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***Asymptotic Behavior of a Multiplexer Fed by a
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Asymptotic Behavior of a Multiplexer Fed by a Long-Range Dependent Process

Zhen Liu, Philippe Nain, Don Towsley, Zhi-Li Zhang

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Abstract: In this paper we study the asymptotic behavior of the tail of the stationary backlog distribution in a single server queue with constant service capacity c , fed by the so-called “ $M/G/\infty$ input process” or “Cox input process”. Asymptotic lower bounds are obtained for any distribution G and asymptotic upper bounds are derived when G is a subexponential distribution. We find the bounds to be tight in some instances, e.g., G corresponding to either the Pareto or lognormal distribution and $c - \rho < 1$, where ρ is the arrival rate to the buffer.

Key-words: Asymptotic self-similar process; Long-range dependence; Subexponential distributions; Pareto distribution; Large deviations; Queues.

(Résumé : *tsvp*)

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Comportement Asymptotique d'un Multiplexeur Alimenté par un Processus à Mémoire Longue

Résumé : Nous nous intéressons dans cet article au comportement asymptotique de la distribution de probabilité complémentaire de la charge (W) d'une file d'attente à serveur unique de capacité c , alimentée par un processus d'arrivée de type " $M/G/\infty$ ", dit aussi "processus de Cox". Des bornes asymptotiques inférieures et supérieures sont obtenues quand la distribution de probabilité G est sous-exponentielle. Dans le cas où G suit une loi de Pareto ou une loi *lognormal* le comportement asymptotique de $P(W > x)$ peut être calculé lorsque $c - \rho < 1$, où ρ est le taux des arrivées.

Mots-clé : Processus auto-similaire; Processus à mémoire longue; Loi sous-exponentielle; Loi de Pareto; Grandes déviations; Files d'attente.

1 Introduction

The recent discovery [18, 22, 30] that traffic in networks possess long-range time dependencies that cannot be easily captured by Poisson-based models has motivated queueing theorists to propose and analyze new queueing models that capture these dependencies. One such model that has received attention is a buffer with server having rate c fed by an $M/G/\infty$ input process where G is heavy-tailed (e.g., [1, 13, 20, 27]). This is of interest because of its versatility, i.e., the dependencies over different time-scales can be controlled by varying the tail behavior of G .

In this paper we consider the model introduced by Parulekar and Makowski [27]. A discrete-time single-server queue (called the multiplexer) with infinite waiting room and with service capacity c is fed by an integer-valued process $\{b_t, t \in \mathbb{N}\}$. The r.v. b_t is defined as the number of busy servers at time t in an $M/G/\infty$ queue with arrival intensity $\lambda > 0$ and i.i.d. service times $\{\sigma_n\}_n$ with common probability distribution function (p.d.f.) $G(x) = P(\sigma_n \leq x)$ and finite mean $\bar{\sigma}$. An appealing feature of the (stationary version of the) input process $\{b_t, t \in \mathbb{N}\}$ is that it is a long-range dependent process [3] for some well-chosen *subexponential* p.d.f.'s G (see Section 2).

Let Q_t be the queue-length at the multiplexer at time t . Then, Q_t satisfies the Lindley's equation $Q_{t+1} = \max(0, Q_t + b_t - c)$ for all $t \in \mathbb{N}$, with $Q_0 = 0$. Let Q be the stationary queue-length under the stability condition $c > \rho := \lambda \bar{\sigma}$ (see Section 2). The aim of this paper is to study the behavior of $\log P(Q > x)$ and of $P(Q > x)$ for large x . More precisely, we show that there exist positive and finite constants θ_1 and θ_2 such that

$$-\theta_1 \leq \liminf_{x \rightarrow \infty} \frac{\log P(Q > x)}{-\log \bar{G}_1(x)} \leq \limsup_{x \rightarrow \infty} \frac{\log P(Q > x)}{-\log \bar{G}_1(x)} \leq -\theta_2. \quad (1)$$

The lower bound in (1) holds for any p.d.f. G whereas the upper bound holds for any *subexponential* p.d.f. G (to be defined in Section 2). Here \bar{G}_1 is defined as

$$G_1(x) := \frac{1}{\bar{\sigma}} \int_0^x \bar{G}(u) du, \quad x \geq 0 \quad (2)$$

and $\bar{F}(x) = 1 - F(x)$ for any probability distribution F . We also show that the bounds in (1) are tight (i.e. $\theta_1 = \theta_2$) when G is Pareto or lognormal (see Corollary 4.1), provided that $c - \rho < 1$. In the following the bounds in (1) will be referred to as *large deviations* bounds. Asymptotic upper and lower bounds for $P(Q > x)$ are also obtained.

Large deviations bounds were obtained in [29] in the case when G is short-tailed. Duffield observed in [13] that the approach in [27], based on the Gärtner-Ellis theorem, cannot be used to derive large deviations *lower* bounds for heavy-tailed G . By refining Theorem 2.2 in [14] and by using results in [28] Duffield was able to obtain the following large deviations *upper* bound (see [13])

$$\limsup_{x \rightarrow \infty} \frac{\log P(Q > x)}{\log x} \leq 1 - (\alpha - 1)(c - \rho) \quad (3)$$

in the case of the Pareto distribution $\overline{G}(x) \sim x^{-\alpha}$. An asymptotic lower bound for $P(Q > x)$ was obtained by Jelenkovic and Lazar [20] in the case when $c - \rho < 1$ and under a technical condition on G_1 (see comment after the proof of Proposition 3.2).

In this paper we propose an alternative to the approach based on the Gärtner-Ellis theorem that will yield asymptotic lower and upper bounds. We will observe that the large deviations bounds are tight for a number of subexponential distributions when $c - \rho < 1$ and that, in the case of G Pareto, the large deviations upper bound that can be derived from (1) (see Proposition 4.1) is tighter than that of Duffield when $c - \rho \leq \alpha/(\alpha - 1)$; otherwise Duffield's is tighter.

Other models have been proposed for modeling the effects of long-range dependence in arrival processes on buffer occupancy statistics. These include fractional brownian motion [14, 25], fractional gaussian noise [27], and a finite population of on-off sources where the on state holding times are characterized by heavy-tailed distributions [8, 6, 10, 20, 23] (see [7] for a survey paper on fluid queues with long-tailed activity periods).

The rest of the paper is structured as follows. Section 2 contains a characterization of the stationary behavior of the $M/G/\infty$ input process and the definition and characterization of the family of subexponential distributions. Asymptotic lower and upper bounds are established in Sections 3 and 4 respectively. Concluding remarks on the superposition of independent $M/G/\infty$ input processes are given in Section 5.

2 Preliminaries

The lemma below gives a useful characterization of the stationary behavior of the input process $\{b_t, t \in \mathbb{N}\}$. We will assume that customers entering the $M/G/\infty$ queue begin their service upon arrival (see Remark 2.1).

