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# Some Applications of Decomposition Techniques to Systems of Coupled Variational Inequalities

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*Some applications of decomposition techniques  
to systems of coupled variational inequalities.*

Roberto L.V. González , Gabriela F. Reyero

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## Some applications of decomposition techniques to systems of coupled variational inequalities.

Roberto L.V. González , Gabriela F. Reyero

Thème 4 — Simulation et optimisation  
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**Abstract:** We consider a set of problems which consists of systems of coupled variational inequalities. To solve these problems we use a decomposition–coordination method which allows us to solve the coupled problem through the solution of simple independent problems – in general, they are linear problems or simple obstacle problems. In this approach, the original problem is stated in terms of some appropriately defined auxiliary variables. These variables are modified (by an iterative algorithm in the coordination phase of the procedure) until the desired global solution is obtained..

**Key-words:** junction problems, variational inequalities, decomposition methods, convex function, unilateral condition, bilateral condition, numerical solution, iterative algorithm

*(Résumé : tsvp)*

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## Applications de techniques de décomposition aux systèmes d'inéquations variationnelles couplées.

**Résumé :** On considère ici un ensemble de problèmes définis par des systèmes d'inéquations variationnelles couplées. Pour résoudre le problème originel on utilise une méthode de décomposition-coordination. Cette méthode permet de résoudre le problème couplé à travers la solution des simples problèmes indépendants – en général, ce sont des problèmes linéaires ou des problèmes d'obstacle. Dans cette méthode le problème originel est redéfini en termes des variables auxiliaires. Ces variables sont modifiées (par un algorithme itératif dans la phase de coordination du procédé) jusqu'à ce que la solution globale soit obtenue.

**Mots-clé :** problèmes de jonctions, inégalités variationnelles, méthodes de décomposition, fonctions convexes, condition unilatérale, condition bilatérale, solution numérique, algorithmes itératifs.

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## 1 Introduction.

This paper deals with some problems of coupled variational inequalities, where the coupling may take different forms. Sometimes the solution is defined as a  $n$ -uple of functions with different domains, which have some overlapping regions (known as the interface); in that case, the coupling restrictions are defined in terms of the common values of the functions at the interface, but in other cases the coupling involves some global functionals.

In this paper we restrict our attention to the solution of some problems of coupled variational inequalities which were presented in [8].

Starting from the V.I. formulation, we obtain the solution via a decomposition–coordination method (the theoretical justification of the method is given in [5]; the method itself stems from the general methodology presented in [9]).

We use this procedure because it allows us to solve the coupled problem through the solution of simple independent problems – in general, they are linear problems or simple obstacle problems. These problems depend on some auxiliary variables which are modified (by the coordination phase of the procedure) until the desired global solution is obtained.

This rapport is organized in the following way: In this section we give a general introduction to the problem of coupled variational inequalities, Section 2 presents the decomposition method and Section 3 contains the detailed solution of five selected problems

### 1.1 Description of the original problem

#### 1.1.1 A brief presentation of junction problems

This work is originated by what is known as *junction problems* – we can see [1], [2], [4], [5] and the bibliography therein, for a more detailed description of these problems and the analysis of some related topics. Specifically, our work deals with some issues that appear when the variational inequality approach is used to analyze these problems (see [8]). In order to fix the ideas, we consider the geometrical

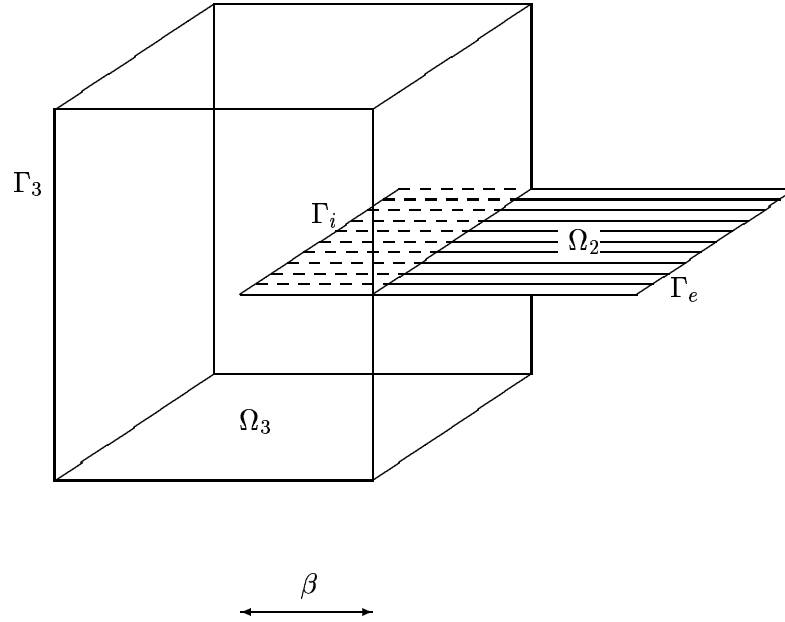


Figure 1: A 2-3 dimensional coupled problem

situation shown in Fig. 1, where

$$\Omega_2 = \{x \in \mathbb{R}^3 : -\beta < x_1 < 1, -1 < x_2 < 1, x_3 = 0\},$$

$$\Phi = \{x \in \mathbb{R}^3 : -1 < x_1 < 0, -1 < x_2 < 1, -1 < x_3 < 1\},$$

$$\Omega_3 = \Phi \setminus \Omega_2.$$

In  $\mathbb{R}^3$  we have two open domains  $\Omega_2$  and  $\Omega_3$ , of dimensions 2 and 3 respectively;  $\Omega_2$  is in the plane  $x_3 = 0$ . The boundary  $\partial\Omega_2$  of  $\Omega_2$  consists of  $\Gamma_{2i}$  (where  $i$  stands for *interface or junctions*) and  $\Gamma_{2e}$  (where  $e$  stands for *exterior*). The domain  $\Omega_3$  is an open domain in  $\mathbb{R}^3$ ; we denote by  $\Gamma_3 = \partial\Omega_3$  its boundary.

**Note 1.1** *Other geometrical situations, and also regions consisting of several pieces of dimensions 1,2,3, can be considered. We want to study boundary value problems in domains of type  $\Omega_2 \cup \Omega_3$ .*



### 1.1.2 System state

The state of the system (which may represent some variables of interest: temperature, displacement, etc) is given by a real function  $(u_2, u_3) : \Omega_2 \cup \Omega_3 \rightarrow \mathfrak{R}$ . We set

$$\left\{ \begin{array}{l} X_2 = H^1(\Omega_2), \quad X_3 = H^1(\Omega_3), \quad X = X_2 \oplus X_3, \\ \partial\Omega_2 = \Gamma_{2i} \cup \Gamma_{2e}, \quad \partial\Omega_3 = \Gamma_3. \end{array} \right. \quad (1)$$

### 1.1.3 Connections: Operators $M_i$

Around the interface  $\Gamma_{2i}$ , both in  $\Omega_2$  and in  $\Omega_3$ , *connections* are established between  $u_2$  and  $u_3$ . These connections may be local or not; they are defined in terms of linear continuous operators  $M_i \in \mathcal{L}(X_i, \mathcal{H})$ ,  $i = 2, 3$ , where  $\mathcal{H}$  is a given Hilbert space. Specifically, we consider a closed convex subset  $\mathcal{K}$  of  $\mathcal{H}$  and we define the *connection* in the following form:

$$M_2 u_2 - M_3 u_3 \in \mathcal{K} \subset \mathcal{H}. \quad (2)$$

We define the set of *admissible states* as the set of states that verify the connection (2), i.e.

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 : M_2 v_2 - M_3 v_3 \in \mathcal{K}\}. \quad (3)$$

Obviously, as  $M_i$  are linear continuous operators, the set  $K$  is a closed convex subset of  $X$ .

### 1.1.4 The energy functional: The bilinear forms

As it is usual in physical problems, we find the state of the system looking for the admissible state that minimizes a functional of energy, which in our problem will be described in terms of a couple of bilinear forms  $a_2, a_3$ . We will suppose that these bilinear forms are symmetric and we define the functional  $J : X_2 \oplus X_3 \rightarrow \mathfrak{R}$  in the following way:

$$J(u_2, u_3) = \frac{1}{2} a_2(u_2, u_2) - (f_2, u_2) + \frac{1}{2} a_3(u_3, u_3) - (f_3, u_3), \quad (4)$$

where the bilinear forms have the following expressions ( $\alpha > 0, \beta > 0$ )

$$a_2(u_2, v_2) = \int_{\Omega_2} (\nabla u_2 \nabla v_2 + \alpha u_2 v_2) dx, \quad (5)$$

$$a_3(u_3, v_3) = \int_{\Omega_3} (\nabla u_3 \nabla v_3 + \beta u_3 v_3) dx. \quad (6)$$

We associate to (5), (6) the differential operators  $A_2$  and  $A_3$  :

$$A_2 = -\Delta + \alpha, \quad A_3 = -\Delta + \beta.$$

In the examples analyzed in this paper, we will restrict the study to bilinear forms associated to these simple second order differential operators, although extensions to more general operators are straightforward and without difficulties.

