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Aggregate and Fractal Tessellations

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Thème 1 — Réseaux et systèmes
Projet Mistral

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Abstract: Consider a sequence of stationary tessellations $\{\Theta^n,\ n = 0, 1, \dots\}$ of R^d consisting of cells $\{C^n(x_i^n)\}$ with the nuclei $\{x_i^n\}$. An aggregate cell of level one, $C_0^1(x_i^0)$, is the result of merging the cells of Θ^1 whose nuclei lie in $C^0(x_i^0)$. An aggregate tessellation Θ_0^n consists of the aggregate cells of level n, $C_0^n(x_i^0)$, defined recursively by merging those cells of Θ^n whose nuclei lie in $C_0^{n-1}(x_i^0)$.

We find an expression for the probability for a point to belong to a typical aggregate cell and obtain bounds for the probability of cell's expansion and extinction. We give necessary conditions for the limit tessellation to exist as $n \to \infty$ and provide upper bounds for the Hausdorff dimension of its fractal boundary and for the spherical contact distribution function in the case of Poisson-Voronoi tessellations $\{\Theta^n\}$.

Key-words: Aggregate tessellation – fractals – Voronoi tessellation – Poisson process – Hausdorff measure – hierarchical models – telecommunications

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Tessellations Agrégées et Fractales

Résumé : Soit $\{\Theta^n, n = 0, 1, \dots\}$ une suite de tessellations stationnaires de R^d constituées de cellules $\{C^n(x_i^n)\}$ avec des noyaux $\{x_i^n\}$. Une cellule agrégée du niveau un, $C_0^1(x_i^0)$, est le résultat de l'union des cellules de Θ^1 dont les noyaux sont dans $C^0(x_i^0)$. Une tessellation agrégée Θ_0^n est composée des cellules agrégées du niveau n, $C_0^n(x_i^0)$, qui sont définies comme les unions des cellules de Θ^n dont les noyaux se trouvent dans $C_0^{n-1}(x_i^0)$, etc.

Nous obtenons une expression de la probabilité pour un point fixe d'être couvert par une cellule agrégée typique ainsi que les bornes sur les probabilités d'expansion et d'extinction d'une cellule. Nous donnons des conditions nécessaires de l'existence d'une tessellation limite quand $n \to \infty$. Dans le cas où $\{\Theta^n\}$ sont des tessellations de Poisson-Voronoï nous donnons également des bornes supérieures de la dimension de Hausdorff de sa frontière fractale et de la fonction de contacte sphérique.

Mots-clés : Tessellation agrégée – fractales – tessellation de Voronoï – processus de Poisson – mesure de Hausdorff – modèles hiérarchiques – télécommunications

1 Motivation

A *tessellation* of $\mathbb{R}^d$ is a countable collection of closed Borel sets called *cells* such that

(a) union of all cells is the whole space;

(b) intersection of any two different cells has d-Lebesgue measure zero;

(c) each bounded set intersects a finite number of cells.

Tessellations are widely used to model different cellular systems: body's tissues in biology, granulated materials in material science, crystal arrangements in physics, etc (see, e. g., [10] and references therein).

Many examples of tessellations can be constructed by division of $\mathbb{R}^d$ between the elements of a countable *nuclei set* $\Pi = \{x_i\} \subset \mathbb{R}^d$ according to a certain rule. For example, the *Voronoi tessellation* has cells defined as

$$C(x_i) = \{x \in \mathbb{R}^d \,:\, \|x - x_i\| \leq \|x - x_j\|,\ \forall j \neq i\},$$

where $\|\cdot\|$ is the Euclidean norm. Thus, the cell with nucleus x_i consists of the points that are closer to x_i than to any other nucleus. Reciprocally, given a collection of cells, one can always assign a unique nucleus to each cell by a certain rule, for instance, its center of gravity. In this paper we deal with *stationary* tessellations with the cells being stationary random closed sets. We will always assume that the rule relating the cells with their nuclei set Π satisfies an obvious compatibility condition:

$$C\big(\theta(x_i), \theta(\Pi)\big) = \theta\big(C(x_i, \Pi)\big)$$

for any shift transformation θ in $\mathbb{R}^d$ for which the point process Π is itself stationary.

Recently, tessellations were used as models of service zones of telecommunication stations. Under the reasonable assumption that the network's subscribers are served by the *nearest* to them station and that the whole area is covered by the network's service, the zones form the Voronoi tessellation with the stations being the nuclei set. If the latter is modeled by a point process then the spatial variability of the zones and related performance characteristics of the network may be described in terms of *functionals* of the corresponding random sets and studied by means of stochastic analysis. The main advantage of these models is that they dramatically reduce the number of structuring parameters of the model to just a few parameters of the underlying stochastic process and often allow for an analytical treatment of complex networks characteristics (see [1], [2], [6]).

In many cases, however, models using Voronoi tessellation over-simplify the complex geometry of service zones. First of all, unlike Voronoi cells, real zones are not necessarily convex. The relative sizes of the zones are affected by capacities of different stations. In some cases, a station may not be contained in its proper zone. Last but not the least, the zones' boundaries may be very fuzzy as, for instance, in the case of *wireless* communications. In such systems, the base station that will handle a call from a mobile terminal is decided upon

the signal's strength rather than Euclidean distance to the stations. Affected by the wave attenuation phenomena, the zones' boundaries have extremely irregular, distorted shapes.

