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*Analytic Expansions of  $(\max, +)$  Lyapunov  
Exponents*

François BACCELLI and Dohy HONG

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THÈME 1



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## Analytic Expansions of $(\max, +)$ Lyapunov Exponents

François BACCELLI\* and Dohy HONG†

Thème 1 — Réseaux et systèmes  
Projet Mistral

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**Abstract:** We give an explicit analytic series expansion of the  $(\max, +)$ -Lyapunov exponent  $\gamma(p)$  of a sequence of independent and identically distributed random matrices in this algebra, generated via a Bernoulli scheme depending on a small parameter  $p$ . A key assumption is that one of the matrices has a unique eigenvector. This allows us to use a representation of this exponent as the mean value of a certain random variable, and then a discrete analogue of the so-called light-traffic perturbation formulas to derive the expansion. We show that it is analytic under a simple condition on  $p$ . This also provides a closed form expression for all derivatives of  $\gamma(p)$  at  $p = 0$  and approximations of  $\gamma(p)$  of any order, together with an error estimate for finite order Taylor approximations. Several extensions of this are discussed, including expansions of multinomial schemes depending on small parameters  $(p_1, \dots, p_m)$  and expansions for exponents associated with iterates of a class of random operators which includes the class of so called topical operators. Several examples pertaining to computer and communication sciences are investigated: timed event graphs, resource sharing models and heaps models.

**Key-words:** Taylor series, Lyapunov exponents,  $(\max, +)$  semiring, strong coupling, renovating events, stationary state variables, analyticity, vectorial recurrence relation, network modeling, stochastic Petri nets. *AMS 1991 subject classifications* : Primary 41A58, 34D08, 15A52, 15A18; Secondary 60K05, 60C05, 32D05, 16A78, 41A63.

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# Développements Analytiques d'Exposants de Lyapounov dans l'Algèbre $(\max, +)$

**Résumé :** Nous donnons un développement analytique explicite de l'exposant de Lyapounov  $\gamma(p)$  d'une suite de matrices aléatoires i.i.d. dans le semi-anneau  $(\max, +)$ , tirées selon un mécanisme de Bernoulli dépendant d'un petit paramètre  $p$ . Une hypothèse clef est que l'une des matrices possède un unique vecteur propre. Cette hypothèse nous permet d'utiliser une représentation de  $\gamma(p)$  comme la moyenne d'une variable aléatoire, puis un analogue discret des méthodes de perturbations pour obtenir ce développement. Nous montrons que ce dernier est analytique sous certaines conditions sur  $p$ . Nous donnons aussi une représentation explicite de toutes les dérivées de  $\gamma(p)$  en  $p = 0$ , ainsi que des approximations de Taylor de tous ordres, accompagnées de bornes sur les restes. Nous proposons plusieurs extensions, notamment au cas multinomial et au cas d'itérées d'opérateurs aléatoires non-expansifs. Plusieurs exemples sont étudiés en relation avec des problèmes de modélisation en informatique et dans le domaine des communications: graphes d'événements, modèles de partage de ressources, et modèles d'empilement de pièces.

**Mots-clés :** développement de Taylor, exposant de Lyapounov, algèbre  $(\max, +)$ , couplage fort, événement de renouvellement, régime stationnaire, analyticit , relation de r currence lin aire vectorielle, mod lisation de r seaux, r seau de P tri stochastique. *classification AMS 1991* : Primaire 41A58, 34D08, 15A52, 15A18; Secondaire 60K05, 60C05, 32D05, 16A78, 41A63.

## 1 Introduction

It is well known that under mild conditions, the Perron Frobenius eigenvalue of a non-negative matrix, the parameters of which depend analytically on a parameter  $z$ , also depends analytically on this parameter. This result of the conventional algebra cannot be extended to the  $(\max, +)$  algebra, since the Perron Frobenius eigenvalue, which is given by the maximal cycle mean formula [1], is then the maximum of a finite family of analytic functions of  $z$ , which is not analytic in  $z$  in general.

In the conventional algebra, there are also several well known results on the analyticity of the Lyapunov exponent of i.i.d. matrices, the law of which depends on a parameter  $p$  ([13], [14], [15]). The focus of the present paper is to investigate the analogue of this second type of questions in the  $(\max, +)$  algebra.

Using a simple class of random matrices sampled from a finite set, we will show that, under technical conditions to be specified later, not only the associated  $(\max, +)$ -Lyapunov exponent depends analytically in the probabilities used for the sampling, but also that a constructive approximation scheme can be given, based on a closed form representation of the coefficients of all orders of the analytic expansion, and on error estimates for finite order Taylor approximations. For  $p$  small enough, the error bound decreases geometrically to 0. So for all given intervals, there exists a finite algorithm allowing one to decide whether  $\gamma(p)$  is in this interval. This situation differs significantly with the result of [16] that conventional Lyapunov exponents are *not algorithmically approximable*.

The computational complexity of the computations required to obtain a Taylor expansion of order  $n$  is not a polynomial function of  $n$ . Nevertheless, we will give several examples where this can be used to derive expansions of practical use.

The paper is structured as follows.

In Section 2, we briefly recall tools and results which we will need to state and analyze the problem.

In Section 3, we consider the series expansion of the  $(\max, +)$ -Lyapunov exponent of a sequence of i.i.d. random matrices, where each matrix is sampled among two possible values using a Bernoulli scheme with parameter  $p$ . The main theorem is first given under certain restrictive assumptions (H1) (H2) (H3) defined in Section 3.1 and is illustrated through various examples. All proofs are gathered in §6.

In Section 4, we present three extensions of the main theorem: the multinomial case is considered in §4.1 whereas §4.2 focuses on the weakening of Assumption (H2). The extension of this class of results to iterates of random operators (which are not necessarily linear operators in a semi-ring, as it is the case for the  $(\max, +)$  setting considered above) is given in §4.3 with an example of  $(\min, \max, +)$  system. in §4.4. A key property for such an operator extension is the *finite range coupling of some pattern* of the operators. Section 4.5 summarizes these three extensions into a generic theorem. An example of application to heaps models is given in §4.6.

Section 5 focuses on the interpretation of the results in terms of perturbation type formulas.

The proofs of the main theorems are concentrated in §6.

Section 7 gives further expansions covering some cases with continuous distributions. It also contains comments on the relationship holding between different expansions which may be proposed for the same exponent.

Finally, Section 8 focuses on the comparison with other approaches. In §8.1, the merits of direct simulations are discussed and compared to those of the series expansion. In §8.2,

the regenerative theory approach is shown to allow one to derive the same type of analytical results in certain particular cases.

## 2 Preliminaries on $(\max, +)$ –Lyapunov exponents

### 2.1 Algebraic framework and basic spectral theorems

Most of this paper bears on product of matrices in the so called  $(\max, +)$  algebra, namely over the semi-field  $\mathbb{R}_{max} = \mathbb{R} \cup \{\varepsilon\}$  where  $\varepsilon = -\infty$ , endowed with an addition denoted  $\oplus$ , which is the max operation and with a product, denoted  $\otimes$ , which is the sum. The element  $\varepsilon$  is the neutral element of this semi-field.

We shall denote  $\mathbb{R}_{max}^d$  the set of vectors of dimension  $d$  and  $\mathbb{R}_{max}^{d \times d}$  the set of square matrices of dimension  $d \times d$  over this semi-field. The set  $\mathbb{R}_{max}^{d \times d}$  is endowed with two operations, also denoted  $\oplus$  and  $\otimes$  and defined by

$$(A \otimes B)_{ij} = \bigoplus_{1 \leq k \leq d} A_{ik} \otimes B_{kj}, \quad (A \oplus B)_{ij} = A_{ij} \oplus B_{ij}.$$

$(\mathbb{R}_{max}^{d \times d}, \oplus, \otimes)$  is a semi-ring. The  $n^{\text{th}}$  power of matrix  $A$ , denoted  $A^{\otimes n}$  or  $A^n$ , is to be understood in the  $(\max, +)$  sense i.e.  $A^n = A \otimes \dots \otimes A$ ,  $n$  times. Note that if  $d = 1$ ,  $a^{\otimes n} = na$ :

The following notations and definitions will be used throughout the paper: for all  $Z \in \mathbb{R}_{max}^d$  and  $M \in \mathbb{R}_{max}^{d \times d}$ ,

$$\begin{aligned} \bullet \|Z\|_{\infty} &= \bigotimes_{1 \leq i \leq d} Z_i & \bullet \|M\|_{\infty} &= \bigotimes_{1 \leq i, j \leq d} M_{ij} & \bullet \|Z\| &= \bigotimes_{\substack{1 \leq i \leq d \\ Z_i > \varepsilon}} |Z_i| \\ \bullet \|M\| &= \bigotimes_{\substack{1 \leq i, j \leq d \\ M_{ij} > \varepsilon}} |M_{ij}| & \bullet \|Z\|_{\mathcal{D}} &= \max_{1 \leq i \leq d} Z_i - \min_{1 \leq i \leq d} Z_i. \end{aligned}$$

For  $Z \in \mathbb{R}^d$ , we shall denote  $\bar{Z}$  the equivalence class of  $Z$  for the (colinearity) equivalence relation  $Y \equiv Z$  iff  $Y = Z \otimes \alpha$ , i.e. for all  $i = 1, \dots, d$ ,  $Y_i = Z_i + \alpha$ , for some scalar  $\alpha$ .

By definition, a matrix  $A \in \mathbb{R}_{max}^{d \times d}$  is irreducible if for all  $i$  and  $j$ , there exists a sequence  $i = i_0, i_1, \dots, i_n = j$ , such that  $A_{i_l, i_{l+1}} > \varepsilon$  for all  $l$ .

A key result concerning irreducible matrices is the following theorem:

**Result 1 (cyclicity theorem for deterministic matrices [1])** *For each irreducible matrix  $A$ , there exist uniquely defined integers  $c(A)$ ,  $\sigma(A)$  and a uniquely defined real number  $\gamma(A)$ , such that for all  $n \geq c(A)$  :*

$$A^{\otimes [n + \sigma(A)]} = (\gamma(A))^{\otimes \sigma(A)} \otimes A^{\otimes n}.$$

- $\sigma(A)$  is the cyclicity of matrix  $A$ ;
- $\gamma(A)$  is its eigenvalue;
- $c(A)$  is its coupling time :

$$c(A) = \inf \{ i : \forall n \geq i, A^{n+1} = \gamma(A) \otimes A^n \}.$$

In particular if  $A$  has cyclicity 1 and has a unique eigenvector  $X_o$ , then for all  $n \geq c(A)$  and for all vectors  $X$  in  $\mathbb{R}^d$ :

$$A^{\otimes(n+1)} = \gamma(A) \otimes A^{\otimes n} \quad \text{and} \quad A^{\otimes n} \otimes X = \alpha \otimes X_o,$$

where  $\alpha$  is a scalar which depends on  $X$  and  $n$ , so that  $\overline{A^{\otimes n} \otimes X} = \overline{X_o}$ .

An irreducible matrix always has a single eigenvalue, but it does not always have a unique eigenvector. A sufficient condition for this last uniqueness property to hold is that the critical graph of  $A$  has a single Strongly Connected Subgraph, with a CYClicity equal to 1 (see [1] for the definition of the critical graph and for that of its cyclicity). This property will be referred to as *scs1-cyc1* below.

## 2.2 Stochastic setting and Lyapunov exponents

Let some probability space be given on which all random variables introduced below are defined. A random element of  $\mathbb{R}_{max}^d$  or of  $\mathbb{R}_{max}^{d \times d}$  will be said to have fixed support if each of its entries is either a.s. equal to  $\varepsilon$  or a.s. non equal to  $\varepsilon$ . In this case, it will be said to be integrable if in addition each entry non equal to  $\varepsilon$  is integrable. Note that the definition of irreducibility can directly be extended to a random matrix with fixed support.

The general setting of the paper will be that of a given sequence of random matrices of  $\mathbb{R}_{max}^{d \times d}$ , say  $\{A(n)\}$ , and of the sequence of random vectors  $X_n \in \mathbb{R}_{max}^d$  defined by the recurrence relation

$$X_{n+1} = A(n) \otimes X_n, \quad n \geq 0 \tag{1}$$

and by the initial condition  $X_0 \in \mathbb{R}^d$ .

Here are two general results on this type of sequences:

**Result 2 (Lyapunov exponents via subadditivity [7], [1], [12])** *Assume that  $\{A(n)\}$  is a stationary and ergodic sequence of random matrices of  $\mathbb{R}_{max}^{d \times d}$ , and that  $A(0)$  has fixed support and is irreducible and integrable. Then the following limits exist regardless of the initial condition:*

$$\lim_{n \rightarrow +\infty} \frac{X_n}{n} = \lim_{n \rightarrow +\infty} \mathbb{E} \left( \frac{X_n}{n} \right) = \begin{bmatrix} \gamma \\ \vdots \\ \gamma \end{bmatrix} = \Gamma. \tag{2}$$

The constant  $\gamma$  is referred to as the  $(\max, +)$ -Lyapunov exponent of the sequence of random matrices  $\{A(n)\}$ .

The proof of this result is in two steps: using first the fact that for all  $n > m > l$ ,

$$\|A(n) \otimes \cdots \otimes A(l)\|_\infty \leq \|A(n) \otimes \cdots \otimes A(m)\|_\infty \otimes \|A(m+1) \otimes \cdots \otimes A(l)\|_\infty,$$

it follows from the subadditive ergodic theorem that a.s.

$$\lim_{n \rightarrow +\infty} \frac{\|A(n) \otimes \cdots \otimes A(0)\|_\infty}{n} = \gamma,$$

for some constant  $\gamma$ . The second step consists in showing that all coordinates of  $X_n/n$  have the same limit  $\gamma$ , regardless of the initial condition, which follows from the irreducibility assumption.



**Result 3 (strong coupling [12], 6.8)** *Assume that  $\{A(n)\}$  is an i.i.d. sequence of random matrices, independent of  $X_0$ , and that  $A(0)$  takes its values in a finite set  $\{A_l, l \in \mathcal{L}\}$  of irreducible matrices of  $\mathbb{R}_{max}^{d \times d}$ , where each element of the set has a positive probability of occurrence.*

*If there exists a scs1-cyc1 pattern in  $\{A_l, l \in \mathcal{L}\}$ , namely a product  $A_{l_1} \otimes \dots \otimes A_{l_q}$  of elements of this set, which is irreducible and scs1-cyc1, then  $\{\overline{X}_n\}$  converges with strong coupling to a unique stationary sequence. In particular, there exists a unique random equivalence class  $\overline{X}(\omega)$  such that for all deterministic initial conditions  $X_0$ ,*

- *the law of  $\{\overline{X}_n\}$  converges in total variation to that of  $\overline{X}$ ;*
- *for a.s. all  $\omega$ , there exists  $N(\omega) < \infty$  s.t.*

$$\forall n \geq N(\omega), \overline{A(0, \omega) \otimes A(-1, \omega) \otimes \dots \otimes A(-n, \omega) \otimes X_0} = \overline{X}(\omega). \quad (3)$$

This result, which is stated under much more general assumptions (stationary and ergodic) in [12], is essentially based on Borovkov's renovating events theorem (see [4], where the definition of strong coupling may also be found).

Result 3 allows for another representation of the Lyapunov exponent, which will be crucial in this paper: from (3), under the above assumptions,

$$\exists \lim_{n \rightarrow \infty} A(1) \otimes \dots \otimes A(-n) \otimes X_0 - A(0) \otimes \dots \otimes A(-n) \otimes X_0 = \Delta, \quad a.s.$$

where

$$\Delta = A(1) \otimes X - X \quad (4)$$

is a finite random variable. Therefore, if  $|A(1) \otimes Z - Z|$  is uniformly bounded by an integrable random variable (this is a restrictive hypothesis which we shall partially relax later; see Remark 14 in §6.1.2),

$$\begin{aligned} & \exists \lim_{n \rightarrow +\infty} \mathbb{E} [X_{n+2} - X_{n+1}] \\ &= \lim_{n \rightarrow +\infty} \mathbb{E} [A(1) \otimes \dots \otimes A(-n) \otimes X_0 - A(0) \otimes \dots \otimes A(-n) \otimes X_0] \\ &= \mathbb{E} \left[ \lim_{n \rightarrow +\infty} A(1) \otimes \dots \otimes A(-n) \otimes X_0 - A(0) \otimes \dots \otimes A(-n) \otimes X_0 \right] \\ &= \mathbb{E} [\Delta], \end{aligned}$$

where we used the dominated convergence theorem to get the second equality. Using now Result 2 and a Cesaro averaging argument, we get that

$$\lim_{n \rightarrow +\infty} \mathbb{E} [X_{n+2} - X_{n+1}] = \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \frac{X_n}{n} \right] = \Gamma,$$

where  $\Gamma$  is the Lyapunov exponent of  $\{A(n)\}$ . Therefore under the above assumptions, we also have the following representation:

$$\Gamma = \mathbb{E} [\Delta] = \lim_{n \rightarrow +\infty} \mathbb{E} [X_{n+1} - X_n]. \quad (5)$$

### 3 Bernoulli case

#### 3.1 Assumptions and main results for the Bernoulli case

The setting of this section is the following:  $\{A(n)\}$  is an i.i.d. sequence of matrices of  $\mathbf{R}_{\max}^{d \times d}$ , and  $A(0)$  takes its value in the set  $\{A, A'\}$ : for all  $n$ ,

$$A(n) = \begin{cases} A & \text{with probability } 1 - p \\ A' & \text{with probability } p. \end{cases}$$

In this section, the assumptions on  $A$  and  $A'$  are the following:

(H1) : *The matrix  $A$  is irreducible.*

(H2) : *The matrix  $A$  is scs1-cyc1.*

(H3) : *The matrix  $A'$  has at least one entry different from  $\varepsilon$  on each line.*

If in addition  $A'$  is irreducible, this setting is then a special case of that of Result 3. If  $A$  and  $A'$  have the same support, it is also a special case of that of Result 2. As we shall see below (Lemma 1 in §6), under (H1), (H2) and (H3), both Result 2 and Result 3 hold, so that the Lyapunov exponent

$$\Gamma = \Gamma(p) = \begin{bmatrix} \gamma(p) \\ \vdots \\ \gamma(p) \end{bmatrix}$$

of the above Bernoulli scheme is well defined via Result 2. The main result in this case is the following theorem.