We also define the functional operators  $\mathcal{A}_2$  and  $\mathcal{A}_3$  such that

$$a_2(u_2, v_2) = \langle \mathcal{A}_2 u_2, v_2 \rangle \text{ and } a_3(u_3, v_3) = \langle \mathcal{A}_3 u_3, v_3 \rangle, \quad (7)$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $H^1(\Omega_2)$  or  $H^1(\Omega_3)$ .

## 1.2 Coupled variational inequalities

We must minimize the functional (4) in the set of admissible states  $K$  or, in a equivalent way, we will consider the variational inequality: Find  $u = (u_2, u_3) \in K$  such that

$$a_2(u_2, v_2 - u_2) + a_3(u_3, v_3 - u_3) \geq (f_2, v_2 - u_2) + (f_3, v_3 - u_3), \quad \forall (v_2, v_3) \in K, \quad (8)$$

where  $f_2 \in L^2(\Omega_2)$ ,  $f_3 \in L^2(\Omega_3)$  and  $(v, w)$  denotes the inner product in  $L^2(\Omega_2)$  or  $L^2(\Omega_3)$ . By virtue of (5), (6), this variational inequality has a unique solution (see [10]).

**Remark 1** *By assumption, the bilinear forms  $a_2$  and  $a_3$  are symmetric and in consequence, the inequality (8) is the necessary condition that must hold at the point that realizes the minimum of the functional  $J$  on the set  $K$ .*

Using the general framework above presented, we introduce de following example which will be completely defined once we have specified  $\mathcal{H}$ ,  $\mathcal{K}$ ,  $M_2$  and  $M_3$ . This problem will be solved, among others, by decomposition techniques in Section 3.

**Example 1** *We take  $\mathcal{H} = H^{1/2}(\Gamma_3)$ ,  $\mathcal{K} = 0$ , and we define the operator  $M_3$  to be*

$$M_3 v_3 = \gamma_3 v_3,$$

*where  $\gamma_3$  is the trace of  $v_3$  on  $\Gamma_3$ .*

We consider now the trace operator  $T \in H^1(\Gamma_3) \longrightarrow H^{1/2}(\Gamma_{2i})$  such that  $T \phi = \phi|_{\Gamma_{2i}}$ . This operator has inverse, an operator  $R$  such that

$$R \in \mathcal{L}(H^{1/2}(\Gamma_{2i}); H^1(\Gamma_3)), \quad T R \phi = \phi \quad \forall \phi \in H^{1/2}(\Gamma_{2i}). \quad (9)$$

We introduce now the operator  $E \in \mathcal{L}(H^{1/2}(\Gamma_{2i}); H^{1/2}(\Gamma_3))$ ; we take, for example,  $E = R$ . Let  $\gamma_{2i} v_2$  be the trace of  $v_2$  on  $\Gamma_{2i}$ , and so we have  $\gamma_{2i} v_2 \in H^{1/2}(\Gamma_{2i})$ . We define  $M_2 v_2 = E \gamma_{2i} v_2$  and we consider the convex set of admissible states

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 : M_2 v_2 - M_3 v_3 = 0\}. \quad (10)$$

In this case, the connection condition

$$M_2 v_2 = M_3 v_3, \quad (11)$$

means that the extension to  $\Gamma_3$  of the trace  $\gamma_{2i} v_2$  must be equal to the trace of  $v_3$ .

## 2 A decomposition method

The idea of solving (8) by hierarchical optimization and decomposition stems from the fact that the solution  $u = (u_2, u_3)$  depends on the values of a function defined on some intermediate space and if we choose the correct (and unique) intermediate value, then the original problem (8) is solved.

The procedure starts choosing a value  $u_I$  and then – through the solution of some ad-hoc defined separated problems – that value is corrected until condition (8) is verified.

### Decomposition of the convex $K$

We introduce a linear space  $X_I$  and a convex set  $K_I$ . Also,  $\forall u_I \in K_I$ , the associated convex closed sets  $K_2(u_I)$  and  $K_3(u_I)$  with the property

$$K = \bigcup_{u_I \in K_I} (K_2(u_I) \oplus K_3(u_I)). \quad (12)$$

**Definition 1** *We introduce the notation*

$$\varphi_2(u_I) = \min_{u_2 \in K_2(u_I)} J(u_2, 0), \quad (13)$$

$$\varphi_3(u_I) = \min_{u_3 \in K_3(u_I)} J(0, u_3). \quad (14)$$

**Definition 2** **The separated problems**

*To compute the functions  $\varphi_2$  and  $\varphi_3$  we define the problems  $\mathcal{P}_2(u_I)$  and  $\mathcal{P}_3(u_I)$ :*

**Problem  $\mathcal{P}_2(u_I)$**

$$\text{Find } \bar{u}_2(u_I) \in K_2(u_I) \text{ such that } J(\bar{u}_2, 0) = \varphi_2(u_I). \quad (15)$$

**Problem  $\mathcal{P}_3(u_I)$**

$$\text{Find } \bar{u}_3(u_I) \in K_3(u_I) \text{ such that } J(0, \bar{u}_3) = \varphi_3(u_I). \quad (16)$$

**Remark 2** *By (5) and (6), there exists unique solution of (15) and (16).*

**Definition 3** **The hierarchical optimization problem**

*Let us define the auxiliary function  $\varphi$*

$$\varphi(u_I) = \varphi_2(u_I) + \varphi_3(u_I). \quad (17)$$

*By virtue of (12) we have*

$$\min_{(u_2, u_3) \in K} J(u_2, u_3) = \min_{u_I \in K_I} \left( \min_{K_2(u_I) \oplus K_3(u_I)} J(u_2, u_3) \right). \quad (18)$$

*But from (4) we have*

$$\begin{aligned} \min_{K_2(u_I) \oplus K_3(u_I)} J(u_2, u_3) &= \left( \min_{u_2 \in K_2(u_I)} J(u_2, 0) \right) + \left( \min_{u_3 \in K_3(u_I)} J(0, u_3) \right) \\ &= \varphi_2(u_I) + \varphi_3(u_I) = \varphi(u_I). \end{aligned} \quad (19)$$

Then

$$\min_{(u_2, u_3) \in K} J(u_2, u_3) = \min_{u_I \in K_I} \varphi(u_I), \quad (20)$$

and we conclude that problem (8) is equivalent to the following:

**Problem  $\mathcal{P}_I$**

$$\text{Find } \bar{u}_I \in K_I \text{ such that } \varphi(\bar{u}_I) = \min_{u_I \in K_I} \varphi(u_I). \quad (21)$$

**Remark 3** From (18)-(21) it follows that the new problem  $\mathcal{P}_I$  is a decomposable hierarchical optimization problem [9].

The keystone of the decomposition method is the fact that the function  $\varphi$  appearing in (21) verify (17) and that  $(\varphi_2, \varphi_3)$  can be computed by solving two separated optimization problems.

**Remark 4**  $(\bar{u}_2(\bar{u}_I), \bar{u}_3(\bar{u}_I))$  is the solution of the variational inequality (8) when  $\bar{u}_I$  realizes the minimum of  $\varphi$ .

**Remark 5** In general, the problem  $\mathcal{P}_I$  is easier than (8), because

1. The dimension of  $\Omega_I$  (where  $X_I$  is defined) is smaller than the dimensions of  $\Omega_2, \Omega_3$ .
2.  $X_I$  has a simpler structure.
3. Problem  $\mathcal{P}_I$  is the minimum of a convex function on the set  $K_I$  which involves sometimes simpler restrictions.
4. By solving the problems  $\mathcal{P}_2$  and  $\mathcal{P}_3$  it is possible to compute a descent direction for  $\varphi$ .

In the following we will suppose that the hypotheses of Theorem 1 hold and in consequence,  $\varphi_2$  and  $\varphi_3$  are convex functions. Finally, from (17),  $\varphi$  is also convex.

**Theorem 1** Let  $X_I, X$  be vector spaces,  $K_I$  a closed convex subset of  $X_I$ ,  $\phi(\cdot)$  a strictly convex continuous function defined on  $X$  and (for each  $v \in K_I$ ) a closed convex subset  $K^v \subset X$  such that we can define on  $K_I$  the function

$$\psi(v) = \inf_{x \in K^v} \phi(x) \quad (22)$$

and such that there exists a unique  $\bar{u}(v) \in K^v$  with the property

$$\phi(\bar{u}(v)) = \psi(v). \quad (23)$$

Assume the multivalued mapping  $v \rightarrow K^v$  is a convex mapping in the sense that

$$\lambda K^v + (1 - \lambda) K^{\tilde{v}} \subset K^{\lambda v + (1 - \lambda)\tilde{v}}, \quad (24)$$

then  $\psi$  is a convex function.