This calls for development of more complex tessellation models that are still described in terms of a small number of parameters and simple enough to be analytically treatable.

For this we introduce an operation of *aggregation* on tessellations equipped with nuclei. Let $\Theta^0 = \{C^0(x_i^0)\}$ and $\Theta^1 = \{C^1(x_i^1)\}$ be two such tessellations. Define the cells of a new tessellation $\Theta_0^1 = \Theta^0 \circ \Theta^1$ as

$$C_0^1(x_i^0) = \bigcup_{j:\ x_j^1 \in C^0(x_i^0)} C^1(x_j^1)\,.$$

In words, $C_0^1(x_i^0)$ is the union of all the cells of Θ^1 whose nuclei lie in $C^0(x_i^0)$ (see Figure 1). The result of such operation is again a tessellation though some $C_0^1(x_i^0)$ may be empty. Due to the independence and stationarity assumptions, a.s. no x_j^1 lies on the boundary $\Gamma(\Theta^1)$, and hence each Θ^1-cell belongs to a unique Θ_0^1-cell. It is easy to verify that the operation of aggregation is associative

$$\Theta^0 \circ \left(\Theta^1 \circ \Theta^2\right) = \left(\Theta^0 \circ \Theta^1\right) \circ \Theta^2\,,$$

and that the aggregate tessellation is stationary provided the initial tessellations are stationary. Let $\{\Theta^n\}_{n\in\mathbb{N}}$ be a sequence of tessellations with the nuclei sets $\Pi_n = \{x_i^n\}$, $n \in \mathbb{N}$. Consequent aggregation of the first n terms of the sequence yields the *aggregate tessellation of order n*

$$\Theta_0^n = \Theta^0 \circ \Theta^1 \circ \ldots \circ \Theta^n$$

with the nuclei set $\Pi_0 = \{x_i^0\}$. The cells of this tessellation will be called *aggregate n-cells* and denoted by $C_0^n(x_i^0)$. Of course, aggregate cells constructed in this way depend on positions of the nuclei and are, in general, neither convex nor connected; nor do they need to contain the nucleus. Figure 2 shows several simulated aggregate cells with nucleus at the origin generated by independent Poisson-Voronoi plane tessellations with exponentially growing intensities.

It is clear that more the intensities of the consequent processes differ, less the boundary of cell $C_0^{n+1}(x_i^0)$ deviates from the boundary of $C_0^n(x_i^0)$. On the other hand, for close intensity values the boundary becomes very irregular, cells are more likely to split and quite often there is no points of Π_{n+1} in $C_0^n(x_i^0)$, so that $C_0^{n+1}(x_i^0)$ is empty. Using an analogy with branching processes, we may think of the nuclei of the Θ^n-cells that make up the aggregate cell $C_0^n(x_i^0)$, as of n-generation *offspring* of a 0-level parent nucleus x_i^0. If we connect by segments the nuclei Π_n of each level n with their offspring in the next level nuclei Π_{n+1}, we will obtain a family of *spanning trees* studied for Poisson-Voronoi case in [3]. In the present paper we address properties of the aggregate *cells* rather than those of the spanning trees.

New phenomena appear in the limit, when n tends to infinity. As we have seen above, there exist models in which with positive probability some of the aggregate cells are empty, which means that the densities of 0-level nuclei that have not died out till the n-th generation

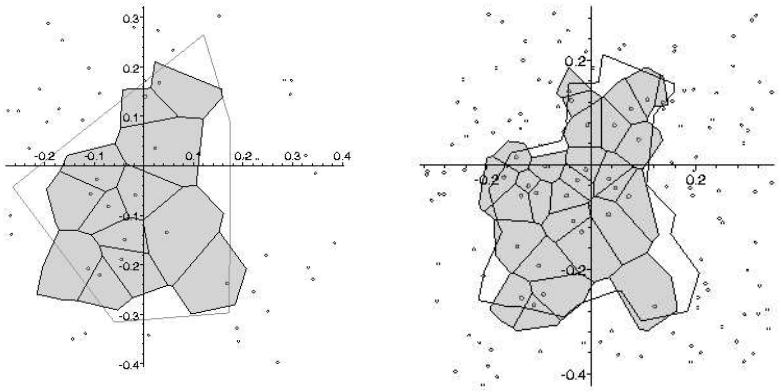

Figure 1: First and second iterations in the construction of aggregate cells. The boundary of the previous level cell is countered.