Lemma 2.1 *The distribution of the sequence $\{b_{t+k}, t \in \mathbb{N}\}$ converges monotonically for $k \rightarrow \infty$ to that of a proper stationary and ergodic sequence $\{b^t, t \in \mathbb{N}\}$ such that*

$$b^t \stackrel{\text{st}}{=} \sum_{j=0}^{b^0} I(\hat{\sigma}_j > t) + \sum_{s=0}^{t-1} \sum_{s \leq T_j < s+1} I(\sigma_j > t - T_j), \quad t \in \mathbb{N} \quad (4)$$

where

- (i) $0 \leq T_1 \leq T_2 \leq \dots$ are the successive jump times of a Poisson process with intensity λ , independent of the service times $\{\sigma_n, n = 1, 2, \dots\}$;
- (ii) b^0 is a Poisson r.v. with parameter $\rho := \lambda \bar{\sigma}$;
- (iii) conditioned on the event $\{b^0 = k\}$, $k \geq 1$, the r.v.'s $\{\hat{\sigma}_1, \dots, \hat{\sigma}_k\}$ are i.i.d. with common p.d.f. G_1 as defined in (2), namely,

$$P(\hat{\sigma}_1 \leq x_1, \dots, \hat{\sigma}_k \leq x_k | b^0 = k) = \prod_{j=1}^k G_1(x_j).$$

Further, the r.v.'s $\{T_j, \sigma_j, j = 1, 2, \dots\}$ are independent of the r.v.'s $\{b^0, \hat{\sigma}_j, j = 1, 2, \dots\}$.

The proof of this lemma follows from [5, Chapter 6] and [32, pp. 160-162] (see also [27]). The interpretation of (4) is the following: given that the $M/G/\infty$ queue is in steady-state at time $t = 0$, the first sum in the r.h.s. gives the number of busy servers at time $t > 0$ among all servers busy at time 0; the second sum gives the number of servers that became busy at time s , $0 \leq s \leq t - 1$, and that are still busy at time t .

Assume that $\rho < c$. Since the process $\{b_{t+k}, t \in \mathbb{N}\}$ converges to the stationary and ergodic process $\{b^t, t \in \mathbb{N}\}$ (see Lemma 2.1) then it is well-known (see e.g. [5, Theorem 6, p. 12]) that there exists a proper r.v. Q such that

$$P(Q > x) = \lim_{t \rightarrow \infty} P(Q_t > x) = P\left(\sup_{t \in \mathbb{N}} \left(\sum_{s=0}^{t-1} b^{-s} - ct\right) > x\right), \quad x \in \mathbb{N} \quad (5)$$

where $\{b^t, -\infty < t < \infty\}$ is a stationary and ergodic process obtained by supplementing $\{b^t, t \in \mathbb{N}\}$. We will however prefer the following representation for the stationary queue length distribution:

$$P(Q > x) = P\left(\sup_{t \in \mathbb{N}} \left(\sum_{s=0}^{t-1} b^s - ct\right) > x\right), \quad x \in \mathbb{N}, \quad (6)$$

which follows from (5) together with the property that the number of busy servers in a stationary $M/G/\infty$ queue is a reversible stochastic process [21, Theorem 3.11].

The rest of this paper is devoted to the computation of asymptotic lower and upper bounds for $P(Q > x)$. Particular attention will be devoted to the case when the p.d.f. G of the service times is *subexponential*. Recall that a probability distribution F on $[0, \infty)$ is subexponential ($F \in \mathcal{S}$) if $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$ where F^{*2} denotes the 2nd convolution of F with itself, namely, $F^{*2}(x) = \int_0^\infty F(x-u)F(du)$. As usual, the notation $f(x) \sim g(x)$ stands for $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ and $f(x) = o(g(x))$ stands for $\lim_{x \rightarrow \infty} f(x)/g(x) = 0$. The class of subexponential distributions was introduced by Chistakov [9] and contains lognormal, Pareto and Weibull distributions (see Section 3), among others. A probability distribution F on $[0, \infty)$ belongs to the class \mathcal{D} of dominated-variation distributions if $\limsup_{x \rightarrow \infty} \overline{F}(x)/\overline{F}(2x) < \infty$ and to the class \mathcal{L} of long-tailed distributions if $\lim_{x \rightarrow \infty} \overline{F}(x-y)/\overline{F}(x) = 1$ for all $y \in (-\infty, \infty)$.

For any p.d.f. F on $[0, \infty)$ with finite expectation μ , (i.e. $\mu := \int_0^\infty u F(du) < \infty$), define the integrated tail distribution F_1 by

$$F_1(x) := \frac{1}{\mu} \int_0^x \overline{F}(u) du, \quad x \geq 0.$$

Note that G_1 in (2) is the integrated tail distribution of σ_n .

The next lemma reports basic properties of subexponential probability distributions.

Lemma 2.2 *The following statements hold:*

- (a) $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$ [16, 19];
- (b) If F has finite expectation and if $F \in \mathcal{D}$ then $F_1 \in \mathcal{D} \cap \mathcal{L}$ [16].
- (c) If $F \in \mathcal{S}$ and $G(x) \sim C F(x)$ where $0 < C < \infty$, then $G \in \mathcal{S}$ [33].

In particular, we see from properties (a) and (b) that if $F \in \mathcal{D} \cap \mathcal{L}$ and if F has finite expectation then $F, F_1 \in \mathcal{S}$.

We conclude this section by pointing out an interesting feature (already observed in [27, p. 1455]) of the process $\{b^t, t \in \mathbb{N}\}$ defined in (4). First, it has been shown in [12, formula (5.39)] that $\text{cov}(b^t, b^{t+h}) = \rho \overline{G_1}(h)$ for all $t, h \in \mathbb{N}$. Therefore, the stationary process $\{b^t, t \in \mathbb{N}\}$ will be long-range dependent [3] if $\sum_{h=0}^{\infty} \overline{G_1}(h) = \infty$, which will occur, for instance, when G is Pareto (i.e. $\overline{G}(x) \sim x^{-\alpha}$) with parameter $1 < \alpha < 2$.

Remark 2.1 *By taking integer-valued service times our model reduces to that in [27]. This follows from the fact that in the case of integer-valued service times the number of busy servers at time $t+1$ is the same whether customers entering the $M/G/\infty$ queue in $(t, t+1)$ begin their service upon arrival (as in our model) or begin their service at time $t+1$ (as in [27]).*

3 Lower Bounds

The following representation of $A(0, t) := \sum_{s=0}^{t-1} b^s$ will prove useful:

$$\begin{aligned}
A(0, t) &= \sum_{s=0}^{t-1} b^s \\
&= \sum_{s=0}^{t-1} \sum_{j=1}^{b^0} I(\hat{\sigma}_j > s) + \sum_{s=0}^{t-1} \sum_{k=0}^{s-1} \sum_{k \leq T_j < k+1} I(\sigma_j > s - T_j) \\
&= \sum_{j=1}^{b^0} \sum_{s=0}^{t-1} I(\hat{\sigma}_j > s) + \sum_{k=0}^{t-2} \sum_{k \leq T_j < k+1} \sum_{s=k+1}^{t-1} I(\sigma_j > s - T_j) \\
&= \sum_{j=1}^{b^0} \min(\lceil \hat{\sigma}_j \rceil, t) + \sum_{k=0}^{t-2} \sum_{k \leq T_j < k+1} \sum_{s=k+1}^{t-1} I(\sigma_j > s - T_j). \tag{7}
\end{aligned}$$