**Proof:** Let  $v$  and  $\tilde{v}$  be elements of  $K_I$  and  $u, \tilde{u}$  such that

$$u = \bar{u}(v), \quad \tilde{u} = \bar{u}(\tilde{v}), \quad (25)$$

then,  $\psi$  is convex because

$$\begin{aligned} \psi(\lambda v + (1 - \lambda)\tilde{v}) &\leq \phi(\lambda u + (1 - \lambda)\tilde{u}) && \text{by (22) and (24)} \\ &\leq \lambda \phi(u) + (1 - \lambda)\phi(\tilde{u}) && \text{by the convexity of } \phi \\ &= \lambda \psi(v) + (1 - \lambda)\psi(\tilde{v}) && \text{by (23) and (25)}. \end{aligned}$$

□

**Definition 4 The auxiliary function  $g$**

Let us suppose that  $\varphi_2, \varphi_3$  are differentiable and let us denote  $g_2, g_3$  the corresponding derivatives. For each  $v_I \in X_I$ , we define

$$g(v_I) = g_2(v_I) + g_3(v_I). \quad (26)$$

**Definition 5 The equilibrium condition**

The optimality condition (that we will refer as the equilibrium condition) for  $\bar{u}_I$  to be the solution of Problem  $\mathcal{P}_I$  is:

$$(g(\bar{u}_I), v_I - \bar{u}_I) \geq 0, \quad \forall v_I \in K_I. \quad (27)$$

**Remark 6** As  $\varphi$  is convex this condition is also a sufficient condition of optimality. In addition, if  $\varphi$  is strictly convex, the point  $\bar{u}_I$  that verifies this optimality condition is unique.

**Definition 6 An iterative algorithm**

The above mentioned unique value  $\bar{u}_I$  can be found iteratively using the algorithm described below, which employs the information given by the function  $g$ . The algorithm generates a sequence  $(\bar{u}_2(v_I^\nu), \bar{u}_3(v_I^\nu))$ , which starts at an initial pair  $\bar{u}_2(v_I^0), \bar{u}_3(v_I^0)$  and that converges to the solution  $(u_2, u_3)$  of (8).

Let  $\{\gamma_\nu : \nu = 0, \dots\}$  be a sequence of positive numbers such that

$$\lim_{\nu \rightarrow +\infty} \gamma_\nu = 0 \text{ and } \sum_{\nu=0}^{\infty} \gamma_\nu = +\infty.$$

Let us denote by  $P_{K_I}(q)$  the projection of  $q$  on  $K_I$ .

**ALGORITHM**

Step 1 :  $\nu = 0, v_I^0 \in K_I$ .

Step 2 : Solve problems  $\mathcal{P}_2(v_I^\nu)$  and  $\mathcal{P}_3(v_I^\nu)$ . Set  $\gamma = \gamma_\nu$ .

Step 3 :  $\hat{v}_I = P_{K_I}(v_I^\nu - \gamma g(v_I^\nu))$ .

Step 4 : If  $\varphi(\hat{v}_I) < \varphi(v_I^\nu) + \frac{1}{2}(g(v_I^\nu), \hat{v}_I - v_I^\nu)$ , set  $v_I^{\nu+1} = \hat{v}_I, \nu = \nu + 1$   
and go to Step 2;

else,  $\gamma = \frac{\gamma}{2}$ , and go to Step 3.

**3 Applications**

We give in this section some examples where we apply the methodology described in the previous section. In the first four cases, we obtain the solution in an explicit form and in the last example, the solution is reduced to the application of a simple iterative method to find the zero of a monotone real function.

### 3.1 Example 1

#### 3.1.1 The original problem

The description of this problem was given in the section 1.2; so here we have

$$\left\{ \begin{array}{l} \mathcal{H} = H^{1/2}(\Gamma_3), \quad \mathcal{K} = 0, \\ (M_2 v_2)(x) = (E \gamma_{2i} v_2)(x), \quad x \in \Gamma_3, \\ (M_3 v_3)(x) = (\gamma_3 v_3)(x), \quad x \in \Gamma_3. \end{array} \right. \quad (28)$$

In this case we have

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 : E \gamma_{2i} v_2 = \gamma_3 v_3\}. \quad (29)$$

#### 3.1.2 Solution by decomposition

##### Elements of the decomposition

We define:

$$\left\{ \begin{array}{l} X_I = H^{1/2}(\Gamma_{2i}), \\ K_2(v_I) = \{v_2 \in X_2 : \gamma_{2i} v_2(x) = v_I(x), \text{ a.e. } x \in \Gamma_{2i}\}, \\ K_3(v_I) = \{v_3 \in X_3 : \gamma_3 v_3(x) = (E v_I)(x), \text{ a.e. } x \in \Gamma_3\}. \end{array} \right. \quad (30)$$

##### The separated optimization problems

With these data, we consider the problem  $\mathcal{P}_2(v_I)$

$$\boxed{\mathcal{P}_2(v_I)} : \quad \varphi_2(v_I) = \min_{u_2 \in K_2(v_I)} \frac{1}{2} \langle \mathcal{A}_2 u_2, u_2 \rangle - \langle \hat{f}_2, u_2 \rangle, \quad (31)$$



which is equivalent to find the solution  $\bar{u}_2(v_I)$  of the Dirichlet-Neumann problem:

$$\begin{cases} A_2 \bar{u}_2 = f_2, \\ \bar{u}_2|_{\Gamma_{2i}} = v_I, \\ \frac{\partial \bar{u}_2}{\partial n}|_{\Gamma_{2e}} = 0. \end{cases} \quad (32)$$

To make explicit the dependence on  $v_I$  of the solution  $\bar{u}_2(v_I)$ , we introduce the following definitions: Let  $w_2$  be the solution of

$$\begin{cases} A_2 w_2 = f_2, \\ w_2|_{\Gamma_{2i}} = 0, \\ \frac{\partial w_2}{\partial n}|_{\Gamma_{2e}} = 0 \end{cases} \quad (33)$$

and let  $G_2$  be the operator

$$\begin{aligned} G_2 : H^{1/2}(\Gamma_{2i}) &\longrightarrow H^1(\Omega_2), \\ \phi &\longrightarrow G_2 \phi, \end{aligned} \quad (34)$$

such that

$$\begin{cases} A_2(G_2 \phi) = 0, \\ (G_2 \phi)|_{\Gamma_{2i}} = \phi, \\ \frac{\partial(G_2 \phi)}{\partial n}|_{\Gamma_{2e}} = 0. \end{cases} \quad (35)$$

By virtue of (32), (33) and (35), we have

$$\bar{u}_2(v_I) = G_2 v_I + w_2. \quad (36)$$

In consequence, for the auxiliary function  $\varphi_2(v_I)$  we have

$$\varphi_2(v_I) = \frac{1}{2} \langle \mathcal{A}_2(G_2 v_I + w_2), G_2 v_I + w_2 \rangle - \langle \hat{f}_2, G_2 v_I + w_2 \rangle \quad (37)$$

and then, for the derivative (in the Frechet sense) we obtain:  $\forall \zeta \in X_I$

$$\langle D\varphi_2(v_I), \zeta \rangle = \langle G_2^* \mathcal{A}_2 G_2 v_I + G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2), \zeta \rangle. \quad (38)$$

Now, we consider the problem  $\mathcal{P}_3(v_I)$

$$\boxed{\mathcal{P}_3(v_I)} : \quad \varphi_3(v_I) = \min_{u_3 \in K_3(v_I)} \frac{1}{2} \langle \mathcal{A}_3 u_3, u_3 \rangle - \langle \hat{f}_3, u_3 \rangle. \quad (39)$$

In this case, the solution  $\bar{u}_3(v_I)$  is given by the solution of the Dirichlet problem:

$$\begin{cases} A_3 \bar{u}_3 = f_3, \\ \bar{u}_3|_{\Gamma_3} = Ev_I. \end{cases} \quad (40)$$

Let be  $w_3$  the solution of

$$\begin{cases} A_3 w_3 = f_3, \\ w_3|_{\Gamma_3} = 0 \end{cases} \quad (41)$$

and let  $G_3$  be the operator

$$\begin{aligned} G_3 : H^{1/2}(\Gamma_3) &\longrightarrow H^1(\Omega_3), \\ \phi &\longrightarrow G_3 \phi, \end{aligned} \quad (42)$$

such that

$$\begin{cases} A_3(G_3 \phi) = 0, \\ (G_3 \phi)|_{\Gamma_3} = \phi. \end{cases} \quad (43)$$

By virtue of (40), (41) and (43), we have

$$\bar{u}_3(v_I) = G_3 Ev_I + w_3. \quad (44)$$

Then, we obtain

$$\varphi_3(v_I) = \frac{1}{2} \langle \mathcal{A}_3(G_3 Ev_I + w_3), G_3 Ev_I + w_3 \rangle - \langle \hat{f}_3, G_3 Ev_I + w_3 \rangle. \quad (45)$$