**Theorem 1 1.** *Under assumptions (H1), (H2) and (H3),  $\gamma(p)$  is analytic at point 0, with a radius of convergence larger than or equal to  $\frac{1}{2c}$ , where  $c$  denotes the coupling time of  $A$ :  $c = c(A)$ . The coefficients of the analytic expansion are given by the following formula, where  $X_o$  denotes the unique eigenvector of  $A$ :*

$$\begin{aligned} \frac{1}{l!} \frac{d^l}{dp^l} [\Gamma(p)]_{p=0} &= \begin{bmatrix} \pi(l) \\ \vdots \\ \pi(l) \end{bmatrix} \\ &= (-1)^l \left\{ \binom{c}{l-1} X_o + \binom{c+1}{l} \Gamma(0) \right\} + \sum_{k=1}^l \\ &\left\{ (-1)^{l-k} \sum_{j_1, \dots, j_{k-1}=0}^{c-1} \binom{2c+j_1+\dots+j_{k-1}}{l-k} A^c \otimes A' \otimes A^{j_1} \otimes A' \dots A^{j_{k-1}} \otimes A' \otimes X_o \right\}. \end{aligned} \quad (6)$$

2. *For all  $p \in [0, \delta]$ , with  $\delta < \frac{1}{2c}$ , the error term in the Taylor expansion of  $\gamma(p)$  of order  $l$  is bounded from above by:*

$$\frac{D(2c\delta)^{l+1}[1 + l(1 - 2c\delta)]}{(1 - 2c\delta)^2},$$

where  $D = [2(\|A\| \vee \|A'\|) + \|X_o\|_{\mathcal{D}}](c + 1)$ .

In (6), we adopted the following conventions:  $\binom{n}{p} = 0$  if  $p > n$  and for  $k = 1$  :

$$\begin{aligned} & \sum_{j_1, \dots, j_{k-1}=0}^{c-1} \binom{2c+j_1+\dots+j_{k-1}}{l-k} A^c \otimes A' \otimes A^{j_1} \otimes A' \dots A^{j_{k-1}} \otimes A' \otimes X_o \\ &= \binom{2c}{l-1} A^c \otimes A' \otimes X_o. \end{aligned}$$

The first three coefficients of the expansion of  $\Gamma(p)$  are the following:

$$\begin{aligned} \begin{bmatrix} \pi(0) \\ \vdots \\ \pi(0) \end{bmatrix} &= \begin{bmatrix} \gamma(0) \\ \vdots \\ \gamma(0) \end{bmatrix} = \Gamma(0), \\ \begin{bmatrix} \pi(1) \\ \vdots \\ \pi(1) \end{bmatrix} &= A^c \otimes A' \otimes X_o - X_o - (c+1)\Gamma(0), \\ \begin{bmatrix} \pi(2) \\ \vdots \\ \pi(2) \end{bmatrix} &= \sum_{j=0}^{c-1} \{A^c \otimes A' \otimes A^j \otimes A' \otimes X_o\} \\ &\quad - 2cA^c \otimes A' \otimes X_o + cX_o + \binom{c+1}{2} \Gamma(0). \end{aligned}$$

**Remark 1** An estimate of the computational cost to evaluate  $\pi(l)$  using Equation 6 is :  $d^2lc^l$ . For more details see §6.1.4.

**Remark 2** Note that due to Result 1, the generic term in (6) can be rewritten as follows :

$$A^c \otimes A' \otimes A^{j_1} \otimes A' \dots A^{j_{k-1}} \otimes A' \otimes X_o = \lambda(j_1, \dots, j_{k-1}) \otimes X_o,$$

where  $\lambda(j_1, \dots, j_{k-1})$  is a scalar. We know that (6) is not modified if we replace  $X_o$  by  $X_o \otimes \alpha$  for some scalar  $\alpha$ . Since  $A \otimes (X_o \otimes \alpha) = (A \otimes X_o) \otimes \alpha$ , this property implies that we can rewrite Equation (6) as follow :

$$\begin{aligned} & \frac{1}{l!} \frac{d^l}{dp^l} [\Gamma(p)]_{p=0} \\ &= \sum_{k=1}^l \left\{ (-1)^{l-k} \sum_{j_1, \dots, j_{k-1}=0}^{c-1} \binom{2c+j_1+\dots+j_{k-1}}{l-k} \lambda(j_1, \dots, j_{k-1}) \right\} \\ &\quad + (-1)^l \binom{c+1}{l} \gamma(0). \end{aligned}$$

**Remark 3** Note also that in (6), it is enough to consider the indices  $k$  larger than or equal to  $\alpha$ , with  $\alpha$  the integer part of  $\left\lfloor \frac{l-2}{c} \right\rfloor$ . It is easy to see that for  $k < \alpha$ , the binomial coefficients in the sum are equal to zero.

The proof of Theorem 1 is given in Section 6.1.

## 3.2 Examples

### 3.2.1 A simple case of closed cyclic Jackson network

We consider a closed Jackson network with 2 single server FIFO stations. Assume that there

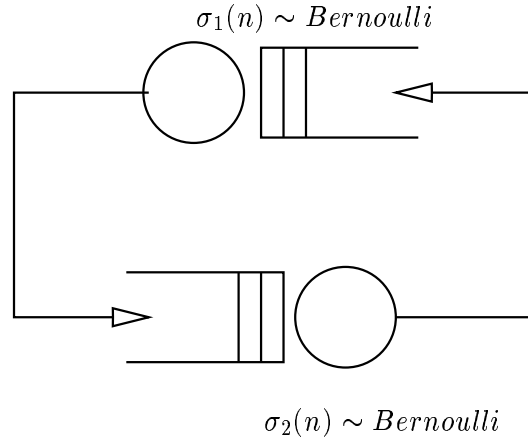


Figure 1: A cyclic Jackson network with 2 stations.

are exactly 2 customers and that there is initially one customer in each station. All service times are independent with a Bernoulli distribution :

$$\begin{aligned} \forall n, \sigma_i(n) &= \sigma_i \text{ with probability } 1 - p. \\ &= \sigma'_i \text{ with probability } p. \end{aligned}$$

The evolution of such a model can be captured via the following  $(\max, +)$  recurrence :

$$\begin{aligned} X_{n+1} &= A(n) \otimes X_n, \text{ where} \\ A_n &= \begin{pmatrix} \sigma_1(n) & \sigma_1(n) \\ \sigma_2(n) & \sigma_2(n) \end{pmatrix}. \end{aligned}$$

Here, we take for state vector  $X_n = (X_n^1, X_n^2)^t$ , where  $X_n^i$  is the epoch of the  $n$ -th departure from station  $i$ , and we take for initial condition  $X_0 = (0, 0)$ , which corresponds to the case when each station starts its very first service when the evolution begins.

We can evaluate the Lyapunov exponent characterizing its stationary throughput using Theorem 1. Assume  $\sigma_1 > \sigma_2$  (this is not a restrictive assumption as in a cyclic network, the choice of the first coordinate is arbitrary). Then we have  $X_o = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$  and  $c = 1$ .

We denote  $\gamma = \sigma_1 \vee \sigma_2$  and  $\gamma' = \sigma'_1 \vee \sigma'_2$ . Formula (6) gives:

$$\pi(l) = \sum_{k=1}^l \left\{ (-1)^{l-k} \binom{2}{l-k} A \otimes (A')^k \otimes X_o \right\} + (-1)^l \left\{ X_o + \binom{2}{l} \Gamma \right\}.$$

A direct evaluation shows that :

$$\pi(0) = \gamma$$

$$\begin{aligned}\pi(1) &= \gamma' - \gamma \\ \pi(l) &= 0 \text{ for } l > 1.\end{aligned}$$

Then we have

$$\gamma(p) = \gamma + (\gamma' - \gamma)p = \gamma'p + \gamma(1 - p). \quad (7)$$

**Remark 4** *One may wonder why (7) holds and when this occurs. The easy case is when  $X_o$  is also an eigenvector of  $A'$  for the eigenvalue  $\gamma'$ . This implies that  $\mathbb{E}[X_{n+1} - X_n]$  does not depend on  $X_n$ , and  $\mathbb{E}[X_{n+1} - X_n] = \mathbb{E}[A_1 \cdot X_o - X_o] = (1 - p)\gamma + p\gamma'$ . A slightly more general case is when there is at least one component of  $A' \cdot X_o$  which increases like a common eigenvector of  $A$  and  $A'$ , when multiplied by a finite number of these matrices.*

### 3.2.2 Network with breakdowns

Consider a network with two stations, where the first one is a single server station as above, and the second one has two servers. The network has three customers (see the Petri net of Figure 2).

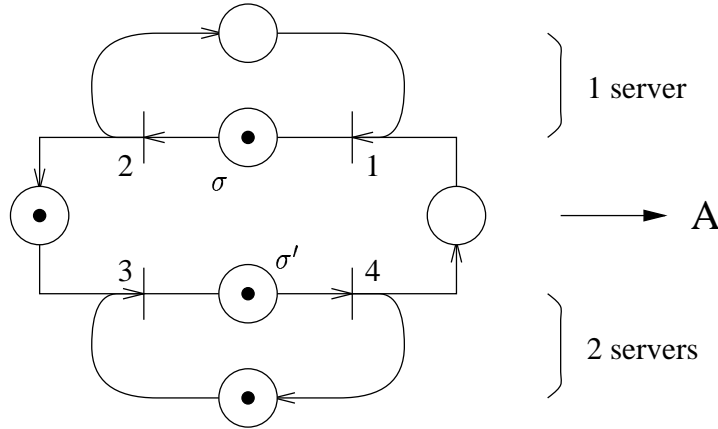


Figure 2: A network with 3 servers modeled by  $A$

This network can be described as a  $(\max, +)$ -linear systems with matrix  $A$  given by the formula  $A = \mathbb{P}_0^* \otimes \mathbb{P}_1$  (see [1]) with :

$$\mathbb{P}_0 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \quad \mathbb{P}_1 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix},$$

$$\mathbb{P}_0^* = \bigoplus_{n \geq 0} \mathbb{P}_0^n = \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix},$$

so that

$$A = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}.$$

Let  $Y_n^i$  denote the epoch of the  $n$ th firing of transition  $i$  in the Petri net of Figure 2, and let  $Y_n = (Y_n^i)$ . It is easy to check that if one takes the initial condition  $Y_0 = (0, 0, 0, 0)^t$ , then  $\{Y_n\}$  is the solution of the evolution equation

$$Y_{n+1} = A \otimes Y_n, \quad n \geq 1$$

and is coordinatewise non-decreasing. Notice for further use that this recurrence equation reads

$$\begin{aligned} Y_{n+1}^1 &= \sigma \otimes Y_n^1 \oplus \sigma' \otimes Y_n^3 \\ Y_{n+1}^2 &= \sigma \otimes Y_n^1 \\ Y_{n+1}^3 &= Y_n^2 \oplus Y_n^4 \\ Y_{n+1}^4 &= \sigma' \otimes Y_n^3. \end{aligned}$$

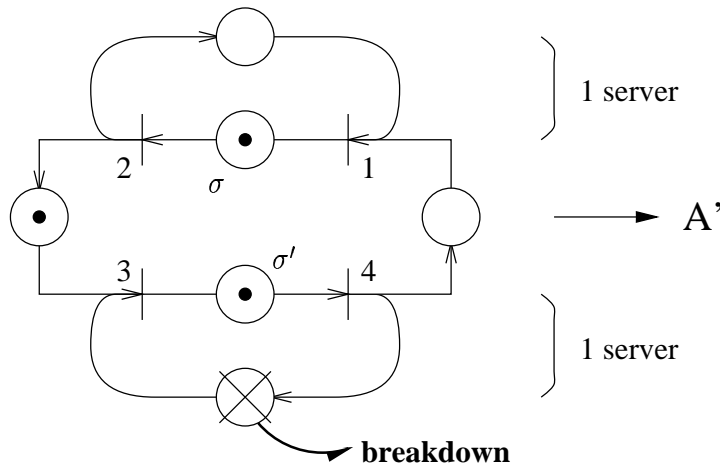


Figure 3: Breakdown case modeled by  $A'$ .

Consider now another network with one server less in station 2 (Figure 3). By similar arguments,

$$\mathbf{P}'_0 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \quad \mathbf{P}'_1 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix},$$

$$\mathbf{P}'_0^* = \bigoplus_{n \geq 0} \mathbf{P}'_0^n = \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & 0 \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix},$$

that is,

$$A' = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \sigma' & \varepsilon \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}.$$

The same observations as above can be made, and in this case,

$$\begin{aligned} Y'_{n+1}{}^1 &= \sigma \otimes Y'_n{}^1 \oplus \sigma' \otimes Y'_n{}^3 \\ Y'_{n+1}{}^2 &= \sigma \otimes Y'_n{}^1 \\ Y'_{n+1}{}^3 &= Y'_n{}^2 \oplus Y'_n{}^3 \otimes \sigma' = Y'_n{}^2 \oplus Y'_{n+1}{}^4 \\ Y'_{n+1}{}^4 &= \sigma' \otimes Y'_n{}^3. \end{aligned}$$

Let us now consider a third network with an initial condition as above, namely one customer starting its service in station 1, and two customers in station 2, one starting its service and the other one in the buffer, and with the following type of breakdowns: when the  $n$ -th customer enters station 2,  $n \geq 1$ ,

- either there is a breakdown (which takes place with probability  $p$ ), and this customer can only start its service there after  $n$  departures have taken place from this station (like in the one server case);
- or there is no breakdown (which takes place with probability  $1 - p$ ) and he only has to wait for  $n - 1$  departures to have taken place (like in the two server case).

To describe the evolution of a such system, we put:

$$X_{n+1} = A(n) \otimes X_n,$$

where  $X_n^i$  denote the  $n$ -th epochs when a customer leaves the transition  $i$  (i.e.  $X_n^1$  (resp.  $X_n^3$ ) is the  $n$ -th epochs when a customer starts its service in station 1 (resp. 2) and  $X_n^2$  (resp.  $X_n^4$ ) is the epochs when a customer leaves station 1 (resp. 2)).

Then we can use the above framework with  $A(n)$  equal to  $A$  or  $A'$  with probability  $(1 - p)$  and  $p$  respectively to describe the network with breakdown (for justifications and details, see

Appendix §10). We check that  $c = 4$ ,  $\gamma(0) = \sigma$  and  $X_o = \begin{pmatrix} 2\sigma \\ 2\sigma \\ \sigma \\ \sigma' \end{pmatrix}$  for  $A$ .

1. For  $\sigma' < 2\sigma$ , there are two regimes :

- if  $2\sigma' < 3\sigma$ ,  $\gamma(p) = \sigma$
- if  $2\sigma' > 3\sigma$ , we find by direct computations the following series expansion :

$$\begin{aligned} \gamma(p) &= \sigma + (2\sigma' - 3\sigma)p + (5\sigma - 3\sigma')p^2 + (4\sigma' - 6\sigma)p^3 + (7\sigma - 5\sigma')p^4 \\ &\quad + (9\sigma' - 13\sigma)p^5 + (26\sigma - 17\sigma')p^6 + \dots \end{aligned}$$

2. For  $\sigma' > 2\sigma$ ,  $A$  becomes 2-periodic (see Remark 10 in §4.5).

### 3.2.3 Window flow control

Now we proceed in the same way as the previous example except that instead of breaking down servers, we cut down the number of customers allowed in the system. This is what happens in the TCP/IP protocol, where the window size is reduced in case of overload. We consider here the case where there are 4 or 2 customers in the network.

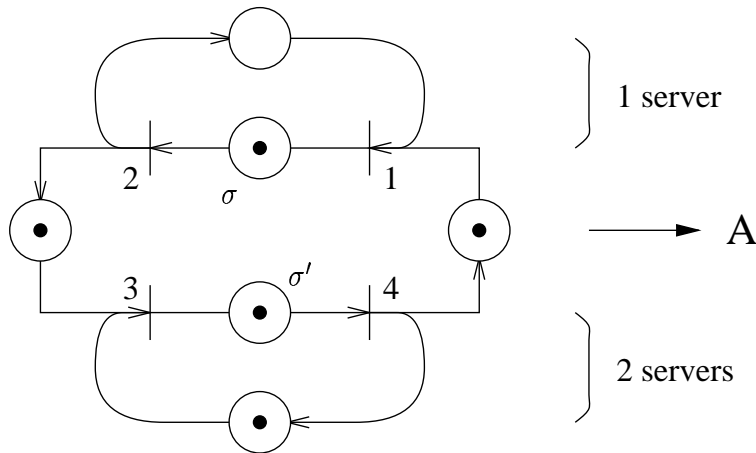


Figure 4: A network with 4 customers modeled by  $A$

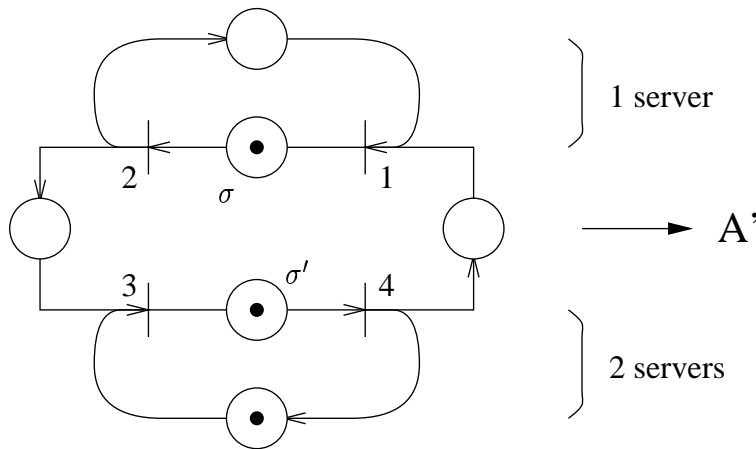


Figure 5: 2 customers only allowed case modeled by  $A'$

Let  $A$  denote the matrix of the system with 4 customers and  $A'$  that of the system with 2. We have :

$$A = \begin{pmatrix} \sigma & \varepsilon & \varepsilon & 0 \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}, \quad A' = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}.$$



Here,  $c = 7$ ,  $\gamma(0) = \sigma$  and  $X_o = \begin{pmatrix} 2\sigma \\ 2\sigma \\ \sigma \\ \sigma' \end{pmatrix}$ .

We will only consider the case  $\sigma' \in ]\sigma, 2\sigma[$ . Direct computations give the following series expansion :

$$\gamma(p) = \sigma + (-\sigma + \sigma')p^2 + (\sigma - \sigma')p^3 + (-\sigma + \sigma')p^4 + \dots$$

Note that here, there is no direct interpretation of this representation as certain epochs of the system, as we did in the previous example.

## 4 Extensions

In this section, we give three extensions of the Bernoulli case.

### 4.1 Multinomial case

We first extend Theorem 1 to the multinomial case. That is, instead of having one perturbation possibility by  $A'$ , we allow  $m$  types of perturbations through  $m$  matrices  $A'_1, \dots, A'_m$ .

#### 4.1.1 Notations and assumptions

Let  $\{A(n)\}$  be an i.i.d. sequence of matrices of  $\mathbb{R}_{\max}^{d,d}$ , following a discrete distribution :

$$A(n) = \begin{cases} A & \text{with probability } 1 - p_1 - p_2 - \dots - p_m \\ A'_1 & \text{with probability } p_1 \\ A'_2 & \text{with probability } p_2 \\ \vdots & \vdots \\ A'_m & \text{with probability } p_m. \end{cases}$$

The sequence  $\{X_n\}$  is defined by the same linear recurrence equation as in (1). This will be referred to as  $(m+1)$ -nomial scheme in what follows.

