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Roberto L.V. González, Edmundo Rofman

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**HAL Id: inria-00073762**

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

## *On Some Junction Problems*

Roberto L.V. González - Edmundo Rofman

N° 2937

Juillet 1996

THÈME 4

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# ON SOME JUNCTION PROBLEMS

## Sur quelques problèmes de jonctions

Roberto L.V. González,\*      Edmundo Rofman†

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## Abstract

We consider two junction problems involving bilateral restrictions. These restrictions are defined in terms of the common values of the functions at the interface. Starting from a variational inequality formulation, we obtain a characterization which comprises local differential relations and a global equilibrium condition. To solve the original problem, a decomposition–coordination method is proposed, where the global equilibrium condition plays an essential role in the coordination phase.

## Résumé

Nous étudions ici deux problèmes de jonctions qui comprennent restrictions bilatérales. Ces restrictions sont définies en fonction des valeurs communes que les fonctions prennent à l'interface. En employant une formulation avec des inéquations variationnelles nous obtenons une caractérisation de la solution qui est composée des relations différentielles et d'une condition d'équilibre globale. Pour résoudre le problème originel, nous proposons une méthode de décomposition-coordination dans laquelle la condition d'équilibre globale joue un rôle essentiel dans la face de coordination.

**Keywords:** *junction problems, variational inequalities, decomposition methods, convex function, unilateral condition, bilateral condition, numerical solution.*

**Mots-Clés:** *jonctions, inéquations variationnelles, méthodes de décomposition, fonction convexe, condition unilatérale, condition bilatérale, solution numérique.*

**AMS Classification:** Primary: 35A15, 35J65, 35J50, 49A29  
Secondary: 49B27, 52A40, 58E35, 49D35

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# 1 Introduction

This paper deals with the issue of coupled physical systems – sometimes of different dimensions – where the coupling may take different forms.

Examples of these systems are the so called multistructures. A multistructure is a multidimensional assembly of beams, plates, shells or three dimensional structures, attached together with junctions. The analysis of these assemblages has received a lot of attention in recent years, as we can see in [1], [2], [4], [7], [8] and the bibliography therein.

A fruitful way to analyze those systems is the variational inequality (V.I.) approach, specially when there are state constraints or connections involving unilateral or bilateral restrictions. This approach can be seen in [8], where several cases are modeled and analyzed with the V.I. method.

In this paper we restrict our attention to the analysis of two junction problems involving bilateral restrictions. These restrictions are defined in terms of the common values of the functions at the interface. Starting from a V.I. formulation, we obtain a characterization which comprises local differential relations and a global equilibrium condition.

To solve the original problem, a decomposition–coordination method is proposed. We use this procedure because it allows us to solve the coupled problem through the solution of two simple independent problems – one of them a bilateral obstacle problem and the other a linear problem. These problems depend on some auxiliary variables which are modified (by a coordination procedure) until the desired solution is obtained. An important issue of this approach is that the global equilibrium condition plays an essential role in the coordination phase.

The contents of this paper can be outlined as follows:

Section 2 analyses almost thoroughly a 1-1 dimensional problem in order to understand the principal features of these types of couplings.

Section 3 contains the study of a 2-3 dimensional problem. The analytical characterization of the solution is given and it is presented a special implementation of the decomposition–coordination method.



## 2 An illustrative example

In order to grasp the ideas involved in the characterization of the solution  $u$  of problem (68) – which deals with a junction problem of dimensions 2-3 – we will present in this section a simplified problem and we will analyze almost thoroughly some aspects of its solution. In that way, the basic elements which gives rise to the characterization given in (85), (86) will be easily understood.

### 2.1 Elements of the problem

#### 2.1.1 The coupled domains

$$\Omega_1 = (-1, 0), \quad \Omega_2 = (0, 1). \quad (1)$$

#### 2.1.2 The convex $K$

$$K = \{(v_1, v_2) : v_1 \in H^1(\Omega_1), v_2 \in H^1(\Omega_2), (3)\}, \quad (2)$$

$$m(x)v_2(0) \leq v_1(x) \leq M(x)v_2(0) \quad \forall x \in \bar{\Omega}_1, \quad (3)$$

where

$$\left\{ \begin{array}{l} m(\cdot) \in H^2(\Omega_1), M(\cdot) \in H^2(\Omega_1), \\ 0 \leq m(x) \leq M(x) \leq 1, \forall x \in \bar{\Omega}_1, \\ m(0) = M(0) = 1. \end{array} \right. \quad (4)$$

#### 2.1.3 The bilinear forms

Let be  $\alpha > 0, \beta > 0$

$$a_1(u_1, v_1) = \int_{\Omega_1} (\nabla u_1(x) \nabla v_1(x) + \alpha u_1(x) v_1(x)) dx, \quad (5)$$

$$a_2(u_2, v_2) = \int_{\Omega_2} (\nabla u_2(x) \nabla v_2(x) + \beta u_2(x) v_2(x)) dx. \quad (6)$$

We associate to (5), (6) the differential operators  $A_1, A_2$

$$A_1 = -\frac{\partial}{\partial x^2} + \alpha,$$

$$A_2 = -\frac{\partial}{\partial x^2} + \beta.$$

### 2.1.4 The variational inequality

Find  $u = (u_1, u_2) \in K$  such that

$$a_1(u_1, v_1 - u_1) + a_2(u_2, v_2 - u_2) \geq (f_1, v_1 - u_1) + (f_2, v_2 - u_2), \quad \forall (v_1, v_2) \in K, \quad (7)$$

where  $f_1 \in L^2(\Omega_1)$ ,  $f_2 \in L^2(\Omega_2)$ , and  $(v, w)$  denote the scalar product in  $L^2(\Omega_1)$  or  $L^2(\Omega_2)$ .

### 2.1.5 An equivalent formulation

Since  $K$  is a cone, (7) is equivalent to the following conditions

$$a_1(u_1, u_1) + a_2(u_2, u_2) = (f_1, u_1) + (f_2, u_2), \quad (u_1, u_2) \in K, \quad (8)$$

$$a_1(u_1, v_1) + a_2(u_2, v_2) \geq (f_1, v_1) + (f_2, v_2), \quad \forall (v_1, v_2) \in K. \quad (9)$$

## 2.2 Characterization of the solution

As  $\alpha > 0$  and  $\beta > 0$ , the bilinear forms  $a_1, a_2$  are coercive and in consequence, there exists a unique solution  $u = (u_1, u_2)$  of (7). Here we will present the necessary conditions verified by  $(u_1, u_2)$ . It can be proved that these conditions are also sufficient to characterize the solution.

### 2.2.1 Case $u_2(0) = 0$

Here we have

$$u_2(0) = 0 \Rightarrow u_1 \equiv 0. \quad (10)$$

In this case, (9) is reduced to

$$a_2(u_2, v_2) \geq (f_2, v_2) \quad \forall (v_1, v_2) \in K. \quad (11)$$

In case of choosing an arbitrary  $v_2$  such that  $v_2(0) = 0$ , we obtain

$$a_2(u_2, v_2) \geq (f_2, v_2). \quad (12)$$

This condition implies (in the distribution sense)

$$A_2 u_2 = f_2 \quad (13)$$

and

$$\frac{\partial u_2}{\partial n}(1) = 0. \quad (14)$$

By virtue of (9), for an arbitrary function  $v_2$  verifying  $v_2(0) \geq 0$ , it must be also satisfied

$$\int_{\Omega_2} (A_2 u_2 - f_2)(x) v_2(x) dx + \frac{\partial u_2}{\partial n}(0) v_2(0) + \frac{\partial u_2}{\partial n}(1) v_2(1) \geq (f_1, v_1); \quad (15)$$

then, by virtue of (13) and (14), (15) is reduced to

$$\frac{\partial u_2}{\partial n}(0) v_2(0) \geq (f_1, v_1). \quad (16)$$

Taking now an element  $v_2$  such that  $v_2(0) = 1$ , we have

$$\frac{\partial u_2}{\partial n}(0) \geq (f_1, v_1) \quad \forall v_1 / v_1(x) \in [m(x), M(x)] \quad \forall x \in \Omega_1. \quad (17)$$

The maximum on the right of (17) is achieved at

$$\begin{cases} v_1(x) = M(x) & \text{if } f_1(x) \geq 0, \\ v_1(x) = m(x) & \text{if } f_1(x) \leq 0, \end{cases} \quad (18)$$

then

$$\sup \{(f_1, v_1) : (v_1, v_2) \in K, v_2(0) = 1\} = \int_{\Omega_1} (f_1^+(x) M(x) - f_1^-(x) m(x)) dx. \quad (19)$$

In consequence, the necessary conditions (for the case of solution  $u_1 \equiv 0$ ) can be summarized in the following form

$$\begin{cases} A_2 u_2 = f_2, \\ \frac{\partial u_2}{\partial n}(1) = 0, \\ u_2(0) = 0, \end{cases} \quad (20)$$

$$\frac{\partial u_2}{\partial n}(0) \geq (f_1^+, M) - (f_1^-, m). \quad (21)$$

### 2.2.2 Case $u_2(0) \neq 0$

In this case, from (9) we also arrive to relations (13) and (14). Taking into account that

$$a_1(u_1, v_1) = (A_1 u_1, v_1) + \frac{\partial u_1}{\partial n}(-1) v_1(-1) + \frac{\partial u_1}{\partial n}(0) v_1(0), \quad (22)$$

from (9) we obtain the following conditions

$$\begin{cases} \frac{\partial u_1}{\partial n}(-1) = 0 & \text{if } u_1(-1) \in (m(-1) u_2(0), M(-1) u_2(0)), \\ \frac{\partial u_1}{\partial n}(-1) \geq 0 & \text{if } u_1(-1) = m(-1) u_2(0), \\ \frac{\partial u_1}{\partial n}(-1) \leq 0 & \text{if } u_1(-1) = M(-1) u_2(0). \end{cases} \quad (23)$$