Therefore, for the derivative (in the Frechet sense) we have:  $\forall \zeta \in X_I$

$$\langle D\varphi_3(v_I), \zeta \rangle = \langle (E^* G_3^* \mathcal{A}_3 G_3 E)v_I + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3), \zeta \rangle. \quad (46)$$

### Coordination phase

The solution  $\bar{u}_I$  is given by solving the equation  $g(v_I) = 0$ , where  $g(v_I) = D\varphi_2(v_I) + D\varphi_3(v_I)$ . By virtue of (38) and (46), we have

$$G_2^* \mathcal{A}_2 G_2 \bar{u}_I + G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + (E^* G_3^* \mathcal{A}_3 G_3 E) \bar{u}_I + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3) = 0. \quad (47)$$

In consequence we get

$$\bar{u}_I = Q \left( G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3) \right), \quad (48)$$

where

$$Q = -(G_2^* \mathcal{A}_2 G_2 + E^* G_3^* \mathcal{A}_3 G_3 E)^{-1}. \quad (49)$$

So, the solution is given by

$$\boxed{\begin{cases} u_2(\bar{u}_I) = G_2 Q \left( G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3) \right) + w_2, \\ u_3(\bar{u}_I) = G_3 E Q \left( G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3) \right) + w_3. \end{cases}} \quad (50)$$

## 3.2 Example 2

### 3.2.1 The original problem

We take  $X_2 = H^1(\Omega_2)$ ,  $X_3 = H^1(\Omega_3)$ ,  $X = X_2 \oplus X_3$ ,  $\mathcal{H} = \mathfrak{R}^p$ ,  $\mathcal{K} = 0$ .

We define the operators  $W_2 : X_2 \mapsto \mathfrak{R}^p$  such that  $(W_2 u_2)_j = (w_{2j}, u_2)$  and  $W_3 : X_3 \mapsto \mathfrak{R}^p$  such that  $(W_3 u_3)_j = (w_{3j}, u_3)$ , with  $w_{2j} \in L^2(\Omega_2)$  and  $w_{3j} \in L^2(\Omega_3)$ ,  $\forall j = 1, \dots, p$ .

We define the operators  $M_i$  by

$$M_2 v_2 = W_2 v_2, \quad M_3 v_3 = W_3 v_3.$$

In this case we have

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 : M_2 v_2 - M_3 v_3 = 0\}, \quad (51)$$

or

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 : (v_2, w_{2j}) = (v_3, w_{3j}), 1 \leq j \leq p\}. \quad (52)$$

**Note 3.1** *In this example we will suppose that the operators  $W_2, W_3$  defined above verify:  $\text{rank}(W_i) = p$ .*

### 3.2.2 Solution by decomposition. Case of dimension 1

$$\left\{ \begin{array}{l} X_I = R, \\ K_2(\xi) = \{u_2 \in X_2 : (w_2, u_2) = \xi\}, \\ K_3(\xi) = \{u_3 \in X_3 : (w_3, u_3) = \xi\}. \end{array} \right. \quad (53)$$

We compute  $\varphi_2$  and  $\varphi_3$  by solving the decoupled optimization problems  $\mathcal{P}_2(\xi)$  and  $\mathcal{P}_3(\xi)$ , with solutions  $u_2(\xi)$  and  $u_3(\xi)$  respectively.

**The optimization problem  $\mathcal{P}_2(\xi)$**

$$\boxed{\mathcal{P}_2(\xi)} : \quad \varphi_2(\xi) = \min_{u_2 \in K_2(\xi)} \frac{1}{2} a_2(u_2, u_2) - (f_2, u_2). \quad (54)$$

The necessary condition of optimality in the convex set  $K_2(\xi)$  is:

$$\nabla \varphi_2^0(u_2)(v_2 - u_2) \geq 0 \quad \forall v_2 \in K_2(\xi). \quad (55)$$

Let  $\mathcal{A}_2$  be the operator associated to  $a_2$ , i.e.  $a_2(u, v) = \langle \mathcal{A}_2 u, v \rangle$  and let  $\hat{f}$  be the element given by the Riesz representation theorem, i.e.  $\langle \hat{f}_2, v_2 \rangle = (f_2, v_2)$ ,  $\forall v_2 \in X_2$ . Then (55) is equivalent to

$$\langle \mathcal{A}_2 u_2 - \hat{f}_2, v_2 - u_2 \rangle \geq 0 \quad \forall v_2 / \langle \hat{w}_2, v_2 \rangle = \xi. \quad (56)$$

So, we have

$$\langle \mathcal{A}_2 u_2 - \hat{f}_2, z_2 \rangle \geq 0 \quad \forall z_2 / \langle \hat{w}_2, z_2 \rangle = 0, \quad (57)$$

that is

$$\langle \mathcal{A}_2 u_2 - \hat{f}_2, z_2 \rangle = 0 \quad \forall z_2 / \langle \hat{w}_2, z_2 \rangle = 0, \quad (58)$$

therefore  $\exists \lambda_2 \in \Re$  such that

$$\begin{cases} \mathcal{A}_2 u_2 - \hat{f}_2 = \lambda_2 \hat{w}_2, \\ \langle \hat{w}_2, u_2 \rangle = \xi. \end{cases} \quad (59)$$

We obtain  $u_2(\xi)$  in the following form

$$\mathcal{A}_2 u_2 = \hat{f}_2 + \lambda_2 \hat{w}_2. \quad (60)$$

To compute  $\lambda_2$ , we consider that

$$\begin{aligned} \xi &= \langle \hat{w}_2, u_2 \rangle = \langle \hat{w}_2, \mathcal{A}_2^{-1} \mathcal{A}_2 u_2 \rangle = \langle \hat{w}_2, \mathcal{A}_2^{-1} (\hat{f}_2 + \lambda_2 \hat{w}_2) \rangle \\ &= \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle + \lambda_2 \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle \end{aligned} \quad (61)$$

and then, we have

$$\begin{cases} \lambda_2 = \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle^{-1} \left( \xi - \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle \right), \\ u_2(\xi) = \mathcal{A}_2^{-1} \hat{f}_2 + \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle^{-1} \left( \xi - \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle \right) \mathcal{A}_2^{-1} \hat{w}_2. \end{cases} \quad (62)$$

Finally, from (62) we get

$$\varphi_2(\xi) = \frac{1}{2} \langle \mathcal{A}_2 u_2, u_2 \rangle - \langle \hat{f}_2, u_2 \rangle. \quad (63)$$

**The optimization problem  $\mathcal{P}_3(\xi)$**

$$\boxed{\mathcal{P}_3(\xi)} : \quad \varphi_3(\xi) = \min_{u_3 \in K_3(\xi)} \frac{1}{2} a_3(u_3, u_3) - (f_3, u_3). \quad (64)$$

With a similar procedure to that one used above, we obtain  $\lambda_3$  and  $u_3(\xi)$  in the following form

$$\begin{cases} \lambda_3 = \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \rangle^{-1} (\xi - \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle), \\ u_3(\xi) = \mathcal{A}_3^{-1} \hat{f}_3 + \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \rangle^{-1} (\xi - \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle) \mathcal{A}_3^{-1} \hat{w}_3. \end{cases} \quad (65)$$

In consequence, we get the following explicit expression of  $\varphi_3$

$$\varphi_3(\xi) = \frac{1}{2} \langle \mathcal{A}_3 u_3, u_3 \rangle - \langle \hat{f}_3, u_3 \rangle. \quad (66)$$

### Computation of $\varphi'_2$ and $\varphi'_3$

We denote

$$\boxed{\begin{array}{ll} \vartheta_2 = \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle^{-1} & \vartheta_3 = \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \rangle^{-1} \\ \kappa_2 = \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle & \kappa_3 = \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle \end{array}} \quad (67)$$

then, from (62) and (63), we have

$$u_2(\xi) = \mathcal{A}_2^{-1} \hat{f}_2 - \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 + \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \quad (68)$$

and so,

$$\begin{aligned}
\varphi_2(\xi) &= \frac{1}{2} \left\langle \mathcal{A}_2 \left( \mathcal{A}_2^{-1} \hat{f}_2 - \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 + \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \right), \mathcal{A}_2^{-1} \hat{f}_2 - \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 + \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \right\rangle \\
&\quad - \left\langle \hat{f}_2, \mathcal{A}_2^{-1} \hat{f}_2 - \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 + \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \right\rangle \\
&= \frac{1}{2} \left\langle \hat{f}_2 - \vartheta_2 \kappa_2 \hat{w}_2 + \vartheta_2 \hat{w}_2 \xi, \mathcal{A}_2^{-1} \hat{f}_2 - \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 + \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \right\rangle \\
&\quad - \left\langle \hat{f}_2, \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle + \left\langle \hat{f}_2, \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 \right\rangle - \left\langle \hat{f}_2, \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \right\rangle \\
&= \frac{1}{2} \left\langle \hat{f}_2, \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle - \frac{1}{2} \left\langle \hat{f}_2, \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 \right\rangle + \frac{1}{2} \left\langle \hat{f}_2, \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \right\rangle \\
&\quad - \frac{1}{2} \left\langle \vartheta_2 \kappa_2 \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle + \frac{1}{2} \left\langle \vartheta_2 \kappa_2 \hat{w}_2, \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 \right\rangle - \frac{1}{2} \left\langle \vartheta_2 \kappa_2 \hat{w}_2, \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \right\rangle \\
&\quad + \frac{1}{2} \left\langle \vartheta_2 \hat{w}_2 \xi, \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle - \frac{1}{2} \left\langle \vartheta_2 \hat{w}_2 \xi, \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 \right\rangle + \frac{1}{2} \left\langle \vartheta_2 \hat{w}_2 \xi, \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \right\rangle \\
&\quad - \left\langle \hat{f}_2, \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle + \left\langle \hat{f}_2, \vartheta_2 \kappa_2 \mathcal{A}_2^{-1} \hat{w}_2 \right\rangle - \left\langle \hat{f}_2, \vartheta_2 \mathcal{A}_2^{-1} \hat{w}_2 \xi \right\rangle .
\end{aligned}$$