In this section, the assumptions are the following :

**(H1)'** : *The matrix  $A$  is irreducible.*

**(H2)'** : *The matrix  $A$  is scs1-cyc1.*

**(H3)'** : *Each of the  $m$  matrices  $A'_i$  has at least one entry different from  $\varepsilon$  on each line.*

By the same arguments as in the binomial case, under these assumptions, both Result 2 and Result 3 still hold, and in particular the Lyapunov exponent  $\Gamma(P)$  is well defined.

For stating the main theorem, we will need the following notations :

- $P = (p_1, \dots, p_m)$ .
- $K = (k_1, \dots, k_m)$ .
- If  $B = \{A(n)\}$ ,  $|B| = K$  means that in the sequence  $B$ ,  $\text{Card} \{ i : A_i = A'_i \} = k_i$ , for  $1 \leq i \leq m$ .

- $|K| = \sum_{i=1}^m k_i$
- $K! = \prod_{i=1}^m k_i!$
- $\binom{n}{L} = \frac{n!}{L!(n-L)!}$
- $\binom{L}{K} = \frac{L!}{K!(L-K)!}$
- $K \leq L$  means that  $\forall i, k_i \leq l_i$

Note that all indices in capital letters are vector indices.

**Theorem 2** Under assumptions  $(H1)'$ ,  $(H2)'$  and  $(H3)'$ , the Lyapunov exponent of the  $m+1$ -nomial scheme,  $\Gamma(P)$ , is analytic at point 0, with a radius of convergence larger than or equal to  $\frac{1}{2cm}$ , where  $c$  is the coupling time of  $A$ . The coefficients of the expansion are given by the following formula, where  $X_o$  is the eigenvector of  $A$ :

$$\begin{aligned} \frac{1}{L!} \frac{d^L}{dP^L} [\Gamma(P)]_{P=0} &= (-1)^{|L|} \left\{ \sum_{|K|=1} \left\{ \binom{c}{L-K} X_o \right\} + \binom{c+1}{L} \Gamma(0) \right\} \\ &+ \sum_{\substack{1 \leq |K| \\ K \leq L}} \left\{ (-1)^{|L|-|K|} \sum_{j_1 \dots j_{|K|-1}=0}^{c-1} \sum_{\substack{C_n, n=1, \dots, |K| \\ \text{with } C_n = A'_i \text{ for some } i \\ \text{Card}\{n: C_n = A'_i\} = k_i}} \binom{2c+j_1+\dots+j_{|K|-1}}{L-K} \right. \\ &\left. A^c \otimes C_1 \otimes A^{j_1} \otimes C_2 \dots A^{j_{|K|-1}} \otimes C_{|K|} \otimes X_o \right\}. \end{aligned} \quad (8)$$

For all  $p \in [0, \delta]$ , with  $\delta < \frac{1}{2cm}$ , the error term in the Taylor expansion of  $\Gamma(P)$  of order  $l$  is bounded from above by:

$$\frac{D(2c\delta)^{l+1}[1 + l(1 - 2c\delta)]}{(1 - 2c\delta)^2},$$

where  $D = \left[ 2(\|A\| \vee \|A'_1\| \vee \dots \vee \|A'_m\|) + \|X_o\|_{\mathcal{D}} \right] (c+1)$ .

The proof is given in §6.2.

#### 4.1.2 Example

Here is an example of a 3-nomial scheme, i.e. of an i.i.d. sequence  $\{A(n)\}$  sampled from three values  $A$ ,  $A'_1$  and  $A'_2$ , with respective probabilities  $p_0$ ,  $p_1$  and  $p_2$ .

The three matrices are chosen as follow:

$$A = \begin{pmatrix} 1 & \varepsilon & 0 \\ 2 & 0 & 1 \\ \varepsilon & 0 & 0 \end{pmatrix}, \quad A'_1 = \begin{pmatrix} 0 & \varepsilon & 2 \\ 1 & 0 & 2 \\ \varepsilon & 0 & 3 \end{pmatrix}, \quad A'_2 = \begin{pmatrix} 2 & \varepsilon & 0 \\ 1 & 0 & 3 \\ \varepsilon & 1 & 0 \end{pmatrix}.$$

$A$  is scs1-cycl,  $c = 2$ ,  $X_0 = (0, 1, 0)^t$  and  $\gamma(0) = 1$ . We can then compute the coefficient of Taylor series expansion from Formula (8), which gives:

$$\begin{aligned} \gamma(p_1, p_2) = & 1 \\ & + p_1 + p_2 \\ & + p_1^2 \\ & + 0 \\ & + p_1^2 p_2^2 - p_1^3 p_2 \\ & - p_1^3 p_2^2 \\ & + p_1^2 p_2^4 - 3p_1^3 p_2^3 + p_1^4 p_2^2 \\ & - 2p_1^3 p_2^4 + 2p_1^4 p_2^3 \\ & - 5p_1^3 p_2^5 + 6p_1^4 p_2^4 - p_1^5 p_2^3 \\ & + 7p_1^4 p_2^5 - 3p_1^5 p_2^4 \\ & - 10p_1^5 p_2^5 + o\left(\|(p_1, p_2)\|_\infty^{10}\right). \end{aligned}$$

In this example, the random sequence has a fixed structure but this is not necessary for this type of computations. This case was chosen in such a way that when starting from the initial condition  $X_0$ , the Markov chain  $X_n$  evolves on a finite number of states, which allows us to obtain the exact value of the Lyapunov exponent:

$$\gamma(p_1, p_2) = 1 + p_1 + p_2 + \frac{p_1^2(1 - p_2 + p_1 p_2 - p_1 p_2^2)}{1 - p_2 - p_2^2 + p_2^3 + 2p_1 p_2 - p_1 p_2^2 - p_1 p_2^3 + p_1^2 p_2^2}.$$

Of course this is generally not the case that such a direct computation can be made. The interest of a series expansion stems from the fact that it also holds when the above finiteness property is not satisfied (a sufficient condition of this property can be given solving the well known *Burnside problem*, see [8]).

## 4.2 Weakening of the (H2) assumption

All assumptions are as in the binomial case, but for (H2) which is replaced by the weaker assumption that there is a pattern of  $\{A, A'\}$  of length  $q$  which is scs1-cycl and irreducible (see definition in Result 3).

Let  $\tilde{A}$  denote the pattern,  $\tilde{A}(n)$  be defined by the relation

$$\tilde{A}(n) = A((n+1)q-1) \otimes \cdots \otimes A(nq)$$

and let  $\tilde{X}_n$  be defined by the recurrence relation

$$\tilde{X}_{n+1} = \tilde{A}(n) \otimes \tilde{X}_n.$$

The sequence  $\{\tilde{A}(n)\}$  is i.i.d. and each matrix in this sequence can take at most  $2^q$  values. If all these matrices have at least one entry different from  $\varepsilon$  per line, we can check that :

$$\tilde{\Gamma}(p) = q\Gamma(p),$$

where  $\tilde{\Gamma}(p)$  is the Lyapunov exponent of the  $2^q$ -nomial scheme.

### 4.3 The operator case

An operator  $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is 1-homogeneous if for all vectors  $X \in \mathbb{R}^d$ , and all scalars  $\alpha$ , if  $Y = X + \alpha \mathbf{1}$  is the vector with coordinates  $Y_i = X_i + \alpha$ , then  $A(Y) = A(X) + \alpha \mathbf{1}$ , namely  $A(Y)_i = A(X)_i + \alpha$  for all  $i$ .

Consider the following setting: let  $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $A' : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be two deterministic 1-homogeneous operators. Let  $A(n)$  be an i.i.d. sequence of operators defined the same way as in the  $(\max, +)$ -binomial case, namely  $A(n)$  is equal to  $A$  with probability  $p$  and to  $A'$  with probability  $(1 - p)$ . Let

$$X_n = A(n-1) \circ \cdots \circ A(0)(X_o),$$

where  $X_o$  is some deterministic vector. Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , be such that for all  $X$  and  $Y$  as above,  $f(Y) = \alpha + f(X)$ , and consider the random variables

$$\Psi(B^n) = f \circ A(1) \circ A(0) \circ \cdots \circ A(-n)(X_o) - f \circ A(0) \circ \cdots \circ A(-n)(X_o), \quad (9)$$

where  $B^n$  denotes the truncation of the sequence  $B = \{A(n)\}_{n \leq 1}$ .

The results of the binomial setting can be generalized to such an operator setting under the condition that the properties (P1)–(P3) below are satisfied :

**(P1) [strong law of large numbers]**

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[f(X_n)]}{n} = \gamma_f(p) \left( \text{and } \frac{f(X_n)}{n} \xrightarrow{a.s.} \gamma_f(p) \right),$$

where  $\gamma_f(p) \in \mathbb{R}$ .

**(P2) [uniform coupling property]**  $A^n$ , the  $n$ -th iterate of  $A$  is such that

$$\exists c \text{ s.t. } \forall n \geq c, \forall Y \in \mathbb{R}^d \ A^n(Y) = \alpha(n, Y) + X_o,$$

where  $\alpha(n, Y)$  is a scalar.

In the  $(\max, +)$ -algebraic framework which was considered so far, the last property was derived from (H1) and (H2).

**(P3) [growth rate condition]**  $\forall l, |f(X_l)| \leq g(A, A', X_o, l)$ , for some non-negative function  $g$  such that

$$\frac{g(A, A', X_o, l+1)}{g(A, A', X_o, l)} \xrightarrow{l \rightarrow +\infty} \eta \in \mathbb{R}.$$

An example of a class of operators for which these properties have been studied is that of topical operators, which includes the class of  $(\min, \max, +)$  functions. For the deterministic theory of such operators, see [10] and [6]; for the random case, see [17], where an analogue of Result 2 can be found, and [3], which contains an analogue of Result 3.

**Theorem 3** *Under (P1), (P2) and (P3),  $\gamma_f(p)$  is analytic at point 0; the radius of convergence is larger than or equal to  $\frac{1}{2c\eta}$  and the coefficients of the analytic expansion are given by the*

following formula:

$$\begin{aligned} \pi_f(l) = & (-1)^l \left\{ \binom{c}{l-1} f(X_o) + \binom{c+1}{l} \gamma_f(0) \right\} \\ & + \sum_{k=1}^l \left\{ (-1)^{l-k} \sum_{j_1, \dots, j_{k-1}=0}^{c-1} \binom{2c+j_1+\dots+j_{k-1}}{l-k} \right. \\ & \left. f(A^c \circ A' \circ A^{j_1} \circ A' \dots A^{j_{k-1}} \circ A'(X_o)) \right\} \end{aligned} \quad (10)$$

The proof is given in §6.3.

**Remark 5** In the binomial  $(\max, +)$  framework described in the previous sections, (P2) means that  $A^c$  is of rank one, which means that  $A^c$  is a scs1-cyc1 pattern. In this case, (P2) is equivalent to (H2) when the pattern is irreducible. (P2) includes cases of reducible matrices.

**Remark 6** As a direct application of Theorem 3 to the  $(\max, +)$  case, one can evaluate all components of the first order limits like  $\lim_n \frac{(X_n)_i}{n}$  and in particular,

$$\lim_{n \rightarrow +\infty} \frac{\max_{1 \leq i \leq d} (X_n)_i}{n} = \overline{\gamma}(p), \quad \text{and} \quad \lim_{n \rightarrow +\infty} \frac{\min_{1 \leq i \leq d} (X_n)_i}{n} = \underline{\gamma}(p),$$

when taking  $f(X_n) = \max_i (X_n)_i$  (resp.  $\min_i (X_n)_i$ ). These questions are only of interest in the non irreducible case, otherwise these two limits are equal to  $\gamma(p)$ .

More generally, all Lyapunov exponents can be expanded in analytic series using this method, and not only the maximal one.

#### 4.4 Example : random $(\min, \max, +)$ operators

Here we give an example of random topical operators belonging to the  $(\min, \max, +)$  class of functions which includes that of  $(\max, +)$  functions (see [10]).

For operator  $A$ , we take:

$$A(x_1, x_2, x_3) = \left[ \begin{array}{l} (x_1 \vee (x_2 + 1) \vee (x_3 + 1)) \wedge ((x_1 + 1) \vee (x_2 + 1) \vee (x_3 + 1)) \\ ((x_1 + 1) \vee (x_2 + 2) \vee (x_3 + 1)) \wedge (x_1 \vee (x_2 + 1) \vee x_3) \\ ((x_1 + 1) \vee x_2 \vee (x_3 + 2)) \wedge (x_1 \vee (x_2 + 1) \vee (x_3 + 2)) \end{array} \right].$$

For this operator, it was shown in [6] that an analogue of Result 1 holds, with  $c = 2$ ,  $X_o = (1, 0, 2)^t$  and  $\gamma = 2$ .

For  $A'$  we take operator  $A$  of the last  $(\max, +)$  example, i.e.:

$$A'(x_1, x_2, x_3) = \left[ \begin{array}{l} (x_1 + 1) \vee x_3 \\ (x_1 + 2) \vee x_2 \vee (x_3 + 1) \\ x_2 \vee x_3 \end{array} \right].$$

From Formula (10), we get:

$$\gamma(p) = 2 - 2p + p^2 + 2p^3 - p^4 - 4p^5 + 7p^7 + 3p^8 + o(p^8),$$

a formula which can be checked by simulation.

## 4.5 General extension theorem

The three extensions we have presented above are all compatible: we can consider a multinomial case with a pattern of length larger than one and satisfying (H2)', or consider a multinomial scheme in a more general operator framework, or such a framework with a general pattern etc.

Let  $\mathcal{L} = \{A, A'_1, \dots, A'_m\}$  be a finite set of operators from  $\mathbb{R}^d$  to  $\mathbb{R}^d$  and let  $\{A(n)\}$  be a sequence of i.i.d.  $\mathcal{L}$ -valued operators, where  $A(n)$  is equal to  $A'_l$  with probability  $p_l$  and to  $A$  with probability  $(1 - \sum p_l)$ . Let

$$X_n = A(n-1) \circ \dots \circ A(0)(X_o),$$

where  $X_o$  is some vector. Let  $\mathcal{M}$  denote the space of realizations of the sequence  $\{A(n)\}_{n \leq 1}$ , and let  $\Psi : \mathcal{M} \rightarrow \mathbb{R}^d$  be a measurable mapping. Consider the following set of assumptions:

**(A2) [memory loss property]** There exists a vector  $X_o \in \mathbb{R}^d$  and a sequence  $M[\mathcal{L}, q] = (C_1^0, C_2^0, \dots, C_q^0)$  of elements of  $\mathcal{L}$ , such that for all  $A(1), A(0), \dots, A(-n), A(-n-q-1), A(-n-q-2), \dots$  in  $\mathcal{L}$ ,

$$\begin{aligned} \Psi(A(1), A(0), \dots, A(-n), M[\mathcal{L}, q], A[-n-q-1, -\infty]) \\ = \Phi_n(A(1), \dots, A(-n)), \end{aligned}$$

where  $A[-n-q-1, -\infty]$  denotes the sequence of matrices  $A(i)$  for  $i \leq -n-q-1$ , and  $\Phi_n$  is some function depending only on  $(A(1), \dots, A(-n))$ .

**(A3) [growth rate condition]** There exists a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\Psi(B^n) = [f(X_{n+2}) - f(X_{n+1})] \circ \theta^{-n}$  and that  $|f(X_l)| \leq g(\mathcal{L}, X_o, l)$ , for some non-negative function  $g$  such that

$$\frac{g(\mathcal{L}, X_o, l+1)}{g(\mathcal{L}, X_o, l)} \xrightarrow{l \rightarrow +\infty} \eta \in \mathbb{R}.$$

**Remark 7** (A2) is satisfied under the assumption of §4.3 thanks to (P2) and the homogeneity property.

We can embed this problem in a  $(m+1)^q$ -nomial scheme as in Section 4.2 as follows. Let

$$\tilde{A} = C_1^0 \circ C_2^0 \dots \circ C_q^0$$

and let  $\tilde{\mathcal{L}}$  be the set of all compositions of  $q$  operators of  $\mathcal{L}$ , such that the composition is different from  $\tilde{A}$ . This set has at most  $\tilde{m} = (m+1)^q$  elements. If we define

$$\tilde{A}(n) = A((n+1)q-1) \circ \dots \circ A(nq),$$

then the sequence  $\{\tilde{A}(n)\}$  is i.i.d. and the law of  $\tilde{A}(n)$  is multinomial on the finite set  $\tilde{\mathcal{L}} \cup \{\tilde{A}\}$ . Let  $\tilde{p}_i$ , for  $i$  ranging from 1 to  $\tilde{m} = (m+1)^q - 1$ , denote the probabilities of the elements of  $\tilde{\mathcal{L}}$ . We shall use the following notations :

$$\tilde{P} = (\tilde{p}_1, \dots, \tilde{p}_{\tilde{m}});$$

for all infinite sequences  $\tilde{B}$  of elements of  $\tilde{\mathcal{L}} \cup \{\tilde{A}\}$ ,

$$\begin{aligned}\tilde{\Psi}(\tilde{B}) &= \Psi(\tilde{A}(1), \tilde{A}(0), \dots, \tilde{A}(-n), \dots); \\ \tilde{\Psi}(\tilde{B}^n) &= \Psi(\tilde{A}(1), \tilde{A}(0), \dots, \tilde{A}(-n), \tilde{A}, \tilde{A}, \dots).\end{aligned}$$

For all subsets  $S$  of  $\{1, 0, -1, \dots\}$ ,  $\tilde{B}_S$  denotes the set of all infinite sequences of elements of  $\mathcal{L} \cup \tilde{A}$  such that all terms of the sequence are equal to  $\tilde{A}$ , but for those with indices in  $S$ , which belong to  $\mathcal{L}$ .