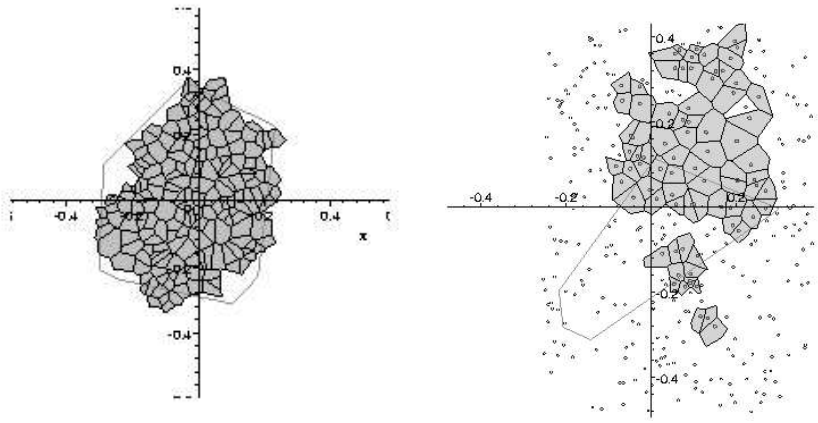

Figure 2: Aggregate cells $C_0^n(0)$ in **PVAT** model and the initial $C^0(0)$ Voronoi cell. Left image: $n = 2, \lambda = 10$. Right image: $n = 20, \lambda = 1.1$.

form a decreasing sequence. *A priori* it is not clear if the limit is positive, i.e., that we do not end up with all the 0-nuclei Π_0 dying out with probability 1. Next, even if the limit *is* positive, will Θ_0^n converge in some sense to a limit, say, Θ_0^∞ that *is* a tessellation? It is easy to imagine that Θ_0^∞ may have a fractal boundary (it also may not! – see Example 1

in Section 2) and thus it is unclear if the boundary has d-Lebesgue measure 0 and whether only a finite number of the limit aggregate cells hits a bounded set. Finally, if the boundary does have a positive measure and thus the cells overlap, does a typical cell contain with a positive probability a "core" — an open set with all the points belonging only to this cell?

As the examples in the text readily show, different models may manifest very different behavior. We illustrate this on two basic models: cubic lattices and *Poisson-Voronoi aggregate tessellations*, quoted as **PVAT** in the sequel, for which the elements of the sequence $\{\Theta^n\}_{n\in\mathbb{N}}$ are all Voronoi tessellations generated, respectively, by mutually independent homogeneous Poisson point processes $\Pi_n,\ n \in \mathbb{N}$.

The structure of the paper is the following. In Section 2 we find an expression for the *coverage probability* function for a typical aggregate n-cell via the corresponding characteristics of $\Theta^0, \dots, \Theta^n$. This result is valid for *any* independent sequence of stationary tessellations although a closed form expression can be obtained only in a few cases. In section 3 we find a uniform upper bound on the diameter of a typical cell in **PVAT** (a *typical* cell is the one with nucleus at the origin under the Palm distribution of the process Π_0). Here we also give estimates for the probability of cell's extinction and show that with positive probability there is a ball contained in *all* n-level aggregate cells $C_0^n(0)$. In the next Section we prove that this property is sufficient in general case for the limit cells, defined as the set lower limit of $\{C_0^n(x_i^0)\}_{n\in\mathbb{N}}$, to form a tessellation, so that their boundaries have almost surely d-Lebesgue measure 0. Defined by a simple recursive procedure, the boundary of the limit **PVAT** has an intricate auto-similar structure at any scale allowing us to call it fractal. To characterize its degree of irregularity, we provide in Section 6 an upper bound for its Hausdorff dimension. It is based on the analysis of the boundary's contact distribution function in the preceding Section 5. Note that the parts of the cell's fractal boundary are highly dependent making most of previously developed techniques for random fractals inapplicable in our case (a presentation of modern methods used in studying fractals can be found, e. g., in Kenneth Falconer's book [5]).

The following notation is used throughout the paper. By $\mathbf{P}$ we denote the *distribution* in a probability space carrying the sequence of *independent stationary* point processes $\Pi_0, \Pi_1, \dots$, and by $\mathbf{P}_n^0$, the *Palm distribution* with respect to the process of nuclei Π_n of level n, $n = 0, 1, \dots$ Most frequently we consider the Palm distribution with respect to Π_0 for which we simply write $\mathbf{P}^0$ instead of $\mathbf{P}_0^0$. Similar notation is used for the corresponding expectations. $\Pi_n(B)$ stands for the number of points of Π_n in a Borel set $B \subset \mathbb{R}^d$. The *intensity* of Π_n is denoted by λ_n, so that $\mathbf{E}_n\,\Pi_n(B) = \lambda_n|B|$ and it is the only parameter characterizing a homogeneous Poisson process. We also assume that $\lambda_0 = 1$, which is just a matter of scale choice.

