



# On the Marginal Laws of One-Dimensional Stochastic Integrals with Uniformly Elliptic Integrand

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***On the Marginal Laws of One-Dimensional  
Stochastic Integrals with Uniformly Elliptic  
Integrand***

Claude MARTINI

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\_\_\_\_\_ THÈME 4 \_\_\_\_\_

 ***apport  
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## On the Marginal Laws of One-Dimensional Stochastic Integrals with Uniformly Elliptic Integrand

Claude MARTINI

Thème 4 — Simulation et optimisation  
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**Abstract:** We show that the marginal law of the stochastic integral with respect to a standard Brownian motion of a uniformly elliptic and bounded progressive process doesn't weight points.

**Key-words:** Absolute continuity, marginal law, stochastic integral

(Résumé : *tsvp*)

To appear in *Annales de l'I.H.P. Probabilités et Statistiques*

## Sur les lois marginales d'une intégrale stochastique a intégrande uniformément elliptique en dimension 1

**Résumé :** On montre que la loi marginale de l'intégrale stochastique par rapport au mouvement Brownien standard d'un processus progressif uniformément elliptique et borné est diffuse.

**Mots-clé :** Absolue continuité, loi marginale, intégrale stochastique

## 1 Introduction

Let us consider on some  $P$ -complete filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F})_{u \geq 0}, P)$  the stochastic integral

$$M_t = \int_0^t \sigma_u dB_u$$

where  $B$  is a one-dimensional standard  $P-(\mathcal{F})_{u \geq 0}$  Brownian motion and  $\sigma$  is a  $(\mathcal{F})_{u \geq 0}$ -progressive process such that

$$0 < \underline{\sigma} < \sigma_u < \bar{\sigma} < \infty \text{ } du \text{ } dP - a.s. \quad (1)$$

for two real numbers  $(\underline{\sigma}, \bar{\sigma})$ , and  $t > 0$ .

Let  $A$  be a Lebesgue null set. Then

$$\begin{aligned} 0 \leq \underline{\sigma}^2 \int_0^T 1_A(M_u) du &\leq \int_0^T 1_A(M_u) d\langle M \rangle_u \\ &= \int 1_A(a) L_T^a(M) da = 0 \text{ } P a.s. \end{aligned}$$

where  $L_T^a(M)$  denotes the local time of  $M$ . This entails

$$\int_0^T P(M_u \in A) du = 0$$

which says that the set of time indexes  $u$  at which the law of  $M_u$  gives a weight to a given Lebesgue null set  $A$  is not a big one, at least of zero measure with respect to the Lebesgue measure on  $[0, T]$ .

Thus the following question is very natural: for every  $t > 0$ , is the law of  $M_t$  absolutely continuous? This is a much stronger statement: in the reasoning above it could happen that for a fixed  $u$  the law of  $M_u$  weights  $A$ . In fact such a phenomenon does happen for some processes  $\sigma$ : Fabes and Kenig<sup>1</sup> in [2] have designed a uniformly continuous function  $\sigma : [0, T] \times \mathbb{R}_+ \rightarrow [1, 2]$  such that the law of the solution of the s.d.e.

$$X_0 = 0, dX_u = \sigma(u, X_u) dB_u$$

is singular at time  $T$ , such yielding a negative answer to the issue of absolute continuity.

The purpose of this paper is to show the following:

**Theorem 1.1** *For every  $t > 0$ , the law of  $M_t$  doesn't weight points.*

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<sup>1</sup>I thank Yuyin Hu for this reference.

First of all let us remark that because of assumption (1),  $\sigma$  is a progressive process with respect to the natural filtration of  $M$  and  $B$  a standard Brownian motion with respect to the same filtration. Therefore we may “normalize” the situation by considering the image law of  $M$  on the canonical space  $\left(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R})), (G_u)_{u \geq 0}\right)$  where  $(G_u)_{u \geq 0}$  is the coordinate filtration. Let still denote by  $P$  the image law of  $M$ . Then under  $\bar{P}$  the canonical process  $(\omega_u)_{u \geq 0}$  is a  $(G_u)_{u \geq 0}$ -martingale such that for some  $(G_u)_{u \geq 0}$ -progressive process  $\sigma$  and some  $(G_u)_{u \geq 0}$ -standard Brownian motion  $B$

$$\omega = \int_0^\cdot \sigma_u dB_u$$

Let us fix  $(\underline{\sigma}, \bar{\sigma})$  and denote by  $\mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)$  the set of probability laws on

$$\left(C(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(C(\mathbb{R}_+, \mathbb{R})), (G_u)_{u \geq 0}\right)$$

for which the canonical process may be written this way, or equivalently (cf [3] chapter 5) for which the canonical process is a martingale with bracket almost surely equivalent to  $du$  satisfying:

$$0 < \underline{\sigma}^2 < \frac{d\langle \omega \rangle_u}{du} < \bar{\sigma}^2 < \infty \quad du \, dP - a.s.$$