In a similar form, from (7), (22) and (23) we obtain

$$(A_1 u_1 - f_1, v_1 - u_1) \geq 0, \quad (24)$$

in consequence (a.e.  $x \in \Omega_1$ )

$$\begin{cases} A_1 u_1(x) - f_1(x) = 0 & \text{if } u_1(x) \in (m(x) u_2(0), M(x) u_2(0)), \\ A_1 u_1(x) - f_1(x) \geq 0 & \text{if } u_1(x) = m(x) u_2(0), \\ A_1 u_1(x) - f_1(x) \leq 0 & \text{if } u_1(x) = M(x) u_2(0). \end{cases} \quad (25)$$

From (8), (13), (14) we obtain the coupling condition in compact form

$$\frac{\partial u_2}{\partial n}(0) + \frac{1}{u_2(0)} (a_1(u_1, u_1) - (f_1, u_1)) = 0. \quad (26)$$

To write it in a more descriptive form, from (22) we obtain

$$\frac{\partial u_2}{\partial n}(0) u_2(0) + (A_1 u_1 - f_1, u_1) + \frac{\partial u_1}{\partial n}(-1) u_1(-1) + \frac{\partial u_1}{\partial n}(0) u_1(0) = 0. \quad (27)$$

From (23) and (25) we obtain, dividing by  $u_2(0)$

$$\begin{aligned} \left( \frac{\partial u_2}{\partial n}(0) \right) + \left( \frac{\partial u_1}{\partial n}(-1) \right)^+ \cdot m(-1) - \left( \frac{\partial u_1}{\partial n}(-1) \right)^- \cdot M(-1) \\ + \frac{\partial u_1}{\partial n}(0) + ((A_1 u_1 - f_1)^+, m) - ((A_1 u_1 - f_1)^-, M) = 0. \end{aligned} \quad (28)$$

### 2.3 A decomposition method

Relations (13), (14) and (23), (25) are the necessary and sufficient conditions verified by the solutions of two separated problem. One is a common elliptic problem and the other one is an obstacle problem. Both problems are related by the coupling conditions (21) and (28).

The idea of solving (7) by hierarchical optimization and decomposition stems from the fact that both problems depend on the level  $u_2(0) = \xi$  and if we choose the correct (and unique)  $\xi$ , then the original problem (7) is solved.

The procedure chooses a value of  $\xi$  and – after solving the separated problems – that level is modified until it verifies conditions (21) and (28).

#### 2.3.1 Description of the procedure

We use the real parameter  $\xi \geq 0$  and introduce the following problems  $P_1^\xi, P_2^\xi$ , with solutions  $u_1^\xi$  and  $u_2^\xi$  respectively

#### Definition 2.1 The separated problems

**Problem  $P_1^\xi$**

$$u_1^\xi \in K_1^\xi, \quad a_1(u_1^\xi, v_1 - u_1^\xi) \geq (f_1, v_1 - u_1^\xi) \quad \forall v_1 \in K_1^\xi, \quad (29)$$

$$K_1^\xi = \{v_1 \in H^1(\Omega_1) : m(x) \cdot \xi \leq v_1(x) \leq M(x) \cdot \xi \quad \forall x \in \Omega_1\}. \quad (30)$$

**Problem  $P_2^\xi$**

$$\left\{ \begin{array}{l} u_2^\xi(0) = \xi, \\ \frac{\partial u_2^\xi}{\partial n}(1) = 0, \\ A_2 u_2^\xi - f_2 = 0. \end{array} \right. \quad (31)$$

#### Definition 2.2 The auxiliary function $\Psi$

$$\Psi(\xi) = \Psi_1(\xi) + \Psi_2(\xi), \quad (32)$$

where

$$\Psi_1(\xi) = \begin{cases} \frac{1}{\xi} \left( a_1(u_1^\xi, u_1^\xi) - (f_1, u_1^\xi) \right) & \text{if } \xi > 0, \\ ((f_1)^-, m) - ((f_1)^+, M) & \text{if } \xi = 0, \end{cases} \quad (33)$$

$$\Psi_2(\xi) = \frac{\partial n_2^\xi}{\partial n}(0). \quad (34)$$

**Remark 2.1** *It is clear that only one value of  $\xi$  verifies  $\min(\xi, \Psi(\xi)) = 0$ , because if two values of  $\xi$  verify this condition, then there would be two solutions of  $(\gamma)$ , which has a unique solution. This unique value of level  $\xi$  is found iteratively by the algorithm described below, using the information given by the function  $\Psi$ .*

### Definition 2.3 An iterative algorithm

Let be  $\{\gamma_\nu : \nu = 0, \dots\}$  a sequence verifying  $\gamma_\nu > 0 \ \forall \nu = 0, \dots$ , and

$$\sum_{\nu=0}^{\infty} \gamma_\nu = +\infty. \quad (35)$$

The following algorithm computes the solution of  $(\gamma)$ , starting at an initial pair  $u_1^{\xi^0}, u_2^{\xi^0}$  (which depend of an arbitrary non-negative initial  $\xi^0$ )

### ALGORITHM

Step 1:  $\nu = 0, \xi^0 \geq 0$ .

Step 2: Solve problems  $P_1^{\xi^\nu}$  and  $P_2^{\xi^\nu}$ .

Step 3: If  $\min(\xi^\nu, \Psi(\xi^\nu)) = 0$ , Stop.

Step 4: If  $\min(\xi^\nu, \Psi(\xi^\nu)) \neq 0$ , set  $\xi^{\nu+1} = \max(\xi^\nu - \gamma_\nu \Psi(\xi^\nu), 0)$  and go to Step 2.

### 2.3.2 A simple example

The problem  $P_1^\xi$

Data

$$\left\{ \begin{array}{l} M(x) \equiv 1, \\ m(x) = \frac{-2 + e^{-\sqrt{\alpha}x} + e^{\sqrt{\alpha}x}}{-2 + e^{-\frac{\sqrt{\alpha}}{2}} + e^{\frac{\sqrt{\alpha}}{2}}}, \\ f_1(x) \equiv \frac{-2\alpha}{-2 + e^{-\frac{\sqrt{\alpha}}{2}} + e^{\frac{\sqrt{\alpha}}{2}}}. \end{array} \right.$$

Explicit solution  $u_1$

$$\left\{ \begin{array}{l} u_1(x) = m(x) \times \xi, \quad \text{if } \xi \leq 1, \\ u_1(x) = \frac{-2}{-2 + e^{-\frac{\sqrt{\alpha}}{2}} + e^{\frac{\sqrt{\alpha}}{2}}} \\ \quad + \frac{(e^{-\sqrt{\alpha}(0.5+x)} + e^{\sqrt{\alpha}(0.5+x)})}{\left(e^{-\frac{\sqrt{\alpha}}{2}} + e^{\frac{\sqrt{\alpha}}{2}}\right)} \times \left( \xi + \frac{2}{-2 + e^{-\frac{\sqrt{\alpha}}{2}} + e^{\frac{\sqrt{\alpha}}{2}}} \right), \quad \text{if } \xi \geq 1. \end{array} \right.$$

The function  $\Psi_1$

$$\left\{ \begin{array}{l} \frac{1}{\xi}(a_1(u_1, u_1) - (f_1, u_1)) = a_1(m, m) \times \xi - (f_1, m), \quad \text{if } \xi \leq 1, \\ \frac{1}{\xi}(a_1(u_1, u_1) - (f_1, u_1)) = \frac{2\sqrt{\alpha} \left( e^{\frac{\sqrt{\alpha}}{2}} - e^{-\frac{\sqrt{\alpha}}{2}} \right)}{\left( e^{-\frac{\sqrt{\alpha}}{2}} + e^{\frac{\sqrt{\alpha}}{2}} \right)} \left( \xi + \frac{2}{-2 + e^{-\frac{\sqrt{\alpha}}{2}} + e^{\frac{\sqrt{\alpha}}{2}}} \right), \quad \text{if } \xi \geq 1. \end{array} \right.$$

The problem  $P_2^\xi$

Explicit solution  $u_2$

We suppose  $f_2$  constant. In that case, the general solution of  $u_2$  is

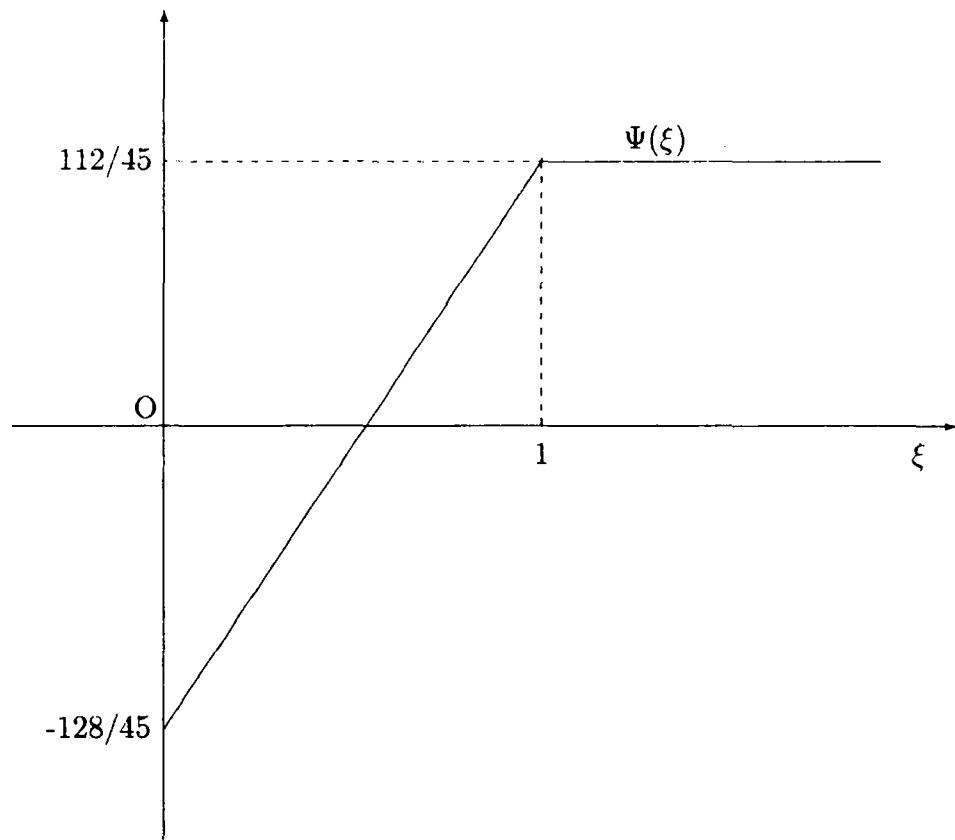
$$u_2(x) = \frac{f_2}{\beta} + \frac{\left(-\frac{f_2}{\beta} + \xi\right)}{\left(e^{\sqrt{\beta}} + e^{-\sqrt{\beta}}\right)} \times \left( e^{\sqrt{\beta}(1-x)} + e^{\sqrt{\beta}(x-1)} \right). \quad (36)$$