**Theorem 4** *For all  $n$ ,  $\mathbb{E}\tilde{\Psi}(\tilde{B}^n)$  is an analytic function of  $\tilde{P}$ . Let  $\tilde{\pi}_{\tilde{\Psi}}^n(L)$  be the coefficient of  $\tilde{P}^L$  in the analytic expansion of  $\mathbb{E}\tilde{\Psi}(\tilde{B}^n)$ . Under (A2),  $\tilde{\pi}_{\tilde{\Psi}}^n(L)$  tends to a limit when  $n$  tends to  $\infty$ , which is equal to*

$$\begin{aligned}\tilde{\pi}_{\tilde{\Psi}}(L) &= (-1)^{|L|} \binom{2}{L} \tilde{\Psi}(M_\emptyset) \\ &+ \sum_{\substack{1 \leq |K| \\ \bar{K} \leq L}} \left\{ (-1)^{|L|-|K|} \sum_{j=0}^1 \binom{1+j}{L-K} \sum_{\substack{M \in B_{1-j, -j, \dots, -j-|K|+2} \\ |M|=K}} \tilde{\Psi}(M) \right\}.\end{aligned}\tag{11}$$

**Theorem 5** *Under (P1), (A2) and (A3),*

$$\tilde{\gamma}_f(\tilde{P}) = q\gamma_f(\tilde{P})$$

and the function  $\gamma_f(\tilde{P})$  is analytic at least in the open ball of radius  $\frac{1}{2\tilde{m}\eta}$ ; the coefficients of its series expansion are given by

$$\begin{aligned}\tilde{\pi}_f(L) &= (-1)^{|L|} \left\{ \sum_{|K|=1} \left\{ \binom{1}{L-K} f(X_o) \right\} + \binom{2}{L} \tilde{\gamma}_f(0) \right\} \\ &+ \sum_{\substack{1 \leq |K| \\ \bar{K} \leq L}} \left\{ (-1)^{|L|-|K|} \sum_{\substack{C_n \in \tilde{\mathcal{L}}, n=1, \dots, |K| \\ \text{Card}\{n: C_n = \tilde{A}'_i\} = k_i \\ i=1 \text{ to } (m+1)^q - 1}} \binom{2}{L-K} \right. \\ &\left. f(\tilde{A} \circ C_1 \circ \dots \circ C_{|K|}(X_o)) \right\}.\end{aligned}\tag{12}$$

**Remark 8** *Note that in (12), we only need to sum over the set  $\{\max(1, |L| - 2) \leq |K|, K \leq L\}$ .*

**Remark 9** *If there exists a constant  $c'$  with  $q = c'r$  s.t.  $(M[\mathcal{L}, r])^{c'} = M[\mathcal{L}, q]$  for  $c \geq 2$  and  $r \geq 1$ , we can construct a reduced scheme by embedding the problem in a  $(m+1)^r$ -nomial scheme with  $c = c'$ . This is really a reduced scheme if this leads to a decrease of the complexity of the algorithm.*

*In the induced  $(m+1)^q$ -nomial scheme, the complexity is (see 24) :*

$$d^2 |L| ((m+1)^q - 1)^{|L|}.$$

*In the reduced  $(m+1)^r$ -nomial scheme, the complexity is :*

$$d^2 |L| ((m+1)^r - 1)^{|L|} c'^{|L|}.$$

Then the ratio is :

$$\frac{((m+1)^q - 1)^{|L|}}{((m+1)^r - 1) c'}^{|L|}$$

This ratio is larger than 1 when  $(m+1)^{rc'} - 1 > ((m+1)^r - 1) c'$ . To prove this, put  $g(x) = (x+1)^{rc'} - 1 - ((x+1)^r - 1) c'$ , differentiate it to show that  $g'(x) \geq 0$  and check that  $g(0) = 0$ .

**Remark 10** In Assumption (H2), the cyclicity assumption could be relaxed, because if the pattern is *scs1-cyc h*, the *h*-composition of the pattern is *scs h-cyc1* and reducible. Then, we have to look for *h* classes of periodicity, each irreducible, which means at most *h* Lyapunov exponents to expand.

**Remark 11** The formulation with the distribution on  $A'$  may be obtained on each extension results in the same way that we did in (25) in §7.1.

### 4.6 Example : stochastic task resource model

We consider the following task resource model described in [9]. We recall some notations :

- $\mathcal{A}$  is a finite set of pieces.
- $\mathcal{R}$  is a finite set of slots.
- $R : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{R})$  gives the subset of slots covered by a piece.
- $h : \mathcal{A} \rightarrow \mathbb{R}^+$  gives the execution time of a task.

We assume :  $\mathcal{A} = \{a, a_1, a_2\}$ ,  $\mathcal{R} = \{r_1, r_2\}$ ,  $R(a) = \{r_1, r_2\}$ ,  $R(a_1) = \{r_1\}$ ,  $R(a_2) = \{r_2\}$ .  $h \equiv 1$ . That is : the matrices associated with this resource automaton are :

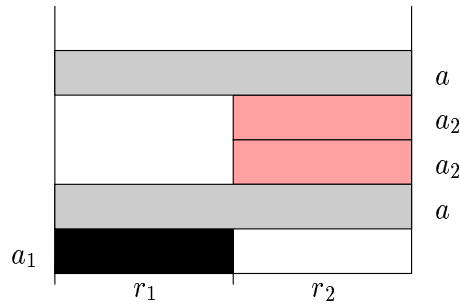


Figure 6: 2 resources model.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad A'_1 = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \quad A'_2 = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}.$$

We assume that the sequence of tasks is given by a sequence of i.i.d. variables  $a(n) \in \mathcal{A}$  with :

$$\begin{aligned} p_1 &= P(a(n) = a_1) \\ p_2 &= P(a(n) = a_2) \\ 1 - p_1 - p_2 &= P(a(n) = a). \end{aligned}$$



In this case, the Lyapunov exponent  $\gamma(p)$  was considered and calculated explicitly in [5] :

$$\gamma(p_1, p_2) = \frac{1}{2} \frac{p_1 + p_2 - 4p_1p_2 + (2 - p_1 - p_2)\sqrt{1 - 4p_1p_2}}{\sqrt{1 - 4p_1p_2}}. \quad (13)$$

By application of (12) for  $q = 1$ , we get the formula :

$$\begin{aligned} \pi(L) = & \sum_{\substack{1 \leq |K| \\ K \leq L}} \left\{ (-1)^{|L|-|K|} \sum_{\substack{A'=A'_i \\ \text{Card}\{n:A_n=A'_i\}=k_i}} \binom{2}{L-K} A \otimes (A')^{|K|} \otimes X_o \right\} \\ & + (-1)^{|L|} \left\{ \sum_{|K|=1} \left\{ \binom{1}{L-K} X_o \right\} + \binom{2}{L} \Gamma(0) \right\}, \end{aligned}$$

with  $\gamma(0) = 1$  and any vector  $X_o$ , for instance  $X_o = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ . Then after some elementary calculations,

$$\begin{aligned} \gamma(p_1, p_2) = & 1 - 2 \sum_{n=1}^{+\infty} \binom{2n-2}{n-1} (p_1p_2)^n + \sum_{n=0}^{+\infty} \binom{2n-1}{n} [p_1(p_1p_2)^n + p_2(p_1p_2)^n] \end{aligned}$$

which is equal to (13) for  $p_1 + p_2 < 1$ .

In case of generalized task resource models (Tetris type [9]), the formulas of Theorem 5 hold if the memory loss property (A2) is satisfied, that is if there is a piece of Tetris that occupies all resources or a sequence of pieces of rank one. Here,  $a$  guarantees this property.

## 5 Perturbation representation of the coefficients of the expansion

In this section we show how Formula (12) can be interpreted as a perturbation formula. This representation will allow us to get a new formulation of (12) which is both more compact and more easy to understand. The setting is that of §4.5.

For any  $K \in \mathbb{R}^m$ , we define:

$$S^f(K) = \sum_{\substack{C_n \in \{A'_1, \dots, A'_m\}, n=1, \dots, |K| \\ \text{Card}\{n:C_n=A'_i\}=k_i \\ i=1 \text{ to } m}} f(A \circ C_1 \circ \dots \circ C_{|K|}(X_o)),$$

with

$$S^f((0, \dots, 0)) = f(A(X_o)).$$

Let

$$\delta_i S^f(K) = S^f(K - e_i),$$

where  $e_i$  is the vector with all components equal to zero except the  $i$ th one which is equal to 1, and by convention  $S^f(K) = 0$  if at least one component of  $K$  is negative.

Finally, we define:

$$\Delta S^f(K) = S^f(K) - \sum_{i=1}^m \delta_i S^f(K).$$

Then we get :

$$\Delta^2 S^f(K) = \Delta(\Delta S^f(K)) = S^f(K) - 2 \sum_{i=1}^m \delta_i S^f(K) + \sum_{i,j=1}^m \delta_i \delta_j S^f(K).$$

We now come back to Formula (12). Without loss of generality, we can assume that  $q = 1$  and rewrite it, when  $|L| > 2$ , as:

$$\begin{aligned} \pi_f(L) &= (-1)^{|L|} \left\{ \sum_{|K|=1} \left\{ \binom{1}{L-K} f(X_o) \right\} + \binom{2}{L} \tilde{\Psi}(M_\emptyset) \right\} \\ &+ \sum_{\substack{|L|-2 \leq |K| \\ K \leq L}} \left\{ (-1)^{|L|-|K|} \sum_{\substack{C_n \in \{A'_1, \dots, A'_m\}, n=1, \dots, |K| \\ \text{Card}\{n: C_n = A'_i\} = k_i \\ i=1 \text{ to } m}} \binom{2}{L-K} f(A \circ C_1 \circ \dots \circ C_{|K|}(X_o)) \right\} \\ &= \sum_{\substack{C_n \in \{A'_1, \dots, A'_m\}, n=1, \dots, |L| \\ \text{Card}\{n: C_n = A'_i\} = l_i \\ i=1 \text{ to } m}} f(A \circ C_1 \circ \dots \circ C_{|L|}(X_o)) \\ &- 2 \sum_{j=1}^m \sum_{\substack{C_n \in \{A'_1, \dots, A'_m\}, n=1, \dots, |L|-1 \\ \text{Card}\{n: C_n = A'_i\} = l_i - \delta_{ij} \\ i=1 \text{ to } m}} f(A \circ C_1 \circ \dots \circ C_{|L|-1}(X_o)) \\ &+ 2 \sum_{j \neq h=1}^m \sum_{\substack{C_n \in \{A'_1, \dots, A'_m\}, n=1, \dots, |L|-2 \\ \text{Card}\{n: C_n = A'_i\} = l_i - \delta_{ij} - \delta_{ih} \\ i=1 \text{ to } m}} f(A \circ C_1 \circ \dots \circ C_{|L|-2}(X_o)) \\ &+ \sum_j^m \sum_{\substack{C_n \in \{A'_1, \dots, A'_m\}, n=1, \dots, |L|-2 \\ \text{Card}\{n: C_n = A'_i\} = l_i - 2\delta_{ij} \\ i=1 \text{ to } m}} f(A \circ C_1 \circ \dots \circ C_{|L|-2}(X_o)) \end{aligned}$$

with  $\delta_{jh} = 0$  if  $j \neq h$  and  $\delta_{jj} = 1$ . It is then easy to verify that:

$$\pi_f(L) = \Delta^2 S^f(L). \quad (14)$$

This relation is verified for any  $L$  if we put  $\Delta^2 S^f((0, \dots, 0)) = \Delta f(A(X_o)) = f(A(X_o)) - f(X_o) = \gamma_f(0)$ .

## 6 Proofs

### 6.1 Bernoulli case

This section focuses on the proof of Theorem 1. This proof is based on a light traffic type expansion in the spirit of that of [2]: the expansion is first derived in the finite memory case; the infinite memory expansion is then obtained via some direct analytical convergence arguments.

Its first part is proved in three steps (6.1.1, 6.1.2, 6.1.3). The notations and assumptions are those of §3.1.

**Lemma 1** 1. Under (H1) and (H3), Result 2 holds.

2. Under (H1), (H2) and (H3), Result 3 holds.

*Proof*

- Extension of Result 2. The first step of the proof is the same; as for the second one Assumptions (H1) and (H3) are sufficient to prove that all coordinates of  $X_n/n$  again have the same a.s. limit. More precisely, for each  $(i, j)$ , there exists  $n_0$  such that:

$$(X_{n+n_0})_j \geq (A^{n_0})_{ji} + (X_n)_i.$$

This occurs almost surely an infinity of times, such that we can construct a subsequence of  $n$  choosing  $n(\omega)$  for which this is hold, and diving by  $n$ , we get :  $\gamma_j \geq \gamma_i$ . Then, permute  $i$  and  $j$ .

- Extension of Result 3. The proof is the same than in [12] (6.8, 8).

♡

**Remark 12** To prove Theorem 1, only the extension of Result 2 is needed.

In what follows, we will take  $X_0 = X_o$ , where  $X_o$  is the unique eigenvector of  $A$ . Thanks to this and to (3), we can see  $\bar{X}$  (Result 3) as a  $\mathbb{R}^d$ -valued functional of the sequence

$$B = \{A_i\}_{i \leq 1}.$$

We will also use the following truncation of  $B$ :

$$B^n = \{\tilde{A}_i\}_{i \leq 1} \quad \text{with} \quad \begin{cases} \tilde{A}_i = A_i & \text{if } i \geq n \\ \tilde{A}_i = A & \text{if } i < n. \end{cases}$$

We will denote  $\mathcal{M}$  the set of all possible values of  $B$ , and  $\mathcal{M}^n$  that of all possible values of  $B^n$ . Finally, we will denote  $\Psi : \mathcal{M} \rightarrow \mathbb{R}^d$  the mapping

$$\Psi(B) = A(1) \otimes X - X = \Delta,$$

where  $X$  is any vector in the equivalence class of  $\bar{X}$  (see (4)).

Since  $X_o$  is an eigenvector of  $A$ ,

$$\begin{aligned}\Psi(B^n) &= A_1 \otimes A_0 \cdots A_{-n} \otimes X_o - A_0 \cdots A_{-n} \otimes X_o \\ &= A_1 \otimes X_{n+1} \circ \theta^{-n} - X_{n+1} \circ \theta^{-n} \\ &= X_{n+2} \circ \theta^{-n} - X_{n+1} \circ \theta^{-n},\end{aligned}$$

where  $\theta$  is the basic shift of the sequence  $\{A_n\}$  ( $A_n = A_0 \circ \theta^n$  for all  $n$ ). In addition, since Result 3 holds,

$$\Psi(B^n) \rightarrow_n \Psi(B) = \Delta \quad a.s.$$

**Remark 13** *The function  $\Psi$  is not modified if we replace  $X_o$  by  $X_o \otimes \alpha$  for any scalar  $\alpha$ . Besides, since  $X_o$  is an eigenvector of an irreducible matrix, for all  $i \in \{1, \dots, d\}$ ,  $(X_o)_i > \varepsilon$ . Hence taking  $\alpha = |||X_o|||$ , we can assume  $X_o \geq 0$ , i.e. each component of  $X_o$  is positive. In the same way, replacing  $A$  and  $A'$  by  $\tilde{A} = A \otimes \alpha$  and  $\tilde{A}' = A' \otimes \alpha$ , with  $\alpha = |||A||| \oplus |||A'|||$ , we get that  $\tilde{\Psi} = \Psi \otimes \alpha$ . From this, we obtain the initial Lyapunov exponent by the relation :  $\gamma(p) = \tilde{\gamma}(p) - \alpha$ . We conclude that we can assume  $X_o, A$  and  $A'$  positive (namely, all entries of these matrices which are not equal to  $\varepsilon$  are non-negative). Then for all  $n$ ,  $X_n$  is positive too.*

Throughout the section, we will use the following notations :

- In order to simplify notations, we will replace the product operation  $\otimes$  in  $(\max, +)$  by "." and the conventional product operation  $\times$  will be omitted.
- $|B| = k$  means that in the sequence  $B$ ,  $\text{Card} \{ i : A_i = A' \} = k$ .
- $M_{i_1 \dots i_k}$  is a sequence  $B$  where  $A_i = A'$  for  $i = i_1 \dots i_k$  and only for these indices.
- $\mathcal{A}_n^p$  denotes the number of arrangements of  $p$  elements among  $n$ , i.e.

$$\mathcal{A}_n^p = \frac{n!}{(n-p)!}.$$

- $c$  denotes the coupling time of the matrix  $A$ , i.e. (see Result 1)

$$c = \inf \{ i : \forall n \geq i, A^{n+1} = \gamma(0).A^n \}.$$

### 6.1.1 Evaluation of $E\Psi$ – truncated case

First we can obtain an expansion of  $E\Psi$  by the following conditioning :

$$\begin{aligned}\mathbb{E} [\Psi(B^n)] &= \sum_{k=0}^{n+2} \mathbb{E} [\Psi(B^n) \mid |B^n| = k] P(|B^n| = k) \\ &= \sum_{k=0}^{n+2} \binom{n+2}{k} p^k (1-p)^{n+2-k} \left( \frac{1}{\binom{n+2}{k}} \sum_{\substack{M \in \mathcal{M}^n \\ |M|=k}} \Psi(M) \right),\end{aligned}\tag{15}$$

because  $P(|B^n| = k) = \binom{n+2}{k} p^k (1-p)^{n+2-k}$  and  $B^n$  conditioned by  $|B^n| = k$  follows a uniform distribution.

By expanding  $(1-p)^{n+2-k}$  and regrouping the coefficients of  $p^k$ , we get the final equation :

$$\mathbb{E}[\Psi(B^n)] = \sum_{l=0}^{n+2} \left\{ \sum_{k=0}^l \binom{n+2-l+k}{k} (-1)^k \left( \sum_{\substack{M \in \mathcal{M}^n \\ |M|=l-k}} \Psi(M) \right) \right\} p^l.$$

Note that the coefficient  $\pi^n(l)$  of  $p^l$ ,  $0 \leq l \leq n+2$ , in the last expression can be rewritten as follows:

$$\pi^n(l) = \sum_{k=0}^l (-1)^{l-k} \binom{n+2-k}{l-k} \left( \sum_{\substack{M \in \mathcal{M}^n \\ |M|=k}} \Psi(M) \right).$$

Remark that the two following relations hold :

- $\sum_{|M|=k} \Psi(M) = \frac{1}{k!} \sum_{\substack{i_1 \neq \dots \neq i_k \\ -n \leq i_j \leq 1}} \Psi(M_{i_1 \dots i_k})$
- $\mathcal{A}_{n-k}^{l-k} \sum_{\substack{i_1 \neq \dots \neq i_k \\ 1 \leq i_j \leq n}} \Psi(M_{i_1 \dots i_k}) = \sum_{\substack{i_1 \neq \dots \neq i_l \\ 1 \leq i_j \leq n}} \Psi(M_{i_1 \dots i_k}), \quad k \leq l \leq n.$

The second relation comes from the fact that  $l-k$  indices have to be chosen within the  $n-k$  free indices in the R.H.S sum.