After some simplifications, we obtain

$$\varphi_2(\xi) = \frac{1}{2} \vartheta_2^2 \left\langle \hat{w}_2 \xi, \mathcal{A}_2^{-1} \hat{w}_2 \xi \right\rangle - \vartheta_2 \kappa_2 \xi - \frac{1}{2} \left\langle \hat{f}_2, \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle + \frac{1}{2} \vartheta_2 \kappa_2^2, \quad (69)$$

in consequence, we get for  $\varphi_2'$  the expression

$$\varphi_2'(\xi) = \vartheta_2 \xi - \vartheta_2 \kappa_2 = \left\langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \right\rangle^{-1} \left( \xi - \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle \right). \quad (70)$$

In a similar way, we get

$$\varphi_3(\xi) = \frac{1}{2} \vartheta_3^2 \left\langle \hat{w}_3 \xi, \mathcal{A}_3^{-1} \hat{w}_3 \xi \right\rangle - \vartheta_3 \kappa_3 \xi - \frac{1}{2} \left\langle \hat{f}_3, \mathcal{A}_3^{-1} \hat{f}_3 \right\rangle + \frac{1}{2} \vartheta_3 \kappa_3^2, \quad (71)$$

$$\varphi_3'(\xi) = \vartheta_3 \xi - \vartheta_3 \kappa_3 = \left\langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \right\rangle^{-1} \left( \xi - \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle \right). \quad (72)$$

So, by virtue of (26), we have

$$\begin{cases} g_2(\xi) = \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle^{-1} \left( \xi - \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle \right), \\ g_3(\xi) = \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \rangle^{-1} \left( \xi - \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle \right). \end{cases} \quad (73)$$

### Coordination phase

We obtain the solution by solving the equation  $g(\xi) = 0$ . So, we have

$$\langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle^{-1} \left( \xi - \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle \right) + \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \rangle^{-1} \left( \xi - \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle \right) = 0 \quad (74)$$

and then

$$\xi = \frac{\langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \rangle + \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle}{\langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \rangle + \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle}. \quad (75)$$

Therefore, the solution is given by

$$\begin{cases} u_2(\xi) = \mathcal{A}_2^{-1} \hat{f}_2 + \frac{\langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle - \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle}{\langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle + \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \rangle} \mathcal{A}_2^{-1} \hat{w}_2, \\ u_3(\xi) = \mathcal{A}_3^{-1} \hat{f}_3 + \frac{\langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle - \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle}{\langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle + \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \rangle} \mathcal{A}_3^{-1} \hat{w}_3. \end{cases} \quad (76)$$

### 3.2.3 Solution by decomposition. Case of dimension $p$

We take

$$\begin{cases} X_I = \mathbb{R}^p, \quad \mathcal{K} = 0, \\ K_2(\xi) = \{v_2 \in X_2 : (W_2 v_2)_j = \xi_j, \forall j = 1, \dots, p\}, \\ K_3(\xi) = \{v_3 \in X_3 : (W_3 v_3)_j = \xi_j, \forall j = 1, \dots, p\}. \end{cases} \quad (77)$$



We compute  $\varphi_2$  and  $\varphi_3$  by solving the decoupled optimization problems  $\mathcal{P}_2(\xi)$  and  $\mathcal{P}_3(\xi)$ , with solutions  $u_2(\xi)$  and  $u_3(\xi)$  respectively.

**The optimization problem  $\mathcal{P}_2(\xi)$**

$$\boxed{\mathcal{P}_2(\xi)} : \quad \min_{u_2 \in K_2(\xi)} \varphi_2^0(\xi) = \min_{u_2 \in K_2(\xi)} \frac{1}{2} a_2(u_2, u_2) - (f_2, u_2). \quad (78)$$

**Computation of  $\nabla \varphi_2$**

We have

$$\varphi_2(\xi) = \min_{u_2 \in K_2(\xi)} \left\{ \frac{1}{2} \langle \mathcal{A}_2 u_2, u_2 \rangle - \langle \hat{f}_2, u_2 \rangle \right\}. \quad (79)$$

With the same argument used for the case of dimension 1,  $\exists \lambda_2 \in \mathfrak{R}^p$  such that

$$\begin{cases} \mathcal{A}_2 u_2 - \hat{f}_2 + W_2^* \lambda_2 = 0, \\ W_2 u_2 = \xi. \end{cases} \quad (80)$$

Then

$$\begin{cases} u_2 = \mathcal{A}_2^{-1}(\hat{f}_2 - W_2^* \lambda_2), \\ W_2 u_2 = W_2 \mathcal{A}_2^{-1} \hat{f}_2 - (W_2 \mathcal{A}_2^{-1} W_2^*) \lambda_2 = \xi. \end{cases} \quad (81)$$

We denote  $P_2 \in \mathfrak{R}^{p \times p}$  as

$$P_2 = W_2 \mathcal{A}_2^{-1} W_2^*, \quad (82)$$

which is a positive definite matrix. Then, we obtain

$$\lambda_2 = P_2^{-1} (W_2 \mathcal{A}_2^{-1} \hat{f}_2 - \xi) \quad (83)$$

and

$$u_2(\xi) = S_2 \xi + (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2, \quad (84)$$

where

$$S_2 = \mathcal{A}_2^{-1} W_2^* P_2^{-1}. \quad (85)$$

In consequence, we get

$$\begin{aligned}
 \varphi_2(\xi) &= \frac{1}{2} \left\langle \mathcal{A}_2 S_2 \xi + \mathcal{A}_2 (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2, S_2 \xi + (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle \\
 &\quad - \left\langle \hat{f}_2, S_2 \xi + (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle \\
 &= \frac{1}{2} \langle \mathcal{A}_2 S_2 \xi, S_2 \xi \rangle + \frac{1}{2} \langle \mathcal{A}_2 S_2 \xi, (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2 \rangle \\
 &\quad + \frac{1}{2} \langle \mathcal{A}_2 (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2, S_2 \xi \rangle - \langle \hat{f}_2, S_2 \xi \rangle \\
 &\quad + \left\langle \frac{1}{2} \mathcal{A}_2 (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2 - \hat{f}_2, (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle \\
 &= \frac{1}{2} \langle S_2^* \mathcal{A}_2 S_2 \xi, \xi \rangle - \langle \xi, S_2^* \mathcal{A}_2 S_2 W_2 \mathcal{A}_2^{-1} \hat{f}_2 \rangle \\
 &\quad + \left\langle \frac{1}{2} \mathcal{A}_2 (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2 - \hat{f}_2, (I - S_2 W_2) \mathcal{A}_2^{-1} \hat{f}_2 \right\rangle,
 \end{aligned} \tag{86}$$

therefore

$$\begin{aligned}
 \nabla \varphi_2 &= S_2^* \mathcal{A}_2 S_2 \xi - S_2^* \mathcal{A}_2 S_2 W_2 \mathcal{A}_2^{-1} \hat{f}_2 \\
 &= P_2^{-1} W_2 \mathcal{A}_2^{-1} W_2^* P_2^{-1} \xi - P_2^{-1} W_2 \mathcal{A}_2^{-1} W_2^* P_2^{-1} W_2 \mathcal{A}_2^{-1} \hat{f}_2.
 \end{aligned} \tag{87}$$

By virtue of (82) we obtain

$$\nabla \varphi_2 = P_2^{-1} \xi - S_2^* \hat{f}_2. \tag{88}$$

**The optimization problem  $\mathcal{P}_3(\xi)$**

Now, we study the following problem

$$\boxed{\mathcal{P}_3(\xi)} : \quad \min_{u_3 \in K_3(\xi)} \varphi_3^0(\xi) = \min_{u_3 \in K_3(\xi)} \left\{ \frac{1}{2} a_3(u_3, u_3) - (f_3, u_3) \right\}. \tag{89}$$

### Computation of $\nabla\varphi_3$

In a similar way to that one used above, we have

$$\varphi_3(\xi) = \min_{u_3 \in K_3(\xi)} \left\{ \frac{1}{2} \langle \mathcal{A}_3 u_3, u_3 \rangle - \langle \hat{f}_3, u_3 \rangle \right\}. \quad (90)$$

The solution  $u_3$  is given by:

$$u_3(\xi) = S_3 \xi + (I - S_3 W_3) \mathcal{A}_3^{-1} \hat{f}_3, \quad (91)$$

where we define  $P_3 \in \mathfrak{R}^{p \times p}$  as

$$P_3 = W_3 \mathcal{A}_3^{-1} W_3^* \quad (92)$$

and

$$S_3 = \mathcal{A}_3^{-1} W_3^* P_3^{-1}. \quad (93)$$

Therefore

$$\nabla\varphi_3 = P_3^{-1} \xi - S_3^* \hat{f}_3. \quad (94)$$

### Coordination phase

We define  $g(\xi) = \nabla\varphi_2 + \nabla\varphi_3$  and then, the solution is given by solving the equation  $g(\xi) = 0$ .

