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Zakariae Bouziane

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Zakariae Bouziane

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Abstract: The general Petri net reachability problem is shown to be decidable in double exponential space. The previous algorithm is non-primitive recursive.

Key-words: Petri Net, Reachability, Decidability, Polynomial

(Résumé : tsvp)

Unité de recherche INRIA Rennes
IRISA, Campus universitaire de Beaulieu, 35042 RENNES Cedex (France)
Téléphone : 02 99 84 71 00 - International : +33 2 99 84 71 00
Télécopie : 02 99 84 71 71 - International : +33 2 99 84 71 71

Un algorithme primitif récursif pour le problème de l'accessibilité dans les réseaux de Petri

Résumé : On montre que le problème d'accessibilité dans les réseaux de Petri est décidable en espace doublement exponentiel. L'algorithme existant avant est non primitif récursif.

Mots-clé : Réseau de Petri, Accessibilité, Décidabilité, Polynôme

1 Introduction.

Many aspects of the fundamental nature of computation are often studied via formal models, such as Turing machines, finite-state machines, and push-down automata. One formalism that has been used to model parallel computations is the Petri net [7, 8]. As a means of gaining a better understanding of the Petri net model, the decidability and computational complexity of typical automata theoretic problems concerning Petri nets have been examined. Examples of such problems include deadlock freedom, and liveness of the system. Solutions to these example problems proclaim, in a sense, the absence of "difficulties" for all states that are reachable in the system.

Another question is whether some arbitrary state can be reached from a fixed initial state. The later, the so-called general reachability problem, is of basic importance for many others. It is recursively equivalent to the liveness problem [2]. Moreover, a number of other problems in the representation of parallel and concurrent systems, in language generating systems, in algebra and in number theory can be shown to be effectively reducible or equivalent to the reachability problem.

Cardoza, Lipton, Mayr and Meyer [1, 6] have shown exponential space lower bound for the reachability problem, but the only known algorithm is non-primitive recursive [5, 4]. Even the decidability of this problem was an open question for many years.

2 Petri nets and polynomial equations

As usual, $\mathbb{N}$ denotes the set of nonnegative integers and $\mathbb{N}^m$ denotes the m-dimensional column vectors of natural numbers. For any integers $a, b \in \mathbb{N}$ such that $a \leq b$, the interval $[a, b]$ denotes the set $\{a, a+1, \ldots, b\}$. For any vectors $V_1, V_2 \in \mathbb{N}^m$ we write:

- $V_1 = V_2$ if and only if $V_1(i) = V_2(i)$, for every $i \in \{1, 2, \ldots, m\}$,

- $V_1 + V_2$ to denote the vector of $\mathbb{N}^m$ whose i^{th} component is $V_1(i) + V_2(i)$,
- $\sup(V_1, V_2)$ to denote the vector of $\mathbb{N}^m$ whose i^{th} component is $\max(V_1(i), V_2(i))$ and
- $\|V_1\|_\infty = \max\{V_1(i) \mid i \in [1, m]\}$.

A *Petri net* is a tuple $N = (P, T, F, M_0)$ where, $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of *places*, $T = \{t_1, t_2, \ldots, t_n\}$ is a finite set of *transitions*, $F : (P \times T) \cup (T \times P) \longrightarrow \mathbb{N}$ is a *flow function* [1], and $M_0 \in \mathbb{N}^m$ is an *initial marking.* In all this paper m will designate the number of places and n the number of transitions of the Petri net.

Let $N = (P, T, F, M_0)$ be a Petri net and $M_f \in \mathbb{N}^m$ be a marking. M_f is said to be *reachable* in N, and we note it $M_f \in \mathcal{RS}(N)$, if there exist a finite set of vectors $H_1, H_2, \ldots, H_k \in \mathbb{N}^m$ and a sequence of transitions $t_{i_1}, t_{i_2}, \ldots, t_{i_k} \in T$ such that

$$\left\{ \begin{array}{rcll} M_0 & = & H_1 + F(., t_{i_1}) & \\ H_j + F(t_{i_j}, .) & = & H_{j+1} + F(., t_{i_{j+1}}) & \text{for every } j \in [1, k-1] \\ H_k + F(t_{i_k}, .) & = & M_f & \end{array} \right. \tag{1}$$

Note that system (1) has a solution if and only if it has a solution such that, for every $j, j' \in [1, k]$, $j \neq j'$, we have $M_0 \neq H_j + F(., t_{i_j}) \neq H_{j'} + F(., t_{i_{j'}})$.