Using these relations, we get :

$$\begin{aligned} \pi^n(l) &= \sum_{k=0}^l \binom{n+2-k}{l-k} \frac{(-1)^{l-k}}{k!} \left\{ \sum_{\substack{i_1 \neq \dots \neq i_k \\ -n \leq i_j \leq 1}} \Psi(M_{i_1 \dots i_k}) \right\} \\ &= \sum_{k=0}^l (-1)^{l-k} \frac{1}{k!} \binom{n+2-k}{l-k} \frac{(n+2-l)!}{(n+2-k)!} \left\{ \sum_{\substack{i_1 \neq \dots \neq i_l \\ -n \leq i_j \leq 1}} \Psi(M_{i_1 \dots i_k}) \right\} \\ &= \sum_{k=0}^l (-1)^{l-k} \frac{1}{l!} \binom{l}{k} \left\{ \sum_{\substack{i_1 \neq \dots \neq i_l \\ -n \leq i_j \leq 1}} \Psi(M_{i_1 \dots i_k}) \right\}. \end{aligned}$$

Interchanging the summations :

$$\pi^n(l) = \frac{1}{l!} \sum_{\substack{i_1 \neq \dots \neq i_l \\ -n \leq i_j \leq 1}} \left\{ \sum_{k=0}^l (-1)^{l-k} \binom{l}{k} \Psi(M_{i_1 \dots i_k}) \right\} \quad (16)$$

$$= \frac{1}{l!} \sum_{\substack{i_1 \neq \dots \neq i_l \\ -n \leq i_j \leq 1}} \left\{ \binom{l}{l} \Psi(M_{i_1 \dots i_l}) - \binom{l}{l-1} \Psi(M_{i_1 \dots i_{l-1}}) \cdots + (-1)^l \binom{l}{0} \Psi(M_\emptyset) \right\},$$

where  $M_\emptyset$  is the sequence where  $A_i = A$  for all  $i$ .

### 6.1.2 Convergence

**Convergence of  $\pi^n(l)$**  We prove that, for each  $l$ ,  $\pi^n(l) \rightarrow \pi(l)$ , when  $n \rightarrow +\infty$ . In fact this limit exists and is reached in a finite time. This result is a direct consequence of the following lemma.

**Lemma 2** *Under the foregoing assumptions,*

1. *If the  $l$  indices in (16) are all such that  $i_j < -c + 1, \forall j \in \{1 \dots l\}$ , then*

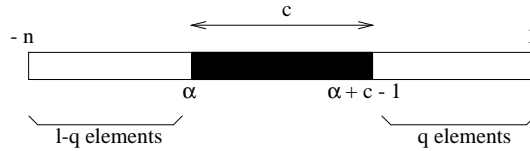
$$\sum_{\substack{i_1 \neq \dots \neq i_l \\ -n \leq i_j \leq -c}} \left\{ \sum_{k=0}^l (-1)^{l-k} \binom{l}{k} \Psi(M_{i_1 \dots i_k}) \right\} = 0.$$

2. *Let  $\mathcal{N} = \{-n, \dots, 1\}$  and let  $\mathcal{C}(\alpha) = \{\alpha, \alpha + 1, \dots, \alpha + c - 1\}$ , where  $\alpha$  is an integer such that  $\alpha + c - 1 < 1$  and  $-n < \alpha$ . Then we have*

$$\sum_{\substack{i_1 \neq \dots \neq i_l \\ i_j \in \mathcal{N} - \mathcal{C}(\alpha)}} \left\{ \sum_{k=0}^l (-1)^{l-k} \binom{l}{k} \Psi(M_{i_1 \dots i_k}) \right\} = 0.$$

*Proof* Property 1 follows from the fact that if  $i_j < -c + 1$  for all  $j$ , then  $\Psi(M_{i_1 \dots i_k})$  is equal to  $\Gamma(0)$  and of the fact that  $\sum_{k=0}^l (-1)^{l-k} \binom{l}{k} = 0$ .

For Property 2, consider the case where there are  $q$  indices  $\alpha_1, \dots, \alpha_q$ , all larger than  $\alpha + c - 1$  and  $l - q$  indices smaller than  $\alpha$ .



In the expression

$$V = \sum_{\substack{i_1 \neq \dots \neq i_l \\ i_j \in \mathcal{N} - \mathcal{C}(\alpha)}} \left\{ \binom{l}{l} \Psi(M_{i_1 \dots i_l}) - \binom{l}{l-1} \Psi(M_{i_1 \dots i_{l-1}}) + \cdots + (-1)^l \binom{l}{0} \Psi(M_\emptyset) \right\} \quad (17)$$

we start by regrouping the terms which contribute to the same value of  $\Psi$ . Because of the gap of length  $c$ , the  $l - q$  indices on the left of  $\alpha$  do not affect the value of  $\Psi$ . Therefore, once the  $l$  indices  $\alpha_1 \cdots \alpha_l$  and the integer  $q$  are given, the different values of  $\Psi$  are :

$$\begin{array}{ll}
\Psi(M_{\alpha_1, \alpha_2, \dots, \alpha_q}) & \rightsquigarrow 1 \text{ possibility} \\
\Psi(M_{\alpha_1, \dots, \alpha_{q-1}}), \Psi(M_{\alpha_1, \dots, \alpha_{q-2}, \alpha_q}), \text{ etc} & \rightsquigarrow q = \binom{q}{1} \text{ possibilities} \\
\Psi(M_{\alpha_1, \dots, \alpha_{q-2}}), \text{ etc} & \rightsquigarrow \binom{q}{2} \text{ possibilities} \\
\vdots & \vdots \\
\Psi(M_\emptyset) & \rightsquigarrow 1 \text{ possibility} .
\end{array}$$

Let us represent  $V$  as the sum

$$V = \sum_{\alpha_1, \dots, \alpha_l} \sum_{q=0}^l V_{\alpha_1, \dots, \alpha_l}^q,$$

where  $V_{\alpha_1, \dots, \alpha_l}^q$  collects all terms with  $q$  indices bigger than  $\alpha + c - 1$ , chosen in the set  $\{\alpha_1, \dots, \alpha_l\}$ . This can be rewritten as

$$V_{\alpha_1, \dots, \alpha_l}^q = \sum_{k=0}^q \beta_k \underbrace{\left\{ \sum_{\alpha(1) < \dots < \alpha(k) \in \{\alpha_1, \dots, \alpha_q\}} \Psi(M_{\alpha(1) \dots \alpha(k)}) \right\}}_{\binom{q}{k} \text{ terms}},$$

since all terms of type  $\Psi(M_{\alpha(1) \dots \alpha(k)})$  depending on  $k$  indices have a common factor  $\beta_k$  by symmetry.

To get  $\beta_q$  we count how many times  $\Psi(M_{\alpha_1, \alpha_2, \dots, \alpha_q})$  appears in  $V_{\alpha_1, \dots, \alpha_l}^q$ . For the first term of (17), we have to count the number of ways of arranging the  $q$  elements  $\{\alpha_1, \dots, \alpha_q\}$  among the  $l$  positions. For the second one, we have to count the number of ways of arranging the  $q$  elements  $\{\alpha_1, \dots, \alpha_q\}$  among the  $l - 1$ -st first positions and so on; in each case we have to multiply this by the number of permutations of the  $l - q$  indices on the left of  $\alpha$ , so that,

$$\beta_q = (l - q)! \left\{ \binom{l}{l} \mathcal{A}_l^q - \binom{l}{l-1} \mathcal{A}_{l-1}^q + \dots + (-1)^{l-q} \binom{l}{q} \mathcal{A}_q^q \right\}.$$

Consider now the general case  $\beta_{q-j}$ . First we must remove at least  $j$  indices from the  $q$  first ones. This means that the first term giving  $\beta_{q-j}$  comes from the  $j$ -th term of (17), that is  $\Psi(M_{i_1, \dots, i_{l-j}})$  : here we have to count the number of ways of arranging the  $q - j$  indices  $\alpha(1), \dots, \alpha(q - j)$  among the  $l - j$  indices showing up in  $\Psi$ , and to multiply this by the number of arrangements of the  $j$  indices  $i_{l-j+1}, \dots, i_l$  among the  $j$  removed indices. For the  $(l - j - 1 + k)$ -th term from  $\Psi(M_{i_1, \dots, i_k})$ , we have to count the number of ways of arranging  $q - j$  elements among  $k$  indices, times the number of ways of arranging  $j$  indices among the  $l - k$  removed indices, and so on. Therefore

$$\frac{1}{(l - q)!} \beta_{q-j} = \sum_{k=q-j}^{l-j} (-1)^{l-k} \binom{l}{k} \mathcal{A}_k^{q-j} \mathcal{A}_{l-k}^j$$

$$= \sum_{k=0}^l (-1)^{l-k} \binom{l}{k} \mathcal{A}_k^{q-j} \mathcal{A}_{l-k}^j,$$

with the convention  $\mathcal{A}_n^p = 0$  if  $p > n$ . Now the lemma is almost proved, because  $\mathcal{A}_k^{q-j} \mathcal{A}_{l-k}^j$  is a polynomial in  $k$ , the degree of which is less than  $q$ , that is less than  $l - 1$ . We just need the following result to conclude the proof :

$$\forall j \in \{1, \dots, l-1\}, \quad \sum_{k=0}^l (-1)^k k^j \binom{l}{k} = 0,$$

and this can be easily proved by induction when differentiating  $j$  times the function  $(1-x)^l$  and evaluating it at point 1.  $\heartsuit$

This lemma enables us to write  $\pi(l)$  as

$$\pi(l) = \frac{1}{l!} \sum_{\substack{i_1 \neq \dots \neq i_l \\ |i_m - i_n| \leq c \\ \text{for all successive indices } (i_m, i_n)}} \sum_{k=0}^l (-1)^{l-k} \binom{l}{k} \Psi(M_{i_1 \dots i_k}), \quad (18)$$

with  $\Psi(M_{i_1 \dots i_k}) = \Psi(M_\emptyset) = \Gamma(0)$  if  $k = 0$ . Remark that the first index should be in  $[-c+1, 1]$  due to a) of Lemma 2. Consequently it is sufficient to look for indices in  $[-(lc-1), 1]$  and this implies that  $\pi_n(l) = \pi(l)$ , for  $n \geq lc-1$ .

**Convergence of  $\mathbb{E}[\Psi(B^n)]$**  Here we focus on the convergence of the series representation of  $\mathbb{E}[\Psi(B^n)]$ , namely

$$S_n = \sum \pi^n(l) p^l,$$

when  $n$  goes to infinity. We show that this series is uniformly convergent, that it converges to  $\Gamma(p)$ , and that

$$S = \sum \pi(l) p^l = \Gamma(p). \quad (19)$$

We first discuss the conditions under which the last series is convergent. First, let us give a simple bound on  $\Psi(M_{i_1 \dots i_k})$ . For all  $i \in \{1, \dots, d\}$ ,  $Z \in \mathbb{R}^d$ ,

$$\begin{aligned} |(A(1).Z - Z)_i| &= \left| \bigoplus_{1 \leq j \leq d} (A(1)_{ij}.Z_j) - Z_i \right| \\ &= \left| \bigoplus_{\substack{1 \leq j \leq d \\ A(1)_{ij} \neq \varepsilon}} (A(1)_{ij}.Z_j) - Z_i \right| \\ &= |A(1)_{ij_o} + Z_{j_o} - Z_i| \quad \text{for some } j_o \\ &\leq |A(1)_{ij_o}| + |Z_{j_o} - Z_i| \\ &\leq F + \|Z\|_{\mathcal{D}}, \end{aligned}$$

with  $F = \|A\| \vee \|A'\|$ . In these inequalities, we have used the assumption that both  $A$  and  $A'$  have at least one non  $\varepsilon$  element per line.

Similarly, for all  $i, j$ , all  $Z \in \mathbb{R}^d$  and all  $n$ ,

$$|(A(n).Z)_i - (A(n).Z)_j| = |A(n)_{ii_o} + Z_{i_o} - A(n)_{jj_o} - Z_{j_o}| \leq 2F + \|Z\|_{\mathcal{D}}$$



(sharper bounds can be derived whenever  $A$  and  $A'$  are positive, in the sense of Remark 13). Hence

$$\|A(n).Z\|_{\mathcal{D}} \leq 2F + \|Z\|_{\mathcal{D}}$$

and by induction

$$\forall k, \forall i_j, \forall l \geq 1 \quad |\Psi(M_{i_1 \dots i_k})| \leq (cl + 1)D' \leq Dl,$$

with

$$D = [2(\|A\| \vee \|A'\|) + \|X_o\|_{\mathcal{D}}] (c + 1).$$

Using this, it follows from (18) that  $|\pi(l)|$  is bounded from above by

$$\frac{l!}{l!} l^c D \sum_{k=0}^l \binom{l}{k} = Dl(2c)^l. \quad (20)$$

This bound also holds for  $\pi^n(l)$ , for all  $n$ . Therefore we have dominated convergence for  $S_n$  on every compact  $[0, \delta]$  with  $\delta < (2c)^{-1}$ , so that the convergence is uniform on every  $[0, \delta]$ , and therefore

$$S = \lim_{n \rightarrow +\infty} S_n = \sum_l \pi(l)p^l$$

is analytic in  $p$  in the open disc of radius  $(2c)^{-1}$ . Now by Cesaro averaging,

$$S = \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \mathbb{E} [\Psi(B^i)]}{n}$$

and since

$$\frac{\sum_{i=1}^n \mathbb{E} [\Psi(B^i)]}{n} = \frac{\mathbb{E} [X_{n+2}] - \mathbb{E} [X_2]}{n} \xrightarrow{n \rightarrow +\infty} \Gamma(p),$$

(where the last limit follows the fact that Result 2 holds from Lemma 1), (19) follows immediately.

**Remark 14** Here are two sufficient conditions where, in addition to (19), we have

$$\Gamma(p) = \mathbb{E}[\Delta] \quad (21)$$

( $\Delta$  is the random variable defined in (4)):

1. If  $A$  and  $A'$  have all their entries different of  $\varepsilon$ , then we can show that  $\|X_n\|_{\mathcal{D}}$  is bounded, which implies the boundedness of  $\|A_1.X_n - X_n\|_{\infty}$ . By dominated convergence, we can then conclude as indicated at the end of §2.2 (see the proof of (5)).

The case when there exists a positive integer  $q$  such that all matrices obtained by products of  $q$  matrices of  $\{A, A'\}$  have all their entries different of  $\varepsilon$  can be handled in the same way.

A more general sufficient condition can be obtained from the finitely generated torsion semigroup property related to the so called Burnside problem ([8]).

2. When  $p$  is small enough, we can bound  $\Psi(B^n)$  uniformly by an integrable random variable, using the same kind of arguments as above. Indeed, let  $l(A')$  be the last index  $n$ ,  $n \in \{1, 0, -1, \dots\}$  such that  $A(n) = A'$  and such that the sequence  $1, 0, -1, \dots, l(A')$  has no subsequence of more than  $c$  consecutive  $A$ . Then,

$$\exists D \text{ such that } \forall n, |\Psi(B^n)| \leq Dl(A').$$

It is easy to show that  $l(A')$  is integrable. So we obtain by dominated convergence that  $\Psi(B^n)$  tends to  $\Psi(B)$  in  $L_1$ , which concludes the proof of (21).

**Remark 15** In practice we find  $c$  by calculating the successive values of  $A^n$  and then evaluate  $\pi(l)$ .

**Remark 16** The exact domain of analyticity is generally larger than the disc of radius  $(2c)^{-1}$ . We found no numerical evidence indicating the existence of a singularity at this point. This question will be examined in a future paper.

### 6.1.3 Simplification of the coefficients

The aim of this section is to simplify the expression in (18). Indeed we can easily see that many terms in (18) are redundant. Let us give an exemple through a simple case.

First remark that there is another way to write our expressions using the notation :

$$A(i_1, \dots, i_k) = A^{i_1}.A'.A^{i_2}.A' \dots A'.A^{i_k}.A'$$

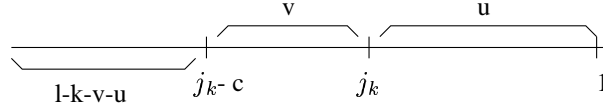
Then

$$\begin{aligned} & \frac{1}{l!} \sum_{i_1 \neq \dots \neq i_l} \Psi(M_{i_1 \dots i_l}) \\ &= \sum_{i_1 > \dots > i_l} \Psi(M_{i_1 \dots i_l}) \\ &= \sum_{i_1 = -c+2}^1 \sum_{i_2 = i_1 - c}^{i_1 - 1} \dots \sum_{i_l = i_{l-1} - c}^{i_{l-1} - 1} \Psi(M_{i_1 \dots i_l}) \\ &= \sum_{\dots} \dots \sum_{\dots} \left\{ A^{1-i_1}.A'.A^{i_1-i_2-1}.A' \dots A'.A^{i_{l-1}-i_l-1}.A'.X_o - A^{-i_1}.A' \dots A'.X_o \right\} \\ & \quad \text{with the convention } A^{-1}.A' = I_d \text{ identity matrix} \\ &= \sum_{i_1=0}^c \sum_{i_2=0}^{c-1} \dots \sum_{i_l=0}^{c-1} \left\{ A^{i_1}.A'.A^{i_2}.A' \dots A'.A^{i_l}.A'.X_o - A^{-i_1-1}.A' \dots A'.X_o \right\} \\ &= \sum_{i_1=0}^c \sum_{i_2=0}^{c-1} \dots \sum_{i_l=0}^{c-1} A^{i_1}.A'.A^{i_2}.A' \dots A'.A^{i_l}.A'.X_o \\ & \quad - \sum_{i_1=-1}^{c-1} \sum_{i_2=0}^{c-1} \dots \sum_{i_l=0}^{c-1} A^{i_1}.A'.A^{i_2}.A' \dots A'.A^{i_l}.A'.X_o \\ &= \sum_{i_2=0}^{c-1} \dots \sum_{i_l=0}^{c-1} \left\{ A^c.A'.A^{i_2}.A' \dots A'.A^{i_l}.A'.X_o - A^{i_2} \dots X_o \right\} \\ &= \sum_{i_1=0}^{c-1} \dots \sum_{i_{l-1}=0}^{c-1} \left\{ A^c.A'.A(i_1 \dots i_{l-1}).X_o - A(i_1 \dots i_{l-1}).X_o \right\}. \end{aligned}$$

Now we would like to operate in the same way for the other terms. But the difficulty is that the general term  $\Psi(M_{i_1 \dots i_k})$  is not invariant by permutation of  $\{i_1 \dots i_l\}$ . This will be taken care of by the two following lemmas.

**Lemma 3** Let  $j_k < \dots < j_1$  be such that  $|j_n - j_{n+1}| \leq c$  for  $n = 1$  to  $k - 1$  (this garantees  $\Psi(M_{j_1 \dots j_k})$  is really depending on  $k$  terms, we exclude cases where  $\Psi(M_{j_1 \dots j_k})$  could have for

value  $\Psi(M_{\text{strictly less than } k \text{ indices}})$ . Assume there are  $v$  indices of  $j_{k+1}, \dots, j_l$  in  $]j_k - c, j_k[$  and  $u$  indices of  $j_{k+1}, \dots, j_l$  between  $j_k$  and 1. We suppose the  $l$  indices  $j_1, \dots, j_k$  and  $j_{k+1}, \dots, j_l$  fixed. Then in the sum (18), all the factors of terms  $\Psi(M_{j_1 \dots j_k})$  where  $v + u$  indices have been



suppressed in  $]j_k - c, 1[$ , sum up to zero if  $l - k - v - u$  is not equal to zero.