Finally,

$$f * g(y) = \int_{\mathbb{R}^d} f(z)\, g(y-z)\, dz$$

is the *convolution* of two functions, $b(x,r)$ is the closed ball centered in x with radius r, and b_d is the volume of a *unit ball* in $\mathbb{R}^d$:

$$b_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}.$$

2 Coverage probability

Consider a tessellation Θ^n of fixed level n. Under Palm probability $\mathbf{P}_n^0$, there is almost surely a point of Π_n at the origin 0. Since the density of cells is λ_n, the volume of a typical cell is $\mathbf{E}_n^0 |C^n(0)| = \lambda_n^{-1}$ (see, e. g., [9], Corollary 5.2, equation (5.6)). Therefore,

$$\lambda_n \int \mathbf{P}_n^0\{y \in C^n(0)\}\, dy = \lambda_n\, \mathbf{E}_n^0 \int \mathbb{1}(y \in C^n(0))\, dy = \lambda_n\, \mathbf{E}_n^0\, |C^n(0)| = 1,$$

so that the function

$$f_n(y) = \lambda_n \mathbf{P}_n^0\{y \in C^n(0)\}$$

is a distribution density in $\mathbb{R}^d$. It has the following interpretation. Given a realization of $C^n(0)$, let ξ_n be a r.v. uniformly distributed in the scaled set $\lambda_n^{1/d} C^n(0)$. Then f_n is the density of the the unconditional distribution of ξ_n in $\mathbb{R}^d$.

The next statement provides a formula for calculating the probability that a point $y \in \mathbb{R}^d$ is covered by a typical aggregate cell of level n.

Theorem 1 *Let $f_{0,n}(y) = \mathbf{P}^0\{y \in C_0^n(0)\}$. Then for each natural n,*

$$f_{0,n}(y) = f_0 * f_1 * \cdots * f_n(y). \tag{1}$$

Proof. By definition, for $n = 0$, obviously,

$$f_0(y) = \mathbf{P}^0\{y \in C^0(0)\}.$$

Suppose, the statement of the Lemma holds for $n-1$. By the Campbell theorem (see, e. g., [11, p. 119]),

$$\mathbf{P}^0\{y \in C_0^n(0)\} = \mathbf{E}^0 \sum_{x_i^n \in \Pi_n} \mathbb{1}\big(x_i^n \in C_0^{n-1}(0)\big)\, \mathbb{1}\big(y \in C^n(x_i^n)\big)$$

$$= \lambda_n \int_{\mathbb{R}^d} \mathbf{P}^0\{z \in C_0^{n-1}(0)\}\, \mathbf{P}_n^0\{y - z \in C^n(0)\}\, dz.$$

It is easy to see that this expression is equivalent to

$$f_{0,n}(y) = \int_{\mathbb{R}^d} f_{0,n-1}(z)\, f_n(y-z)\, dz.$$

Example 1 Consider a stationary tessellation Θ^n of $\mathbb{R}^d$ obtained by shifting the regular mesh of d-hypercubes of side λ_n^{-1} with a vertex at the origin by a random vector uniformly distributed in $[0, \lambda_n^{-1}]^d$. Note that the d-dimensional stationary mesh is simply a Cartesian product of d one-dimensional independent components. Therefore, it is sufficient to study the case $d = 1$ when Θ^n is a stationary sequence of the intervals of length λ_n^{-1} on the line.

Assume that $\{\lambda_n\}$ is a non-increasing sequence and $\lambda_0 = 1$. It is easy to see by induction that the size of any Θ_0^n-cell along each coordinate axis is at least λ_n^{-1}, so there is always at least one nucleus of Π_{n+1} in each cell. As a result, all aggregate cells are rectangles and the cells never die. Moreover, for each n, the sizes of the aggregate cells along the line do not change if λ_{n+1}/λ_n is a natural number and change periodically if λ_{n+1}/λ_n is a rational one. By construction, the boundaries of the Θ^n-cells have coordinates $\lambda_n^{-1}(k + u_n)$, $k \in \mathbb{Z}$, where u_n are independent uniformly distributed in $(0, 1)$ random variables describing the shift. Define the nuclei set as

$$\Pi_n = \{\lambda_n^{-1}(\alpha + k + u_n)\}, \quad k \in \mathbb{Z},\ \alpha \in [0, 1]$$

so that the nucleus of each cell divides it in proportion $\alpha : (1 - \alpha)$ from left to right. According to (1), the characteristic function of $f_0^n(y)$ is given by

$$\chi_0^n(z) = \prod_{k=0}^{n} \frac{\lambda_k}{iz} \left(e^{iz\lambda_k^{-1}(1-\alpha)} - e^{-iz\lambda_k^{-1}\alpha} \right).$$

In fact, a complete analysis of this model is possible. Let a_k^n be the right boundary point of the n-aggregate cell with the nucleus $\alpha + k + u_0$, $k \in \mathbb{Z}$. It is straightforward to verify that the evolution of the boundaries are given by the following recursion

$$\begin{aligned} a_k^n &= a_k^{n-1} + d_k^n, \quad \text{where} \\ d_k^n &= \lambda_n^{-1}\big(1 - \alpha - \langle \lambda_n a_k^{n-1} - \alpha - u_n \rangle\big) \end{aligned}$$

with $\langle z \rangle = z - \max\{k \in \mathbb{Z} :\ k \le z\}$ being the fractional part of a real number z. Note that for any z and for any u uniformly distributed in $(0, 1)$, the r.v. $\langle z + u \rangle$ is again uniform in $(0, 1)$. Therefore,

$$\begin{aligned} \mathbf{E}\, d_k^n &= \lambda_n^{-1}(1/2 - \alpha) \\ \mathbf{var}\, d_k^n &= 1/(12\lambda_n^2), \end{aligned}$$

so that there is a systematic drift to the right or to the left if $\alpha < 1/2$ or $\alpha > 1/2$, respectively. We have $|d_k^n| \le 1$. By the well known theorem on random series convergence (see, e. g., [4, p. 239]), the boundaries of the aggregate cells almost surely stabilize as $n \to \infty$ if and only if both series $(1/2 - \alpha)\sum_n \lambda_n^{-1}$ and $\sum_n \lambda_n^{-2}$ converge. We see a noticeable dependence on the nuclei choice when, say, $\lambda_n = n$. In this case the cells stabilize only if $\alpha = 1/2$ and float to plus or minus infinity depending on whether α is smaller or greater than $1/2$.