The **general Petri net reachability problem** is the problem of deciding for a given a Petri net $N = (P, T, F, M_0)$ and a marking $M_f \in \mathbb{N}^m$ whether $M_f \in \mathcal{RS}(N)$.

Our computation of an upper bound for the reachability problem is based on a reduction of this later to the problem of solving a polynomial equation. For that we need to define the polynomials associated to a Petri net. We first

[1] We write $F(., t)$ (resp. $F(t, .)$) to denote the m-dimensional column vector whose i^{th} component is $F(p_i, t)$ (resp. $F(t, p_i)$).

code the set of vectors into integers by the help of an *injective function* φ defined by $\varphi(V) = 2^{V(1)}.3^{V(2)}.5^{V(3)} \dots q^{V(m)}$ where q is the m^{th} prime number. Then, in order to simplify the notations, we put $\alpha_0 = \varphi(M_f)$, $\beta_0 = \varphi(M_0)$ and, $\alpha_i = \varphi(F(.,t_i))$ and $\beta_i = \varphi(F(t_i,.))$, for $i \in [1,n]$. Finally, let $\mathbf{I}[X]$ be the set of polynomials over one variable X and whose coefficients are in $\{0,1\}$. All the polynomials we will deal with in this paper are in $\mathbf{I}[X]$.

The reduction from the reachability problem to the problem of solving a polynomial system is given by the following lemma.

Lemma 2.1 *$M_f \in \mathcal{RS}(N)$ if and only if there exists a finite sequence of integers $h_{i,j} \in \mathbb{N}$ such that the following polynomial equation is verified*

$$X^{\alpha_0} + \sum_{i=1}^{n}\sum_{j=1}^{k_i} X^{h_{i,j}.\alpha_i} = X^{\beta_0} + \sum_{i=1}^{n}\sum_{j=1}^{k_i} X^{h_{i,j}.\beta_i} \qquad \in \mathbf{I}[X] \tag{2}$$

Proof. For every vectors $V_1, V_2, V_3 \in \mathbb{N}^m$, we have

$$[V_1 = V_2 + V_3] \Longleftrightarrow [\varphi(V_1) = \varphi(V_2).\varphi(V_3)]$$

Thus, for every vectors $V_1, V_2, V_3 \in \mathbb{N}^m$, we have

$$[V_1 = V_2 + V_3] \Longleftrightarrow [X^{\varphi(V_1)} = X^{\varphi(V_2).\varphi(V_3)}]$$

If we put $l_i = \varphi(H_i)$, for $i \in [1,k]$, then, from the reachability definition, we deduce that $M_f \in \mathcal{RS}(N)$ if and only if we have

$$\begin{cases} \alpha_0 & = & l_1.\beta_{i_1} \\ l_j.\alpha_{i_j} & = & l_{j+1}.\beta_{i_{j+1}} \quad \text{for } j \in [1, k-1] \\ l_k.\alpha_{i_k} & = & \beta_0 \end{cases}$$

Hence, $M_f \in \mathcal{RS}(N)$ if and only if we have

$$\begin{cases} X^{\alpha_0} & = & X^{l_1.\beta_{i_1}} \\ X^{l_j.\alpha_{i_j}} & = & X^{l_{j+1}.\beta_{i_{j+1}}} \quad \text{for } j \in [1, k-1] \\ X^{l_k.\alpha_{i_k}} & = & X^{\beta_0} \end{cases}$$

We can choose $\alpha_0 \neq l_j.\alpha_{i_j} \neq l_{j'}.\alpha_{i_{j'}}$, for every $j, j' \in [1, k]$ (This is equivalent to $M_0 \neq H_j + F(t_{i_j}, .) \neq H_{j'} + F(t_{i_{j'}}, .)$).