*Proof* To get  $\Psi(M_{j_1 \dots j_k})$  we have to suppress at least  $v + u$  elements between  $j_k - c$  and 1. Therefore we have to look at the contributions of the following terms :

$$\begin{aligned} & (-1)^{v+u} \binom{l}{l-(v+u)} \Psi(M_{j_1 \dots j_{l-(v+u)}}), \\ & (-1)^{v+u+1} \binom{l}{l-(v+u+1)} \Psi(M_{j_1 \dots j_{l-(v+u+1)}}), \\ & \quad \vdots \\ & (-1)^{l-k} \binom{l}{k} \Psi(M_{j_1 \dots j_k}). \end{aligned}$$

From the first term we get :

$$\frac{1}{l!} (-1)^{v+u} \binom{l}{l-(v+u)} (v+u)! (l-(v+u))! = (-1)^{v+u}$$

because the  $l$  indices are fixed and we can only permute those giving the same value i.e.  $v + u$  and  $l - (v + u)$ .

From the second term :

$$\begin{aligned} & \frac{1}{l!} (-1)^{v+u+1} \binom{l}{l-(v+u+1)} (v+u+1)! (l-(v+u+1))! \binom{l-k-v-u}{1} \\ & = (-1)^{v+u+1} \binom{l-k-v-u}{1} \end{aligned}$$

because the  $(v + u + 1)$ th index to remove can be chosen among the  $l - k - v - u$  indices on the left.

From the  $(n + 1)$ th term :

$$(-1)^{v+u+n} \binom{l-k-v-u}{n}$$

because we can remove  $n$  indices chosen among the  $l - k - v - u$  indices on the left.

So by summation :

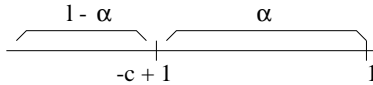
$$(-1)^{v+u} \binom{l-k-v-u}{0} + \dots + (-1)^{l-k} \binom{l-k-v-u}{l-k-v-u} = 0$$

and the lemma is proved. ♡

**Consequence 1** *The sum differs from zero only when  $l - k - v - u = 0$ . Therefore in order to get the factor of  $\Psi(M_{j_1 \dots j_k})$ , we just need to consider cases where only indices between  $j_k - c$  and 1 are suppressed. And for this operation, we choose  $l - k$  indices among  $(1 - (j_k - c) + 1) - k$  so that the factor is :*

$$\frac{1}{l!} (-1)^{l-k} \binom{l}{k} k!(l-k)! \binom{2-j_k+c-k}{l-k} = (-1)^{l-k} \binom{2-j_k+c-k}{l-k}$$

**Lemma 4 (3 bis)** *In the sum (18) all the factors giving  $\Psi(M_\emptyset)$  when suppressing  $\alpha$  indices in  $[-c+1, 1]$  sum up to zero if  $\alpha$  is not equal to  $l$ .*



*Proof* Just do as in the previous lemma to obtain :

$$(-1)^\alpha \binom{l-\alpha}{0} + \dots + (-1)^l \binom{l-\alpha}{l-\alpha} = 0.$$

♡

**Consequence 2** *The factor of  $\Psi(M_\emptyset) = \Gamma(0)$  in  $\pi(l)$  comes from the case  $l - \alpha = 0$ . Hence it is equal to  $(-1)^l \binom{c+1}{l}$  if  $l \leq c+1$  and zero if not.*

Now we can rewrite  $\pi(l)$  as follow :

$$\begin{aligned} \pi(l) = & \sum_{k=1}^l \left\{ \sum_{i_1=1}^{-c+1} \sum_{i_2=i_1-1}^{i_1-c} \dots \sum_{i_k=i_{k-1}-1}^{i_{k-1}-c} (-1)^{l-k} \binom{2-i_k+c-k}{l-k} \Psi(M_{i_1 \dots i_k}) \right\} \\ & + (-1)^l \binom{c+1}{l} \Gamma(0), \end{aligned} \quad (22)$$

with the convention  $\binom{n}{p} = 0$  if  $p > n$ .

Take now as new variables :

$$\begin{aligned} j_1 &= 1 - i_1 \\ j_2 &= i_1 - i_2 - 1 & j_1 + j_2 + \dots + j_k &= -i_k + 2 - k \\ &\vdots \\ j_l &= i_{l-1} - i_l - 1. \end{aligned}$$

Then

$$\begin{aligned} \pi(l) = & \sum_{k=1}^l \sum_{j_1=0}^c \sum_{j_2, \dots, j_k=0}^{c-1} (-1)^{l-k} \binom{c+j_1+\dots+j_k}{l-k} [A(j_1, j_2 \dots j_k).X_o \\ & - A(j_1 - 1 \dots j_k).X_o] + (-1)^l \binom{c+1}{l} \Gamma(0). \end{aligned}$$

For the  $k$  fixed indices, we have, omitting  $X_o$ :

$$\begin{aligned}
& \sum_{\dots} \binom{c+j_1+\dots+j_k}{l-k} [A(j_1 \dots j_k) - A(j_1 - 1 \dots j_k)] \\
&= \sum_{\substack{j_1=0 \\ \dots}}^c \binom{c+j_1+\dots+j_k}{l-k} A(j_1 \dots j_k) - \sum_{\substack{j_1=-1 \\ \dots}}^{c-1} \underbrace{\binom{c+j_1+\dots+j_{k+1}}{l-k}}_{\binom{c+j_1 \dots j_k}{l-k} + \binom{c+j_1 \dots j_k}{l-k-1}} A(j_1 \dots j_k) \\
&= \left\{ \sum_{\substack{j_1=0 \\ \dots}}^c \binom{c+j_1 \dots j_k}{l-k} A(j_1 \dots j_k) - \sum_{\substack{j_1=-1 \\ \dots}}^{c-1} \binom{c+j_1 \dots j_k}{l-k} A(j_1 \dots j_k) \right\} \\
&\quad - \sum_{\substack{j_1=0 \\ \dots}}^{c-1} \binom{c+j_1 \dots j_k}{l-k-1} A(j_1 \dots j_k) - \sum_{\substack{j_2=0 \\ \dots}}^{c-1} \binom{c-1+j_2 \dots j_k}{l-k-1} A(j_1 \dots j_k).
\end{aligned}$$

Simplifying the two first sums by summing w.r.t.  $j_1$ , we get :

$$\begin{aligned}
&= \sum_{\substack{j_1=0 \\ \dots}}^{c-1} \left\{ \binom{2c+j_1 \dots j_{k-1}}{l-k} A^c \cdot A' \cdot A(j_1 \dots j_{k-1}) - \underbrace{\binom{c-1+j_1 \dots j_{k-1}}{l-k} A(j_1 \dots j_{k-1})}_{\dots} \right\} \\
&\quad - \sum_{\substack{j_1=0 \\ \dots}}^{c-1} \binom{c+j_1 \dots j_k}{l-k-1} A(j_1 \dots j_k) - \sum_{\substack{j_1=0 \\ \dots}}^{c-1} \underbrace{\binom{c-1+j_1 \dots j_{k-1}}{l-k-1} A(j_1 \dots j_{k-1})}_{\dots}
\end{aligned}$$

Using the relation  $\binom{n}{p} + \binom{n}{p+1} = \binom{n+1}{p+1}$  in the underlined terms, we get :

$$\begin{aligned}
&= \sum_{\substack{j_1=0 \\ \dots}}^{c-1} \left\{ \binom{2c+j_1 \dots j_{k-1}}{l-k} A^c \cdot A' \cdot A(j_1 \dots j_{k-1}) - \binom{c+j_1 \dots j_{k-1}}{l-k} A(j_1 \dots j_{k-1}) \right\} \\
&\quad - \sum_{\substack{j_1=0 \\ \dots}}^{c-1} \binom{c+j_1 \dots j_k}{l-k-1} A(j_1 \dots j_k) \\
&= \sum_{\substack{j_1=0 \\ \dots}}^{c-1} \binom{2c+j_1 \dots j_{k-1}}{l-k} A^c \cdot A' \cdot A(j_1 \dots j_{k-1}) \\
&\quad - \underbrace{\sum_{\substack{j_1=0 \\ \dots}}^{c-1} \binom{c+j_1 \dots j_k}{l-k-1} A(j_1 \dots j_k)}_{\Sigma_k} - \underbrace{\sum_{\substack{j_1=0 \\ \dots}}^{c-1} \binom{c+j_1 \dots j_{k-1}}{l-k} A(j_1 \dots j_{k-1})}_{\Sigma_{k-1}}.
\end{aligned}$$

Injecting this in the first equation, the terms  $\Sigma_k$  cancel each other except for the last one for  $k = 1$ , which is equal to  $\binom{c}{l-1} X_o$ , so that:

$$\pi(l) = \sum_{k=1}^l \left\{ (-1)^{l-k} \sum_{\substack{j_1=0 \dots c-1, \dots \\ j_{k-1}=0 \dots c-1}} \binom{2c+j_1+\dots+j_{k-1}}{l-k} A^c \cdot A' \cdot A(j_1 \dots j_{k-1}) \cdot X_o \right\}$$

$$-(-1)^{l-1} \binom{c}{l-1} X_o + (-1)^l \binom{c+1}{l} \Gamma(0).$$

This proves the first part of Theorem 1. ♡

### 6.1.4 Complexity and error term

In this section, we give an estimate of the computational cost to evaluate the Lyapunov exponent by Formula (6), when using a Taylor approximation of order  $l$ , (and when supposing that the coupling time  $c$ , the eigen vector  $X_o$ , the eigenvalue  $\gamma(0)$  and all binomial coefficients are given). We also compare this to what would be obtained by Formula (18). The estimates are given up to a multiplicative constant.

To evaluate  $\pi(l)$  by applying (18), or equivalently (16) with  $n = cl$ , we need :

- for  $\Psi(M_{i_1..i_k})$ , we need  $2d^2(2 - \max(i_1..i_k))$  operations (summations and multiplications) since we multiply  $(2 - \max(i_1..i_k))$  matrices of size  $d \times d$  by a vector. Hence we need at least  $2d^2k$  operations.
- for  $\sum_{\substack{i_1 \neq \dots \neq i_l \\ i_j \geq -lc}} \Psi(M_{i_1..i_k})$  :

$$\text{we write this as } k!(l-k)! \underbrace{\sum_{i_1 > \dots > i_k} \binom{lc}{k}}_{\text{terms}} \underbrace{\sum_{i_{k+1} > \dots > i_l} \binom{lc-k}{l-k}}_{\text{terms}} \underbrace{\Psi(M_{i_1..i_k})}_{2d^2k}$$

$$\text{Then we need } \binom{lc}{k} \times \binom{lc-k}{l-k} \times 2d^2k \text{ operations.}$$

$$\implies \text{at least } d^2 l 2^l \binom{lc}{l}.$$

In order to evaluate  $\pi(l)$  using Equation (6) :

- for the generic term, we need :

$$\begin{aligned} 2d^2 \sum_{j_1..j_{k-1}=0}^{c-1} (k + j_1 + \dots + j_{k-1}) &= (kc^{k-1} + (k-1)\frac{c(c-1)}{2}c^{k-2})2d^2 \\ &= c^{k-1} \left[ k + (k-1)\frac{c-1}{2} \right] 2d^2 \\ &\sim kc^k d^2 \end{aligned}$$

$$\implies \sim d^2 lc^l.$$

As for the error term in a Taylor approximation of order  $l$ , since we have a geometrically dominated convergence in the analyticity region from (20), the simplest bound on this error is :

$$D\delta \left( \frac{(2c\delta)^{l+1}}{1-2c\delta} \right)^l = \frac{D(2c\delta)^{l+1}[1+l(1-2c\delta)]}{(1-2c\delta)^2} \text{ for } p \in [0, \delta], \delta < \frac{1}{2c}.$$

And this proves the second part of Theorem 1. ♡

## 6.2 Multinomial case

The proof in multinomial case is an extension of the above calculations.

### 6.2.1 Evaluation of $\Psi$

By conditioning as in the binomial case :

$$\begin{aligned} \mathbb{E} [\Psi(B^n)] &= \sum_{|K| \leq n+2} \mathbb{E} [\Psi(B^n) \mid |B^n| = K] P(|B^n| = K) \\ &= \sum_{|K| \leq n+2} p_1^{k_1} \cdots p_m^{k_m} (1 - p_1 - \cdots - p_m)^{n+2-|K|} \left( \sum_{\substack{M \\ |M|=K}} \Psi(M) \right) \end{aligned}$$

with

$$(1 - p_1 - \cdots - p_m)^{n+2-|K|} = \sum_{|L| \leq n+2-|K|} (-1)^{|L|} p_1^{l_1} \cdots p_m^{l_m} \frac{(n+2-|K|)!}{l_1! \cdots l_d! (n+2-|K|-|L|)!},$$

so that

$$\begin{aligned} \mathbb{E} [\Psi(B^n)] &= \sum_{|K| \leq n+2} \prod_{1 \leq i \leq m} p_i^{k_i} \sum_{|L| \leq n+2-|K|} (-1)^{|L|} \prod_{1 \leq i \leq m} p_i^{l_i} \binom{n+2-|K|}{L} \left( \sum_{\substack{M \\ |M|=K}} \Psi(M) \right) \\ &= \sum_{|K| \leq n+2} \sum_{|L|+|K| \leq n+2} (-1)^{|L|} \prod_{1 \leq i \leq m} p_i^{l_i+k_i} \binom{n+2-|K|}{L} \left( \sum_{\substack{M \\ |M|=K}} \Psi(M) \right). \end{aligned}$$

Taking as new variables  $l_i := l_i + k_i$

$$\begin{aligned} \mathbb{E} [\Psi(B^n)] &= \sum_{|K| \leq n+2} \sum_{\substack{K \leq L \\ |L| \leq n+2}} (-1)^{|L|-|K|} \prod_{1 \leq i \leq m} p_i^{l_i} \binom{n+2-|K|}{L-K} \left( \sum_{\substack{M \\ |M|=K}} \Psi(M) \right) \\ &= \sum_{|L| \leq n+2} \left\{ \sum_{K \leq L} (-1)^{|L|-|K|} \binom{n+2-|K|}{L-K} \left( \sum_{\substack{M \\ |M|=K}} \Psi(M) \right) \right\} \prod_{1 \leq i \leq m} p_i^{l_i}. \end{aligned}$$

We shall denote  $\pi^n(L)$  be the coefficient of  $\prod_{1 \leq i \leq m} p_i^{l_i}$  in the last sum. Below,  $q_j^i \neq$  means that the variables  $q_j^i$  are all different, and  $M_{q_1^1 \cdots q_{k_1}^1 q_1^2 \cdots q_{k_2}^2 \cdots q_1^m \cdots q_{k_m}^m}$  denotes the infinite sequence with all its elements equal to  $A$ , but for those of indices  $q_1^1, \dots, q_{k_1}^1$  which are all equal to  $A'_i$ ,  $\forall i = 1, \dots, m$ . We have :

$$\bullet \quad \sum_{|M|=K} \Psi(M) = \frac{1}{K!} \sum_{\substack{q_j^i \neq \\ -n \leq q_j^i \leq 1}} \Psi(M_{q_1^1 \cdots q_{k_1}^1 q_1^2 \cdots q_{k_2}^2 \cdots q_1^m \cdots q_{k_m}^m}),$$

$$\bullet \quad \mathcal{A}_{n-|K|}^{|L|-|K|} \sum_{\substack{q_j^i \neq \\ 1 \leq q_j^i \leq n \\ 1 \leq j \leq k_j}} \Psi(M_{q_1^1 \dots q_{k_m}^m}) = \sum_{\substack{q_j^i \neq \\ 1 \leq q_j^i \leq n \\ 1 \leq j \leq l_j}} \Psi(M_{q_1^1 \dots q_{k_m}^m}).$$

Applying this and simplifying, we get :

$$\pi^n(L) = \frac{1}{L!} \sum_{\substack{q_j^i \neq \\ -n \leq q_j^i \leq 1 \\ 1 \leq j \leq l_j}} \left\{ \sum_{K \leq L} (-1)^{|L|-|K|} \binom{L}{K} \Psi(M_{q_1^1 \dots q_{k_m}^m}) \right\}. \quad (23)$$

### 6.2.2 Convergence

We proceed exactly in the same way as for Lemmas 2 and 3. Reasoning term by term, one can check that the coefficient  $\beta_{Q-J}$  becomes :

$$\beta_{Q-J} = (L - Q)! \left\{ \sum_{K \leq L} (-1)^{|L|-|K|} \binom{L}{K} \prod_{1 \leq i \leq m} \mathcal{A}_{k_i}^{q_i - j_i} \mathcal{A}_{l_i - k_i}^{j_i} \right\}.$$

Here  $\mathcal{A}_{k_i}^{q_i - j_i} \mathcal{A}_{l_i - k_i}^{j_i}$  is a multivariate polynomial in  $k$ , the degree of which in  $k_i$  is less than  $q_i$ , that is less than  $l_i - 1$ . And we have again :

$$\sum_{K \leq L} (-1)^{|K|} \binom{L}{K} k_1^{\omega_1} \dots k_m^{\omega_m} = 0,$$

if for all  $i$ ,  $\omega_i \leq l_i - 1$ . This can be proved by differentiating  $\omega_i$  times the function  $\prod_{1 \leq i \leq m} (1 - x_i)^{l_i}$  w.r.t  $x_i$  and evaluating it at the point  $(1, \dots, 1)$ .

The variable  $\pi^n(L)$  is bounded from above (23) by  $D|L|(2c)^{|L|} \frac{|L|!}{L!}$  which can be bounded itself by  $Dl(2cm)^l$  when summing on  $|L| = l$  since  $m^n = \sum_{|K|=n} \frac{n!}{K!}$ , so that the convergence region contains the ball of radius  $(2cm)^{-1}$  w.r.t. norm  $\|\cdot\|_\infty$  on  $\mathbb{R}^m$  defined by  $\|P\|_\infty = \max_{1 \leq i \leq m} |p_i|$ .

For the extension of Lemma 3, the coefficients we obtain are :

$$\sum_{L-(V+U) \leq I \leq K} (-1)^{|L|-|I|} \binom{L-I-V-U}{L-K-V-U} = 0.$$

The other steps of the proof of Theorem 2 are very similar to those of the binomial case.

### 6.2.3 Complexity

To evaluate  $\pi(L)$  by applying Equation (8), we need a first summation with :

$$2d^2 \sum_{j_1 \dots j_{|K|-1}=0}^{c-1} (|K| + j_1 + \dots + j_{|K|-1}) \sim |K| c^{|K|} d^2$$

terms.