Let us take a closer look at the coverage probability for Poisson-Voronoi aggregate tessellations. For **PVAT** we have:

$$f_n(y) = \lambda_n \exp\{-\lambda_n b_d \|y\|^d\}\,. \tag{2}$$

The next two statements provide explicit formulas for the coverage probability in $\mathbb{R}^1$ and $\mathbb{R}^2$.

Example 2 Consider **PVAT** in $\mathbb{R}^1$ and assume that all λ_i are pairwise different. Then

$$f_{0,n}(y) = \sum_{i=0}^{n} c_i \lambda_i \exp\{-2\lambda_i |y|\}, \tag{3}$$

where

$$c_i = 2 \prod_{\substack{l=0 \\ l\neq i}}^{n} \left(1 - \frac{\lambda_i}{\lambda_l}\right)^{-1} \sum_{j=0}^{n} \left(1 + \frac{\lambda_i}{\lambda_j}\right)^{-1} \prod_{\substack{m=0 \\ m\neq j}}^{n} \left(1 - \frac{\lambda_j}{\lambda_m}\right)^{-1}$$

Indeed, from Theorem 1 it follows that $f_{0,n}(y)$ is the density of the sum of independent r.v.'s $\sum_{i=0}^{n} \xi_i$ whose densities $f_i(y)$ are given by (2) with $d = 1$. Note that $\xi_i = \nu_i - \nu_i'$, where ν_i, ν_i' are independent exponentially distributed r.v.'s with parameter $2\lambda_i$. The density of $\sum_{i=0}^{n} \nu_i$ equals (see, e. g., [4, p. 170])

$$\sum_{i=0}^{n} 2\lambda_i \exp\{-2\lambda_i y\} \prod_{\substack{l=0 \\ l\neq i}}^{n} \left(1 - \frac{\lambda_i}{\lambda_l}\right)^{-1} \mathbb{1}(y \geq 0).$$

Finding its symmetrization, we obtain (3).

Example 3 Consider **PVAT** in $\mathbb{R}^2$. According to (2), in $\mathbb{R}^2$ the r.v. ξ_i has normal distribution with zero mean and the covariance matrix

$$\begin{pmatrix} (2\pi\lambda_i)^{-1} & 0 \\ 0 & (2\pi\lambda_i)^{-1} \end{pmatrix}. \tag{4}$$

Thus $\sum_{i=0}^{n} \xi_i$ is also normal with zero mean and the covariance matrix being the sum of (4):

$$\begin{pmatrix} (2\pi L_n)^{-1} & 0 \\ 0 & (2\pi L_n)^{-1} \end{pmatrix},$$

where

$$L_n = \left(\sum_{i=0}^{n} \frac{1}{\lambda_i}\right)^{-1}.$$

The corresponding density is therefore

$$f_{0,n}(y) = L_n \exp\{-L_n \pi \|y\|^2\}$$

that is the same as for a typical cell in the ordinary Voronoi tessellation with the nuclei intensity L_n. Such "mean field" approximation is valid only in this planar case, the reason being the stability of the distributions of ξ_i's in $d = 2$.

Example 4 For **PVAT** in $\mathbb{R}^d$, as follows from (1), the characteristic function of $f_{0,n}(y)$ is given by

$$\prod_n \chi\big(t/\lambda_n^{-1/d}\big),$$

where $\chi(t)$ is the characteristic function of the density $f_0(y)$ in (2). It can be shown that

$$\chi(t) = \frac{\Gamma\big(\frac{d}{2}\big)}{\pi^{1/2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \frac{\|t\|^{2m}}{\pi^m} \left[\Gamma\Big(1+\frac{d}{2}\Big)\right]^{2m/d} \frac{\Gamma\big(m+\frac{1}{2}\big)\Gamma\big(1+\frac{2m}{d}\big)}{\Gamma\big(m+\frac{d}{2}\big)}.$$

An alternative representation uses Bessel functions of the first kind (see, e. g., [7]):

$$\chi(t) = \frac{(2\pi)^{d/2}}{\|t\|^{d/2-1}} \int_0^{\infty} \rho^{d/2} e^{-b_d \rho^d} J_{d/2-1}\big(\rho\|t\|\big)\, d\rho.$$