Thus, $M_f \in \mathcal{RS}(N)$ if and only if we have

$$X^{\alpha_0} + \sum_{j=1}^{k} X^{l_j.\alpha_{i_j}} = X^{\beta_0} + \sum_{j=1}^{k} X^{l_j.\beta_{i_j}} \qquad \in \mathbf{I}[X]$$

■

3 Solution of the polynomial equation

From lemma 2.1 we deduce that the reachability problem is equivalent to the problem of deciding whether system (2) has a solution. In this section we show that this later is decidable. Our proof consists in reducing system (2) to an equivalent system made of two equations (lemma 3.1). Then we show that the set of solutions to the first one can be effectively constructed (lemma 3.2). Finally, we construct this set and extract from it the solutions that also verify the second equation (lemma 3.3 and lemma 3.4).

Lemma 3.1 *Equation (2) has a solution if and only if there exist two finite sequences of integers $f_{i,j}, g_{i,j} \in \mathbb{N}$ such that the two following polynomial equations are verified*

$$X^{\alpha_0} + \sum_{i=1}^{n} \sum_{j=1}^{k_i} X^{f_{i,j}.\alpha_i} = X^{\beta_0} + \sum_{i=1}^{n} \sum_{j=1}^{k'_i} X^{g_{i,j}.\beta_i} \qquad \in \mathbf{I}[X] \tag{3}$$

$$\sum_{j=1}^{k_i}(X^{\beta_i})^{f_{i,j}.\alpha_i} = \sum_{j=1}^{k'_i}(X^{\alpha_i})^{g_{i,j}.\beta_i} \quad \textit{for every} \quad i \in [1,n] \tag{4}$$

Proof. Equation (4) is equivalent to the proposition

$$f_{i,j} = g_{i,j} \quad \text{for } i \in [1,n] \text{ and } j \in [1,k_i]$$

Thus, the intersection of equation (3) and equation (4) is equivalent to equation (2). ■

Let E_0 be the set of uplets $(P_1, \ldots, P_{2.n})$ such that, for every $i \in [1,n]$, $P_i \in \mathbf{I}[X^{\alpha_i}]$ and $P_{n+i} \in \mathbf{I}[X^{\beta_i}]$, and $X^{\alpha_0} + \sum_{i=1}^{n} P_i(X) = X^{\beta_0} + \sum_{i=n+1}^{2.n} P_i(X) \in \mathbf{I}[X]$ 2. E_0 is then the set of solutions to equation (3).

Let $l_0 = lcm\{\alpha_i, \beta_i \mid i \in [1,n]\}$ (lmc is the least common multiple) and $l_1 = \min\{\lambda.l_0 \mid \lambda \in \mathbb{N} \text{ and } \max(\alpha_0, \beta_0) < \lambda.l_0\}$.

Let E_1 be the set of uplets $(Q_1, \ldots, Q_{2.n}) \in E_0$ such that, for every $i \in [1, 2.n]$, we have $\deg(Q_i) < l_1$. E_1 is finite and we can write $E_1 = \{(Q_1^1, \ldots, Q_{2.n}^1), \ldots, (Q_1^h, \ldots, Q_{2.n}^h)\}$.

Let E_2 be the set of uplets $(R_1, \ldots, R_{2.n})$ such that for every $i \in [1,n]$, we have $R_i \in \mathbf{I}[X^{\alpha_i}]$, $R_{n+i} \in \mathbf{I}[X^{\beta_i}]$, $\max\{\deg(R_i), \deg(R_{n+i})\} < l_0$ and

$$\sum_{i=1}^{n} R_i(X) = \sum_{i=n+1}^{2.n} R_i(X)$$

E_2 is finite and we can write $E_2 = \{(R_1^1, \ldots, R_{2.n}^1), \ldots, (R_1^k, \ldots, R_{2.n}^k)\}$.

Lemma 3.2 *E_0 is equal to the set of uplets $(P_1, \ldots, P_{2.n})$ such that*

$$(P_1, \ldots, P_{2.n}) = (Q_1, \ldots, Q_{2.n}) + X^{l_1}. \sum_{i=1}^{k} .S_i(X).(R_1^i, \ldots, R_{2.n}^i)$$

for some $(Q_1,\ldots,Q_{2.n}) \in E_1$ *and* $S_1,\ldots,S_k \in \mathbf{I}[X]$ *such that* $\sum_{i=1}^{k} .S_i(X) = \sum_{i=0}^{a} X^{i.l_0}$, *for some* $a \in \mathbb{N}$.