The first sum in the r.h.s. of (7) gives the total number of customers arriving to the multiplexer in $[0, t]$ generated by all servers in the infinite-server queue busy at time 0; the second sum gives the total number of customers arriving to the multiplexer in $(0, t)$ generated by all servers in the infinite-server queue that become active at time $1, 2, \dots, t-1$. Set

$$a_0(t) := \sum_{j=1}^{b^0} \min(\lceil \hat{\sigma}_j \rceil, t) \tag{8}$$

$$a_s(t) := \sum_{s-1 \leq T_j < s} \sum_{i=s}^{t-1} I(\sigma_j > i - T_j) \quad (9)$$

so that

$$A(0, t) = \sum_{s=0}^{t-1} a_s(t). \quad (10)$$

The following asymptotic lower bound for $\log P(Q > x)$ holds:

Proposition 3.1 (Large deviations lower bound)

For any p.d.f. G ,

$$\liminf_{x \rightarrow \infty} \frac{\log P(Q > x)}{-\log \overline{G_1}(x)} \geq - \inf_{\beta > 0} \left\{ (\lfloor c - \rho + \beta \rfloor + 1) \limsup_{x \rightarrow \infty} \frac{\log \overline{G_1}(x)}{\log \overline{G_1}(\beta x)} \right\}. \quad (11)$$

Proof. Fix $\beta > 0$, $\epsilon > 0$, and define $\gamma := c - \rho + \beta + \epsilon$. Note that $\gamma > 0$ under the stability condition $c > \rho$.

We have

$$\begin{aligned} \liminf_{x \rightarrow \infty} \frac{\log P(Q > x)}{-\log \overline{G_1}(x)} &= \liminf_{t \rightarrow \infty} \frac{\log P(Q > \beta t)}{-\log \overline{G_1}(\beta t)} \\ &\geq \liminf_{t \rightarrow \infty} \frac{\log P(A(0, t) - ct > \beta t)}{-\log \overline{G_1}(\beta t)} \end{aligned} \quad (12)$$

$$\begin{aligned} &\geq \liminf_{t \rightarrow \infty} \frac{-1}{\log \overline{G_1}(\beta t)} \log P \left(a_0(t) \geq \gamma t, \sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon)t \right) \\ &= \liminf_{t \rightarrow \infty} \frac{-1}{\log \overline{G_1}(\beta t)} \left[\log P(a_0(t) \geq \gamma t) + \log P \left(\sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon)t \right) \right] \end{aligned} \quad (13)$$

$$\geq \liminf_{t \rightarrow \infty} \frac{\log P(a_0(t) \geq \gamma t)}{-\log \overline{G_1}(\beta t)} + \liminf_{t \rightarrow \infty} \frac{-1}{\log \overline{G_1}(\beta t)} \log P \left(\sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon)t \right). \quad (14)$$

Inequality (12) follows from $P(Q > x) \geq P(A(0, t) - ct > x)$ (see (6)); (13) is a consequence of the independence of the r.v.'s $a_0(t)$ and $\sum_{s=1}^{t-1} a_s(t)$ (see Lemma 2.1); (14) comes from the inequality $\liminf_n (a_n + b_n) \geq \liminf_n a_n + \liminf_n b_n$.

Let us now focus on the first limit in the r.h.s. of (14). We have for $t > 0$

$$\begin{aligned} P(a_0(t) \geq \gamma t) &= P \left(\sum_{j=1}^{b^0} \min(\lceil \hat{\sigma}_j \rceil, t) \geq \gamma t \right) \\ &\geq \sum_{k=\lceil \gamma \rceil}^{\infty} P \left(\sum_{j=1}^k \min(\hat{\sigma}_j, t) \geq \gamma t \mid b^0 = k \right) P(b^0 = k) \end{aligned} \quad (15)$$

$$\begin{aligned}
&\geq \sum_{k=\lceil \gamma \rceil}^{\infty} P(\hat{\sigma}_1 > t, \dots, \hat{\sigma}_{\lceil \gamma \rceil} > t \mid b^0 = k) P(b^0 = k) \\
&= \overline{G}_1(t)^{\lceil \gamma \rceil} P(b^0 \geq \lceil \gamma \rceil)
\end{aligned} \tag{16}$$

where (16) follows from Lemma 2.1(iii).

Since $P(b^0 \geq \lceil \gamma \rceil) > 0$ (see Lemma 2.1(ii)) we deduce from (16) that

$$\liminf_{t \rightarrow \infty} \frac{\log P(a_0(t) \geq \gamma t)}{-\log \overline{G}_1(\beta t)} \geq -\lceil \gamma \rceil \limsup_{t \rightarrow \infty} \frac{\log \overline{G}_1(t)}{\log \overline{G}_1(\beta t)}. \tag{17}$$

Let us show that the second limit in the r.h.s. of (14) is 0. We see from the definition of $A(0, t)$ and from (8)-(10) that

$$\sum_{s=1}^{t-1} a_s(t) \geq \sum_{s=0}^{t-1} b^s - \sum_{j=1}^{b^0} \lceil \hat{\sigma}_j \rceil. \tag{18}$$

On the other hand, the stationarity and ergodicity of the sequence $\{b^t, t \in \mathbb{N}\}$ together with $\rho = E[b^0] < \infty$ (see Lemma 2.1) yields

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=0}^{t-1} b^s = \rho \quad \text{a.s.} \tag{19}$$

from ergodic theory (see e.g. [31, Chapter V]). We therefore deduce from (18)-(19) that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \sum_{s=1}^{t-1} a_s(t) \geq \rho \quad \text{a.s.} \tag{20}$$

since $\sum_{j=1}^{b^0} \hat{\sigma}_j < \infty$ a.s. by Lemma 2.1.

Combining [24, Proposition I-4-3] together with (20) yields

$$1 \geq \liminf_t P\left(\sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon)t\right) \geq P\left(\liminf_t \left\{\sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon)t\right\}\right) = 1 \tag{21}$$

which entails that

$$\liminf_{t \rightarrow \infty} \frac{-1}{\log \overline{G}_1(\beta t)} \log P\left(\sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon)t\right) = 0. \tag{22}$$

In summary, we have shown that (cf. (14), (17), (22))

$$\begin{aligned}
\liminf_{x \rightarrow \infty} \frac{\log P(Q > x)}{-\log \overline{G}_1(x)} &\geq - \inf_{\beta > 0, \epsilon > 0} \left\{ \lceil c - \rho + \beta + \epsilon \rceil \limsup_{t \rightarrow \infty} \frac{\log \overline{G}_1(t)}{\log \overline{G}_1(\beta t)} \right\} \\
&\geq - \inf_{\beta > 0} \left\{ (\lfloor c - \rho + \beta \rfloor + 1) \limsup_{t \rightarrow \infty} \frac{\log \overline{G}_1(t)}{\log \overline{G}_1(\beta t)} \right\}
\end{aligned}$$

which completes the proof. \blacksquare

It is worth noting that the lower bound in (11) is never trivial as it is always larger than or equal to $-(\lfloor c - \rho \rfloor + 2)$ that is obtained for $\beta = 1$.

The next result proposes asymptotic lower bounds for $P(Q > x)$.