So, we must find  $\xi$  such that

$$P_2^{-1} \xi - S_2^* \hat{f}_2 + P_3^{-1} \xi - S_3^* \hat{f}_3 = 0, \quad (95)$$

then

$$\xi = \hat{P} \left( S_2^* \hat{f}_2 + S_3^* \hat{f}_3 \right), \quad (96)$$

where

$$\hat{P} = (P_2^{-1} + P_3^{-1})^{-1}. \quad (97)$$

By virtue of (84) and (91), we have

$$\boxed{\begin{cases} u_2(\xi) = \mathcal{A}_2^{-1} \hat{f}_2 + S_2 \hat{P} \left( S_2^* \hat{f}_2 + S_3^* \hat{f}_3 \right) - S_2 W_2 \mathcal{A}_2^{-1} \hat{f}_2, \\ u_3(\xi) = \mathcal{A}_3^{-1} \hat{f}_3 + S_3 \hat{P} \left( S_2^* \hat{f}_2 + S_3^* \hat{f}_3 \right) - S_3 W_3 \mathcal{A}_3^{-1} \hat{f}_3. \end{cases}} \quad (98)$$

### 3.3 Example 3

#### 3.3.1 The original problem

We take  $\mathcal{H} = L^2(\Omega_3)$ ,  $\mathcal{K} = 0$ , and we define

$$(M_3 v_3)(x) = m_3(x) v_3(x), \quad (99)$$

where  $m_3 \in L^\infty(\Omega_3)$ .

We consider  $\gamma_{2i} v_2$  (trace of  $v_2$  on  $\Gamma_{2i}$ ) and we extend  $\gamma_{2i} v_2$  to  $\Omega_3$ . This can be done in the following ways. Let  $P$  be any extension operator:

$$\begin{aligned} P &\in \mathcal{L}(H^{1/2}(\Gamma_{2i}), H^{3/2}(\Omega_3)), \\ P \varphi|_{\Gamma_{2i}} &= \varphi, \quad \forall \varphi \in H^{1/2}(\Gamma_{2i}) \end{aligned} \quad (100)$$

and we define

$$(M_2 v_2)(x) = (P \gamma_{2i} v_2)(x), \quad x \in \Gamma_{2i}.$$

In this way, (2) becomes

$$m_3 u_3 = P \gamma_{2i} u_2, \quad \text{in } \Omega_3. \quad (101)$$

So, in this case we have

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 : m_3(x) v_3(x) = (P \gamma_{2i} v_2)(x), \text{ a.e. } x \in \Omega_3\}. \quad (102)$$

### 3.3.2 Solution by decomposition

#### Elements of the decomposition

$$\left\{ \begin{array}{l} X_I = H^{1/2}(\Gamma_{2i}), \quad K_I = X_I, \\ K_2(v_I) = \{v_2 \in X_2 : \gamma_{2i} v_2(x) = v_I(x), \text{ a.e. } x \in \Gamma_{2i}\}, \\ K_3(v_I) = \{v_3 \in X_3 : m_3(x) v_3(x) = P(v_I)(x), \text{ a.e. } x \in \Omega_3\}. \end{array} \right. \quad (103)$$

#### Analysis of problems $\mathcal{P}_2$ , $\mathcal{P}_3$

We consider the problem  $\mathcal{P}_2(v_I)$  defined by

$$\boxed{\mathcal{P}_2(v_I)} : \quad \varphi_2(v_I) = \min_{u_2 \in K_2(v_I)} \frac{1}{2} \langle \mathcal{A}_2 u_2, u_2 \rangle - \langle \hat{f}_2, u_2 \rangle. \quad (104)$$

To solve this problem  $\mathcal{P}_2$  we work in a similar way as that one appearing in section 3.1. So, the solution has the following form

$$\bar{u}_2(v_I) = G_2 v_I + w_2. \quad (105)$$

In consequence, for the auxiliary function  $\varphi_2(v_I)$  we have

$$\varphi_2(v_I) = \frac{1}{2} \langle \mathcal{A}_2(G_2 v_I + w_2), G_2 v_I + w_2 \rangle - \langle \hat{f}_2, G_2 v_I + w_2 \rangle, \quad (106)$$

then, for the derivative (in the Frechet sense) we obtain:  $\forall \zeta \in X_I$

$$\langle D\varphi_2(v_I), \zeta \rangle = \langle G_2^* \mathcal{A}_2 G_2 v_I + G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2), \zeta \rangle. \quad (107)$$

Now, we consider the problem  $\mathcal{P}_3(v_I)$  defined by

$$\boxed{\mathcal{P}_3(v_I)} : \quad \varphi_3(v_I) = \min_{u_3 \in K_3(v_I)} \frac{1}{2} \langle \mathcal{A}_3 u_3, u_3 \rangle - \langle \hat{f}_3, u_3 \rangle. \quad (108)$$

Let  $D$  and  $I$  be subsets of  $\Omega_3$  such that

$$D = \{x_3 : m_3(x_3) \neq 0\}, \quad I = \{x_3 : m_3(x_3) = 0\}. \quad (109)$$

Then

$$u_3(x_3) = \frac{1}{m_3} P(v_I)(x_3), \quad \forall x_3 \in D. \quad (110)$$

We define the operator  $G_3$  such that

$$(G_3 v_I)(x_3) = \begin{cases} \frac{1}{m_3} P(v_I)(x_3) & \forall x_3 \in D, \\ 0 & \forall x_3 \in I, \end{cases} \quad (111)$$

and  $w_3$  the solution of

$$\begin{cases} A_3 w_3 = f_3, \\ w_3|_D = 0. \end{cases} \quad (112)$$

Then we can write the solution  $\bar{u}_3$  in the following form

$$\bar{u}_3 = G_3 v_I + w_3, \quad (113)$$

and so,

$$\begin{aligned} \varphi_3(v_I) &= \frac{1}{2} \langle \mathcal{A}_3 u_3, u_3 \rangle - \langle \hat{f}_3, u_3 \rangle \\ &= \frac{1}{2} \langle \mathcal{A}_3 (G_3 v_I + w_3), (G_3 v_I + w_3) \rangle - \langle \hat{f}_3, (G_3 v_I + w_3) \rangle \\ &= \frac{1}{2} \langle \mathcal{A}_3 w_3, w_3 \rangle - \langle \hat{f}_3, w_3 \rangle + \frac{1}{2} \langle \mathcal{A}_3 G_3 v_I, G_3 v_I \rangle - \langle \hat{f}_3, G_3 v_I \rangle \\ &\quad + \frac{1}{2} \langle \mathcal{A}_3 G_3 v_I, w_3 \rangle + \frac{1}{2} \langle \mathcal{A}_3 w_3, G_3 v_I \rangle. \end{aligned} \quad (114)$$

The support of the functions  $G_3 v_I$  and  $w_3$  are disjoint, then we have

$$\langle \mathcal{A}_3 G_3 v_I, w_3 \rangle = \langle \mathcal{A}_3 w_3, G_3 v_I \rangle = \langle \mathcal{A}_3 w_3, G_3 v_I \rangle = 0, \quad (115)$$

therefore, we obtain

$$\varphi_3(v_I) = \bar{\varphi}_3 + \tilde{\varphi}_3(v_I), \quad (116)$$

where  $\bar{\varphi}_3$  is independent of  $v_I$ , that is

$$\bar{\varphi}_3 = \frac{1}{2} \langle \mathcal{A}_3 w_3, w_3 \rangle - \langle \hat{f}_3, w_3 \rangle \quad (117)$$

and

$$\tilde{\varphi}_3(v_I) = \frac{1}{2} \langle \mathcal{A}_3 G_3 v_I, G_3 v_I \rangle - \langle \hat{f}_3, G_3 v_I \rangle = \frac{1}{2} \langle G_3^* \mathcal{A}_3 G_3 v_I, v_I \rangle - \langle G_3^* \hat{f}_3, v_I \rangle. \quad (118)$$