Proof.

- For every $(Q_1,\ldots,Q_{2.n}) \in E_1$, for every $l \in [1,2.n]$ and for every monomial X^c in $Q_l(X)$ we have $0 \leq c < l_1$.
  For every $i \in [1,k]$, for every $l \in [1,2.n]$ and for every monomial X^c in $R_l^i(X)$ we have $0 \leq c < l_0$.
  Hence, for every $i \in [1,k]$, for every $l \in [1,2.n]$ and for every monomial X^c in $X^{l_1}.X^{b.l_0}.R_l^i(X)$ we have $l_1 + b.l_0 \leq c < l_1 + (b+1).l_0$.
  Consequently, $Q_l(X) + X^{l_1}.\sum_{i=1}^{k} .S_i(X).R_l^i(X) \in \mathbf{I}[X]$. Thus the sum

  $$(Q_1,\ldots,Q_{2.n}) + X^{l_1}.\sum_{i=1}^{k} .S_i(X).(R_1^i,\ldots,R_{2.n}^i)$$

  is in E_0.

- Let $(P_1,\ldots,P_{2.n}) \in E_0$ and let $d = \max\{\deg(P_j) \mid j \in [1,2.n]\}$. For every $j \in [1,2.n]$, let Q_j to be the largest sub-polynomial of P_j of degree less than l_1, and let R_j^l to be the largest sub-polynomial of P_j such that any monomial X^c of P_j verify $l_1 + (l-1).l_0 \leq c < l_1 + l.l_0$ (l is such that $l_1 + (l-1).l_0 \leq d$.

  The uplet $(Q_1,\ldots,Q_{2.n})$ is then in E_1.

  For $j \leq n$ we have $P_j \in \mathbf{I}[X^{\alpha_j}]$. Hence every monomial X^c of R_j^l verify

  $$l_1 + (l-1).l_0 \leq c = \lambda.\alpha_j < l_1 + l.l_0 \text{ , for some } \lambda \in \mathbb{N}$$

  Moreover, from the definition of l_0 and l_1, we deduce that $l_0 = \lambda_0.\alpha_j$ and $l_1 = \lambda_1.\alpha_j$, for some $\lambda_0, \lambda_1 \in \mathbb{N}$. Thus

  $$(\lambda_1 + (l-1).\lambda_0).\alpha_j \leq c = \lambda.\alpha_j < (\lambda_1 + l.\lambda_0).\alpha_j$$

Which means that

$$X^c = X^{l_1}.X^{(l-1).l_0}.X^{b.\alpha_j} \text{ for some } b.\alpha_j < l_0$$

In the case where $n < j \leq 2.n$, we use the same proof by replacing α_j by β_{j-n}.
Consequently, $(R_1^l, \ldots, R_{2.n}^l) \in E_1$. Thus $(P_1, \ldots, P_{2.n})$ can be written in the form of the equation in lemma 3.2. ■

Lemma 3.3 *Equation (2) has a solution if and only if there exist* $S_1, \ldots, S_k \in \mathbf{I}[X]$ *such that the two following systems are verified*

$$Q_l(2^{\beta_l}) + 2^{l_1.\beta_l}.\sum_{i=1}^{k} S_i(2^{\beta_l}).R_l^i(2^{\beta_l}) = Q_{n+l}(2^{\alpha_l}) + 2^{l_1.\alpha_l}.\sum_{i=1}^{k} S_i(2^{\alpha_l}).R_{n+l}^i(2^{\alpha_l}) \ , \forall l \in [1, n] \quad (5)$$

and

$$\sum_{i=1}^{k} S_i(X) = \sum_{i=0}^{a} X^{i.l_0} \quad (6)$$

for some $a \in \mathbb{N}$ *and* $(Q_1, \ldots, Q_{2.n}) \in E_1$.