The function  $\Psi_2$

$$\frac{\partial u_2(0)}{\partial n} = \frac{\sqrt{\beta} \left( e^{\sqrt{\beta}} - e^{-\sqrt{\beta}} \right)}{\left( e^{\sqrt{\beta}} + e^{-\sqrt{\beta}} \right)} \times \left( -\frac{f_2}{\beta} + \xi \right). \quad (37)$$

If we use (37) and we analyze – for  $\alpha \simeq 0$ ,  $\beta \simeq 0$  and  $f_2 = 248/45$  – the function  $\Psi(\xi)$  given by

$$\Psi(\xi) = \frac{\partial u_2(0)}{\partial n} + \frac{1}{\xi}(a_1(u_1, u_1) - (f_1, u_1)), \quad (38)$$

Figure 1: The function  $\Psi(\xi)$ 

we obtain the function depicted in the Figure 1, where the uniqueness of the value  $\xi$  which verifies  $\min(\xi, \Psi(\xi)) = 0$  is clearly shown.



### 2.3.3 General shape of the function $\Psi$ . Examples.

When  $M(0) = m(0) = 1$ , problem  $P_1^\xi$  has an unique solution. So, function  $\Psi_1(\xi)$  is well defined and, if  $\alpha \geq 0$ , it is **non-decreasing**.

Example 1:

Data	$M(x) \equiv 1$ $m(x) = (x + 1)^2$ $f_1(x) \equiv 0$ $\alpha \simeq 0$
Explicit solution $u_1^\xi$	$u_1^\xi(x) = \xi \frac{e^{\sqrt{\alpha}(1-x)} + e^{\sqrt{\alpha}(x-1)}}{e^{-\sqrt{\alpha}} + e^{\sqrt{\alpha}}}$
Function $\Psi_1$	$\Psi_1(\xi) = \xi \sqrt{\alpha} \frac{e^{\sqrt{\alpha}} - e^{-\sqrt{\alpha}}}{e^{-\sqrt{\alpha}} + e^{\sqrt{\alpha}}}$

**Remark 2.2** If  $m\{\alpha(x) > 0\} > 0$ , then  $\Psi_1(\xi)$  is strictly increasing, as we have seen in Example 1.

**Remark 2.3** If  $\alpha \equiv 0$ , the function  $\Psi_1(\xi)$  is nondecreasing, as happens in Example 2.

Example 2:

Data	$M(x) \equiv 1$ $m(x) = (x + 1)^2$ $\alpha \equiv 0$ $f_1(x) \equiv 0$
Explicit solution $u_1^\xi$	$u_1^\xi(x) \equiv \xi$
Function $\Psi_1$	$\Psi_1(\xi) \equiv 0$

**Remark 2.4** *Even if  $\alpha \equiv 0$ , the function  $\Psi(\xi)$  can be strictly increasing, as it is shown in Example 3.*

Example 3:

Data	$M(x) = \frac{1}{2} + \frac{(x+1)^2}{2}$ $m(x) = (x+1)^2$ $\alpha \equiv 0$ $f_1(x) \equiv 0$
Explicit solution $u_1^\xi$	$u_1^\xi(x) = \xi \left( \frac{1}{2} + \frac{(x+1)^2}{2} \right)$
Function $\Psi_1$	$\Psi_1(\xi) = \xi$

### 2.3.4 General shape of the function $\Psi$ . Theoretical results.

The key result of this section is the proof that, provided the forms (5)-(6) are symmetric, the function  $\Psi$  is the derivative of a convex function  $\varphi$ . In fact  $\Psi$  is the derivative of an strictly convex function and in consequence, the value  $\bar{\xi}$  that verifies  $\min (\bar{\xi}, \Psi (\bar{\xi})) = 0$  is unique.

We know that when the forms (5)-(6) are symmetric, we can analyze the VI (7) as the optimality condition of the minimization problem:

$$\min_{(v_1, v_2) \in K} \varphi^0(v_1, v_2), \quad (39)$$

where

$$\begin{cases} \varphi^0(v_1, v_2) = \varphi_1^0(v_1, v_1) + \varphi_2^0(v_2, v_2) \\ \varphi_1^0(v_1, v_1) = \frac{1}{2} a_1(v_1, v_1) - (f_1, v_1) \\ \varphi_2^0(v_2, v_2) = \frac{1}{2} a_2(v_2, v_2) - (f_2, v_2) \end{cases} \quad (40)$$

and the set  $K$  is given by (2).

$K$  can be rewritten in the following form:

$$K = \bigcup_{\xi \geq 0} K^\xi, \quad (41)$$

where

$$\begin{cases} K^\xi = K_1^\xi \times K_2^\xi, \\ K_1^\xi = \{v_1 \in H^1(\Omega_1) : m(x) \cdot \xi \leq v_1(x) \leq M(x) \cdot \xi \quad \forall x \in \Omega_1\}, \\ K_2^\xi = \{v_2 \in H^1(\Omega_2) : v_2(0) = \xi\}. \end{cases} \quad (42)$$

In consequence, by virtue of (41)

$$\min_{(v_1, v_2) \in K} \varphi^0(v_1, v_2) = \min_{\xi \geq 0} \varphi(\xi), \quad (43)$$

where

$$\varphi(\xi) = \min_{(v_1, v_2) \in K^\xi} \varphi^0(v_1, v_2). \quad (44)$$

So, the problem (39) can be seen as the following hierarchical optimization problem [9]:

$$\min_{\xi \geq 0} \left( \min_{(v_1, v_2) \in K^\xi} \varphi^0(v_1, v_2) \right). \quad (45)$$

In our problem, by (40), we can compute the function  $\varphi$  by decomposition techniques. Specifically we have

$$\min_{\xi \geq 0} \varphi(\xi) = \min_{\xi \geq 0} (\varphi_1(\xi) + \varphi_2(\xi)). \quad (46)$$

where

$$\begin{cases} \varphi_2(\xi) = \min_{(v_2, v_2) \in K_2^\xi} \varphi_2^0(v_2, v_2) \\ \varphi_1(\xi) = \min_{(v_1, v_1) \in K_1^\xi} \varphi_1^0(v_1, v_1) \\ \varphi(\xi) = \varphi_1(\xi) + \varphi_2(\xi). \end{cases} \quad (47)$$

We see that each function  $\varphi_1$  and  $\varphi_2$  can be computed solving two separated optimization problems.

By the theorem presented below,  $\varphi_1$  and  $\varphi_2$  are convex functions (it is almost obvious that  $\varphi_2$  is strictly convex) and in consequence,  $\varphi$  is strictly convex. Then, the above analyzed uniqueness property holds.

### 2.3.5 An auxiliary convexity result and its applications

**Definition 2.4** The auxiliary function  $\varphi$ .

Let  $V_I$  be a vector space and  $K_I$  a convex subset of  $V_I$ . Let  $X$  be a vector space,  $\phi(\cdot)$  a real function defined on  $X$  and (for each  $v \in K_I$ ) a convex subset  $K^v \subset X$  such that  $\inf_{x \in K^v} \phi(x) > -\infty$ . So, we can define on  $K_I$  the function

$$\varphi(v) = \inf_{x \in K^v} \phi(x). \quad (48)$$

**Definition 2.5** Let  $\phi$  be an strictly convex and continuous function, such that (48) has a unique solution  $\bar{x}(v)$ . In that case, we define the function  $\bar{x}(v)$ , with the property

$$\varphi(v) = \phi(\bar{x}(v)). \quad (49)$$

**Theorem 2.1** If the function  $\phi$  is convex, the conditions of Definition 2.5 are verified and  $v \rightarrow K^v$  is a convex mapping in the sense that

$$\lambda(K^{v_1}) + (1 - \lambda)K^{v_2} \subset K^{\lambda v_1 + (1 - \lambda)v_2} \subset \lambda(K^{v_1}) + (1 - \lambda)K^{v_2}, \quad (50)$$

then  $\varphi$  is a convex function.