3 Evolution of aggregate cells

In this section we will investigate the behavior of the typical aggregate cell $C_0^n(0)$ as n tends to infinity on the Palm space of the process Π_0. The maximal and the minimal distance from a point z to the cell's boundary can be defined, respectively, as

$$R_n(z) = \begin{cases} \min\{r \,:\, b(z,r) \supset C_0^n(0)\} & \text{if } C_0^n(0) \neq \emptyset, \\ 0 & \text{otherwise;} \end{cases}$$

$$r_n(z) = \begin{cases} \max\{r \,:\, b(z,r) \subset C_0^n(0)\} & \text{if } z \in C_0^n(0), \\ 0 & \text{otherwise.} \end{cases}$$

The definition takes into account the fact that the aggregate cell of order $n \geq 1$ might not contain z or might be empty. Our aim is to characterize the distribution of

$$R_\infty(z) = \sup_n R_n(z),$$
$$r_\infty(z) = \inf_n r_n(z).$$

Theorem 2 *Let $\phi(y)$ be the inverse of the function $y(x) = xe^x$. Assume that*

$$c = \sum_{n=1}^{\infty} \Big(\frac{\phi(\lambda_n)}{\lambda_n} \Big)^{1/d} = \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)/d} < \infty \,. \tag{5}$$

Then for **PVAT** *the following inequalities hold for all $\rho > c\sqrt{d}$ and any $z \in \mathbb{R}^d$:*

$$\mathbf{P}^0\big\{R_\infty(z) > \rho + \|z\|\big\} \le a_1 \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)A(\rho)} \,, \tag{6}$$

$$\mathbf{P}^0\big\{r_\infty(z) = 0 \,\big|\, r_0(z) > \rho\big\} \le a_1 \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)A(\rho)} \,, \tag{7}$$

where

$$a_1 = 2b_d d^{d/2+1} c^{d-1} \,,$$
$$A(\rho) = \Big(\frac{\rho}{c\sqrt{d}} \Big)^d + 1/d - 1 \,.$$

Proof. We begin from inequality (6). Since

$$\mathbf{P}^0\big\{R_\infty(z) > \rho + \|z\|\big\} \le \mathbf{P}^0\big\{R_\infty(0) > \rho\big\},$$

it is sufficient to prove (6) for $z = 0$. Let $\{\rho_n\}$ be a monotonously increasing sequence of positive numbers converging to ρ, then

$$\{R_\infty(0) > \rho\} \subset \cup_{n=0}^{\infty} \{R_n(0) > \rho_n\}.$$

Next, we use the following inequality: if $B \subset \cup_{n=0}^{\infty} B_n$, then

$$\mathbf{P}(B) \le \mathbf{P}(B_0) + \sum_{n=0}^{\infty} \mathbf{P}(B_n \cap \bar{B}_{n-1}).$$

Hence,

$$\mathbf{P}^0\big\{R_\infty(0) > \rho\big\} \le \mathbf{P}^0\big\{R_0 > \rho_0\big\} + \sum_{n=0}^{\infty} \mathbf{P}^0\big\{R_n(0) > \rho_n,\, R_{n-1}(0) \le \rho_{n-1}\big\}. \tag{8}$$

The event

$$\{R_n(0) > \rho_n,\, R_{n-1}(0) \le \rho_{n-1}\}$$

implies the existence of a Voronoi cell $C^n(x_i^n)$ with the nucleus inside of the ball $b(0, \rho_{n-1})$ containing some point y on the sphere $\partial b(0, \rho_n)$. Therefore, the interior of the ball $b(y, \|y - x_i^n\|)$ contains no points of Π_n (see Figure 3).

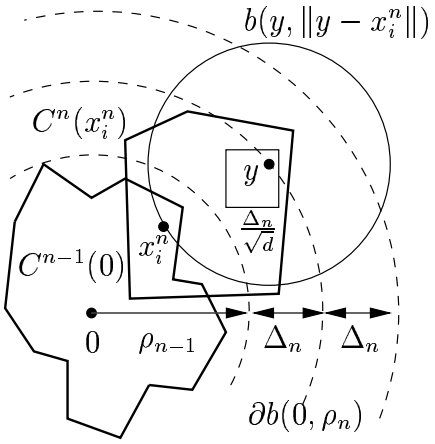

Figure 3: Large increase in $R_n(0)$ implies existence of a large empty ball.

Denote $\Delta_n = \rho_n - \rho_{n-1}$ and consider the collection of mesh cubes of side $\Delta_n/\sqrt{d}$ that lie entirely in the annulus $b(0, \rho_n + \Delta_n) \setminus b(0, \rho_{n-1})$. Denote by N their number. At least one of the mesh cubes lies entirely in the ball $b(y, \|y - x_i^n\|)$ (e.g., the cube containing y). Thus for the summands in (8) we obtain a bound, which is the probability that at least one of the mesh cubes contains no points of Π_n

$$1 - \Big(1 - \exp\big\{ - \lambda_n (\Delta_n/\sqrt{d})^d \big\}\Big)^N . \tag{9}$$

Write the corresponding inequality for the volumes of the union of the cubes and the volume of the annulus:

$$\begin{aligned} N\Big(\frac{\Delta_n}{\sqrt{d}}\Big)^d &< b_d\big[(\rho_n + \Delta_n)^d - (\rho_n - \Delta_n)^d\big] \\ &= 2b_d\Delta_n \sum_{k=0}^{d-1} (\rho_n + \Delta_n)^k (\rho_n - \Delta_n)^{d-k-1} \\ &< 2db_d\Delta_n \sum_{k=0}^{d-1} \rho^k \rho^{d-k-1} < 2db_d\Delta_n\rho^{d-1} . \end{aligned}$$