Proposition 3.2 (Asymptotic lower bound)

For any p.d.f. G ,

$$\liminf_{x \rightarrow \infty} \frac{P(Q > x)}{\overline{G_1}(x)^{\lfloor c - \rho \rfloor + 1}} \geq \sup_{0 < \beta < 1 + \lfloor c - \rho \rfloor - (c - \rho)} \liminf_{x \rightarrow \infty} \left(\frac{\overline{G_1}(x)}{\overline{G_1}(\beta x)} \right)^{\lfloor c - \rho \rfloor + 1} \left(1 - \sum_{k=0}^{\lfloor c - \rho \rfloor} \frac{\rho^k}{k!} e^{-\rho} \right). \quad (23)$$

If $G_1 \in \mathcal{S}$ then

$$\liminf_{x \rightarrow \infty} \frac{P(Q > x)}{\overline{G_1}(x)^{\lfloor c - \rho \rfloor + 1}} \geq (\delta_0 + \rho) \sup_{\substack{\epsilon > 0, \beta > 0 \\ \lceil c - \rho + \beta + \epsilon \rceil = r_0}} \liminf_{x \rightarrow \infty} \left(\frac{\overline{G_1}((c - \rho + \beta + \epsilon)x/r_0)}{\overline{G_1}(\beta x)} \right)^{r_0} \quad (24)$$

with $\delta_0 := e^{-\rho} \left(\sum_{l=1}^{l_0-1} l^{r_0} \sum_{i=l}^{(l+1)r_0-1} \rho^i / i! - \sum_{l=1}^{l_0} \rho^l / (l-1)! \right)$, $r_0 := \lfloor c - \rho \rfloor + 1$, $l_0 := \min\{l \geq 1 : l^{r_0} \geq (l+1)r_0 - 1\}$. In particular, when $c - \rho < 1$ then

$$\liminf_{x \rightarrow \infty} \frac{P(Q > x)}{\overline{G_1}(x)} \geq \rho \sup_{\substack{\epsilon > 0, \beta > 0 \\ \beta + \epsilon < 1 - (c - \rho)}} \liminf_{x \rightarrow \infty} \frac{\overline{G_1}((c - \rho + \beta + \epsilon)x)}{\overline{G_1}(\beta x)}. \quad (25)$$

Proof. The proof of (23) follows the same line of arguments as that of Proposition 3.1. Define $\gamma := c - \rho + \beta + \epsilon$. Let $0 < \beta < 1 + \lfloor c - \rho \rfloor - (c - \rho)$ and pick $\epsilon > 0$ small enough so that $\lceil \gamma \rceil = \lfloor c - \rho \rfloor + 1$.

In direct analogy with the derivation of (14) and by using (16) and (21) we get

$$\liminf_{x \rightarrow \infty} \frac{P(Q > x)}{\overline{G_1}(x)^{\lfloor c - \rho \rfloor + 1}} \geq \liminf_{t \rightarrow \infty} \frac{P(a_0(t) > \gamma t)}{\overline{G_1}(\beta t)^{\lfloor c - \rho \rfloor + 1}} \quad (26)$$

$$\geq \liminf_{t \rightarrow \infty} \left(\frac{\overline{G_1}(t)}{\overline{G_1}(\beta t)} \right)^{\lfloor c - \rho \rfloor + 1} P(b^0 \geq \lfloor c - \rho \rfloor + 1) \quad (27)$$

for all $0 < \beta < 1 + \lfloor c - \rho \rfloor - (c - \rho)$, from which (23) follows.

Assume now that $G_1 \in \mathcal{S}$. Straightforward manipulations in (15) yield

$$\begin{aligned} & P(a_0(t) > \gamma t) \\ & \geq \sum_{l=1}^{\infty} \sum_{m=0}^{\lceil \gamma \rceil - 1} P \left(\sum_{j=1}^{l \lceil \gamma \rceil + m} \min(\hat{\sigma}_j, t) \geq \gamma t \mid b^0 = l \lceil \gamma \rceil + m \right) P(b^0 = l \lceil \gamma \rceil + m) \\ & \geq \sum_{l=1}^{\infty} \sum_{m=0}^{\lceil \gamma \rceil - 1} P \left(\sum_{j=1}^{l \lceil \gamma \rceil} \min(\hat{\sigma}_j, t) \geq \gamma t \mid b^0 = l \lceil \gamma \rceil + m \right) P(b^0 = l \lceil \gamma \rceil + m) \end{aligned}$$

$$\geq \sum_{l=1}^{\infty} \sum_{m=0}^{\lceil \gamma \rceil - 1} P \left(\sum_{j=1}^l \min(\hat{\sigma}_j, t) \geq \frac{\gamma}{\lceil \gamma \rceil} t \mid b^0 = l\lceil \gamma \rceil + m \right)^{\lceil \gamma \rceil} P(b^0 = l\lceil \gamma \rceil + m). \quad (28)$$

It is shown in Lemma A.1 in Appendix A that $P \left(\sum_{j=1}^l \min(\hat{\sigma}_j, t) \geq \theta t \mid b^0 = k \right) \sim l \overline{G_1}(\theta t)$ for all $\theta \in (0, 1]$, $k \geq l$. By applying Fatou's lemma to (28) and by using the latter result we obtain

$$\liminf_{t \rightarrow \infty} \frac{P(a_0(t) > \gamma t)}{\overline{G_1}(\beta t)^{\lceil \gamma \rceil}} \geq \eta \sum_{l=1}^{\infty} l^{\lceil \gamma \rceil} P(l\lceil \gamma \rceil \leq b^0 \leq (l+1)\lceil \gamma \rceil - 1) \quad (29)$$

with $\eta := \liminf_{t \rightarrow \infty} \left(\frac{\overline{G_1}(\gamma t / \lceil \gamma \rceil)}{\overline{G_1}(\beta t)} \right)^{\lceil \gamma \rceil}$.

Define $l_0 = \min \{l \geq 1 : l^{\lceil \gamma \rceil} \geq (l+1)\lceil \gamma \rceil - 1\}$. Observe that $l^{\lceil \gamma \rceil} \geq (l+1)\lceil \gamma \rceil - 1$ for all $l \geq l_0$ since the mapping $l \rightarrow l^{\lceil \gamma \rceil} - (l+1)\lceil \gamma \rceil + 1$ is non-decreasing. The above and (29) yield

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{P(a_0(t) > \gamma t)}{\overline{G_1}(\beta t)^{\lceil \gamma \rceil}} &\geq \eta \left(\sum_{l=1}^{l_0-1} l^{\lceil \gamma \rceil} P(l\lceil \gamma \rceil \leq b^0 \leq (l+1)\lceil \gamma \rceil - 1) + \sum_{l=l_0}^{\infty} l P(b^0 = l) \right) \\ &= \eta \left(\sum_{l=1}^{l_0-1} l^{\lceil \gamma \rceil} P(l\lceil \gamma \rceil \leq b^0 \leq (l+1)\lceil \gamma \rceil - 1) - \sum_{l=1}^{l_0\lceil \gamma \rceil - 1} l P(b^0 = l) + \rho \right). \quad (30) \end{aligned}$$

Substituting $\lceil \gamma \rceil$ for $\lfloor c - \rho \rfloor + 1$ in (30) yields (24). \blacksquare

It is seen from Lemma A.2 in Appendix A that the supremum in the r.h.s. of (23) (resp. (24), (25)) is strictly positive if and only if $G_1 \in \mathcal{D}$. A sufficient condition for $G_1 \in \mathcal{D}$ is that $G \in \mathcal{D}$ (e.g. G Pareto) and G has finite expectation (see Lemma 2.2(b)).