Proof. From lemma 3.1 and lemma 3.2, we deduce that equation (2) has a solution if and only if there exist $S_1, \ldots, S_k \in \mathbf{I}[X]$ and $a \in \mathbb{N}$ such that $\sum_{i=1}^{k} S_i(X) = \sum_{i=0}^{a} X^{i.l_0}$, and

$$Q_l(X^{\beta_l}) + X^{l_1.\beta_l}.\sum_{i=1}^{k} S_i(X^{\beta_l}).R_l^i(X^{\beta_l}) = Q_{n+l}(X^{\alpha_l}) + X^{l_1.\alpha_l}.\sum_{i=1}^{k} S_i(X^{\alpha_l}).R_{n+l}^i(X^{\alpha_l}) \ , \forall l \in [1, n]$$

However, for $l \in [1, n]$, the polynomials $Q_l(X^{\beta_l}) + X^{l_1.\beta_l}.\sum_{i=1}^{k} S_i(X^{\beta_l}).R_l^i(X^{\beta_l})$ and $Q_{n+l}(X^{\alpha_l}) + X^{l_1.\alpha_l}.\sum_{i=1}^{k} S_i(X^{\alpha_l}).R_{n+l}^i(X^{\alpha_l})$ are in $\mathbf{I}[X]$. Hence, these two polynomials are equal if and only if they have the same value on integer 2. This ends the proof. ■

Lemma 3.4 *The system made of system (5) and system (6) has a solution if and only if it has a solution with* $a \leq 2^{2^{c.n.l_1}}$*, for some constant* $c \in \mathbb{N}$.

Proof. For $i \in [1,k]$, $l \in [1,n]$, let $A_{i,l} = 2^{l_1.\beta_l}.R_l^i(2^{\beta_l})$, $A_{i,n+l} = 2^{l_1.\alpha_l}.R_{n+l}^i(2^{\alpha_l})$, $A_{0,l} = Q_l(2^{\beta_l})$ and $A_{0,n+l} = Q_{n+l}(2^{\alpha_l})$. Let $\Sigma = \{Z \in \mathbb{N}^k \mid \sum_{i=1}^k Z(i) = 1\}$. Hence, equation (5) is equivalent to the linear diophantine system

$$A_{0,l} + \sum_{i=1}^{k} A_{i,l}.Y_{i,l} = A_{0,n+l} + \sum_{i=1}^{k} A_{i,n+l}.Y_{i,n+l} \ , \forall l \in [1,n] \tag{7}$$

and equation (6) is equivalent to the system

$$\begin{cases} Y_{i,l} = Z_0(i) + 2^{l_0.\beta_l}.Z_1(i) + \ldots + 2^{a.l_0.\beta_l}.Z_a(i) & , \forall i \in [1,k] \ , \forall l \in [1,n] \\ Y_{i,n+l} = Z_0(i) + 2^{l_0.\alpha_l}.Z_1(i) + \ldots + 2^{a.l_0.\alpha_l}.Z_a(i) & , \forall i \in [1,k] \ , \forall l \in [1,n] \end{cases} \tag{8}$$

for some $Z_0, \ldots, Z_a \in \Sigma$.

However, for every $l \in [1,n]$, the set of solution

$$\mathcal{S}_l = \{(Y_{1,l}, \ldots, Y_{k,l}, Y_{1,n+l}, \ldots, Y_{k,n+l}) \in \mathbb{N}^{2.k} \mid A_{0,l} + \sum_{i=1}^{k} A_{i,l}.Y_{i,l} = A_{0,n+l} + \sum_{i=1}^{k} A_{i,n+l}.Y_{i,n+l}\}$$

is semi-linear, and can be written in the form

$$\mathcal{S}_l = \{V_l + \sum_{j=1}^{h} \lambda_j.W_{j,l} \mid V_l \in \mathcal{S}_{l,0} \ , \ W_{1,l}, \ldots, W_{h,l} \in \mathcal{S}_{l,1} \ , \ \lambda_1, \ldots, \lambda_h \in \mathbb{N}\}$$

where $\mathcal{S}_{l,0} \subset \mathbb{N}^{2.k}$ is the set of *minimal solutions* of the l^{th} equation in system (7) and $\mathcal{S}_{l,1} \subset \mathbb{N}^{2.k}$ is the set of *minimal solutions* of its homogeneous part $\sum_{i=1}^k A_{i,l}.Y_{i,l} = \sum_{i=1}^k A_{i,n+l}.Y_{i,n+l}$.

