 4, 6, 7, 8\}$. Note that G'_3 is the only marked graph component that contains transitions with several inputs. There are 6 different SI-subnets of G'_3 (and therefore, of the whole net) all displayed in Figure 6.

In the following, we will focus on the SI-subnet generated by $S_1(G'_3)$. The associated system

$$\begin{aligned} Y &= C \otimes Z \\ Z &= \mathcal{H}(P'Z + R') \end{aligned} \tag{44}$$

has a realization given in Figure 7, where we forget about the variables Y .

The matrix P' (on transitions 0, 1, 3, 4, 6, 7, 8) is :

$$P' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R' = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

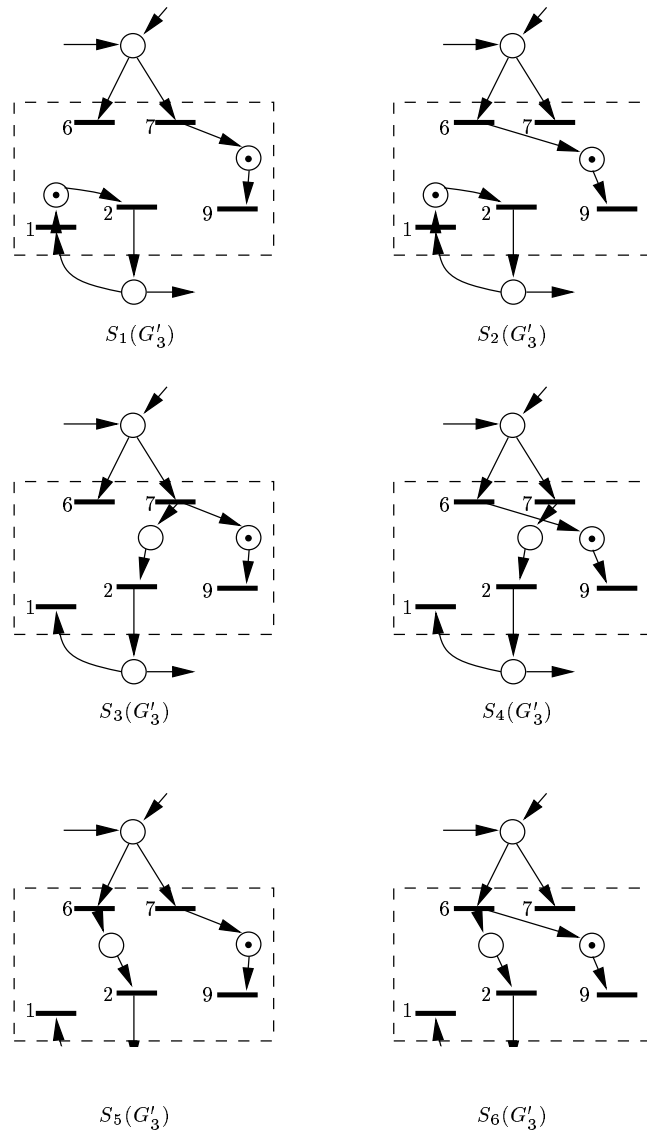
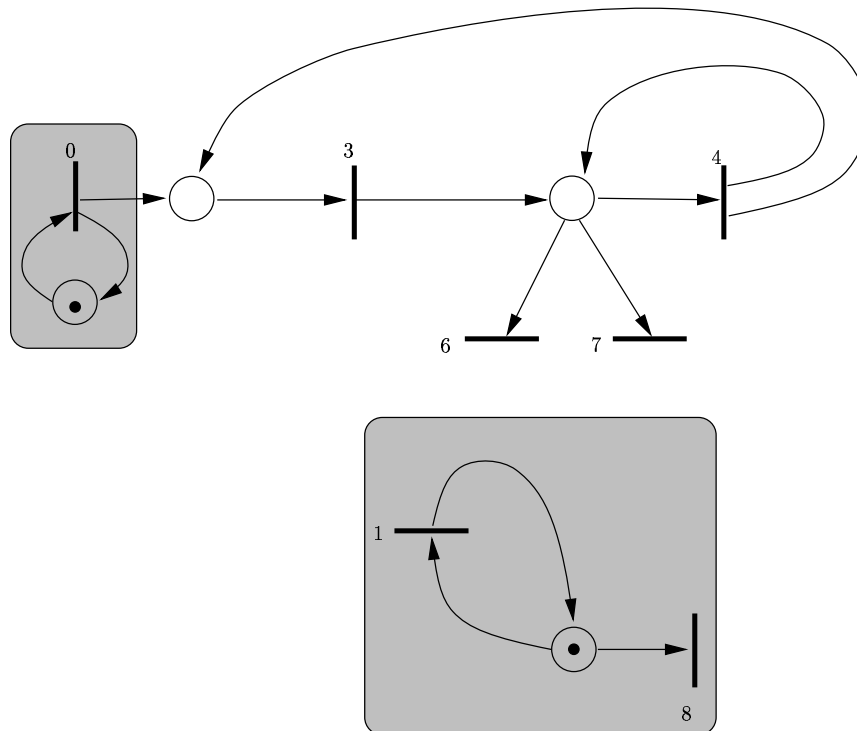