**Proof:** Let  $v_1$  and  $v_2$  be elements of  $K_I$  and  $x_1, x_2$  such that

$$x_1 = \bar{x}(v_1), \quad x_2 = \bar{x}(v_2), \quad (51)$$

then,  $\varphi$  is convex because

$$\begin{aligned} \varphi(\lambda v_1 + (1 - \lambda)v_2) &\leq \phi(\lambda x_1 + (1 - \lambda)x_2) && \text{by (48) and (50)} \\ &\leq \lambda \phi(x_1) + (1 - \lambda)\phi(x_2) && \text{by the convexity of } \phi \\ &= \lambda \varphi(v_1) + (1 - \lambda)\varphi(v_2) && \text{by (51).} \end{aligned}$$

□

## Application

We can apply the previous theorem to the case:

$$\left\{ \begin{array}{l} K_I = \mathfrak{R} \\ K_I = \mathfrak{R}^+ \\ X = H^1(\Omega_1) \\ K_1^\xi = \{v_1 \in X : m(x) \cdot \xi \leq v_1(x) \leq M(x) \cdot \xi \quad \forall x \in \Omega_1\} \\ \phi = \varphi_1^0(v_1, v_1) = \frac{1}{2} a_1(v_1, v_1) - (f_1, v_1) \end{array} \right.$$

and

$$\left\{ \begin{array}{l} K_I = \mathfrak{R} \\ K_I = \mathfrak{R}^+ \\ X = H^1(\Omega_2) \\ K_2^\xi = \{v_2 \in X : v_2(0) = \xi\} \\ \phi = \varphi_2^0(v_2, v_2) = \frac{1}{2} a_2(v_2, v_2) - (f_2, v_2) \end{array} \right.$$

obtaining that  $\varphi_1$  and  $\varphi_2$  are convex.

**Remark 2.5** *In addition, it can be proved that  $\varphi_2(\xi)$  is strictly convex. The proof is simple and it is here omitted.*

**Remark 2.6** *We can prove easily that  $\Psi_1(\xi) = \varphi_1'(\xi)$  and  $\Psi_2(\xi) = \varphi_2'(\xi)$ . In consequence,  $\Psi(\cdot)$  is strictly increasing. In addition, it can be proved that  $\Psi(\xi) = \varphi'(\xi)$  is Lipschitz continuous. The proof is simple and it is here omitted.*

### 2.3.6 Convergence of the procedure

#### Technical assumptions

We assume that the forms (5)-(6) are symmetric and in consequence,  $\Psi$  is an increasing Lipschitz continuous function on  $[0, \infty]$ .

**Theorem 2.1** *The algorithm produces a sequence of values  $(u_1^{\xi^\nu}, u_2^{\xi^\nu})$  such that*

$$u_1^{\xi^\nu} \rightarrow u_1 \text{ in } H^1(\Omega_1), \quad (52)$$

$$u_2^{\xi^\nu} \rightarrow u_2 \text{ in } H^1(\Omega_2). \quad (53)$$

*The values*

$$\min(\xi^\nu, \Psi(\xi^\nu)) \rightarrow 0. \quad (54)$$

**Proof:**

- Case  $\Psi(0) \geq 0$

Here we have

$$\Psi(\xi^\nu) \geq 0 \quad \forall \xi^\nu \Rightarrow \xi^{\nu+1} \leq \xi^\nu, \quad \forall \nu = 0, 1, \dots;$$

so,  $\{\xi^\nu\}$  is nonincreasing. If  $\exists \bar{\nu} / \xi^{\bar{\nu}} = 0 \Rightarrow \min(\xi^{\bar{\nu}}, \Psi(\xi^{\bar{\nu}})) = 0$  and the algorithm stops at Step 3.

If  $\xi^\nu > 0 \quad \forall \nu \geq 0 \Rightarrow \Psi(\xi^\nu) > 0, \quad \forall \nu \geq 0 \Rightarrow$

$$\xi^{\nu+1} = \max(\xi^\nu - \gamma_\nu \Psi(\xi^\nu), 0) = \xi^\nu - \gamma_\nu \Psi(\xi^\nu) < \xi^\nu, \quad (55)$$

then

$$\xi^\nu \rightarrow \check{\xi} \geq 0. \quad (56)$$

If we suppose  $\check{\xi} > 0$ , then

$$\Psi(\xi^\nu) > \Psi(\check{\xi}) > 0, \quad \forall \nu \geq 0. \quad (57)$$

By (55), (57)

$$\xi^{\nu+1} < \xi^\nu - \gamma_\nu \Psi(\check{\xi}),$$

by induction, we have

$$\xi^{\rho+1} \leq \xi^0 - \Psi(\check{\xi}) \left( \sum_{\nu=0}^{\rho} \gamma_\nu \right) \rightarrow -\infty. \quad (58)$$

So, by *reductio ad absurdum* we have  $\check{\xi} = 0$ .

- Case  $\Psi(0) < 0$

Let be  $\bar{\xi}$  the unique point that verifies  $\Psi(\bar{\xi}) = 0$  and  $\hat{\xi} = \max(\xi^0, \bar{\xi} - \Psi(0) \gamma_1)$ . We can see that the sequence  $\xi^\nu$  is bounded, because

$$\xi^1 \leq \hat{\xi}$$

and by induction it also follows

$$\xi^\nu \leq \widehat{\xi}, \quad \forall \nu > 1.$$

By a similar argument as that used to obtain (58), we can see that

$$\underline{\lim} \xi^\nu \leq \bar{\xi} \leq \overline{\lim} \xi^\nu.$$

Finally, taking into account that  $\gamma_\nu \rightarrow 0$ , and that  $\Psi(\xi^\nu) \leq \max(-\Psi(0), \Psi(\widehat{\xi}))$ , we have:

$$\underline{\lim} \xi^\nu = \bar{\xi} = \overline{\lim} \xi^\nu.$$

□

**Remark 2.7** *It can be easily proved that, if the sequence  $\xi^\nu$  is infinite, after a finite number of iterations, the convergence is monotone, i.e.  $\exists \nu(\xi^0)$  such that*

$$\left| \begin{array}{l} \xi^\nu > \xi^{\nu+1}, \quad \forall \nu \geq \nu(\xi^0), \\ \text{or} \\ \xi^\nu < \xi^{\nu+1}, \quad \forall \nu \geq \nu(\xi^0). \end{array} \right.$$



### 3 A 2-3 dimensional coupled problem

#### 3.1 Elements of the problem

##### 3.1.1 The coupled domains

$$\Omega_2 = [0, 1] \times [0, 1], \quad (59)$$

$$\Omega_3 = [-1, 0] \times [0, 1] \times [0, 1]. \quad (60)$$

##### 3.1.2 The convex K

$$K = \{(v_2, v_3) : v_2 \in H^1(\Omega_2), v_3 \in H^1(\Omega_3), (62) \text{ a.e. } x \in \bar{\Omega}_3\}, \quad (61)$$

$$m(x_1, x_2, x_3) v_2(0, x_2) \leq v_3(x_1, x_2, x_3) \leq M(x_1, x_2, x_3) v_2(0, x_2) \text{ a.e. } x \in \bar{\Omega}_3, \quad (62)$$

where

$$\left\{ \begin{array}{l} m(\cdot) \in H^2(\Omega_3), M(\cdot) \in H^2(\Omega_3), \\ 0 \leq m(x) \leq M(x) \leq 1, \quad \forall x \in \bar{\Omega}_3, \\ m(0, x_2, 0) = M(0, x_2, 0) = 1, \quad \forall x_2 \in (0, 1), \\ m(x_1, x_2, x_3) < M(x_1, x_2, x_3), \quad \forall x_1 < 0. \end{array} \right. \quad (63)$$

##### 3.1.3 The bilinear forms

Let be  $\alpha > 0, \beta > 0$

$$a_2(u_2, v_2) = \int_{\Omega_2} (\nabla u_2(x) \nabla v_2(x) + \alpha u_2(x) v_2(x)) dx, \quad (64)$$

$$a_3(u_3, v_3) = \int_{\Omega_3} (\nabla u_3(x) \nabla v_3(x) + \beta u_3(x) v_3(x)) dx. \quad (65)$$

We associate to (64), (65) the differential operators  $A_2, A_3$

$$A_2 = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \alpha, \quad (66)$$

$$A_3 = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + \beta. \quad (67)$$

### 3.1.4 The variational inequality

Find  $u = (u_2, u_3) \in K$  such that

$$a_2(u_2, v_2 - u_2) + a_3(u_3, v_3 - u_3) \geq (f_2, v_2 - u_2) + (f_3, v_3 - u_3), \quad \forall (v_2, v_3) \in K. \quad (68)$$

where  $f_2 \in L^2(\Omega_2)$ ,  $f_3 \in L^2(\Omega_3)$ , and  $(v, w)$  denotes the scalar product in  $L^2(\Omega_2)$  or  $L^2(\Omega_3)$ .

### 3.1.5 An equivalent formulation

Since  $K$  is a cone, (68) is equivalent to the following conditions

$$a_2(u_2, u_2) + a_3(u_3, u_3) = (f_2, u_2) + (f_3, u_3), \quad (u_2, u_3) \in K, \quad (69)$$

$$a_2(u_2, v_2) + a_3(u_3, v_3) \geq (f_2, v_2) + (f_3, v_3), \quad \forall (v_2, v_3) \in K. \quad (70)$$

### 3.1.6 Existence and uniqueness

$K$  is a closed set and the bilinear form

$$a_2(u_2, u_2) + a_3(u_3, u_3) \quad (71)$$

is coercive in  $H^1(\Omega_2) \times H^1(\Omega_3)$ , then there exists an unique solution  $u = (u_2, u_3)$  of (68).

## 3.2 Characterization of the solution

We will establish here a set of necessary conditions verified by the solution  $u$  of (68). It can be also proved that these conditions are sufficient to identify the solution  $u$ .

### 3.2.1 Conditions verified by $u_2$

We will suppose in the following that  $u_2 \in H^2(\Omega_2)$ ,  $u_3 \in H^2(\Omega_3)$ .