We take a stochastic control-looking route. We shall work with the symmetrized laws of  $\mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)$ : consider for any  $P$  in  $\mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)$  the law  $\hat{P}$  which is the image law of the process

$$u \mapsto \frac{1}{\sqrt{2}} (\omega_u - \omega'_u)$$

defined on the product of the canonical spaces. Notice that thanks to the  $\frac{1}{\sqrt{2}}$  factor,  $\hat{P} \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)$ . Set for a function  $f$  bounded and Borel

$$\hat{C}(f)(t, x) = \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^{\hat{P}}[f(x + \omega_t)]$$

If a law  $P$  gives some weight to a point in  $\mathbb{R}$ , then  $\hat{P}$  weights  $\{0\}$  so that we must prove for any  $t > 0$   $\hat{C}(1_{\{0\}})(t, 0) = 0$ .

Observe that by Brownian scaling, for every  $t, x$ ,  $\hat{C}(f)(t, x) = \hat{C}(f)\left(1, \frac{x}{\sqrt{t}}\right)$ , in particular for  $x = 0$

$$\hat{C}(1_{\{0\}})(t, 0) = \hat{C}(1_{\{0\}})(1, 0) \tag{2}$$

We proceed as follows: in the next section we show that if  $f$  satisfies

$$\begin{aligned} \text{For every } x, \quad f(x) &\leq f(0) \\ \lim_{|x| \rightarrow \infty} f(x) &= 0 \end{aligned} \tag{3}$$

then  $\sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^P[f(\omega_t)]$  (with no hat) for big enough  $t$  is smaller than  $\lambda^* f(0)$  with  $\lambda^* < 1$ . This relies on a rough upper bound for  $\sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^P[f(\omega_t)]$  obtained from the Dambins-Dubins-Schwarz representation theorem.

The second ingredient is to show that the function  $x \mapsto \widehat{C}(1_{\{0\}})(1, x)$  satisfies (3). The third ingredient (section 3) is the following super-harmonic type property of  $\widehat{C}(1_{\{0\}})$ :

$$\widehat{C}(1_{\{0\}})(t, 0) \leq \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^P[\widehat{C}(1_{\{0\}})(1, \omega_t)]$$

The result then readily follows from the scaling property (2).

Throughout the paper  $f$  stands for a bounded Borel function.

## 2 A rough upper bound

By the Dambins-Dubins-Schwarz theorem

$$M_t = \int_0^t \sigma_u dB_u = \beta_{\int_0^t \sigma_u^2 du}$$

for some Brownian motion  $\beta$ . Note that  $\int_0^t \sigma_u^2 du$  is a stopping time  $T_t$  of the filtration  $G$  with respect to which  $\beta$  is a Brownian motion with

$$\underline{\sigma}^2 t \leq T_t = \int_0^t \sigma_u^2 du \leq \bar{\sigma}^2 t \text{ a.s.}$$

One can wonder if this property of range of  $T_t$  is enough to grant that the law of  $\beta_{T_t}$  doesn't weight points. Of course it is not: take for instance the crossing time of a fixed level by  $\beta$  between times  $\underline{\sigma}^2 t$  and  $\bar{\sigma}^2 t$ . Clearly the stopping times  $T_t$  are very particular stopping times: they satisfy for instance the property that  $\bar{\sigma}^2 t - T_t = \int_0^t (\bar{\sigma}^2 - \sigma_u^2) du$  is also a stopping time of the same filtration.

Nevertheless, we derive in this section a control of  $E^P[f(\beta_T)]$  where  $T$  ranges over all stopping times between times  $\underline{\sigma}^2 t$  and  $\bar{\sigma}^2 t$ , with a few assumptions on  $f$ , which gives in particular a contraction property for big times (corollary 2.3).

**Proposition 2.1** *The following holds:*

$$\sup_{P \in \mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)} E^P[f(x + \omega_t)] \leq \sup_G \sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} E[f(x + \beta_\tau)]$$

where  $T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)$  is the set of the  $G$ -stopping times with values almost surely in  $[\underline{\sigma}^2 t, \bar{\sigma}^2 t]$



It is easy to give a more explicit bound for this upper bound:

**Proposition 2.2** *Let  $-\infty < a \leq b < \infty$ ,  $M \geq 0, \varepsilon \geq 0$ . Assume  $|f| \leq M \mathbf{1}_{[a,b]} + \varepsilon \mathbf{1}_{[a,b]^c}$ . Then*

$$\begin{aligned} \sup_G \sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} |E[f(\beta_\tau)]| &\leq \varepsilon + M P(\beta_{\underline{\sigma}^2 t} \in [a, b]) \\ &\quad + 2M P(\beta_{\underline{\sigma}^2 t} < a < \beta_{\bar{\sigma}^2 t}) + 2M P(\beta_{\underline{\sigma}^2 t} < b < \beta_{\bar{\sigma}^2 t}) \end{aligned}$$

We have

$$\sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} |E[f(\beta_\tau)]| \leq M \sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} E[\mathbf{1}_{[a,b]}(\beta_\tau)] + \varepsilon$$

Observe now that

$$\begin{aligned} &\sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} E[\mathbf{1}_{[a,b]}(\beta_\tau)] \\ &= E[\mathbf{1}(T_{[a,b]} \leq \bar{\sigma}^2 t)] = P(T_{[a,b]} \leq \bar{\sigma}^2 t) \end{aligned}$$

where  $T_{[a,b]} = \inf \{u \geq \underline{\sigma}^2 t, \beta_u \in [a, b]\}$ . Now by the reflection principle

$$\begin{aligned} P(T_{[a,b]} \leq \bar{\sigma}^2 t) &\leq P(\beta_{\underline{\sigma}^2 t} \in [a, b]) \\ &\quad + 2P(\beta_{\underline{\sigma}^2 t} < a < \beta_{\bar{\sigma}^2 t}) + 2P(\beta_{\bar{\sigma}^2 t} < b < \beta_{\underline{\sigma}^2 t}) \end{aligned}$$

whence the result.

As an application:

**Corollary 2.3** *Assume  $0 \leq f \leq f(0)$  and  $f(x) \xrightarrow{|x| \rightarrow \infty} 0$ . Then*

$$\limsup_{t \rightarrow \infty} \left( \sup_G \sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} |E[f(\beta_\tau)]| \right) \leq \lambda^* f(0)$$

where

$$\lambda^* = 1 - \frac{2}{\pi} \arctan \left( \frac{\sigma}{\sqrt{\bar{\sigma}^2 - \underline{\sigma}^2}} \right)$$

In particular  $\lambda^* < 1$ .

By setting  $a = b = 0$  in the proposition the result is clear with  $2P(\beta_{\underline{\sigma}^2 t} < 0 < \beta_{\bar{\sigma}^2 t}) + 2P(\beta_{\bar{\sigma}^2 t} < 0 < \beta_{\underline{\sigma}^2 t})$  for the value of  $\lambda^*$ . Now

$$\begin{aligned} P(\beta_{\underline{\sigma}^2 t} < 0 < \beta_{\bar{\sigma}^2 t}) &= P(\beta_{\underline{\sigma}^2 t} < 0, -\beta_{\underline{\sigma}^2 t} < \beta_{\bar{\sigma}^2 t} - \beta_{\underline{\sigma}^2 t}) \\ &= P\left(X < 0, -\underline{\sigma}\sqrt{t}X < \sqrt{\bar{\sigma}^2 t - \underline{\sigma}^2 t}Y\right) \end{aligned}$$

whence

$$\begin{aligned} \frac{\lambda^*}{2} &= P\left(X < 0, -\underline{\sigma}\sqrt{t}X < \sqrt{\bar{\sigma}^2 t - \underline{\sigma}^2 t}Y\right) \\ &\quad + P\left(X > 0, -\underline{\sigma}\sqrt{t}X > \sqrt{\bar{\sigma}^2 t - \underline{\sigma}^2 t}Y\right) \\ &= P\left(\frac{Y}{X} < -\frac{\underline{\sigma}}{\sqrt{\bar{\sigma}^2 - \underline{\sigma}^2}}\right) \end{aligned}$$

where  $X$  and  $Y$  are two independant standard gaussian random variables. Then  $\frac{Y}{X}$  is Cauchy, whence

$$\begin{aligned} \frac{\lambda^*}{2} &= \int_{-\infty}^{-\frac{\underline{\sigma}}{\sqrt{\bar{\sigma}^2 - \underline{\sigma}^2}}} \frac{dx}{\pi(1+x^2)} \\ &= \frac{1}{\pi} \left[ \frac{\pi}{2} - \arctan\left(\frac{\underline{\sigma}}{\sqrt{\bar{\sigma}^2 - \underline{\sigma}^2}}\right) \right] \end{aligned}$$

We also get from the proposition, by replacing  $f(\cdot)$  by  $f(x + \cdot)$  :

**Lemma 2.4** *If  $f(x) \xrightarrow{|x| \rightarrow \infty} 0$  then for every  $t$*

$$\sup_G \sup_{\tau \in T(G, \underline{\sigma}^2 t, \bar{\sigma}^2 t)} |E[f(x + \beta_\tau)]| \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

### 3 Superharmonic-type property of $\widehat{C}(1_{\{0\}})$

In fact we prove in this section the announced property for any  $f$ .