Figure 6: All possible SI-subnets of G'_3 .

Figure 7: Petri net realization of (P', R')

Transition 0 is an initial block by itself. Transition 1 forms an initial strongly connected component, but we have to add transition 8 to form the second initial block. Transitions $\{3,4\}$ form a strongly connected component. Transitions 6 and 7 form two degenerated components. After reordering the matrix P' (here, we merely move the last line of the matrix into third position) we obtain:

$$P' = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \boxed{0} & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & \boxed{0} & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & \boxed{0} & 0 \end{pmatrix} \quad R' = \begin{pmatrix} \boxed{1} \\ \boxed{1} \\ \boxed{1} \\ \boxed{0} \\ \boxed{0} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The two initial blocks are marked by shaded areas in the Petri net representation of Figure 7.

If we consider the initial block $P_1 = (1)$, then this is a strongly connected component (with no null lines nor columns). Therefore, it is live if and only if the associated R vector is not null. We have $R_1 = (1)$, so this initial block is live.

If we consider the initial block P_2 , then we see that it has a null column (corresponding to transition 8). Therefore we apply the selection $\mathcal{S}_{[8]}$ and we obtain: $\mathcal{S}_{[8]}(P_2)$, and then we project along transition 8, and we get $\Pi_{[8]} \circ \mathcal{S}_{[8]}(P_2)$. These three matrices are given below, and the respective Petri net realizations are given in Figure 8.

$$P_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \mathcal{S}_{[8]}(P_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \Pi_{[8]} \circ \mathcal{S}_{[8]}(P_2) = (0).$$

Since $\Pi_{[8]} \circ \mathcal{S}_{[8]}(P_2) = 0$, this initial block is not live.

By considering successively the six SI-subnets, we can show that S_2 is similar to S_1 and S_3, \dots, S_6 are live. The final result is obtained using the propagation of liveness mentioned in §4.2.4:

Transitions 0,3,4,5,6,7 are live
Transitions 1,2,8,9 are not live.

Future Research The approach presented here can be extended in various ways. Here is the list of the most obvious extensions that we intend to investigate in the future:

- Extension of the results to non-integer valued, random, unbounded firing times;
- Computation of $\Gamma(0)$ in the MI-separable case;

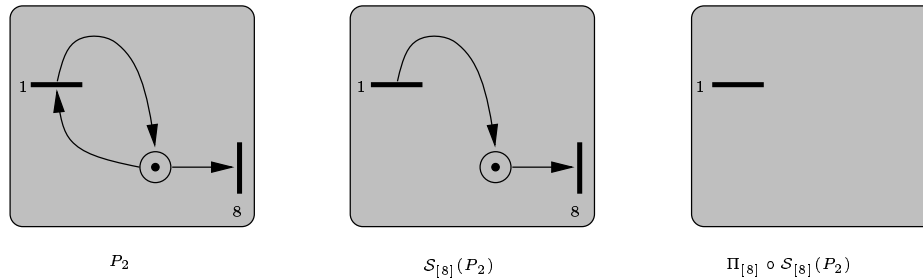


Figure 8: The Petri net realizations of matrices P_2 , $\mathcal{S}_{[8]}(P_2)$, $\Pi_{[8]} \circ \mathcal{S}_{[8]}(P_2)$.

- Extension to the autonomous case (without input process);
- Extension to weighted nets.

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