We define

$$\begin{cases} \partial\Omega_2^e = [0, 1] \times \{0\} \cup [0, 1] \times \{1\} \cup \{1\} \times [0, 1], \\ \partial\Omega_2^i = \{0\} \times [0, 1]. \end{cases} \quad (72)$$

If we take in (68) a function  $v_2$  such that

$$v_2(0, x_2) = 0, \quad \text{a.e. } x_2 \in (0, 1),$$

we get, employing very well known arguments

$$\begin{cases} A_2 u_2 = f_2 & \text{on } \Omega_2, \\ \frac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega_2^\epsilon. \end{cases} \quad (73)$$

### 3.2.2 Conditions verified by $u_3$ : Case $u_2(0, x_2) = 0$

In this case, it is obviously satisfied:

$$u_3(x_1, x_2, x_3) = 0, \quad \forall x_1 \in (0, 1), \quad \forall x_3 \in (0, 1). \quad (74)$$

### 3.2.3 Conditions verified by $u_3$ : Case $u_2(0, x_2) \neq 0$

#### Differential conditions

We define

$$\begin{cases} S_3^- = \{(x_1, x_2, x_3) : u_3(x_1, x_2, x_3) = m(x_1, x_2, x_3) u_2(0, x_2), u_2(0, x_2) > 0\}, \\ S_3^+ = \{(x_1, x_2, x_3) : u_3(x_1, x_2, x_3) = M(x_1, x_2, x_3) u_2(0, x_2), u_2(0, x_2) > 0\}, \\ C_3 = \{x : x \in \Omega_3, u_2(0, x_2) > 0\} \setminus \{S_3^- \cup S_3^+\}. \end{cases} \quad (75)$$

From (68) and using test functions  $v_3 = u_3 + \varphi$  - where  $\varphi$  is an arbitrary smooth function with compact support in  $\Omega_3$  - we get

$$\begin{cases} A_3 u_3 \geq f_3 & \text{a.e. } x \in S_3^-, \\ A_3 u_3 \leq f_3 & \text{a.e. } x \in S_3^+, \\ A_3 u_3 = f_3 & \text{a.e. } x \in C_3. \end{cases} \quad (76)$$

#### Boundary conditions

Let  $\bar{x} = (x_1, x_2, x_3) \in \partial\Omega_3^\epsilon$ , where

$$\begin{aligned} \partial\Omega_3^\epsilon = & \{-1\} \times [0, 1] \times [0, 1] \cup [-1, 0] \times \{0\} \times [0, 1] \cup [-1, 0] \times [0, 1] \times \{1\} \cup \\ & \{0\} \times [0, 1] \times [0, 1] \cup [-1, 0] \times \{1\} \times [0, 1] \cup [-1, 0] \times [0, 1] \times \{0\}. \end{aligned}$$

If  $u_2(0, x_2) > 0$ , we can take a test function  $v_3 = u_3 + \varphi(x) \cdot \check{\varphi}(x)$ , where the support of  $\check{\varphi}$  is the polyhedrum  $Z = ABCDA'B'C'D'$  shown in Figure 2.

It verifies the relations:

$$l = \text{length}(A''B'') = \text{length}(B''C'') = \text{length}(C''D'') = \text{length}(D''A'')$$

$$l = \text{length}(A'A') = \text{length}(B'B') = \text{length}(C'C') = \text{length}(D'D')$$

$$l = \text{length}(A'B') = \text{length}(B'C') = \text{length}(C'D') = \text{length}(D'A')$$

and

$$l + \epsilon = \text{length}(AB) = \text{length}(BC) = \text{length}(CD) = \text{length}(DA).$$

$\varphi$  is a suitable continuous function and the point  $\bar{x}$  is the center of the square ABCD.

The test function  $\check{\varphi}$  is defined in the following way

$$\begin{cases} \check{\varphi}(A) = \check{\varphi}(B) = \check{\varphi}(C) = \check{\varphi}(D) = 0, \\ \check{\varphi}(A') = \check{\varphi}(B') = \check{\varphi}(C') = \check{\varphi}(D') = 0, \\ \check{\varphi}(A'') = \check{\varphi}(B'') = \check{\varphi}(C'') = \check{\varphi}(D'') = 1. \end{cases} \quad (77)$$

Outside the polyhedrum  $Z = ABCDA'B'C'D'$ , the function  $\check{\varphi}$  is zero. Inside Z, it is continuous and locally affine, i.e. it is locally an element of  $P^1$  (see [3]).

We take

$$\epsilon \rightarrow 0, \quad l \rightarrow 0, \quad \epsilon \ll l, \quad (\text{e.g. } l \rightarrow 0, \epsilon = l^2); \quad (78)$$

in that case, from (68) we get,

$$\begin{aligned} 0 &\leq \lim_{l \rightarrow 0} \frac{1}{l^2} (a_3(u_3, v_3 - u_3) - (f_3, v_3 - u_3)) \\ &= \lim_{l \rightarrow 0} \frac{1}{l^2} \int_{\partial\Omega_3} \frac{\partial u_3}{\partial n} \cdot \varphi \, ds = \frac{\partial u_3}{\partial n}(\bar{x}) \cdot \varphi(\bar{x}), \end{aligned} \quad (79)$$

If  $u_3(x) < M(x_1, x_2, x_3) u_2(0, x_2)$ , we can take  $\varphi(\bar{x}) > 0$ , then

$$\frac{\partial u_3}{\partial n}(x) \geq 0. \quad (80)$$

Also, if  $u_3(x) > m(x_1, x_2, x_3) u_2(0, x_2)$ , we take  $\varphi(\bar{x}) < 0$  and we obtain

$$\frac{\partial u_3}{\partial n}(x) \leq 0. \quad (81)$$

Then, in summary:

$$\begin{aligned} & \forall x \in \partial\Omega_3^c / u_2(0, x_2) > 0, \\ & \left\{ \begin{array}{l} \frac{\partial u_3}{\partial n}(x) \geq 0, \quad \text{if } u_3(x) = m(x_1, x_2, x_3) u_2(0, x_2), \\ \frac{\partial u_3}{\partial n}(x) \leq 0, \quad \text{if } u_3(x) = M(x_1, x_2, x_3) u_2(0, x_2), \\ \frac{\partial u_3}{\partial n}(x) = 0, \quad \text{if } m(x_1, x_2, x_3) u_2(0, x_2) < u_3(x) < M(x_1, x_2, x_3) u_2(0, x_2). \end{array} \right. \end{array} \quad (82)$$

**Remark 3.1**  $\forall x \in \partial\Omega_3^c / u_2(0, x_2) = 0$ , it is obviously verified  $\frac{\partial u_3}{\partial n}(x) = 0$ , because in that case,

$$u_3(x_1, x_2, x_3) = 0 \quad \forall (x_1, x_3) \in (-1, 0) \times (0, 1).$$

### 3.2.4 The coupling equilibrium conditions at the interaction boundary $\partial\Omega_2^i$

We present here the coupling conditions that hold at each point  $(0, x_2)$  of the interaction boundary  $\partial\Omega_2^i$ . We denote

$$\Omega_3(x_2) = [-1, 0] \times \{x_2\} \times [0, 1]. \quad (83)$$

$$\begin{aligned} \partial\Omega_3(x_2) = & [-1, 0] \times \{x_2\} \times \{0\} \cup \{0\} \times \{x_2\} \times [0, 1] \cup \\ & [-1, 0] \times \{x_2\} \times \{1\} \cup \{-1\} \times \{x_2\} \times [0, 1]. \end{aligned} \quad (84)$$

With this notation, we will prove that if  $u_2(0, x_2) > 0$ , it is verified

$$\begin{aligned} & -\frac{\partial u_2}{\partial x_1}(0, x_2) + \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3))^+ \cdot m(x_1, x_2, x_3) dx_1 dx_3 \\ & - \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3))^- \cdot M(x_1, x_2, x_3) dx_1 dx_3 \\ & + \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^+ \cdot m(x_1, x_2, x_3) dl \\ & - \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^- \cdot M(x_1, x_2, x_3) dl = 0, \end{aligned} \quad (85)$$

and for the case  $u_2(0, x_2) = 0$

$$\begin{aligned}
& -\frac{\partial u_2}{\partial x_1}(0, x_2) + \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3))^+ \cdot m(x_1, x_2, x_3) dx_1 dx_3 \\
& - \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3))^- \cdot M(x_1, x_2, x_3) dx_1 dx_3 \\
& + \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^+ \cdot m(x_1, x_2, x_3) dl \\
& - \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^- \cdot M(x_1, x_2, x_3) dl \geq 0.
\end{aligned} \tag{86}$$

To obtain the coupling conditions (85), (86), that holds at almost every point  $(0, x_2) \in \partial\Omega_2^i$ , we take into account the inequality (68) and conditions (73), (76), (82).

In this way, we obtain,  $\forall (v_2, v_3) \in K$

$$\int_{\partial\Omega_2^i} \left( \frac{\partial u_2}{\partial n}(0, x_2) \right) \cdot ((v_2 - u_2)(0, x_2)) dx_2 + a_3(u_3, v_3 - u_3) - (f_3, v_3 - u_3) \geq 0. \tag{87}$$

Let  $l$  be a real positive parameter and  $\bar{x}_2$  an arbitrary point of  $\partial\Omega_2^i$  such that

$$u_2(0, \bar{x}_2) > 0. \tag{88}$$

We choose a collection of functions  $v_2^l : v_2^l = u_2 \cdot (1 + \tilde{\varphi}_l)$ , where  $\tilde{\varphi}_l$  has the shape shown in Figure 3. In that figure we can also see the form of  $\tilde{\varphi}_l(0, \cdot)$ .

We take

$$\epsilon \rightarrow 0, l \rightarrow 0, h \rightarrow 0, h \ll l, \epsilon \ll h, \text{ (e.g. } l \rightarrow 0, h = l^2, \epsilon = l^3); \tag{89}$$

in this form we obtain, a.e.  $\bar{x}_2 \in (0, 1)$

$$\lim_{l \rightarrow 0} \frac{1}{l} \int_{\partial\Omega_2^i} \left( \frac{\partial u_2}{\partial x_1}(0, x_2) \right) \cdot ((v_2^l - u_2)(0, x_2)) dx_2 = \left( \frac{\partial u_2}{\partial x_1}(0, \bar{x}_2) \right) \cdot u_2(0, \bar{x}_2). \tag{90}$$

Now, we analyze (87) using functions  $v_3 = u_3 + \psi$  compatible with  $v_2$  (i.e., functions  $v_3$  that

verify (62)).