- The sum  $\sum_{\substack{C_n, n=1, \dots, |K| \\ \text{with } C_n = A'_i \text{ for some } i \\ \text{Card}\{n: C_n = A'_i\} = k_i}} c_n$ , multiplies this by  $\frac{|K|!}{K!}$ ;
- The sum  $\sum_{\substack{1 \leq |K| \\ \bar{K} \leq L}} \frac{|K|!}{K!}$ , multiplies this by  $m^{|L|}$  (because  $\sum_{\substack{1 \leq |K| \\ \bar{K} \leq L}} \frac{|K|!}{K!} \leq \sum_{|K| \leq |L|} \frac{|K|!}{K!}$  that is of the order  $m^{|L|}$ ).

Hence, for expansion of order  $|L|$  we get the estimate :

$$d^2 |L| (cm)^{|L|}. \quad (24)$$

### 6.3 The operator case

We first consider the *finite horizon* expansion for which we can write :

$$\begin{aligned} \mathbb{E}[\Psi(B^n)] &= E[f \circ A(1) \circ A(0) \circ \dots \circ A(-n)(X_o) - f \circ A(0) \circ \dots \circ A(-n)(X_o)] \\ &= \sum_{l=0}^{n+2} \pi_f^n(l) p^l, \end{aligned}$$

when conditioning w.r.t. the choices made for the random variables  $A(l)$ ,  $1 \geq l \geq -n$ .

**Lemma 5** *Under (P2), we have,*

$$\lim_{n \rightarrow \infty} \pi_f^n(l) = \pi_f(l),$$

where the limit  $\pi_f(l)$  is reached in a finite time and is given by :

$$\begin{aligned} \pi_f(l) &= \sum_{k=1}^l \left\{ \sum_{i_1=1}^{-c+1} \sum_{i_2=i_1-1}^{i_1-c} \dots \sum_{i_k=i_{k-1}-1}^{i_{k-1}-c} (-1)^{l-k} \binom{2-i_k+c-k}{l-k} \Psi(M_{i_1 \dots i_k}) \right\} \\ &\quad + (-1)^l \binom{c+1}{l} \Psi(M_\emptyset), \end{aligned}$$

where  $M_{i_1 \dots i_k} = (A(1), A(2), \dots, A(i_k))$  is the sequence with  $A(i) = A$  for all  $i = 1, \dots, i_k$ , but for  $i_1, \dots, i_k$  where  $A(i) = A'$ .

*Proof*

One easily checks that (P2) is a sufficient condition to obtain Lemmas 2, 3, 4. ♡

**Lemma 6** *Under (P2),  $\pi_f(l)$  is given by (10).*

*Proof*

Just repeat the last calculations of Section 6.1.3 replacing all terms  $A(j_1, \dots, j_k) \otimes X_o$  by  $f(\Xi(j_1, \dots, j_k)(X_o))$ , where

$$\Xi(j_1, \dots, j_k) = A^{j_1} \circ A' \circ \dots \circ A^{j_k} \circ A'.$$

♡

*Proof of Theorem 3*

On  $[0, \frac{1}{2c\eta}[$ , we have the equality :

$$\lim_{n \rightarrow +\infty} \mathbf{E} [\Psi(B^n)] = \sum_l \pi_f(l) p^l$$

and this convergence is dominated and hence uniform on every compact in  $[0, \frac{1}{2c\eta}[$ .

(P3) implies that  $\mathbf{E} [f(X_n)]$  is finite for  $n \geq 0$ . Now using (P1) and a Cesaro averaging argument,

$$\frac{\sum_{i=n_0-1}^n \mathbf{E} [\Psi(B^i)]}{n} = \frac{\mathbf{E} [f(X_{n+2})] - \mathbf{E} [f(X_{n_0})]}{n} \xrightarrow{n \rightarrow +\infty} \gamma_f(p).$$

♡

## 6.4 The general case

The proofs are similar to those of the Bernoulli case for  $c = 1$ . The main property leading to (18) from the first relation (15) is indeed only based the fact that the function  $\Psi$  *loses memory of the past*.

## 7 Other expansions and relationship between expansions

### 7.1 Yet another binomial expansion

There is an immediate extension of Theorem 1 when each occurrence of matrix  $A'$  is replaced by that of an independent random matrix  $A'(\omega)$ , which admits a discrete or continuous distribution. More precisely,  $A_n$  is equal to  $A$  with probability  $1 - p$  and to  $B_n$  with probability  $p$ , where the sequence  $\{B_n\}$  is i.i.d. and independent of the sequence used to chose between  $A$  and the alternative. Then, if we note  $E'$  the expectation w.r.t. this distribution, (6) can be reformulated as follows:

$$\begin{aligned} \begin{bmatrix} \pi(l) \\ \vdots \\ \pi(l) \end{bmatrix} &= (-1)^l \left\{ \binom{c}{l-1} X_o + \binom{c+1}{l} \Gamma(0) \right\} + \sum_{k=1}^l \\ &\left\{ (-1)^{l-k} \sum_{j_1, \dots, j_{k-1}=0}^{c-1} \binom{2c+j_1+\dots+j_{k-1}}{l-k} E' [A^c \otimes A' \otimes A^{j_1} \otimes A' \dots A^{j_{k-1}} \otimes A' \otimes X_o] \right\}. \end{aligned}$$

Similar extensions can of course be contemplated for all the extensions considered in §4.1–4.5.

### 7.2 Binomial versus Multinomial

An interesting special case of what is considered in the previous subsection is that when the law of  $B_n$  is discrete and with finite support. Assume for instance that we are in the setting of §4.1, namely  $A'(\omega)$  is equal to  $A'_j$  with probability  $p_j$ ,  $j = 1, \dots, m$ , and each of the matrices in question have at least one non  $\varepsilon$  element per line. In this case, we have both an expansion in the parameter  $p = p_1 + p_2 + \dots + p_m$  which is that of (25) above, and an expansion in the multiparameter  $P' = (pp_1, \dots, pp_m)$  which is that of Theorem 2.

### 7.3 Symmetrical versus Asymmetrical

In the multinomial case, we obtained an analytic expansion of the Lyapunov exponent w.r.t. the parameters  $(p_1, \dots, p_m)$  of the law, which are such that  $p_0 = 1 - p_1 - \dots - p_m$  represents the probability of an event which should be frequent enough to entail the memory loss property (e.g. in the  $(\max, +)$ - algebra case the sampling of the scs1-cyc1 matrix  $A$ ). This is an asymmetrical expansion in that it is not made w.r.t. all parameters  $(p_0, \dots, p_m)$ .

Within the general setting of §4.5, symmetrical expansions can also be derived in complement to those obtained so far in the following way: for all integers  $l \geq 0$ , we have the following representation:

$$\begin{aligned} \mathbb{E}[X_l] &= \sum_{k_0 + \dots + k_m = l} \left\{ \sum_{\substack{|C_1, \dots, C_l| = (k_0, \dots, k_m) \\ C_i \in \mathcal{L}}} C_1 \circ \dots \circ C_l(X_o) \right\} p_0^{k_0} \dots p_m^{k_m} \\ &= \sum_{|\bar{K}|=l} \left\{ \sum_{\substack{|C|=\bar{K} \\ C_i \in \mathcal{L}}} C_1 \circ \dots \circ C_l(X_o) \right\} \bar{P}^{\bar{K}}, \end{aligned}$$

where  $\bar{K} = (k_0, k_1, \dots, k_m)$  and  $\bar{P} = (p_0, \dots, p_m)$ . So for  $l \geq 2$ ,

$$\begin{aligned} \mathbb{E}[X_l - X_{l-1}] - \mathbb{E}[X_{l-1} - X_{l-2}] &= \sum_{|\bar{K}|=l} \\ &\left\{ \sum_{\substack{|C|=\bar{K} \\ C_i \in \mathcal{L} \\ i=1..l}} C_1 \circ \dots \circ C_l(X_o) - 2C_1 \circ \dots \circ C_{l-1}(X_o) + C_1 \circ \dots \circ C_{l-2}(X_o) \right\} \bar{P}^{\bar{K}}. \end{aligned}$$

So, if we define

$$\pi(\bar{L}) = \Delta^2 S(\bar{L}),$$

where

$$S(\bar{L}) = \sum_{\substack{|C|=(l_0, \dots, l_m) \\ C_i \in \mathcal{L}}} \{C_1 \circ \dots \circ C_{|L|}(X_o)\},$$

we obtain:

$$\mathbb{E}[X_{l+1} - X_l] - \mathbb{E}[X_l - X_{l-1}] = \sum_{|\bar{L}|=l} \pi(\bar{L}) \bar{P}^{\bar{L}}.$$

This also gives the terms of order  $l$  in the symmetrical expansion of

$$\mathbb{E}[X_{n+1} - X_n] = \sum_{j=0}^{n+1} \{\mathbb{E}[X_j - X_{j-1}] - \mathbb{E}[X_{j-1} - X_{j-2}]\}$$

with  $X_{-1} = X_{-2} = 0$ .

We can then use the convergence of  $\mathbb{E}[X_{n+1} - X_n]$  to  $\Gamma$  to derive the coefficients of the symmetrical expansion of  $\Gamma$ .

Let us comment on how this result is related to the asymmetrical expansion with  $c = 1$ . Consider the multinomial  $(\max, +)$ -setting of §4.1; we associate with this a  $(m + 2)$ -nomial scheme with parameters  $(p, p_0 - \frac{p}{m+1}, \dots, p_m - \frac{p}{m+1})$ , where  $p$  is a real number such that this vector is a probability law. In this scheme, matrix  $A'_i$  is sampled with probability  $p'_i = p_i - \frac{p}{m+1}$ , for all  $i = 0, \dots, m$  (we take  $A'_0 = A$ ), and with probability  $p$ , the matrix  $\mathbf{0}$  which has all its entries equal to 0 is sampled. We can then apply the asymmetrical expansion w.r.t.  $(p, p'_0, \dots, p'_m)$ , when taking as scs1 cyc1 matrix the matrix  $\mathbf{0}$ , which has for unique eigenvector the vector  $\mathbf{e}$  with all its coordinates equal to 0, and for which  $c = 1$ . It is then easy to check that the coefficient of  $(p'_0)^{l_0} \dots (p'_m)^{l_m}$  in the asymmetrical expansion for this  $(m + 2)$ -nomial scheme is equal to the coefficient of  $(p_0)^{l_0} \dots (p_m)^{l_m}$  in the above symmetrical expansion.

## 8 Comparison with other approaches

### 8.1 Simulation

In this section, we compare the asymptotic expansions of the previous sections with direct simulations based on a sampling of the random matrices.

An important preliminary remark is in order. Even if we would know how to compute a confidence interval  $I(n)$  for the estimator  $X_n/n$  of  $\gamma(p)$ , based on such a simulation, there would of course only be a high probability, say 95%, that  $\gamma(p)$  is in the interval  $[X_n/n - I(n), X_n/n + I(n)]$ . In contrast, when it can be used, the expansion gives certitude that  $\gamma(p)$  is within a certain interval determined by an expansion of finite order and the associated error bound.

#### 8.1.1 Example 1

This example is a continuation of that of §3.2.3. A simulation based on the sampling of Bernoulli variables (in Maple) gives the following expressions for  $X_n/n$  when  $p = \frac{1}{12}$  :

$$\begin{aligned} &\text{for } n = 1000, \quad 0,998\sigma + 0,005\sigma' \\ &\text{for } n = 2000, \quad 0,995\sigma + 0,006\sigma'. \end{aligned}$$

The expansion of order 3 yields :

$$\gamma\left(\frac{1}{12}\right) = 0,994\sigma + 0,006\sigma' + o(p^3).$$

For this sample, the precision looks better than the one we have via the expansion (which is of the form  $\frac{D(14\delta)^4[1+3(1-14\delta)]}{(1-14\delta)^2}$ ), and particularly here since we do not have  $p < 1/2c$ .

### 8.1.2 Example 2

Here we take two arbitrary and unrelated matrices  $A, A'$ . To make sure that (H1) and (H2) hold,  $A$  is taken of the same form as above. We take :

$$A = \begin{pmatrix} 3 & \varepsilon & \varepsilon & 0 \\ 3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & 4 & \varepsilon \end{pmatrix}, \quad A' = \begin{pmatrix} 1 & 1 & \varepsilon & \varepsilon \\ \varepsilon & 2 & 4 & \varepsilon \\ 0 & \varepsilon & \varepsilon & 2 \\ 1 & 2 & \varepsilon & \varepsilon \end{pmatrix}.$$

Then we still have  $c = 7$  and  $\gamma(0) = 3$  and  $X_o = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 1 \end{pmatrix}$ .

From Theorem 1, we get :

$$\gamma(p) = 3 - 2p + p^2 + 2p^3 + 7p^4 + 5p^5 - 61p^6 - 15p^7 \dots$$

Now for  $p = \frac{1}{10}$ , we have :

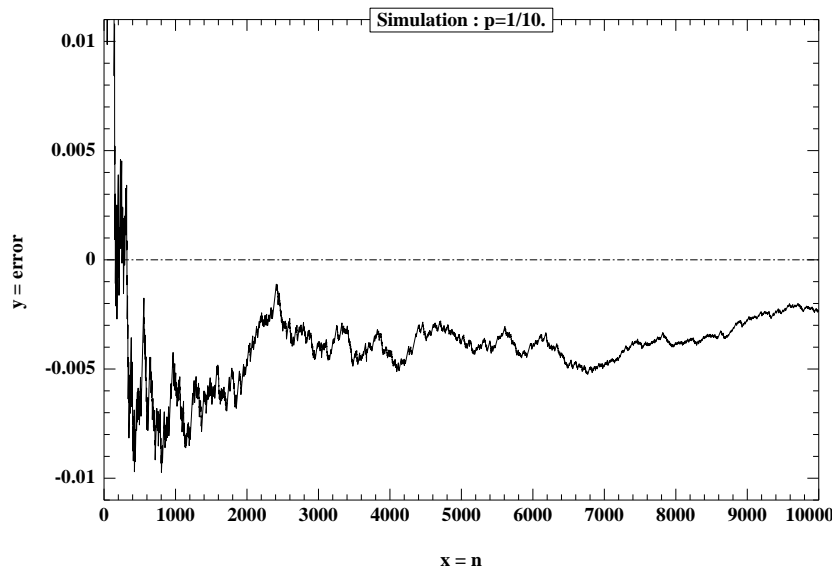
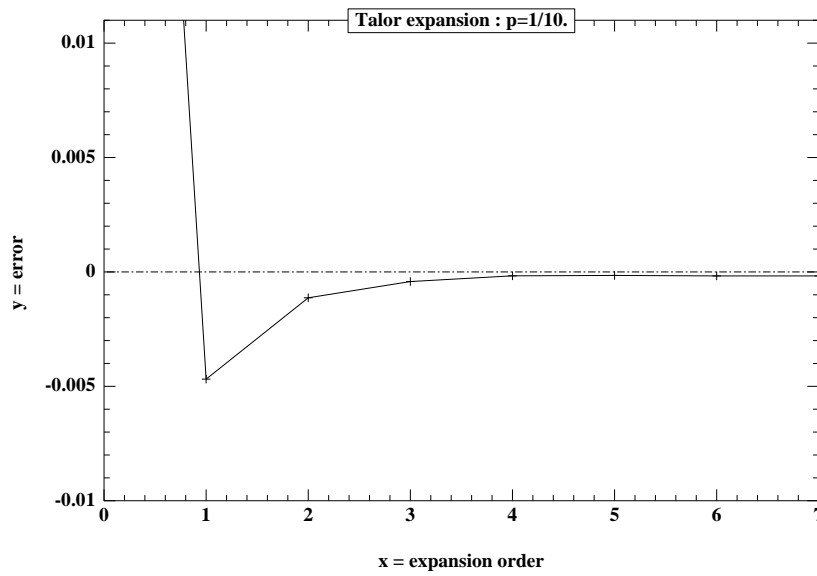
order	0	1	2	3	4	5	6	7
Series	3	2.8	2.81	2.812	2.813	2.813	2.813	2.813
Error(%)	6.64	-0.47	-0.11	-0.04	-0.02	-0.02	-0.02	-0.02

The error is estimated w.r.t. the simulation  $n = 100000$  (2.81318).

In comparison, a direct simulation gives :

n	100	200	300	400	500	1000	2000
Simulation	2.910	2.815	2.820	2.793	2.794	2.798	2.800
Error(%)	3.44	0.06	0.24	-0.74	-0.68	-0.54	-0.49

$\mathbf{n}$  is the number of sampled random matrices and the error is estimated w.r.t. 2.81318 ( $n = 100000$ ).



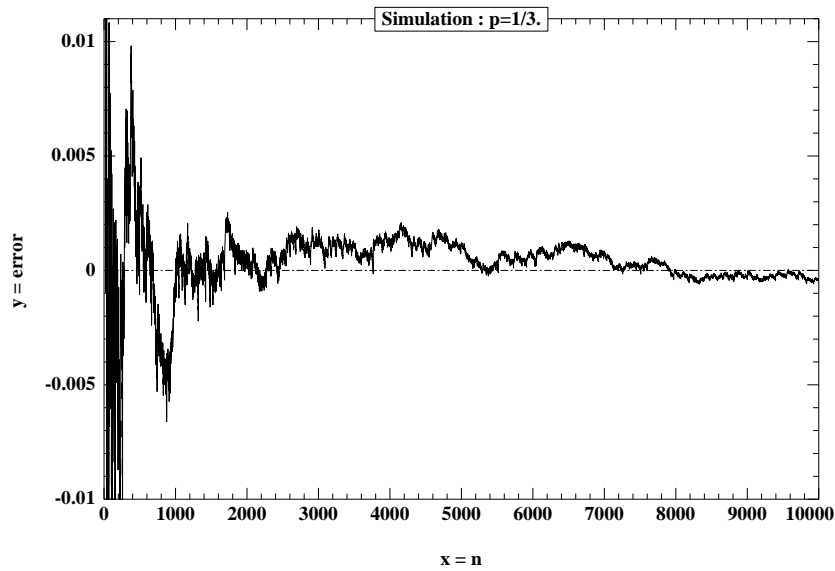
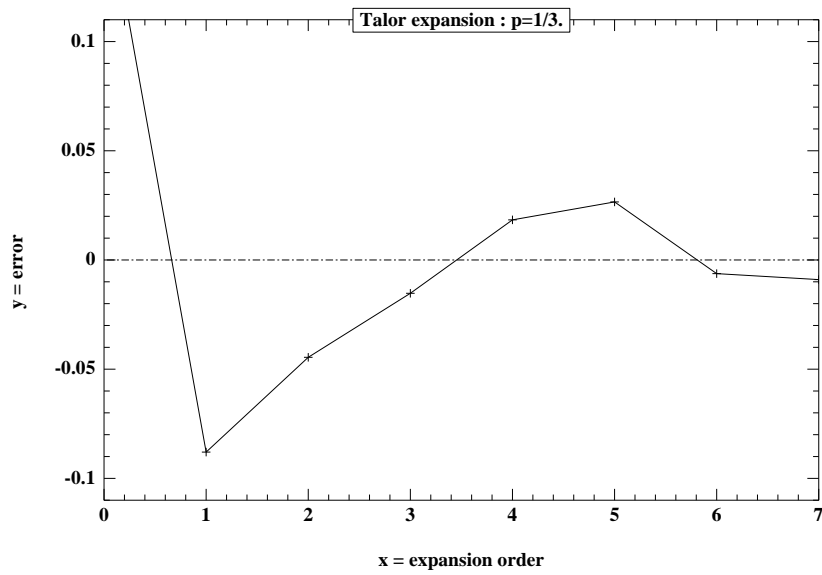
The direct simulation shows fluctuations which make the estimation of  $\gamma$  difficult, whereas the power series converges very quickly to  $\gamma$ .