We only want the solutions $Y_l = (Y_{1,l}, \ldots, Y_{k,l}, Y_{1,n+l}, \ldots, Y_{k,n+l}) \in \mathcal{S}_l$ that verify system (8). Let $Y_l^0 = V_{l,0} + \sum_{j=1}^h \lambda_j.W_{j,l}$ be such a solution. It can be written in the form

$$Y_l^0 = V_{l,0} + \sum_{j=1}^{h} \lambda_{j,1}.W_{j,l} + 2^{l_0.\gamma_l}.\sum_{j=1}^{h} \lambda_{j,2}.W_{j,l}$$

where, $\gamma_l = \max\{\alpha_l, \beta_l\}$ and $0 \leq \lambda_{j,1} < 2^{l_0.\gamma_l}$.

Thus, for every $i \in [1,k]$, we have

$$\begin{cases} V_{l,0}(i) + \sum_{j=1}^{h} \lambda_{j,1}.W_{j,l}(i) & = & Z_0(i) + 2^{l_0.\beta_l}.V'_{l,0}(i) \\ V_{l,0}(k+i) + \sum_{j=1}^{h} \lambda_{j,1}.W_{j,l}(k+i) & = & Z_0(i) + 2^{l_0.\alpha_l}.V'_{l,0}(k+i) \end{cases}$$

for some $V'_{l,0} \in \mathbb{N}^{2.k}$. Let $\delta_l = \min\{\alpha_l, \beta_l\}$ and let $V_{l,1} \in \mathbb{N}^{2.k}$, such that, for every $i \in [1,k]$, we have $V_{l,1}(i) = \frac{2^{l_0.\beta_l}}{2^{l_0.\delta_l}}.V'_{l,0}(i)$ and $V_{l,1}(k+i) = \frac{2^{l_0.\alpha_l}}{2^{l_0.\delta_l}}.V'_{l,0}(k+i)$. Hence, $Y_l^1 = V_{l,1} + \frac{2^{l_0.\gamma_l}}{2^{l_0.\delta_l}}.\sum_{j=1}^{h} \lambda_{j,2}.W_{j,l}$ verifies system (8).

By repeating the previous process we will obtain a sequence of vectors $V_{l,0}$, $V_{l,1}$, ..., $V_{l,a}$ in $\mathbb{N}^{2.k}$. Moreover, we have

$$\|V_{l,j+1}\|_\infty \leq \frac{1}{2^{l_0.\delta_0}}.[\|V_{l,j}\|_\infty + h.2^{l_0.\gamma_l}.\max\{\|W_{j,l}\|_\infty \mid j \in [1,h]\}]$$

Hence,

$$\|V_{l,j+1}\|_\infty \leq \max(\|V_{l,0}\|_\infty, h.2^{l_0.\gamma_l}.\max\{\|W_{j,l}\|_\infty \mid j \in [1,h]\}).\frac{2^{l_0.\delta_l}}{2^{l_0.\delta_l} - 1}$$

From [3], we deduce that, for every $W_{j,l} \in \mathcal{S}_{l,1}$ and for every $V_{l,0} \in \mathcal{S}_{l,0}$, we have

$$\begin{cases} \|W_{j,l}\|_\infty & \leq & \max\{A_{i,l}, A_{i,n+l} \mid i \in [1,k]\} & \leq & 2^{\gamma_l.(l_0+l_1)} \\ \|V_{l,0}\|_\infty & \leq & 2.\max\{A_{i,l}, A_{i,n+l} \mid i \in [0,k]\} & \leq & 2^{\gamma_l.(l_0+l_1)+1} \end{cases}$$

Hence, the number h of elements in $\mathcal{S}_{l,1}$ verifies

$$h \leq 2^{2.k.\gamma_l.(l_0+l_1)}$$

However, the number k of elements in E_2 verifies

$$k \leq 2^{n.l_0}$$

Consequently,

$$h \leq 2^{2.\gamma_l.(l_0+l_1).2^{n.l_0}} \leq 2^{2^{c_0.n.l_1}}$$

for some constant $c \in \mathbb{N}$.
Which means that

$$\|V_{l,j}\|_\infty \leq 2^{2^{c_1.n.l_1}}$$

for some constant c.