When $c - \rho < 1$, Jelenkovic and Lazar [20, Theorem 11] have derived a tighter lower bound with the same decay function $\overline{G_1}(x)$ but with a larger coefficient. The bound in [20] holds provided that $L := \lim_{\delta \downarrow 1} \liminf_{x \uparrow \infty} \overline{G_1}(\delta x) / \overline{G_1}(x) > 0$ (Jelenkovic and Lazar [20] actually assume that $L = 1$ but this assumption can be weakened to $L > 0$; if so, then the coefficient of their lower bound in Theorem 11 has to be multiplied by L). Since $\overline{G_1}$ is non-increasing, it is easy to see from Lemma A.2 that $L > 0$ is equivalent to $G_1 \in \mathcal{D}$. Hence, the bounds in Proposition 3.2 and in [20] are non-trivial if and only if $G_1 \in \mathcal{D}$.

Corollary 3.1 *When $G_1 \in \mathcal{D}$ then*

$$\liminf_{x \rightarrow \infty} \frac{\log P(Q > x)}{-\log \overline{G_1}(x)} \geq -\lfloor c - \rho \rfloor - 1. \quad (31)$$

When Corollary 3.1 applies the lower bound in the r.h.s. of (31) is easier to compute than the lower bound in Proposition 3.1 but may not be as tight (for G Pareto both bounds in (11) and in (31) are the same as reported below).

We conclude this section by addressing the cases when G is (i) geometric, (ii) Pareto, (iii) Weibull, and (iv) lognormal.

- (i) **G is geometric.** We have $G(r) = (1 - q)q^{r-1}$ for $r = 1, 2, \dots$ with $q := (\bar{\sigma} - 1)/\bar{\sigma} \in (0, 1)$. Hence, $\overline{G}_1(r) = q^r$ for $r = 1, 2, \dots$

Proposition 3.2 yields a trivial lower bound ($= 0$). From Proposition 3.1 we find

$$\liminf_{x \rightarrow \infty} \frac{1}{x} \log P(Q > x) \geq \log q \inf_{\beta > 0} \frac{\lfloor c - \rho + \beta \rfloor + 1}{\beta} = \log q. \quad (32)$$

The r.h.s. of (32) follows from the inequalities

$$\frac{c - \rho + \beta + 1}{\beta} \geq \frac{\lfloor c - \rho + \beta \rfloor + 1}{\beta} \geq 1$$

together with $\lim_{\beta \rightarrow \infty} (c - \rho + \beta + 1)/\beta = 1$.

- (ii) **G is Pareto.** We have $\overline{G}(x) \sim x^{-\alpha}$ for some $\alpha > 1$. We assume that $\alpha > 1$ so that G has finite expectation $\bar{\sigma}$. We have

$$\overline{G}_1(x) \sim x^{-\alpha+1}/(\alpha - 1). \quad (33)$$

From (24) we get

$$\liminf_{x \rightarrow \infty} \frac{P(Q > x)}{x^{(-\alpha+1)(\lfloor c-\rho \rfloor+1)}} \geq (\delta_0 + \rho) \left(\frac{\lfloor c - \rho \rfloor + 1}{c - \rho + 1} \right)^{(\alpha-1)(\lfloor c-\rho \rfloor+1)}. \quad (34)$$

In particular, if $c - \rho < 1$ then (cf. (25))

$$\liminf_{x \rightarrow \infty} \frac{P(Q > x)}{x^{-\alpha+1}} \geq \rho \left(\frac{1}{c - \rho + 1} \right)^{\alpha-1}. \quad (35)$$

From (34) (or Proposition 3.1/ Corollary 3.1) we get

$$\liminf_{x \rightarrow \infty} \frac{1}{\log x} \log P(Q > x) \geq (-\alpha + 1)(\lfloor c - \rho \rfloor + 1). \quad (36)$$

- (iii) **G is Weibull.** We have $\overline{G}(x) \sim e^{-x^\nu}$ for some $0 < \nu < 1$ and $\nu > 0$. Simple algebra yield

$$\overline{G}_1(x) \sim e^{-x^\nu} x^{1-\nu}/\nu. \quad (37)$$

Proposition 3.2 yields a trivial lower bound. By Proposition 3.1 we get (Corollary 3.1 does not apply since $G_1 \notin \mathcal{D}$)

$$\begin{aligned} \liminf_{x \rightarrow \infty} \frac{1}{x^\nu} \log P(Q > x) &\geq - \inf_{\beta > 0} \frac{\lfloor c - \rho + \beta \rfloor + 1}{\beta^\nu} \\ &= \begin{cases} - \min \left\{ \frac{\lfloor c - \rho \rfloor + \lfloor a \rfloor}{(\lfloor a \rfloor - q)^\nu}; \frac{\lfloor c - \rho \rfloor + \lceil a \rceil}{(\lceil a \rceil - q)^\nu} \right\}, & \text{if } a \geq 1 \\ - \frac{\lfloor c - \rho \rfloor + 1}{(1 - q)^\nu}, & \text{if } a < 1 \end{cases} \end{aligned} \quad (38)$$

with $a := (\nu \lfloor c - \rho \rfloor + q)/(1 - \nu)$ and $q := c - \rho - \lfloor c - \rho \rfloor$.

[Hint for the derivation of (38): note that

$$\inf_{\beta > 0} \frac{\lfloor c - \rho + \beta \rfloor + 1}{\beta^\nu} = \min_{i=1,2,\dots} \frac{\lfloor c - \rho \rfloor + i}{(i - q)^\nu}. \quad (39)$$

The mapping $g(x) := (\lfloor c - \rho \rfloor + x)/(x - q)^\nu$ being strictly decreasing in $(0, a)$ and strictly increasing in (a, ∞) , the minimum in (39) is reached when $\beta = \lfloor a \rfloor$ or when $\beta = \lceil a \rceil$ if $a \geq 1$ and when $\beta = 1$ if $a < 1$.]

- (iv) **G is lognormal.** The p.d.f. G of a r.v. σ is *lognormal* if $\sigma \stackrel{\text{st}}{=} \exp(Y)$ where Y is a Gaussian r.v. with mean μ and variance δ^2 . Then, $\overline{G}(x) \sim (2\pi)^{-1/2} (\delta/(\log x - \mu)) e^{-(\log x - \mu)^2/(2\delta^2)}$. From this we get

$$\overline{G_1}(x) \sim \frac{\sigma^3 x e^{-(\log x - \mu)^2/(2\delta^2)}}{\sqrt{2\pi} (\log x - \mu)^2}. \quad (40)$$

Proposition 3.2 yields a trivial lower bound. From Proposition 3.1 (Corollary 3.1 does not apply since $G_1 \notin \mathcal{D}$) we have

$$\liminf_{x \rightarrow \infty} \frac{1}{(\log x)^2} \log P(Q > x) \geq -\frac{\lfloor c - \rho \rfloor + 1}{2\delta^2}. \quad (41)$$

4 Upper Bounds

We begin this section by stating two lemmas that will be used in the derivation of asymptotic upper bounds in the case when G and G_1 are subexponential probability distributions.