Notice first:

**Lemma 3.1** *For every  $f$ ,  $t > 0$ , the function  $x \mapsto \widehat{C}(f)(t, x)$  is Borel.*

Take first a (bounded) Lipschitz  $f$  with Lipschitz constant  $k$ . Then for any  $P$  in  $\mathcal{P}(\underline{\sigma}^2, \bar{\sigma}^2)$

$$\left| E^{\widehat{P}}[f(x + \omega_t)] - E^{\widehat{P}}[f(x + \varepsilon + \omega_t)] \right| \leq k\varepsilon$$

which entails

$$\left| \widehat{C}(f)(t, x) - \widehat{C}(f)(t, x + \varepsilon) \right| \leq k\varepsilon$$

so that the function  $\widehat{C}(f)$  is Lipschitz, therefore Borel. The result follows by a monotone class argument, more precisely by the version given in theorem 21, chapter 2 of [1] of the monotone class theorem.

**Lemma 3.2** *For any  $f, x, s > 0, t > 0$*

$$\widehat{C}(f)(t+s, x) \leq \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)} E^P \left[ \widehat{C}(f)(t, x + \omega_s) \right]$$

For any  $P$  in  $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$ , with transparent notations

$$\begin{aligned} & E^{\widehat{P}} [f(x + \omega_{t+s})] \\ = & E^{P \otimes P} [f(x + \omega'_{t+s} - \omega''_{t+s})] \\ = & E^{P \otimes P} [E^{P \otimes P} [f(x + (\omega'_s - \omega''_s) + (\omega'_{t+s} - \omega''_{t+s}) - (\omega'_s - \omega''_s)) \mid F'_s \otimes F''_s]] \end{aligned}$$

Observe now that the conditional law with respect to  $F'_s \otimes F''_s$  is the product of the conditional laws, therefore it is a symmetrized law of  $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$  so that

$$\begin{aligned} & E^{P \otimes P} [f(x + (\omega'_s - \omega''_s) + (\omega'_{t+s} - \omega''_{t+s}) - (\omega'_s - \omega''_s)) \mid F'_s \otimes F''_s] \\ \leq & \widehat{C}(f)(t, x + (\omega'_s - \omega''_s)) \quad P \otimes P \text{ a.s.} \end{aligned}$$

Whence  $E^{\widehat{P}} [f(x + \omega_{t+s})] \leq E^{\widehat{P}} [\widehat{C}(f)(t, x + \omega_s)]$  so that

$$E^{\widehat{P}} [f(x + \omega_{t+s})] \leq \sup_{P \in \mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)} E^P [\widehat{C}(f)(t, x + \omega_s)]$$

The result follows.

## 4 Conclusion

In order to apply corollary 2.3 to the function  $\widehat{C}(1_{\{0\}})$  we need the following property:

**Lemma 4.1** *For every  $t > 0$ ,  $\widehat{C}(1_{\{0\}})(t, x) \leq \widehat{C}(1_{\{0\}})(t, 0)$*

Indeed the result is true if  $\widehat{C}(1_{\{0\}})(t, x) = 0$ . If not, take  $P$  such that  $P$  weights some points, let  $I$  be the (denumerable) set of such points. Then  $\widehat{P}(x) = \sum_{y \in I / y-x \in I} P(y-x) P(y)$  so that

$$\begin{aligned} 2\widehat{P}(x) &= 2 \sum_{y \in I / y-x \in I} P(y-x) P(y) \\ &\leq \sum_{y \in I / y-x \in I} P^2(y) + \sum_{y-x \in I / y \in I} P^2(y-x) \\ &\leq \sum_{y \in I} P^2(y) + \sum_{z \in I} P^2(z) \\ &= 2\widehat{P}(0) \end{aligned}$$

The result follows.

In fact, the above property of the set of symmetrized laws explains why we work with these instead of  $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$  : we were not able to prove the corresponding a priori inequality for  $\mathcal{P}(\underline{\sigma}^2, \overline{\sigma}^2)$ .

It is easy to conclude now: by lemma 2.4 and lemma 4.1 we can apply corollary 2.3 to the function  $x \mapsto C(1_{\{0\}})(1, x)$ . Next lemma 3.2 altogether with the scaling property (2) yield

$$C(1_{\{0\}})(1, 0) = \lim_{t \rightarrow \infty} \sup C(1_{\{0\}})(t, 0) \leq \lambda^* C(1_{\{0\}})(1, 0)$$

whence  $C(1_{\{0\}})(1, 0) = 0$  since  $\lambda^* < 1$ .

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