Thus, the elements of $\mathcal{S}_l$ that verify equation (8) can be generated by a union of $|\mathcal{S}_{l,0}|$ automaton $\mathcal{A}_l = \bigcup\{\mathcal{A}_l(V_{l,i}) \mid V_{l,i} \in \mathcal{S}_{l,0}\}$. Each automata $\mathcal{A}_l(V_{l,i})$ is over the alphabet Σ and has nodes in the set of vectors

$$\{V \in \mathbb{N}^k \mid \|V\|_\infty \leq 2^{2^{c_1.n.l_1}}\}$$

the initial marking is $V_{l,i}$ the final marking is $\mathbf{0} = (0,\ldots,0)$ and there is an arc from V_1 to V_2 labeled Z if and only if

$$\begin{cases} V_1(i) + \sum_{j=1}^{h} \lambda_j.W_{j,l}(i) = Z(i) + 2^{l_0.\delta_l}.V_2(i) \\ V_1(k+i) + \sum_{j=1}^{h} \lambda_j.W_{j,l}(k+i) = Z(i) + 2^{l_0.\delta_l}.V_2(k+i) \end{cases}$$

for $i \in [1,k]$ and $\lambda_1,\ldots,\lambda_h < 2^{l_0.\gamma_l}$.

Finally, the set of solutions of equation (7) that verify equation (8) is completely determined by the intersection

$$\mathcal{L} = \bigcap_{l=1}^{n} \mathcal{L}(\mathcal{A}_l)$$

where $\mathcal{L}(\mathcal{A}_l)$ is the language recognized by $\mathcal{A}_l$.

The number of nodes of every automaton $\mathcal{A}_l(V_{l,j})$ is less than $2^{2^{c_2.n.l_1}}$, for some constant c_2. Hence, $\mathcal{L} \neq \emptyset$ if and only if it contains a sequence of length no longer than $2^{n.2^{c_2.n.l_1}} \leq 2^{2^{c_3.n.l_1}}$, for some constant $c_3 \in \mathbb{N}$. Thus, the system made of system (7) and system (8) has a solution if and only if it has a solution with $a < 2^{2^{c.n.l_1}}$, for some constant $c \in \mathbb{N}$. ■

4 The general Petri net reachability problem is primitive recursive

Now, we are ready to compute a primitive recursive algorithm for the general Petri net reachability problem.

Theorem 4.1 *There exist a constant c such that, for every Petri net $N = (P, T, F, M_0)$ and every marking $M_f \in \mathbb{N}^m$, we have*

$M_f \in \mathcal{RS}(N)$ if and only if there exist $H_1, H_2, \ldots, H_k \in \mathbb{N}^m$ and there exist $t_{i_1}, t_{i_2}, \ldots, t_{i_k} \in T$ such that

$$\begin{cases} M_0 & = & H_1 + F(., t_{i_1}) \\ H_j + F(t_{i_j}, .) & = & H_{j+1} + F(., t_{i_{j+1}}) \quad \text{for every } j \in [1, k-1] \\ M_f & = & H_k + F(t_{i_k}, .) \end{cases}$$

and for every $j \in [1, k]$ we have

$$\varphi(H_j + F(t_{i_j}, .)) \leq 2^{2^{c.n.l}}$$

where $l = \min\{\lambda.l_0 \mid \lambda \in \mathbb{N}\ ,\ \max\{\alpha_0, \beta_0\} < \lambda.l_0\}$, $\alpha_0 = \varphi(M_0)$, $\beta_0 = \varphi(M_f)$, and $l_0 = \varphi(\sup\{F(t_i, .), F(., t_i) \mid i \in [1, n]\})$,

Proof. This theorem is a consequence of the previous section. ∎

5 Open problems

- Reduction of the gap between our upper bound and Cardoza, Lipton, Mayr and Meyer's bound.
- Complexity of :
  - The *liveness* problem.
  - The *regularity* and the *context-freedom* problems.
  - The *semi-linearity* problem.

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Unit´e de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unit´e de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unit´e de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unit´e de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unit´e de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

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