We take,  $\forall(x_1, x_2, x_3) \in \Omega_3$

$$\psi_l(x_1, x_2, x_3) = u_3(x_1, x_2, x_3) \cdot \tilde{\varphi}_l(0, x_2), \quad (91)$$

So,  $v_3^l = u_3 + \psi_l$  verifies

$$m(x_1, x_2, x_3) \cdot v_2^l(x_1, x_2, x_3) \leq v_3^l(x_1, x_2, x_3) \leq M(x_1, x_2, x_3) \cdot v_2^l(x_1, x_2, x_3). \quad (92)$$

The support of  $\psi_l$  is included in the set  $S_\psi^l$ :

$$S_\psi^l = (-1, 0) \times (\bar{x}_2 - l - \epsilon, \bar{x}_2 + l + \epsilon) \times (0, 1). \quad (93)$$

We can easily see that a.e.  $\bar{x}_2 \in (0, 1)$ , the following relations hold

$$\begin{aligned} \lim_{l \rightarrow 0} \frac{1}{l} \int_{\Omega_3} ((A_3 u_3 - f_3)(x)) \cdot \psi_l(x) dx \\ = \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3)) \cdot u_3(x_1, x_2, x_3) dx_1 dx_3 \end{aligned} \quad (94)$$

$$\begin{aligned} \lim_{l \rightarrow 0} \frac{1}{l} \int_{\partial\Omega_3^e} \left( \frac{\partial u_3}{\partial n}(x) \right) \cdot \psi_l(x) d\sigma \\ = \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right) \cdot u_3(x_1, x_2, x_3) dl, \end{aligned} \quad (95)$$

where  $d\sigma$  is the differential area element in  $\partial\Omega_3^e$  and  $dl$  is the infinitesimal length on  $\partial\Omega_3(x_2)$ . In consequence, taking limit in (87), we obtain

$$\begin{aligned} -\frac{\partial u_2}{\partial x_1}(0, x_2) \cdot u_2(0, \bar{x}_2) + \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3)) \cdot u_3(x_1, x_2, x_3) dx_1 dx_3 \\ + \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right) \cdot u_3(x_1, x_2, x_3) dl \geq 0 \end{aligned} \quad (96)$$

Taking now

$$\tilde{\psi}_l(x_1, x_2, x_3) = -u_3(x_1, x_2, x_3) \cdot \tilde{\varphi}_l(0, x_2), \quad (97)$$

and

$$\tilde{v}_2^l = u_2(1 - \tilde{\varphi}_l), \quad \tilde{v}_3^l = u_3 + \tilde{\psi}_l, \quad (98)$$

we get

$$\begin{aligned} \lim_{l \rightarrow 0} \frac{1}{l} \int_{\partial\Omega_2^l} \left( \frac{\partial u_2}{\partial n}(0, x_2) \right) \cdot ((\tilde{v}_2^l - u_2)(0, x_2)) dx_2 &= \frac{\partial u_2}{\partial x_1}(0, \bar{x}_2) \cdot u_2(0, \bar{x}_2) \\ \lim_{l \rightarrow 0} \frac{1}{l} \int_{\partial\Omega_3^l} \left( \frac{\partial u_3}{\partial n}(x) \right) \cdot \tilde{\psi}_l(x) d\sigma &= - \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right) \cdot u_3(x_1, x_2, x_3) dl \\ \lim_{l \rightarrow 0} \frac{1}{l} \int_{\Omega_3} ((A_3 u_3 - f_3)(x)) \cdot \tilde{\psi}_l(x) dx &= - \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3)) \cdot u_3(x_1, x_2, x_3) dx_1 dx_3. \end{aligned}$$

Then, taking limit in (87), we obtain

$$\begin{aligned} - \frac{\partial u_2}{\partial x_1}(0, x_2) \cdot u_2(0, \bar{x}_2) + \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3)) \cdot u_3(x_1, x_2, x_3) dx_1 dx_3 \\ + \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right) \cdot u_3(x_1, x_2, x_3) dl \leq 0. \end{aligned} \tag{99}$$

From (96), (99), we obtain

$$\begin{aligned} - \frac{\partial u_2}{\partial x_1}(0, x_2) + \frac{1}{u_2(0, \bar{x}_2)} \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3)) \cdot u_3(x_1, x_2, x_3) dx_1 dx_3 \\ + \frac{1}{u_2(0, \bar{x}_2)} \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right) \cdot u_3(x_1, x_2, x_3) dl = 0. \end{aligned} \tag{100}$$



We can write (100) in a more detailed form because (76) and (82) imply that

$$\begin{aligned}
-\frac{\partial u_2}{\partial x_1}(0, x_2) &= \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3))^- \cdot M(x_1, x_2, x_3) dx_1 dx_3 \\
&+ \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3))^+ \cdot m(x_1, x_2, x_3) dx_1 dx_3 \\
&- \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^- \cdot M(x_1, x_2, x_3) dl \\
&+ \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^+ \cdot m(x_1, x_2, x_3) dl = 0.
\end{aligned} \tag{101}$$

We introduce now the following notation. We define, if  $u_2(0, x_2) = 0$ ,

$$\begin{aligned}
(F(u_2, u_3))(x_2) &= -\frac{\partial u_2}{\partial x_1}(0, x_2) \\
&- \int_{\Omega_3(x_2)} (f_3(x_1, x_2, x_3))^+ \cdot M(x_1, x_2, x_3) dx_1 dx_3 \\
&+ \int_{\Omega_3(x_2)} (f_3(x_1, x_2, x_3))^- \cdot m(x_1, x_2, x_3) dx_1 dx_3
\end{aligned} \tag{102}$$

and if  $u_2(0, x_2) > 0$ ,

$$\begin{aligned}
(E(u_2, u_3))(x_2) = & -\frac{\partial u_2}{\partial x_1}(0, x_2) \\
& - \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3))^- \cdot M(x_1, x_2, x_3) dx_1 dx_3 \\
& + \int_{\Omega_3(x_2)} ((A_3 u_3 - f_3)(x_1, x_2, x_3))^+ \cdot m(x_1, x_2, x_3) dx_1 dx_3 \\
& - \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^- \cdot M(x_1, x_2, x_3) dl \\
& + \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^+ \cdot m(x_1, x_2, x_3) dl.
\end{aligned} \tag{103}$$

Now, (101) also can be written in the following way

$$(E(u_2, u_3))(x_2) = 0. \tag{104}$$

In the case  $u_2(0, x_2) = 0$ , there are only allowed local positive perturbations of  $u_2$  and the coupling conditions are reduced to

$$(E(u_2, u_3))(x_2) \geq 0. \tag{105}$$

Both condition can be summarized in the following way

$$\min(u_2(0, x_2), (E(u_2, u_3))(x_2)) = 0. \tag{106}$$

### 3.3 Solution by decomposition

#### 3.3.1 An equivalent problem

To solve (68) by decomposition techniques, we proceed in a similar way as to that used in subsection 2.3.

We suppose that the bilinear forms  $a_1$  and  $a_2$  are symmetric and in consequence, the inequality (68) is the necessary condition that must hold at the point that realizes the minimum of the functional  $J$  on the set  $K$ , where  $K$  is given by (2) and

$$J(u_2, u_3) = \frac{1}{2} a_2(u_2, u_2) - \langle f_2, u_2 \rangle + \frac{1}{2} a_3(u_3, u_3) - \langle f_3, u_3 \rangle. \quad (107)$$

We define the set  $K_I$  and,  $\forall u_I \in K_I$ , the associated sets  $K_2(u_I)$ ,  $K_3(u_I)$

$$K_I = \left\{ u_I \in H^{\frac{1}{2}}(0, 1) : u_I(x) \geq 0, \forall x \in (0, 1) \right\}, \quad (108)$$

$$K_2(u_I) = \left\{ u_2 \in H^1(\Omega_2) : u_2(0, x_2) = u_I(x_2), \forall x_2 \in (0, 1) \right\}, \quad (109)$$

$$K_3(u_I) = \left\{ u_3 \in H^1(\Omega_3) : (111) \text{ a.e. } (x_1, x_2, x_3) \in \Omega_3 \right\}, \quad (110)$$

$$m(x_1, x_2, x_3) u_I(x_2) \leq u_3(x_1, x_2, x_3) \leq M(x_1, x_2, x_3) u_I(x_2). \quad (111)$$

We introduce the notation

$$\varphi_2(u_I) = \min_{u_2 \in K_2(u_I)} J(u_2, 0), \quad (112)$$

$$\varphi_3(u_I) = \min_{u_3 \in K_3(u_I)} J(0, u_3), \quad (113)$$

$$\varphi(u_I) = \varphi_2(u_I) + \varphi_3(u_I). \quad (114)$$

and to compute the functions  $\varphi_2$ ,  $\varphi_3$  we define the problems:

$$P_2(u_I) : \text{Find } \bar{u}_2(u_I) \text{ such that } J(\bar{u}_2, 0) = \varphi_2(u_I), \quad (115)$$

$$P_3(u_I) : \text{Find } \bar{u}_3(u_I) \text{ such that } J(0, \bar{u}_3) = \varphi_3(u_I). \quad (116)$$

From (61), (108), (109), (110), we can write

$$K = \bigcup_{u_I \in K_I} (K_2(u_I) \times K_3(u_I))$$

and in consequence

$$\min_{(u_2, u_3) \in K} J(u_2, u_3) = \min_{u_I \in K_I} \left( \min_{K_2(u_I) \times K_3(u_I)} J(u_2, u_3) \right). \quad (117)$$

But from (107) we have

$$\begin{aligned} \min_{K_2(u_I) \times K_3(u_I)} J(u_2, u_3) &= \left( \min_{u_2 \in K_2(u_I)} J(u_2, 0) \right) + \left( \min_{u_3 \in K_3(u_I)} J(0, u_3) \right) \\ &= \varphi_2(u_I) + \varphi_3(u_I) = \varphi(u_I). \end{aligned} \quad (118)$$

Then

$$\min_{(u_2, u_3) \in K} J(u_2, u_3) = \min_{u_I \in K_I} \varphi(u_I) \quad (119)$$

and we conclude that problem (68) is equivalent to the following problem  $P_I$ :

$$P_I : \text{Find } \bar{u}_I \text{ such that } \varphi(\bar{u}_I) = \min_{u_I \in K_I} \varphi(u_I). \quad (120)$$

**Remark 3.2** From (117)-(120) it follows that the new problem  $P_I$  is a decomposable hierarchical optimization problem [9]. The keystone of the decomposition method is the fact that the function  $\varphi$  appearing in (120) verifies (114) and that  $(\varphi_2, \varphi_3)$  can be computed solving two separated optimization problems.