Then, for the derivative (in the Frechet sense) we obtain:  $\forall \zeta \in X_I$

$$\langle D\varphi_3(v_I), \zeta \rangle = \langle G_3^* \mathcal{A}_3 G_3 v_I - G_3^* \hat{f}_3, \zeta \rangle. \quad (119)$$

### Coordination phase

The solution  $\bar{u}_I$  is given by solving the equation  $g(v_I) = 0$ , where  $g(v_I) = D\varphi_2(v_I) + D\varphi_3(v_I)$ . By virtue of (107) and (119), we have

$$G_2^* \mathcal{A}_2 G_2 \bar{u}_I + G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + G_3^* \mathcal{A}_3 G_3 \bar{u}_I - G_3^* \hat{f}_3 = 0. \quad (120)$$

In consequence we get

$$\bar{u}_I = R \left( -G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + G_3^* \hat{f}_3 \right), \quad (121)$$

where

$$R = (G_2^* \mathcal{A}_2 G_2 + G_3^* \mathcal{A}_3 G_3)^{-1}. \quad (122)$$

So, the solution is given by

$$\begin{cases} u_2(\bar{u}_I) = G_2 R \left( -G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + G_3^* \hat{f}_3 \right) + w_2, \\ u_3(\bar{u}_I) = G_3 R \left( -G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + G_3^* \hat{f}_3 \right) + w_3. \end{cases} \quad (123)$$

## 3.4 Example 4

### 3.4.1 The original problem

We consider here  $\mathcal{H} = \mathfrak{R}$ ,  $\mathcal{K} = \mathfrak{R}^+$ , and we define

$$M_2 v_2 = \int_{\Omega_2} v_2 dx, \quad M_3 v_3 = \int_{\Omega_3} v_3 dx. \quad (124)$$

**Note 3.2** We denote

$$\bar{v}_i = (v_i, e_i) = \int_{\Omega_i} v_i dx, \quad (125)$$

where  $e_i$  is the constant function  $\mathbf{1}$  in  $\Omega_i$ .

So, in this case we have

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 : \bar{v}_2 - \bar{v}_3 \geq 0\}. \quad (126)$$

### 3.4.2 Solution by decomposition

$$\left\{ \begin{array}{l} X_I = \mathfrak{R}^2, \\ K_I = \{(x_2, x_3) \in \mathfrak{R}^2 : x_2 - x_3 \geq 0\}, \\ K_2(x_2, x_3) = \{v_2 \in X_2 : \bar{v}_2 = x_2\}, \\ K_3(x_2, x_3) = \{v_3 \in X_3 : \bar{v}_3 = x_3\}. \end{array} \right. \quad (127)$$

We have the problem  $\mathcal{P}_2(x_2)$

$$\boxed{\mathcal{P}_2(x_2)} : \quad \varphi_2(x_2) = \min_{u_2 \in K_2} \frac{1}{2} \langle \mathcal{A}_2 u_2, u_2 \rangle - \langle \hat{f}_2, u_2 \rangle \quad (128)$$

and the problem  $\mathcal{P}_3(x_3)$

$$\boxed{\mathcal{P}_3(x_3)} : \quad \varphi_3(x_3) = \min_{u_3 \in K_3} \frac{1}{2} \langle \mathcal{A}_3 u_3, u_3 \rangle - \langle \hat{f}_3, u_3 \rangle. \quad (129)$$

From (127)-(129) it is easy to check that there exist  $c_i \in \mathfrak{R}$  such that

$$\left\{ \begin{array}{l} \mathcal{A}_i u_i - \hat{f}_i = c_i \hat{e}_i, \quad i = 2, 3, \\ \langle u_i, \hat{e}_i \rangle = x_i, \end{array} \right. \quad (130)$$

then

$$u_i = \mathcal{A}_i^{-1}(\hat{f}_i + c_i \hat{e}_i). \quad (131)$$

To get the value  $c_i$  we take into account that

$$x_i = \langle \hat{e}_i, u_i \rangle = \langle \hat{e}_i, \mathcal{A}_i^{-1}(\hat{f}_i + c_i \hat{e}_i) \rangle = \overline{\mathcal{A}_i^{-1} \hat{f}_i} + c_i \overline{\mathcal{A}_i^{-1} \hat{e}_i}, \quad (132)$$



where

$$\overline{\mathcal{A}_i^{-1} \hat{f}_i} = \langle \hat{e}_i, \mathcal{A}_i^{-1} \hat{f}_i \rangle \quad \text{and} \quad \overline{\mathcal{A}_i^{-1} \hat{e}_i} = \langle \hat{e}_i, \mathcal{A}_i^{-1} \hat{e}_i \rangle. \quad (133)$$

Therefore

$$c_i = \frac{1}{\overline{\mathcal{A}_i^{-1} \hat{e}_i}} (x_i - \overline{\mathcal{A}_i^{-1} \hat{f}_i}) \quad (134)$$

and in consequence

$$u_i = \mathcal{A}_i^{-1} \hat{f}_i + \frac{1}{\overline{\mathcal{A}_i^{-1} \hat{e}_i}} (x_i - \overline{\mathcal{A}_i^{-1} \hat{f}_i}) \mathcal{A}_i^{-1} \hat{e}_i. \quad (135)$$

From these relations we get

$$\varphi_i(x_i) = \frac{1}{2} \frac{1}{\overline{\mathcal{A}_i^{-1} \hat{e}_i}} (x_i - \overline{\mathcal{A}_i^{-1} \hat{f}_i})^2 - \frac{1}{2} \langle \hat{f}_i, \mathcal{A}_i^{-1} \hat{f}_i \rangle \quad (136)$$

and

$$\frac{\partial \varphi_i}{\partial x_i} = \frac{(x_i - \overline{\mathcal{A}_i^{-1} \hat{f}_i})}{\overline{\mathcal{A}_i^{-1} \hat{e}_i}}. \quad (137)$$

We have two alternatives:

1.  $x_2 - x_3 > 0$

In this case the necessary condition that must be satisfied at this interior point is:  $\frac{\partial \varphi_i}{\partial x_i} = 0$ . This condition implies

$$(x_i - \overline{\mathcal{A}_i^{-1} \hat{f}_i}) = 0.$$

Then, by virtue of (135), we have the solution that we may call “*free solution*”:

$$\boxed{u_i = \mathcal{A}_i^{-1} \hat{f}_i \quad i = 2, 3.} \quad (138)$$

So, if

$$\overline{\mathcal{A}_2^{-1} \hat{f}_2} - \overline{\mathcal{A}_3^{-1} \hat{f}_3} > 0, \quad (139)$$

the solution is given by the *free solution*. Otherwise, we will have the following case:

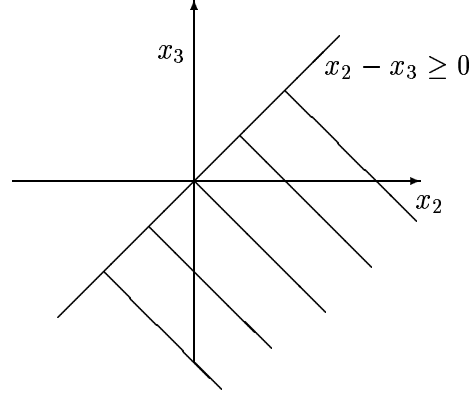


Figure 2:  $K_I = \{(x_2, x_3) \in \mathbb{R}^2 : x_2 - x_3 \geq 0\}$

2.  $x_2 - x_3 = 0$

In this case the necessary condition that must be satisfied at this boundary point is:  $\exists \lambda \neq 0$  such that

$$\begin{pmatrix} \frac{\partial \varphi_2}{\partial x_2} \\ \frac{\partial \varphi_3}{\partial x_3} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \quad (140)$$

This condition implies that

$$\frac{\partial \varphi_2}{\partial x_2} = -\frac{\partial \varphi_3}{\partial x_3}. \quad (141)$$

By virtue of (137) we get

$$\frac{\overline{(x_2 - \mathcal{A}_2^{-1} \hat{f}_2)}}{\overline{\mathcal{A}_2^{-1} \hat{e}_2}} + \frac{\overline{(x_3 - \mathcal{A}_3^{-1} \hat{f}_3)}}{\overline{\mathcal{A}_3^{-1} \hat{e}_3}} = 0, \quad (142)$$

then, as  $x_2 = x_3$  we have

$$\frac{\left(x_2 - \overline{\mathcal{A}_2^{-1} \hat{f}_2}\right)}{\overline{\mathcal{A}_2^{-1} \hat{e}_2}} + \frac{\left(x_2 - \overline{\mathcal{A}_3^{-1} \hat{f}_3}\right)}{\overline{\mathcal{A}_3^{-1} \hat{e}_3}} = 0 \quad (143)$$

and therefore

$$x_2 = \frac{\overline{\mathcal{A}_3^{-1} \hat{e}_3} \overline{\mathcal{A}_2^{-1} \hat{f}_2} + \overline{\mathcal{A}_2^{-1} \hat{e}_2} \overline{\mathcal{A}_3^{-1} \hat{f}_3}}{\left(\overline{\mathcal{A}_2^{-1} \hat{e}_2} + \overline{\mathcal{A}_3^{-1} \hat{e}_3}\right)}. \quad (144)$$