Lemma 4.1 *Let F, F^1, \dots, F^k be probability distributions such that $\overline{F}^j(x) \sim c_j \overline{F}(x)$, $c_j > 0$, for all $j = 1, 2, \dots, k$. If $F \in \mathcal{S}$ then*

- (a) $\overline{F^1 \star \dots \star F^k}(x) \sim \sum_{j=1}^k c_j \overline{F}(x)$
 (b) *for each $\epsilon > 0$ there exists some constant $K_\epsilon < \infty$, independent of k , such that for all $x \geq 0$,*
- $$\overline{F^1 \star \dots \star F^k}(x) \leq K_\epsilon (1 + \epsilon)^k \overline{F}(x). \quad (42)$$

Proof. Statement (a) is due to Cline [11] and (b) to Athreya and Ney [2, p. 149]. ■

Lemma 4.2 (Pakes [26]) *Consider a GI/GI/1 queue with i.i.d. service times $\{\sigma_n\}_n$ with common p.d.f. F and i.i.d. interarrival times $\{\tau_n\}_n$. Assume that $E[\sigma_n] < E[\tau_n]$.*

If $F, F_1 \in \mathcal{S}$, then

$$P(W > x) \sim \frac{E[\sigma_n]}{E[\tau_n] - E[\sigma_n]} \overline{F_1}(x)$$

where $W := \sup_{n \in \mathbb{N}} \left(\sum_{m=0}^{n-1} (\sigma_m - \tau_m) \right)$ is the stationary waiting time.

We are now in position to derive the following asymptotic upper bounds for $P(Q > x)$ and for $\log P(Q > x)$ when G and G_1 are in \mathcal{S} .

Proposition 4.1 (Upper bounds)

Assume that $G, G_1 \in \mathcal{S}$. Then,

$$\limsup_{x \rightarrow \infty} \frac{P(Q > x)}{\overline{G_1}(x)} \leq \rho + \frac{\rho}{c - \rho}. \quad (43)$$

In particular, (43) implies that

$$\limsup_{x \rightarrow \infty} \frac{\log P(Q > x)}{-\log \overline{G_1}(x)} \leq -1. \quad (44)$$

Proof. Define

$$a_0 = \sum_{j=1}^{b^0} (\hat{\sigma}_j + 1) \quad (45)$$

$$a_s = \sum_{j=1}^{v_{s-1}} (\sigma_j + 1), \quad s = 1, 2, \dots \quad (46)$$

where $v_s := \sum_{j=1}^{\infty} I(s \leq T_j < s+1)$ denotes the number of arrivals in the M/G/ ∞ queue in the interval of time $[s, s+1)$, $s \in \mathbb{N}$. Since the arrival process in this queue is Poisson with rate λ , $\{v_s, s \in \mathbb{N}\}$ constitutes an i.i.d. sequence of Poisson r.v.'s with parameter λ , namely, $P(v_s = k) = \lambda^k e^{-\lambda} / k!$ for all $k \in \mathbb{N}$.

From (8)-(9) and (45)-(46) we see that

(a1) $a_0(t) \leq a_0$ and $a_s(t) \leq a_s$ for all $s, t = 1, 2, \dots$;

(a2) the r.v.'s a_s , $s = 1, 2, \dots$ are i.i.d. and independent of the r.v. a_0 .

To get the second inequality in (a1) observe from (9) that

$$a_s(t) \leq \sum_{s-1 \leq T_j < s} \sum_{l=0}^{t-1-s} I(\sigma_j > l) = \sum_{s-1 \leq T_j < s} \min([\sigma_j], t-s) = \sum_{j=1}^{v_{s-1}} \min([\sigma_j], t-s) \leq a_s.$$

We have, cf. (6), (10) and (a1),

$$\begin{aligned} P(Q > x) &= P\left(\sup_{t \in \mathbb{N}} \left(a_0(t) + \sum_{s=1}^{t-1} a_s(t) - ct\right) > x\right) \\ &\leq P\left(a_0 + \sup_{t \in \mathbb{N}} \left(\sum_{s=1}^t a_s - ct\right) > x\right). \end{aligned} \quad (47)$$

Let us show that $P(a_0 > x) \sim \rho \overline{G_1}(x)$. From the definition of a_0 (see (45)) and Lemma 2.1(iii) we find

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{P(a_0 > x)}{\overline{G_1}(x)} &= \lim_{x \rightarrow \infty} \left(\sum_{k=1}^{\lfloor x \rfloor} \frac{\overline{G_1^{*k}}(x-k)}{\overline{G_1}(x)} P(b^0 = k) + \frac{P(b^0 > \lfloor x \rfloor)}{\overline{G_1}(x)} \right) \\ &= \lim_{x \rightarrow \infty} \sum_{k=1}^{\infty} I(k \leq \lfloor x \rfloor) \frac{\overline{G_1^{*k}}(x-k)}{\overline{G_1}(x)} P(b^0 = k) \end{aligned} \quad (48)$$

where (48) follows from the fact that b^0 decreases exponentially fast to 0 (hint: by Chernoff's bound $P(b^0 > x) \leq \exp(\rho(\exp(\theta) - 1)) \exp(-\theta x)$ for all $\theta > 0, x \geq 0$) which implies that $\lim_{x \rightarrow \infty} P(b^0 > \lfloor x \rfloor)/\overline{G_1}(x) = 0$ by [15, Lemma 1(b)]. By applying Lemma 4.1(a) with $F^j(x) = G_1(x-1)$ and $F \equiv G_1$ (note that this lemma applies with $c_j = 1$ since $\overline{F^j}(x)/\overline{F}(x) \sim 1$ since $G_1 \in \mathcal{L}$) we have $f_x(k) := I(k \leq \lfloor x \rfloor) \overline{G_1^{*k}}(x-k)/\overline{G_1}(x) \rightarrow k$ as $x \rightarrow \infty$ for each $k \geq 1$; furthermore, we see from Lemma 4.1(b) that there exists a constant $c_0 < \infty$, independent of k , such that $|f_x(k)| \leq c_0 2^k$ for all $x \geq 0, k = 1, 2, \dots$. Since $\sum_{k=1}^{\infty} 2^k P(b^0 = k) = e^\rho$ is finite we may therefore apply the bounded convergence theorem to the r.h.s. of (48) to finally get

$$\lim_{x \rightarrow \infty} \frac{P(a_0 > x)}{\overline{G_1}(x)} = \sum_{k=1}^{\infty} k P(b^0 = k) = \rho. \quad (49)$$

Let us now focus on $P\left(\sup_{t \in \mathbb{N}} \left(\sum_{s=1}^t a_s - ct\right) > x\right)$ when x is large.

Define $W = \sup_{t \in \mathbb{N}} \left(\sum_{s=1}^t a_s - ct\right)$. Since the r.v.'s $a_s, s = 1, 2, \dots$ are i.i.d. it is seen that $P(W \leq x)$ is the waiting time distribution in a stable (since $E[a_s] = \rho < c$) $D/GI/1$ queue with constant interarrival times c and service times $\{a_s\}_s$. By applying again Lemma 4.1 (with $F^j(x) = G(x-1)$, $F \equiv G$) and the bounded convergence theorem we obtain

$$\lim_{x \rightarrow \infty} \frac{P(a_s > x)}{\overline{G}(x)} = \sum_{k=0}^{\infty} \lim_{x \rightarrow \infty} \frac{\overline{G^{*k}}(x-k)}{\overline{G}(x)} P(v_0 = k) \quad (50)$$

$$= \sum_{k=1}^{\infty} k P(v_0 = k) = \lambda. \quad (51)$$

Define $K(x) := P(a_s \leq x)$. From $\overline{K}(x) \sim \lambda \overline{G}(x)$ (see (51)) and $\overline{K_1}(x) \sim \overline{G_1}(x)$ (which is an easy consequence of (51)) we may conclude from Lemma 2.2(c) that $K, K_1 \in \mathcal{S}$ since $G, G_1 \in \mathcal{S}$.