**Remark 3.3** The sets  $K_2$ ,  $K_3$  and  $K_I$  verify the hypotheses of Theorem 1.1. Then  $\varphi_2$  and  $\varphi_3$  are convex. In fact they are strictly convex and continuous functions. In addition,  $K_I$  is closed, then Problem  $P_I$  has a unique solution  $\bar{u}_I$ .

**Remark 3.4**  $(\bar{u}_2(\bar{u}_I), \bar{u}_3(\bar{u}_I))$  is the solution of the variational inequality (68).

**Remark 3.5** The problem  $P_I$  is easier than (68) because

1. The dimension of  $\Omega_I := (0, 1)$  (where  $K_I$  is defined) is 1.
2.  $K_I$  has a very simple structure (the cone of positive elements of  $H^{\frac{1}{2}}(0, 1)$ ).
3. Problem  $P_I$  is the minimum of a convex function on the set  $K_I$  which involves only an unilateral restriction.
4. By solving the problems  $P_2$  and  $P_3$  it is possible to compute a descent direction for  $\varphi$ .

### 3.3.2 An iterative procedure

The fundamental step to solve  $P_I$  by iteration is to find a way to reduce the value of  $\varphi$ .

Although the function  $\varphi$  can be non-differentiable, when condition (106) is not verified, it is possible to find a descent direction for the function  $\varphi$  and to compute suitable increments  $\delta u_I$ .

### 3.3.3 Computation of a descent direction

Given  $u_I$  and the associated  $u_2$  (solution of problem  $P_2$ ), we will suppose that it is possible to define a continuous linear operator  $T_2$  which carries the admissible increments  $\delta u_I$  (such that  $u_I + \delta u_I \in K_I$ ) to vectors  $T_2 \delta u_I$  such that  $u_2 + T_2 \delta u_I \in K_2(u_I + \delta u_I)$ ; then we have

$$\varphi_2(u_I + \delta u_I) \leq J(u_2 + T_2 \delta u_I, 0). \quad (121)$$

But the function  $J(\cdot, 0)$  is differentiable (Frechet). Then,  $\exists G_2(u_2) \in H^1(\Omega_2)$  such that

$$J(u_2 + T_2 \delta u_I, 0) = \varphi_2(u_I) + (G_2(u_2), T_2 \delta u_I) + o(T_2 \delta u_I).$$

As the operator  $T_2$  is continuous, we have

$$\varphi_2(u_I + \delta u_I) \leq \varphi_2(u_I) + (T_2^* G_2(u_2), \delta u_I) + o(\delta u_I).$$

In a similar form we have

$$\varphi_3(u_I + \delta u_I) \leq \varphi_3(u_I) + (T_3^* G_3(u_3), \delta u_I) + o(\delta u_I).$$

So,

$$\varphi(u_I + \delta u_I) \leq \varphi(u_I) + \langle T_2^* G_2(u_2) + T_3^* G_3(u_3), \delta u_I \rangle + o(\delta u_I).$$

Then, if we find a point  $\hat{u}_I \in K_I$  such that

$$\langle T_2^* G_2(u_2) + T_3^* G_3(u_3), \hat{u}_I - u_I \rangle < 0,$$

we have that  $\hat{u}_I - u_I$  is a descent direction of the functional  $\varphi$ .

As  $K_I$  is unbounded, to obtain a descent direction given by a finite vector, we introduce the problem:

$$\hat{P}_I(\hat{T}, u_I) : \text{Find } \hat{u}_I \in K_I \text{ that verifies : } \Xi_{\hat{T}, u_I}(\hat{v}_I) = \min_{v_I \in K_I} \Xi_{\hat{T}, u_I}(v_I), \quad (122)$$

where

$$\Xi_{\hat{T}, u_I}(v_I) = \frac{1}{2} \|v_I - u_I\|_{H^{\frac{1}{2}}(\Omega_I)}^2 + \langle \hat{T}, v_I - u_I \rangle, \quad (123)$$

$$\hat{T} = T_2^* G_2(u_2) + T_3^* G_3(u_3). \quad (124)$$

**Remark 3.6** For any  $u_I \in K_I$  the problem  $\hat{P}_I$  has a unique solution  $\hat{u}_I$ .

**Remark 3.7** If

$$m \{x_2 : \min (u_2(0, x_2), (E(u_2, u_3))(x_2)) \neq 0\} > 0, \quad (125)$$

then the solution of problem  $P_I(u_I, \phi)$  is non-trivial and it gives a descent direction for the function  $\varphi$ .

We define

$$\underline{\Psi} = \min_{v_I \in K_I} \left( \frac{1}{2} \|v_I - u_I\|_{H^{\frac{1}{2}}(\Omega_I)}^2 + \langle \hat{T}, v_I - u_I \rangle \right). \quad (126)$$

**Remark 3.8** The following alternative holds:  $\exists$  a descent direction iff  $\underline{\Psi} < 0$ .

**Remark 3.9** If  $\hat{u}_I \neq u_I$  then  $\hat{u}_I$  brings a descent direction because

$$\langle \hat{T}, \hat{u}_I - u_I \rangle < \underline{\Psi} - \frac{1}{2} \|\hat{u}_I - u_I\|^2 < 0.$$

**Remark 3.10** If  $\underline{\Psi} = 0$ , then  $\hat{u}_I = u_I$  and  $\langle \hat{T}, v_I - u_I \rangle \geq 0 \quad \forall v_I \in K_I$ . So, the vectors  $\bar{u}_2(u_I)$ ,  $\bar{u}_3(u_I)$  associated to  $u_I$  verifies the conditions to be the unique solution of the original VI (68). Conversely, by Remark 3.7, if the optimality conditions are not verified, there exists a descent direction.

So, we have seen that the basic step to solve iteratively the problem  $P_I$  is to solve a sequence of problems  $\hat{P}_I$ . To do that, it is necessary to compute the linear operator  $\hat{T}$ . In the following sections, we will present some suitable operators  $T_2$ ,  $T_3$  and we will give the explicit expression of  $T_2^* G_2$  and  $T_3^* G_3$ .

### 3.3.4 Calculus of $T_2^* G_2$

#### The operator $T_2$

We define  $T_2 \delta u_1$  as the function  $w_2$  which is the unique solution of the problem

$$\begin{cases} \Delta w_2(x_2) = 0 & a.e. x \in \Omega_2, \\ w_2(0, x_2) = \delta u_1(x_2) & \forall x_2 \in (0, 1), \\ \frac{\partial w_2}{\partial n}(x) = 0 & a.e. x \in \partial\Omega_2 \setminus (\{0\} \times (0, 1)). \end{cases}$$

#### The operator $G_2(u_2)$

By (107) we have

$$J(u_2 + w_2, 0) = \int_{\Omega_2} \left( \frac{1}{2} (\nabla u_2 + \nabla w_2, \nabla u_2 + \nabla w_2) + \frac{\alpha}{2} (u_2 + w_2)^2 - f_2 (u_2 + w_2) \right) dx,$$

then

$$\langle G_2(u_2), w_2 \rangle = \int_{\Omega_2} ((\nabla u_2, \nabla w_2) + \alpha u_2 w_2 - f_2 w_2) dx.$$

If  $u_2 \in H^2(\Omega_2)$

$$\langle G_2(u_2), w_2 \rangle = \int_{\Omega_2} (-\Delta u_2 + \alpha u_2 - f_2) w_2 dx + \int_0^1 \frac{\partial u_2}{\partial n}(0, x_2) \cdot w_2(0, x_2) dx_2. \quad (127)$$

#### Calculus of $T_2^* G_2(u_2)$

But  $w_2 = T_2 \delta u_1$ , and from (73)  $(-\Delta u_2 + \alpha u_2 - f_2) = 0$ , then

$$\langle G_2(u_2), T_2 \delta u_1 \rangle = \int_0^1 \frac{\partial u_2}{\partial n}(0, x_2) \cdot w_2(0, x_2) dx_2.$$

Finally, as  $w_2(0, x_2) = \delta u_1(x_2)$ , we have

$$\langle T_2^* G_2(u_2), \delta u_1 \rangle = \int_0^1 \frac{\partial u_2}{\partial n}(0, x_2) \cdot \delta u_1(x_2) dx_2. \quad (128)$$

3.3.5 Calculus of  $T_3^*G_3$ **The operator  $T_3$** 

We define

$$S_M = \{x \in \Omega_3 : u_3(x_1, x_2, x_3) = M(x_1, x_2, x_3) \cdot u_I(0, x_2)\}, \quad (129)$$

$$S_m = \{x \in \Omega_3 : u_3(x_1, x_2, x_3) = m(x_1, x_2, x_3) \cdot u_I(0, x_2)\}, \quad (130)$$

and we define – partially – the auxiliary function  $t : \bar{\Omega}_3 \rightarrow \mathfrak{R}$

$$t = \begin{cases} M & \text{in } S_M, \\ m & \text{in } S_m. \end{cases} \quad (131)$$

We will suppose that it is possible to extend the function  $t$  from  $(S_M \cup S_m)$  to  $\bar{\Omega}_3 \setminus (S_M \cup S_m)$  in such a way that  $t \in H^2(\bar{\Omega}_3)$ .