This yields

$$N < 2b_d d^{d/2+1} \Big(\frac{\rho}{\Delta_n}\Big)^{d-1} .$$

Since for any $0 < a < 1$ and $N \geq 1$,

$$1 - (1 - a)^N < aN ,$$

the value of (9) does not exceed

$$2b_d d^{d/2+1}\Big(\frac{\rho}{\Delta_n}\Big)^{d-1}\exp\big\{-\lambda_n(\Delta_n/\sqrt{d})^d\big\}. \tag{10}$$

Next, choose a special sequence $\{\rho_n\}$ with the increments Δ_n given by

$$\Delta_n = \frac{\rho}{c}\,e^{-\phi(\lambda_n)/d}$$

with c defined in (5). It is easy to see that ρ_n monotonously converges to ρ. For such choice of $\{\rho_n\}$, from (10) it follows that the right-hand side of (8) is bounded by

$$2b_d d^{d/2+1}\sum_{n=0}^{\infty}\exp\Big\{-\Big(\frac{\rho}{c\sqrt{d}}\Big)^d\phi(\lambda_n)+(d-1)\big(\log c+\frac{\phi(\lambda_n)}{d}\big)\Big\}, \tag{11}$$

which is equivalent to (6). We have used here the definition of ϕ, due to which $e^{-\phi(\lambda_n)} = \phi(\lambda_n)/\lambda_n$. The function $A(\rho) = O(\rho^d)$ increases to infinity and is greater than $1/d$ for all $\rho > c\sqrt{d}$. Therefore, for such ρ, the series in (6) converges and the whole bound tends to 0 as $\rho \to \infty$ providing the almost sure finiteness of $R_\infty(z)$.

Inequality (7) is proved much in the same manner. Fix a small $0 < \varepsilon < 1$ and consider a sequence $\{\rho'_n\}$ with $\rho'_0 = \rho$ that monotonously decreases to $\varepsilon\rho$. First, from

$$\{r_\infty(z) = 0,\, r_0(z) > \rho\} \subset \cup_{n=1}^{\infty}\{r_n(z) < \rho'_n,\, r_0(z) > \rho\}$$

it follows that

$$\mathbf{P}^0\big\{r_\infty(z) = 0 \,\big|\, r_0(z) > \rho\big\} \le \sum_{n=1}^{\infty}\mathbf{P}^0\big\{r_n(z) < \rho'_n,\, r_{n-1}(z) \ge \rho'_{n-1} \,\big|\, r_0(z) > \rho\big\}. \tag{12}$$

The event

$$\{r_n(z) < \rho'_n,\, r_{n-1}(z) \ge \rho'_{n-1}\}$$

implies the existence of a Voronoi cell $C^n(x_i^n)$ with the nucleus outside of the ball $b(z, \rho'_{n-1})$ having some point $y \in C^n(x_i^n)$ inside $b(z, \rho'_n)$. Hence, there exists a ball of radius at least $\Delta'_n = \rho'_{n-1} - \rho'_n$ centered on the sphere $\partial b(z, \rho'_{n-1})$. Note that this event is independent of the event $\{r_0(z) > \rho\}$. Thus the summands in (12) can be bounded as in (9) with $\Delta'_n = (1-\varepsilon)\Delta_n$. With that choice of Δ'_n the right hand side of (12) is bounded by an expression similar to (11) with c replaced by $(1-\varepsilon)c$. Due to the arbitrariness of ε, expression (11) also provides an upper bound. The rest of the proof remains unchanged.

Remark 1 By definition, $\log\phi(x) + \phi(x) = \log x$. Therefore, for $x \ge 1$ one has $\phi(x) \le \log x$. Since $e^{-\phi(\lambda_n)} = \phi(\lambda_n)/\lambda_n$ and $\lambda_n > 1$ for all sufficiently large n, the condition

$$\sum_{n=1}^{\infty}\Big(\frac{\log\lambda_n}{\lambda_n}\Big)^{1/d} < \infty \quad \text{or} \quad \sum_{n=1}^{\infty}\lambda_n^{1/d-\varepsilon} < \infty$$

for some $0 < \varepsilon < 1/d$ is sufficient for (5) to hold. Moreover, $\phi(x) \geq (1 + \log x)/2$ for all x. Thus

$$a_1 \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)A(\rho)} \leq a_1 e^{-A(\rho)/2} \sum_{n=1}^{\infty} \lambda_n^{-A(\rho)/2}$$

and the last function can also be used in (6-7).