In order to check what happens with the formula of Theorem 1 when  $p$  is larger, namely when the expansion method is expected to be less effective, here is what we get for  $p = \frac{1}{3}$  via the same method,

order	0	1	2	3	4	5	6	7
Simulation	3.000	2.333	2.444	2.519	2.605	2.626	2.542	2.535
Error(%)	17.3	-8.80	-4.46	-1.53	1.84	2.66	-0.63	-0.90

The error is estimated w.r.t. the simulation  $n = 100000(2.55801)$ . The simulation gives (with the same rules as above) :

<b>n</b>	100	200	300	400	500	1000	2000
Simulation	2.650	2.550	2.570	2.575	2.564	2.558	2.559
Error(%)	0.08	-0.31	0.47	0.64	0.23	-0.00	0.04



## 8.2 The regenerative theory approach

The aim of this subsection is to investigate another potential way of obtaining the main result, based on regenerative theory. The main conclusion will be that the approach via perturbation formulas which was followed above seems to be more effective.

As in Section 6, we use here the simplified notation "." instead of  $\otimes$ .

We start with the following basic observation:

**Theorem 6** *Assume (H1), (H2) and (H3) hold. Let*

$$\begin{aligned} T_1 &= \inf\{n \geq c : X_n = A^c \cdot A_{n-1-c} \cdots A_0 \cdot X_0\} \\ T_k &= \inf\{n \geq T_{k-1} + c : X_n = A^c \cdot A_{n-1-c} \cdots A_0 \cdot X_0\} \quad \forall k \geq 2, \end{aligned}$$

with the conventions :  $T_0 = 0$  and  $A_{-1} \cdot A_0 = \text{identity matrix in } (\max, +)$ .

For all  $n \geq 0$ ,  $S_n = T_{n+1} - T_n$ ,  $\Lambda_n = X_{n+1} - X_n$ ,  $Z = f(\Lambda_0, \dots, \Lambda_{T_1-1})$  (here we only need  $Z = \Lambda_0 + \dots + \Lambda_{T_1-1}$ ).

Then  $\{T_n, \Lambda_n\}_{n \geq 0}$  is a regenerative process i.e.  $\{S_n\}_{n \geq 0}$  and  $\{Z \circ \Theta_{T_i}\}_{i \geq 0}$  are i.i.d.

Moreover we have for  $p \in ]0, 1[$ :

$$\Gamma(p) = \frac{\mathbb{E}[X_{T_1} - X_0]}{\mathbb{E}[T_1]}, \quad \mathbb{E}[T_1] = \frac{1 - (1-p)^c}{p(1-p)^c}. \quad (25)$$

**Remark 17** *The relation (25) is valid for  $p = 0$  or  $p = 1$  by taking the proper limits.*

**Remark 18** *From the first equality in (25), the property of analyticity could be derived using results obtained by M.A. Zazanis ([18]) who studied Poisson process depending functional under certain moment conditions. His approach consisting in a change of measure argument can be extended to the Bernoulli scheme considered here.*

*Proof*

- 1) The fact that the random variables  $\{S_n\}$  are i.i.d. is immediate.
- 2) One checks that the random variables  $T_n$  are stopping times such that  $T_n = T_{n-1} + T_1 \circ \Theta_{T_{n-1}}$ . The random variable  $Z$  is  $\mathcal{F}_{T_1}$ -measurable, where  $\mathcal{F}_{T_1}$  is the  $\sigma$ -algebra generated by  $\{X_0, X_1, \dots, X_{T_1}\}$ . That the random variables  $\{Z \circ \Theta_{T_i}\}_{i \geq 0}$  are i.i.d. follows from the Strong Markov property for discrete time Markov chains.
- 3) Take  $g(X_k) = X_{k+1} - X_k$ . From the ergodic theorem :

$$\frac{1}{m} \sum_{0 \leq k \leq m-1} g(X_k) \xrightarrow{a.s.} \frac{1}{\mathbb{E}[T_1]} \mathbb{E}[X_{T_1} - X_0]$$

and

$$\frac{1}{m} \sum_{0 \leq k \leq m-1} g(X_k) = \frac{X_m}{m} \xrightarrow{a.s.} \Gamma(p).$$

For proving the last equation, we use the relations :

$$\begin{aligned} P(T_1 = k) &= 0 \quad \text{for } k < c \\ P(T_1 = c) &= (1-p)^c \\ P(T_1 = k) &= p(1-p)^c P(T_1 > k - c - 1) \quad \text{for } k > c \end{aligned}$$



Thus :

$$\begin{aligned}
\sum_{k \geq 0} P(T_1 = k) &= (1-p)^c + \sum_{k > c} p(1-p)^c P(T_1 > k - c - 1) \\
&= (1-p)^c + p(1-p)^c \left( \sum_{k \geq 0} P(T_1 > k) \right) \\
\implies 1 &= (1-p)^c + p(1-p)^c \mathbb{E}[T_1].
\end{aligned}$$

This concludes the proof. ♡

In order to rederive the result of Theorem 1 via this representation, we now need a direct evaluation of the numerator of the first equation of (25). We have :

$$\begin{aligned}
\mathbb{E}[X_{T_1}] &= \sum_{n \geq 0} \mathbb{E}[X_{T_1}/T_1 = n] P[T_1 = n] \\
&= \mathbb{E}[X_{T_1}/T_1 = c] P[T_1 = c] + \sum_{n \geq c+1} \mathbb{E}[X_{T_1}/T_1 = n] P[T_1 = n] \\
&= (1-p)^c A^c . X_o + \sum_{n \geq 0} \mathbb{E}[X_{T_1}/T_1 = n + c + 1] P[T_1 = n + c + 1],
\end{aligned}$$

with

$$\mathbb{E}[X_{T_1}/T_1 = n + c + 1] = \sum_{\substack{i_1 + \dots + i_{l+1} + l = n \\ i_j \leq c-1}} \binom{n}{l} p^l (1-p)^{n-l} A^c . A' . A^{i_1} \dots A' . A^{i_{l+1}} . X_o.$$

Substituting this expression in the upper equality :

$$\begin{aligned}
\mathbb{E}[X_{T_1}] &= (1-p)^c A^c . X_o + \tag{26} \\
&\sum_{n \geq 0} \sum_{\substack{i_1 + \dots + i_{l+1} + l = n \\ i_j \leq c-1}} \binom{n}{l} A^c . A' . A^{i_1} \dots A' . A^{i_{l+1}} . X_o p^l (1-p)^{n-l} P[T_1 = n + c + 1].
\end{aligned}$$

In order to obtain the law of  $T_1$ , we use generating functions as follows :

$$\begin{aligned}
\mathbb{E}[z^{T_1}] &= \sum_{k=0}^{+\infty} P(T_1 = k) z^k \\
&= (1-p)^c z^c + p(1-p)^c \sum_{k \geq c+1} P(T_1 > k - c - 1) z^k \\
&= (1-p)^c z^c + p(1-p)^c \sum_{k \geq 0} P(T_1 > k) z^{c+1+k} \\
&= (1-p)^c z^c + p(1-p)^c z^{c+1} \sum_{k \geq 0} P(T_1 > k) z^k \\
&= (1-p)^c z^c + p(1-p)^c z^{c+1} \sum_{k \geq 1} P(T_1 = k) \underbrace{(z^0 + \dots + z^{k-1})}_{\frac{z^k - 1}{z - 1}} \\
&= (1-p)^c z^c + p(1-p)^c \frac{z^{c+1}}{z - 1} \left( \mathbb{E}[z^{T_1}] - 1 \right),
\end{aligned}$$

thus,

$$\Phi(z) = \mathbb{E} \left[ z^{T_1} \right] = \frac{(1-p)^c z^c \left( 1 + \frac{pz}{1-z} \right)}{1 + p(1-p)^c \left( \frac{z^{c+1}}{1-z} \right)}.$$

Now we have :

$$P(T_1 = n) = \frac{1}{n!} \Phi^{(n)}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(e^{jw}) e^{-jnw} dw. \quad (27)$$

Let's see first what  $\Phi^{(n)}(0)$  looks like :

$$\Phi(z) = \frac{(1-p)^c z^c (1-z) + p(1-p)^c z^{c+1}}{(1-z) + p(1-p)^c z^{c+1}} = a \frac{(p-1)z^{c+1} + z^c}{1 - (z - paz^{c+1})},$$

with  $a = (1-p)^c$ .

We can now expand this as a power series using the relation :

$$\frac{1}{1 - (z - paz^{c+1})} = \sum_{k=0}^{+\infty} z^k (1 - paz^c)^k$$

that is

$$\frac{1}{1 - (z - paz^{c+1})} = \sum_{k=0}^{+\infty} \sum_{i=0}^k \binom{k}{i} (-pa)^i z^{ci+k}.$$

So,

$$\begin{aligned} \Phi(z) &= a \sum_{k=0}^{+\infty} \sum_{i=0}^k \binom{k}{i} (p-1)(-pa)^i z^{c(i+1)+k+1} + a \sum_{k=0}^{+\infty} \sum_{i=0}^k \binom{k}{i} (-pa)^i z^{c(i+1)+k} \\ &= a \sum_{k>0} \sum_{i=0}^{k-1} \binom{k-1}{i} (p-1)(-pa)^i z^{c(i+1)+k} + idem. \\ &= a \sum_{k>0} \sum_{i=0}^k \left( \binom{k}{i} + (p-1) \binom{k-1}{i} \right) (-pa)^i z^{c(i+1)+k} + az^c \\ &= a \sum_{k>0} \sum_{i=0}^k \left( \binom{k-1}{i-1} + p \binom{k-1}{i} \right) (-pa)^i z^{c(i+1)+k} + az^c. \end{aligned}$$

To get the coefficient of  $z^{c(m+1)+r}$  in the last expression, we have to take :

$$\begin{aligned} i &= m & , k &= r \\ i &= m-1 & , k &= c+r \\ & \vdots & & \\ i &= 0 & , k &= cm+r. \end{aligned}$$

Therefore,

$$\begin{aligned} \Phi(z) &= a \sum_{\substack{n=c(m+1)+r \\ n>c}} \left\{ \left( \binom{r-1}{m-1} + p \binom{r-1}{m} \right) (-pa)^m + \dots \right. \\ &\quad \left. + \left( \binom{cm+r-1}{-1} + p \binom{cm+r-1}{0} \right) (-pa)^0 \right\} z^{c(m+1)+r} + az^c. \end{aligned}$$

and for  $n \geq c + 1$  :

$$\frac{1}{n!} \Phi^{(n)}(0) = a \left\{ \sum_{i=0}^m \left( \binom{(m-i)c+r-1}{i-1} + p \binom{(m-i)c+r-1}{i} \right) (-pa)^i \right\}.$$

Since in (26)  $n$  is bounded for a given  $l$ , we deduce from this expression that there is a finite number of terms in  $\pi(l)$  indeed. Unfortunately it seems difficult to derive an explicit expression from there. From the expression :

$$\begin{aligned} \mathbb{E}[X_{T_1}] &= (1-p)^c A^c . X_o + \sum_{n \geq 0} \sum_{\substack{i_1 + \dots + i_{l+1} = n \\ i_j \leq c-1}} \sum_{i=0}^{\lfloor \frac{n+1}{c} \rfloor} \\ &(-1)^i \binom{n}{l} \left( \binom{n-ic}{i-1} + p \binom{n-ic}{i} \right) p^{l+i} (1-p)^{n-l+c(i+1)} A^c . A' . A^{i_1} \dots A' . A^{i_{l+1}} . X_o \end{aligned}$$

one can indeed find back the result of Theorem 1 with this method in the particular case  $c = 1$ . The general cases ( $c > 1$ ) seems much harder.

Starting with the second formula in (27), difficulties are not least. Put  $f(z) = \Phi(z)z^{-(n+1)}$  with  $z = e^{j\omega}$ . Then  $P(T_1 = n) = \frac{1}{2\pi j} \int_{\partial\Omega} f(z) dz$ , where  $\partial\Omega$  is the unit circle covered from  $-\pi$  to  $\pi$  and from the theory of holomorphic functions :

$$\int_{\partial\Omega} f(z) dz = 2\pi j \sum_k \text{Res} f(z_k),$$

where  $z_k$  are the poles of  $f$ , namely 0 (if  $n > c$ ) and those included in  $\Omega$  among the  $c + 1$  complex roots of  $pa z^{c+1} - z + 1 = 0$ . Here difficulties stem from the determination of  $z_k$ .

There is a third way to reach  $P(T_1 = n)$ . Put  $p_n = P(T_1 = n)$ . For  $n \geq c + 1$ , we have :

$$\begin{aligned} p_n &= p(1-p)^c P(T_1 > n - c - 1) \\ &= p(1-p)^c [p_{n-c} + p_{n-c+1} + \dots] \\ p_{n+1} &= p(1-p)^c [p_{n-c+1} + p_{n-c+2} + \dots]. \end{aligned}$$

So,

$$p_{n+1} - p_n = -p(1-p)^c p_{n-c}.$$

The characteristic equation is :

$$z^{c+1} - z^c + p(1-p)^c = 0,$$

which has  $c + 1$  simple complex roots for  $p < \frac{c}{c+1}$  (as it can be checked by differentiating once). Therefore :

$$p_n = \sum_{i=1}^{c+1} \alpha_i (z_i)^n,$$

with  $c + 1$  initial conditions  $\{p_{c+1} = \dots = p_{2c} = p(1-p)^c \text{ and } p_{2c+1} = p(1-p)^c [1 - (1-p)^c]\}$ . The problem here is that we only know one explicit root, namely  $1 - p$ .

## 9 Conclusion

This paper bears on  $(\max, +)$ -Lyapunov exponents obtained by sampling matrices from a finite set. Two results which were derived should be stressed in contrast with the properties of Lyapunov exponents in the conventional algebra:

1. Closed form formulas can be obtained for the coefficients of Taylor approximations of all orders, provided one of the matrices in this set has the so called scs1-cyc1 property.
2. If the probability of occurrence of this specific matrix is large enough, the Lyapunov exponent is analytic in the parameters  $(p_1, \dots, p_m)$ , which give the probabilities of the other matrices in the set; as a result, the exponent is also computationally approximable, in that a geometric error bound can be derived on its approximation by finite order Taylor expansions.

Future research will bear on better estimates of the radius of convergence and of the error bounds for the proposed computation method.

## 10 Appendix

Let us denote  $A_n^1$ ,  $B_n^1$  and  $D_n^1$  the  $n$ -th epochs when a customer enters station 1, starts its service there, and leaves station 1, respectively, and  $\beta_n^1$  and  $\delta_n^1$  the epoch when the customer which enters at time  $A_n^1$  begins its service in station 1, and leaves station 1, respectively. For the same reasons as above, whenever the initial condition is with one customer starting its service in station 1, then for all  $n \geq 0$

$$\begin{aligned} B_{n+1}^1 &= D_{n+1}^1 \oplus A_{n+1}^1 \\ D_{n+1}^1 &= \sigma \cdot B_n^1, \end{aligned}$$

with an initial conditions  $B_0^1 = 0$ , and with  $A_0^1$  and  $D_0^1$  undetermined yet. Notice that as above,  $B_n^1 = \beta_n^1$  and  $D_n^1 = \delta_{n-1}^1$ .

In the same way, let  $A_n^2$ ,  $B_n^2$ ,  $D_n^2$ ,  $\beta_n^2$  and  $\delta_n^2$  be the corresponding quantities for station 2 (i.e.  $\delta_n^2$  is the departure time of the customer entering station 2 at  $A_n^2$ ), and let  $b_n$  be the event that there is a breakdown for the customer entering at time  $A_n^2$ . Then in view of the way breakdowns take place, for all  $n \geq 1$ ,

$$\begin{aligned} \beta_n^2 &= \begin{cases} A_n^2 \oplus D_n^2 & \text{on } \bar{b}_n; \\ A_n^2 \oplus D_{n+1}^2 & \text{on } b_n \end{cases} \\ \delta_n^2 &= \sigma' \cdot \beta_n^2, \end{aligned}$$

with  $A_0^2 = 0$  and  $D_0^2 = 0$ . Let us now prove that for all  $n$ ,  $\beta_{n+1}^2 \geq \beta_n^2$ . Indeed, using the fact that each of the sequences  $\{A_n^2\}$  and  $\{D_n^2\}$  is non-decreasing, we obtain that on  $b_{n+1}$ ,

$$\beta_{n+1}^2 = A_{n+1}^2 \oplus D_{n+2}^2 \geq A_n^2 \oplus D_{n+1}^2 \geq \beta_n^2.$$

On  $\bar{b}_{n+1} \cap \bar{b}_n$ ,

$$\beta_{n+1}^2 = A_{n+1}^2 \oplus D_{n+1}^2 \geq A_{n+1}^2 \oplus D_n^2 = \beta_n^2$$

and finally, on  $\bar{b}_{n+1} \cap b_n$ ,

$$\beta_{n+1}^2 = A_{n+1}^2 \oplus D_{n+1}^2 \geq A_n^2 \oplus D_{n+1}^2 = \beta_n^2.$$

Therefore, for all  $n$ ,  $\beta_n^2 = B_{n+1}^2$  and  $\delta_n^2 = D_{n+2}^2$ . We conclude from this that the following equalities must hold:

$$\begin{aligned} D_n^2 &= A_n^1 \triangleq X_n^4 \\ D_n^1 &= A_n^2 \triangleq X_n^2, \end{aligned}$$

and if we take  $X_n^3 \triangleq B_n^2$  and  $X_n^1 \triangleq B_n^1$ , we finally get

$$\begin{aligned} X_{n+1}^1 &= X_{n+1}^2 \oplus X_{n+1}^4 = \sigma.X_n^1 \oplus \sigma'.X_n^3 \\ X_{n+1}^2 &= \sigma.X_n^1 \\ X_{n+1}^3 &= \begin{cases} X_n^2 \oplus X_n^4 & \text{on } \bar{b}_n; \\ X_n^2 \oplus X_{n+1}^4 = X_n^2 \oplus \sigma'.X_n^3 & \text{on } b_n \end{cases} \\ X_{n+1}^4 &= \sigma'.X_n^3, \end{aligned}$$

with initial condition  $X_0 = (0, 0, 0, 0)^t$ .

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