Finally, the solution is given by

$$\left\{ \begin{array}{l} u_2 = \mathcal{A}_2^{-1} \left( \hat{f}_2 - \frac{\overline{\mathcal{A}_2^{-1} \hat{f}_2}}{\overline{\mathcal{A}_2^{-1} \hat{e}_2}} \hat{e}_2 \right) + \frac{1}{\overline{\mathcal{A}_2^{-1} \hat{e}_2}} \frac{\overline{\mathcal{A}_3^{-1} \hat{e}_3} \overline{\mathcal{A}_2^{-1} \hat{f}_2} + \overline{\mathcal{A}_2^{-1} \hat{e}_2} \overline{\mathcal{A}_3^{-1} \hat{f}_3}}{\left(\overline{\mathcal{A}_2^{-1} \hat{e}_2} + \overline{\mathcal{A}_3^{-1} \hat{e}_3}\right)} \mathcal{A}_2^{-1} \hat{e}_2, \\ u_3 = \mathcal{A}_3^{-1} \left( \hat{f}_3 - \frac{\overline{\mathcal{A}_3^{-1} \hat{f}_3}}{\overline{\mathcal{A}_3^{-1} \hat{e}_3}} \hat{e}_3 \right) + \frac{1}{\overline{\mathcal{A}_3^{-1} \hat{e}_3}} \frac{\overline{\mathcal{A}_3^{-1} \hat{e}_3} \overline{\mathcal{A}_2^{-1} \hat{f}_2} + \overline{\mathcal{A}_2^{-1} \hat{e}_2} \overline{\mathcal{A}_3^{-1} \hat{f}_3}}{\left(\overline{\mathcal{A}_2^{-1} \hat{e}_2} + \overline{\mathcal{A}_3^{-1} \hat{e}_3}\right)} \mathcal{A}_3^{-1} \hat{e}_3. \end{array} \right. \quad (145)$$

### 3.5 Example 5

#### 3.5.1 The original problem

We will deal now with an example in a simpler geometry. We consider (cf. Fig. 3) a body which consists of domains  $\Omega_1$  and  $\Omega_2$  which are respectively of dimensions 1 and 2. In the present situation

$$\partial\Omega_1 = \Gamma_{1i} \cup \Gamma_{1e}, \quad \Gamma_{1i} = 0, \quad \Gamma_{1e} = 1.$$

We consider  $\mathcal{H} = L^2(\Omega_2)$ ,  $\mathcal{K} = (L^2(\Omega_2))^+$ , and we define the operators  $M_i$  by

$$(M_1 v_1)(x) = v_1(0), \quad \forall x \in \Omega_2 \quad \text{and} \quad (M_2 v_2)(x) = v_2(x), \quad \forall x \in \Omega_2; \quad (146)$$

then

$$K = \{(v_1, v_2) \in X_1 \oplus X_2 : v_2(x) \leq v_1(0), \text{ a.e. in } \Omega_2\}. \quad (147)$$

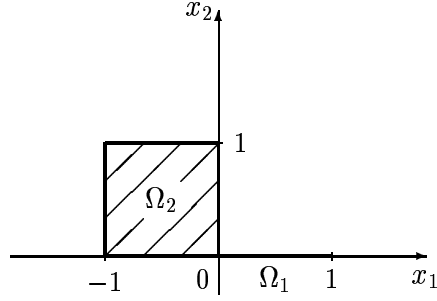


Figure 3: A 1-2 dimensional coupled problem

### 3.5.2 Properties of the solution

As  $K$  is a cone with vertex 0, the V.I. (8) is equivalent to (148) and (149)

$$a_1(u_1, u_1) + a_2(u_2, u_2) = (f_1, u_1) + (f_2, u_2), \quad (u_1, u_2) \in K, \quad (148)$$

and

$$a_1(u_1, v_1) + a_2(u_2, v_2) \geq (f_1, v_1) + (f_2, v_2), \quad \forall (v_1, v_2) \in K. \quad (149)$$

From these relations, the following characterization of the solution is given in [8]:

$$A_1 u_1 = f_1, \quad \text{in } \Omega_1, \quad (150)$$

$$\frac{\partial u_1}{\partial n}(1) = 0, \quad \frac{\partial u_1}{\partial n}(0) \geq 0, \quad (151)$$

$$A_2 u_2 - f_2 \leq 0, \quad u_2 - u_1(0) \leq 0, \quad \text{in } \Omega_2, \quad (152)$$

$$(A_2 u_2 - f_2) \times (u_2 - u_1(0)) = 0, \quad \text{in } \Omega_2, \quad (153)$$

$$\frac{\partial u_2}{\partial n} = 0, \quad \text{in } \partial\Omega_2. \quad (154)$$

It is also given the following relation, which characterizes the equilibrium between both systems:

$$\frac{\partial u_1}{\partial n}(0) + (A_2 u_2 - f_2, 1) = 0. \quad (155)$$

We will prove that

$$\frac{\partial u_1}{\partial n}(0) = \int_S (f_2(x) - \beta u_2(x)) dx, \quad (156)$$

expression which gives the equilibrium condition (155) in terms of the data at the *interaction zone*  $S$ , where

$$S := \{x : u_2(x) = u_1(0)\}. \quad (157)$$

By virtue of (154), we have

$$\int_{\Omega_2} \Delta u_2(x) dx = 0 \quad (158)$$

and from (157), it is verified

$$\Delta u_2(x) = 0, \quad a.e. x \in S. \quad (159)$$

So

$$\int_{CS} \Delta u_2(x) dx = \int_{\Omega_2} \Delta u_2(x) dx = 0. \quad (160)$$

By virtue of (153) we have

$$\int_{CS} (-\Delta u_2 + \beta u_2 - f_2)(x) dx = 0 \quad (161)$$

and from relations (160) and (161), we obtain

$$(\beta u_2 - f_2, 1) = \int_S (\beta u_2 - f_2)(x) dx. \quad (162)$$

Finally, from (155) and (162), we obtain (156).

### 3.5.3 Solution by decomposition techniques

We can solve the V.I. (8) by decomposition techniques. Let us introduce a real parameter  $\xi$ . We take

$$\left\{ \begin{array}{l} X_I = \mathfrak{R}, \quad K_I = \mathfrak{R}^+, \\ K_1(\xi) = \{v_1 \in X_1 : v_1(0) = \xi\}, \\ K_2(\xi) = \{v_2 \in X_2 : v_2(x) \leq \xi, \text{ a.e. } x \in \Omega_2\}. \end{array} \right. \quad (163)$$

For each  $\xi$  we define  $u_1^\xi$  as the unique solution of the following V.I.

$$\left\{ \begin{array}{l} a_1(u_1, v_1 - u_1) \geq (f_1, v_1 - u_1), \\ \forall v_1 \in K_1(\xi), u_1 \in K_1(\xi), \end{array} \right. \quad (164)$$

and  $u_2^\xi$  as the unique solution of the following V.I.

$$\left\{ \begin{array}{l} a_2(u_2, v_2 - u_2) \geq (f_2, v_2 - u_2), \\ \forall v_2 \in K_2(\xi), u_2 \in K_2(\xi), \end{array} \right. \quad (165)$$

In this case, from (164) and (165) we get for the derivative of the auxiliary function (defined in (17))  $\varphi(\xi) = \varphi_1(\xi) + \varphi_2(\xi)$

$$\frac{\partial \varphi}{\partial \xi} := \frac{\partial u_1^\xi}{\partial n}(0) - \int_S (f_2(x) - \beta u_2(x)) dx. \quad (166)$$

From (166), it can be easily seen that  $\frac{\partial \varphi}{\partial \xi}$  is an increasing continuous function with the property

$$\lim_{\xi \rightarrow \pm\infty} \frac{\partial \varphi}{\partial \xi} = \pm\infty. \quad (167)$$

Then, solving the original V.I. (8) is equivalent to solve the equation

$$\frac{\partial \varphi}{\partial \xi} = 0, \quad (168)$$

which can be solved by any iterative method. In particular, using that one described in section 2.

## Conclusions

In this paper we have analyzed some issues concerning the solution of coupled variational inequalities. Specifically, we have studied the solution of some junction problems presented in [8]. Starting from a V.I. formulation, we obtain the solution via a decomposition–coordination method. We have used this procedure because it allows us to solve the coupled problems through the solution of simple independent problems – in general, they are linear problems or simple obstacle problems. These problems depend on some auxiliary variables which are modified by an iterative algorithm. The convergence of this algorithm can be improved using techniques of Newton’s type. The study of this methodology will be contained in [11].

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