Therefore, Lemma 4.2 applies to this $D/GI/1$ queue (with $F = K$) to yield

$$P(W > x) \sim \frac{\rho}{c - \rho} \overline{G_1}(x). \quad (52)$$

The proof is concluded by applying Lemma 4.1(a) (with $F = G_1$, $F^1(x) = P(a_0 > x)$ and $F^2(x) = P(W > x)$) to the independent r.v.'s a_0 and W and by using (49) and (52). ■

We conclude this section by specializing Proposition 4.1 to p.d.f.'s G that are (i) Pareto, (ii) Weibull, and (iii) lognormal. It is known that both G and G_1 belong to \mathcal{S} when G is Pareto, Weibull or lognormal.

(i) **G is Pareto.** From (33) and (44) we get

$$\limsup_{x \rightarrow \infty} \frac{1}{\log x} \log P(Q > x) \leq -\alpha + 1. \quad (53)$$

Also note that the bound in (53) is tighter than Duffield's corresponding bound (3) when $c - \rho \leq \alpha/(\alpha - 1)$; otherwise Duffield's is tighter.

(ii) **G is Weibull.** From (37) and (44) we get

$$\limsup_{x \rightarrow \infty} \frac{1}{x^\nu} \log P(Q > x) \leq -1. \quad (54)$$

(iii) **G is lognormal.** From (40) and (44) we get

$$\limsup_{x \rightarrow \infty} \frac{1}{(\log x)^2} \log P(Q > x) \leq -\frac{1}{2\delta^2}. \quad (55)$$

We observe from (36), (53) and (41), (55) that the bounds are tight when $c - \rho < 1$:

Corollary 4.1 *Assume that $c - \rho < 1$. If G is Pareto then*

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \log P(Q > x) = -\alpha + 1 \quad (56)$$

and if G is lognormal then

$$\lim_{x \rightarrow \infty} \frac{1}{(\log x)^2} \log P(Q > x) = -\frac{1}{2\delta^2}. \quad (57)$$

5 Concluding Remarks

We conclude this paper by addressing the situation when the multiplexer is fed by N independent M/G/ ∞ input processes, with arrival rate λ_i and p.d.f. of the service times G^i for the system i ($i = 1, 2, \dots, N$). Because the arrivals are Poisson this is equivalent to considering a single M/G/ ∞ queueing system with arrival intensity $\lambda := \sum_{i=1}^N \lambda_i$ and p.d.f. G of the service time given by $G(x) = \sum_{i=1}^N (\lambda_i/\lambda) G^i(x)$. All of the results in the paper therefore apply to this pair (λ, G) . Of particular interest is the case when one p.d.f. of the service times, say G^1 , has a heavier tail than the others, namely, $\overline{G^i}(x) = o(\overline{G^1}(x))$ for all

$i = 2, 3, \dots, N$. Then, $\overline{G}_1(x) \sim (\lambda_1/\lambda) \overline{G}_1^1(x)$ and we may conclude from the results in Sections 3-4 that the source with the heaviest tail dominates the other sources. In particular, we see from (11) and (44) that

$$-\theta_1 \leq \liminf_{x \rightarrow \infty} \frac{\log P(Q > x)}{-\log \overline{G}_1^1(x)} \leq \limsup_{x \rightarrow \infty} \frac{\log P(Q > x)}{-\log \overline{G}_1^1(x)} \leq -1$$

where the upper bound holds if $G^1, G_1^1 \in \mathcal{S}$, with $\theta_1 := \inf_{\beta > 0} \left\{ h(\beta) \limsup_{x \rightarrow \infty} \frac{\log \overline{G}_1^1(x)}{\log \overline{G}_1^1(\beta x)} \right\}$, $h(\beta) := \lfloor c - \rho + \beta \rfloor + 1$ and $\rho = \sum_{i=1}^N (\lambda_i/\lambda) \int_0^\infty x G^i(dx)$.

A Appendix

Lemma A.1 Assume that $G_1 \in \mathcal{S}$. Then, for every $k = 1, 2, \dots, l \geq k$, and $0 < \theta \leq 1$,

$$P \left(\sum_{j=1}^k \min(\hat{\sigma}_j, x) \geq \theta x \mid b^0 = l \right) \sim k \overline{G}_1(\theta x). \quad (58)$$

Proof. Clearly, for all $k = 1, 2, \dots, l \geq k$,

$$\limsup_{x \rightarrow \infty} \frac{P \left(\sum_{j=1}^k \min(\hat{\sigma}_j, x) \geq \theta x \mid b^0 = l \right)}{\overline{G}_1(\theta x)} \leq \limsup_{x \rightarrow \infty} \frac{P \left(\sum_{j=1}^k \hat{\sigma}_j \geq \theta x \mid b^0 = l \right)}{\overline{G}_1(\theta x)} = k$$

from Lemma 4.1(a) and Lemma 2.1(iii).

Let us now show that

$$\liminf_{x \rightarrow \infty} \frac{P \left(\sum_{j=1}^k \min(\hat{\sigma}_j, x) \geq \theta x \mid b^0 = l \right)}{\overline{G}_1(\theta x)} \geq k \quad (59)$$

for all $k = 1, 2, \dots, l \geq k$, which will conclude the proof.

Inequality (59) is true for $k = 1, l \geq 1$, since $P(\min(\hat{\sigma}_1, x) \geq \theta x \mid b^0 = l) = P(\hat{\sigma}_1 \geq \theta x \mid b^0 = l) = \overline{G}_1(\theta x)$ from Lemma 2.1(iii). Assume that (59) is true for $k = 1, 2, \dots, n-1, l \geq k$, and let us show that it still holds for $k = n, l \geq k$.

We have

$$\begin{aligned} P \left(\sum_{j=1}^n \min(\hat{\sigma}_j, x) \geq \theta x \mid b^0 = l \right) &\geq P \left(\sum_{j=1}^{n-1} \min(\hat{\sigma}_j, x) \geq \theta x \mid b^0 = l \right) \\ &+ P(\min(\hat{\sigma}_n, x) \geq \theta x \mid b^0 = l) - P \left(\sum_{j=1}^{n-1} \min(\hat{\sigma}_j, x) \geq \theta x, \min(\hat{\sigma}_n, x) \geq \theta x \mid b^0 = l \right) \end{aligned}$$

$$\geq P \left(\sum_{j=1}^{n-1} \min(\hat{\sigma}_j, x) \geq \theta x \mid b^0 = l \right) + \overline{G_1}(\theta x) - P \left(\sum_{j=1}^{n-1} \min(\hat{\sigma}_j, x) \geq \theta x \mid b^0 = l \right) \overline{G_1}(\theta x) \quad (60)$$

where (60) follows from the conditional independence of the r.v.'s $\{\hat{\sigma}_j\}_j$ given b^0 . The inequality (59) (with $k = n$) now follows from the induction hypothesis together with

$$\lim_{x \rightarrow \infty} P \left(\sum_{j=1}^{n-1} \min(\hat{\sigma}_j, x) \geq \theta x \mid b^0 = l \right) = 0.$$

■

Lemma A.2 (Feller [17], Bingham et al. [4, Corollary 2.0.6])

Let F be a probability distribution. If $\liminf_{x \rightarrow \infty} \overline{F}(x)/\overline{F}(\delta_0 x) > 0$ for some $\delta_0 \in (0, 1)$ then $\liminf_{x \rightarrow \infty} \overline{F}(x)/\overline{F}(\delta x) > 0$ for all $\delta \in (0, 1)$.