In fact, the unique utility we get from  $t$  is its role in the computation of the auxiliary function  $\zeta$  (defined in (135)) and that result is independent of the value of  $t$  on  $\bar{\Omega}_3 \setminus (S_M \cup S_m)$ .

We define now,  $\forall (x_1, x_2, x_3) \in \bar{\Omega}_3$

$$(T_3 w_3)(x_1, x_2, x_3) = t(x_1, x_2, x_3) \cdot w_3(x_2). \quad (132)$$

**The operator  $G_3(u_3)$** 

$$J(0, u_3 + w_3) = \int_{\Omega_3} \left( \frac{1}{2} (\nabla u_3 + \nabla w_3, \nabla u_3 + \nabla w_3) + \frac{\beta}{2} (u_3 + w_3, u_3 + w_3) - f_3(u_3 + w_3) \right) dx,$$

then

$$\langle G_3(u_3), w_3 \rangle = \int_{\Omega_3} ((\nabla u_3, \nabla w_3) + \beta u_3 w_3 - f_3 w_3) dx.$$

If  $u_3 \in H^2(\Omega_3)$

$$\langle G_3(u_3), w_3 \rangle = \int_{\Omega_3} (-\Delta u_3 + \beta u_3 - f_3) w_3 dx + \int_{\partial\Omega_3} \frac{\partial u_3}{\partial n} w_3 ds \quad (133)$$



### Calculus of $T_3^* G_3(u_3)$

When  $w_3 = T_3 \delta u_I$ , by (132) and (133) we have

$$\begin{aligned}
\langle G_3(u_3), T_3 \delta u_I \rangle &= \int_{\Omega_3} ((-\Delta u_3 + \beta u_3 - f_3)(x_1, x_2, x_3)) \cdot t(x_1, x_2, x_3) \cdot \delta u_I(x_2) dx \\
&\quad + \int_{\partial\Omega_3} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right) \cdot t(x_1, x_2, x_3) \cdot \delta u_I(x_2) ds \\
&= \int_0^1 \left( \int_{\Omega_3(x_2)} ((-\Delta u_3 + \beta u_3 - f_3)(x_1, x_2, x_3)) \cdot t(x_1, x_2, x_3) dx_1 dx_3 \right) \delta u_I(x_2) dx_2 \\
&\quad + \int_0^1 \left( \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right) \cdot t(x_1, x_2, x_3) dx_1 dx_3 \right) \delta u_I(x_2) dx_2.
\end{aligned}$$

Then, if we take into account (132) we obtain

$$\langle T_3^* G_3(u_3), \delta u_I \rangle = \int_0^1 \zeta(x_2) \cdot \delta u_I(x_2) dx_2, \quad (134)$$

where

$$\begin{aligned}
\zeta(x_2) &= \int_{\Omega_3(x_2)} ((-\Delta u_3 + \beta u_3 - f_3)(x_1, x_2, x_3))^+ \cdot M(x_1, x_2, x_3) dx_1 dx_3 \\
&\quad + \int_{\Omega_3(x_2)} ((-\Delta u_3 + \beta u_3 - f_3)(x_1, x_2, x_3))^- \cdot m(x_1, x_2, x_3) dx_1 dx_3 \\
&\quad + \left( \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^+ \cdot M(x_1, x_2, x_3) dx_1 dx_3 \right) \\
&\quad + \left( \int_{\partial\Omega_3(x_2)} \left( \frac{\partial u_3}{\partial n}(x_1, x_2, x_3) \right)^- \cdot m(x_1, x_2, x_3) dx_1 dx_3 \right). \quad (135)
\end{aligned}$$

### 3.3.6 Calculus of $\hat{T} = T_2^* G_2 + T_3^* G_3$

Taking into account (103), (128) and (134) we obtain the following explicit expression of  $\hat{T}$  :

$$\hat{T}(\delta u_I) = \int_0^1 (E(u_2, u_3))(x_2) \cdot \delta u_I(x_2) dx_2. \quad (136)$$

**Note 3.1** In the following, we will identify the functional  $\hat{T}$  with the function  $E(u_2, u_3)$  (obviously through the action given by (136)).

### 3.3.7 The auxiliary algorithm $A_I$

#### Algorithm $A_I$

Step 1) Choose any  $u_I \in K_I$  and  $\hat{\rho} > 0$ . Set  $\nu = 0$  and  $u_I^0 = u_I$ .

Step 2) Set  $\rho_\nu = \hat{\rho}$ . Compute the solutions  $u_2^\nu(u_I^\nu)$  of  $P_2$  and  $u_3^\nu(u_I^\nu)$  of  $P_3$ .

Step 3) Compute  $\hat{T} = E(u_2^\nu(u_I^\nu), u_3^\nu(u_I^\nu))$ .

Step 4) Find the solution  $\bar{v}$  of the problem  $\hat{P}_I(\hat{T}, u_I^\nu)$ .

Step 5) Set  $\hat{u}_I = u_I^\nu + \rho_\nu (\bar{v} - u_I^\nu)$ .

Step 6) If  $\phi(\hat{u}_I) < \phi(u_I^\nu) + \frac{1}{2} \rho_\nu \langle \hat{T}, \hat{u}_I - u_I^\nu \rangle$  set  $u_I^{\nu+1} = \hat{u}_I$ ,  $\nu = \nu + 1$   
and go to step 2;

else, set  $\rho_\nu = \frac{1}{2} \rho_\nu$  and go to step 5.

#### The issue of convergence.

The algorithm  $A_I$  generates a sequence  $\{u_I^\nu, u_2^\nu, u_3^\nu\}$  such that  $(u_2^\nu, u_3^\nu)$  converges to the unique solution of (68). In order to shorten the length of this report, the proof of this property will not be included here.

## 4 Conclusions

We have solved in this paper two junction problems involving bilateral restrictions. Using a variational inequality formulation, we have obtained both the differential equations and the global equilibrium condition which identify the solution. In order to have a practical methodology to solve the original problem, we have developed an iterative method of decomposition–coordination type. We have used this procedure because it allows us to solve the coupled problem through the solution of two simple independent problems – one of them a bilateral obstacle problem and the other a linear problem.

Although convergent, the performance of the procedure can be further improved choosing a suitable operator in the problem:

$$\min_{v_I \in K_I} \left( \frac{1}{2} \|v_I - u_I\|_{H^{\frac{1}{2}}(\Omega_I)}^2 + \langle \hat{T}, v_I - u_I \rangle \right).$$

ie., we replace that problem by the problem:

$$\min_{v_I \in K_I} \left( \frac{1}{2} \|Q(v_I - u_I)\|_{H^{\frac{1}{2}}(\Omega_I)}^2 + \langle \hat{T}, v_I - u_I \rangle \right).$$

This behavior is related to the preconditioning of the auxiliary problem:

$$P_I : \text{Find } \bar{u}_I \text{ such that } \varphi(\bar{u}_I) = \min_{u_I \in K_I} \varphi(u_I).$$

This phenomenon has been recognized and analyzed in [4]. In case the operator  $Q$  is correctly chosen, we can get a procedure of Newton type. The analysis of this feature is treated in [6]. Sometimes, the separated problems  $P_1$  and  $P_2$  can be solved in a faster way by ad-hoc techniques, v.g. FFT methods. Some applications concerning this method are developed in [6].

## Acknowledgements

The authors would like to thank:

- J.L. Lions for suggesting this problem and for several useful discussions.
- P.A. Lotito and G.F. Reyero for their careful revision of the manuscript.
- The University of Rosario for the financial support given to the Project "Control and Optimization. Theory and Applications".

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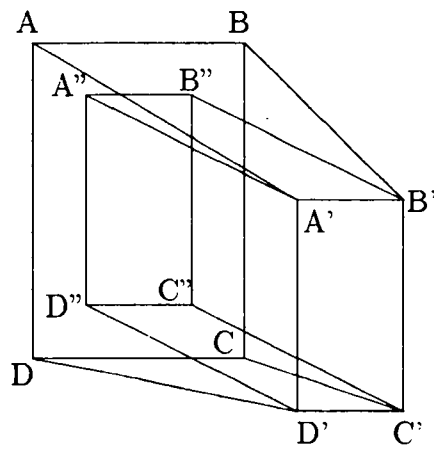


Figure 2: The support of the test function  $\tilde{\varphi}$

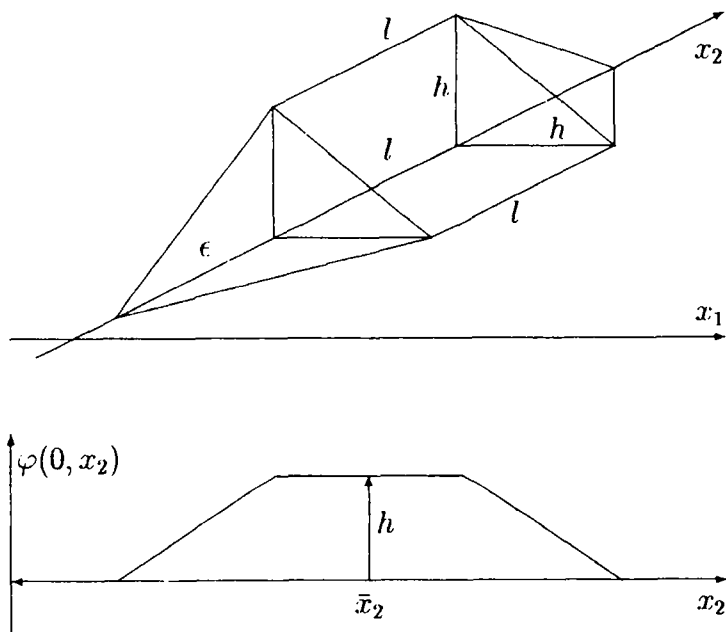


Figure 3: The shape of the test function  $\varphi$



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Éditeur

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ISSN 0249 - 6399



\* R R . 2 9 3 7 \*