As a consequence, $F \in \mathcal{D}$ if and only if $\liminf_{x \rightarrow \infty} \overline{F}(x)/\overline{F}(\delta x) > 0$ for all $\delta > 0$.

We give below a proof of this result for the sake of completeness.

Proof. Assume that there exists $\delta_0 \in (0, 1)$ such that $\liminf_{x \rightarrow \infty} \overline{F}(x)/\overline{F}(\delta_0 x) > 0$. Fix $\delta > 0$. Since $\delta_0 < 1$ there exists $n \geq 1$ such that $\delta/\delta_0^n > 1$. From

$$\frac{\overline{F}(x)}{\overline{F}(\delta x)} = \left(\prod_{i=1}^n \frac{\overline{F}(\delta_0^{i-1} x)}{\overline{F}(\delta_0^i x)} \right) \frac{\overline{F}(\delta_0^n x)}{\overline{F}(\delta x)}$$

we readily deduce that

$$\liminf_{x \rightarrow \infty} \frac{\overline{F}(x)}{\overline{F}(\delta x)} \geq \left(\liminf_{x \rightarrow \infty} \frac{\overline{F}(x)}{\overline{F}(\delta_0 x)} \right)^n \liminf_{x \rightarrow \infty} \frac{\overline{F}(x)}{\overline{F}(\delta/\delta_0^n x)}. \quad (61)$$

The first factor in the r.h.s. of (61) is strictly positive from the definition of δ_0 ; the second factor too since $\overline{F}(x) \geq \overline{F}(\delta/\delta_0^n x)$ for all x . This proves the first statement.

Let us now prove the second statement. The “ \Leftarrow ” part is clearly true (take $\delta = 1/2$). The “ \Rightarrow ” part follows from the first statement by taking $\delta_0 = 1/2$. ■

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References

- [1] V. Anantharam, “On the sojourn time of sessions at an ATM buffer with long-range dependent input traffic,” *Proc. of the 34th IEEE Conf. on Decision & Control*, New Orleans, LA, Dec. 13-15, 1995, Vol. 1, 859-864.

- [2] K. B. Athreya, P. E. Ney, *Branching Processes*. Springer, Berlin, 1972.
- [3] J. Beran, *Statistics for long-memory processes*. Chapman and Hall, New York, 1994.
- [4] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular Variation*. Cambridge University Press, Cambridge, 1987.
- [5] A. A. Borovkov, *Stochastic processes in queueing theory*. Springer-Verlag, New York, 1976.
- [6] O. J. Boxma, "Fluid queues and regular variation," *Performance Evaluation*, **27&28**:699-712, 1996.
- [7] O. J. Boxma, V. Dumas, "Fluid queues with long-tailed activity period distributions." Preprint 1997. To appear in a Special Issue of *Computer Communications* on "Stochastic Analysis and Optimisation of Communication Systems".
- [8] F. Brichet, J. W. Roberts, A. Simonian, D. Veitch, "Heavy traffic analysis of a storage model with long range dependent on/off sources", Preprint 1996. To appear in *QUESTA*.
- [9] V. P. Chistakov, "A theorem on sums of independent positive random variables and its application to branching random processes," *Theory Prob. Appl.*, **9**:640-648, 1964.
- [10] G. L. Choudhury, W. Whitt. "Long-tail buffer-content distributions in broadband networks". To appear in *Performance Evaluation*.
- [11] D. B. H. Cline, "Convolution tails, product tails and domains of attraction," *Prob. Theory Related Fields*, **72**:529-557, 1986.
- [12] D. R. Cox, V. Isham, *Point processes*. Chapman and Hall, New York, 1980.
- [13] N. G. Duffield, "On the relevance of long-tailed durations for the statistical multiplexing of large aggregations," *Proc. of the 34th Annual Allerton Conf. on Communication, Control and Computing*, Oct. 2-4, 1996.
- [14] N. G. Duffield, N. O'Connell, "Large deviations and overflow probabilities for the general single-server queue with applications," *Math. Proc. Cam. Phil. Soc.*, **118**:363-374, 1995.
- [15] P. Embrechts, C. M. Goldie, N. Veraverbeke, "Subexponentiality and infinite divisibility," *Z. Wahrschein. verw. Gebiete*, **49**:335-347, 1979.
- [16] P. Embrechts, E. Omey, "A property of longtailed distributions," *J. Appl. Prob.*, **21**:80-87, 1984.
- [17] W. Feller, "One-sided analogues of Karamata's regular variation," *Enseign. Math.*, **15**, 107-121, 1969.

- [18] M. Garrett, W. Willinger, "Analysis, modeling and generation of self-similar VBR video traffic," *Proc. SIGCOMM'94*, 269-280, Sep. 1994.
- [19] C. M. Goldie, "Subexponential distributions and dominated-variation tails," *J. Appl. Prob.*, **15**:440-442, 1978.
- [20] P. R. Jelenkovic, A. A. Lazar, "Asymptotic results for multiplexing subexponential on-off sources." Submitted for publication.
- [21] F. P. Kelly, *Reversibility and Stochastic Networks*. John Wiley & Sons, New York, 1979.
- [22] W. Leland, M. Taqqu, W. Willinger, D. Wilson, "On the self-similar nature of ethernet traffic(extended version)," *IEEE/ACM Trans. on Networking* **2**:1-15, 1994.
- [23] N. Likhanov, B. Tsybakov, N. D. Georganas, "Analysis of an ATM buffer with self-similar ("fractal") input traffic", *Proc. of the IEEE Infocom'95 Conf.*, Boston, MA, Apr. 4-6, 1995, 985-992.
- [24] J. Neveu, *Bases Mathématiques du Calcul des Probabilités*. Masson & Cie, Paris, 1979.
- [25] I. Norros, "A storage model with self-similar input," *QUESTA*, **16**:387-396, 1994.
- [26] A. G. Pakes, "On the tails of waiting time distributions," *J. Appl. Prob.*, **12**:555-564, 1975.
- [27] M. Parulekar, A. M. Makowski, "Tail probabilities for a multiplexer with self-similar traffic," *Proc. of the IEEE Infocom'96 Conf.*, San Francisco, CA, Mar. 26-28, 1996, 1452-1459.
- [28] M. Parulekar, A. M. Makowski, "Tail probabilities for M/G/ ∞ input processes (I): Preliminary asymptotics." Preprint, University of Maryland, 1996.
- [29] M. Parulekar, A. M. Makowski, "M/G/ ∞ input processes: A versatile class of models for network traffic." Preprint, University of Maryland, 1996.
- [30] V. Paxson, S. Floyd, "Wide area traffic: The failure of Poisson modeling," *IEEE/ACM Trans. on Networking*, **3**:226-244, 1993.
- [31] A. N. Shiriyayev, *Probability*. Springer-Verlag, New York, 1984.
- [32] L. Takács, *Theory of queues*. Oxford University Press, New York, 1962.
- [33] J. L. Teugels, "The class of subexponential distributions," *Ann. Probability*, **3**:1000-1011, 1975.



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