Corollary 1 *For* **PVAT** *with exponentially growing intensities* $\lambda_n = \lambda^n$ *for some* $\lambda > 1$, *one has for all* $\rho > c_*$ *and any* $z \in \mathbb{R}^d$,

$$\mathbf{P}^0\{R_\infty(z) > \rho + \|z\|\} \leq c_1 \exp\{-c_2\rho^d\}, \tag{13}$$

$$\mathbf{P}^0\{r_\infty(z) = 0 \,|\, r_0(z) > \rho\} \leq c_1 \exp\{-c_2\rho^d\}. \tag{14}$$

One may take

$$c_1 = a_1\left(1 + d\frac{1+\phi(\lambda)}{\phi(\lambda)\log\lambda}\right)e^{\phi(\lambda)},$$
$$c_2 = \phi(\lambda)\,(c\sqrt{d})^{-d},$$
$$c_* = ((\log c_1)/c_2)^{1/d}.$$

Proof. Recall the following integral estimate:

$$\sum_{n=1}^{\infty} h(n) \leq h(1) + \int_1^{\infty} h(x)dx$$

for any non-increasing positive function $h(x)$. We have

$$\sum_{n=1}^{\infty} e^{-\phi(\lambda^n)A(\rho)} \leq e^{-\phi(\lambda)A(\rho)} + \int_1^{\infty} e^{-\phi(\lambda^x)A(\rho)}dx$$
$$= e^{-\phi(\lambda)A(\rho)} + \frac{1}{\log\lambda}\int_{\phi(\lambda)}^{\infty}(1+y^{-1})e^{-yA(\rho)}dy$$

after the variable change $y = \phi(\lambda^x)$. Next, since $1 + y^{-1} \leq 1 + \phi(\lambda)^{-1}$ on the integration domain, the whole expression can be bounded by

$$e^{-\phi(\lambda)A(\rho)} + \frac{1+\phi(\lambda)^{-1}}{A(\rho)\log\lambda}e^{-\phi(\lambda)A(\rho)} < \left(1 + \frac{1+\phi(\lambda)^{-1}}{d^{-1}\log\lambda}\right)e^{-\phi(\lambda)A(\rho)},$$

so that (6-7) become (13-14), respectively. It can be immediately verified that these estimates become nontrivial if $\rho > c_*$ and that $c_* > c\sqrt{d}$.

Corollary 2 *For* **PVAT** *with polynomially growing intensities* $\lambda_n = (1+\alpha n)^\beta$ *for some* $\alpha > 0,\ \beta > d$, *one has*

$$\sum_{n=1}^{\infty} e^{-\phi(\lambda_n)A(\rho)} \le \frac{A(\rho)}{A(\rho)-\beta'}\left(e^{-\phi(\lambda_1)A(\rho)} + \frac{\Gamma\big(\beta', \phi(\lambda_1)(A(\rho)-\beta')\big)}{\alpha\beta(A(\rho)-\beta')^{\beta'}}\right),$$

where $\beta' = \beta^{-1}$ *and* $\Gamma(n,x) = \int_x^\infty z^{n-1}e^{-z}dz$ *is the incomplete Gamma-function.*

Proof. is the same as in Corollary 1 with the variable change $z = (A(\rho)-\beta^{-1})\phi((1+\alpha x)^\beta)$ in the corresponding integral.

Corollary 3 *Under conditions of Theorem 2, with probability one, each family of cells* $\{C_0^n(x_i^0)\}_{n\in\mathbb{N}}$ *is uniformly bounded in* $\mathbb{R}^d$.

Proof. Let τ_x be the stationary shift defined on the probability space Ω such that

$$\Pi_n(\tau_x\omega)(B) = \Pi_n(\omega)(B-x)$$

for any Borel set $B \subset \mathbb{R}^d$ and any n. In this notation, $\tau_{x_i^0}R_n(0)$ for $x_i^0 \in \Pi_0$ is the maximal distance from x_i^0 to the boundary of $C_0^n(x_i^0)$ that corresponds to the above definition of $R_n(0)$ with $C_0^n(0)$ replaced with $C_0^n(x_i^0) = \tau_{x_i^0}C_0^n(0)$. The probability that there exists an unbounded family of cells with the nucleus in a ball $b(0,N)$ equals

$$\begin{aligned}\mathbf{P} \bigcup_{x_i\in\Pi_0\cap b(0,N)} &\{\tau_{x_i^0}R_\infty(0) = \infty\} \\ &\le \mathbf{E}\sum_{x_i\in\Pi_0\cap b(0,N)} \mathbb{1}\{\tau_{x_i^0}R_\infty(0) = \infty\} \\ &= b_d N^d\, \mathbf{E}^0\, \mathbb{1}\{R_\infty(0) = \infty\} = 0\,,\end{aligned}$$

where we have used the Campbell theorem and (6). Letting N grow to infinity proves the assertion.

The following fact will be used later to show that the boundaries of the limit tessellation have zero Lebesgue measure.

Corollary 4 *Under conditions of Theorem 2, for any* $y \in \mathbb{R}^d$ *with positive probability, there exists* $x_i^0 \in \Pi_0$ *such that*

$$y \in \operatorname{int}\big(\cap_n C_0^n(x_i^0)\big).$$

The lower bound for such probability is given in (17).

Proof. Indeed, it is sufficient that the distance from y to the boundary of the Π_0-cell containing y is sufficiently large so that the boundaries of the progressing n-cells never reach y. The probability of the latter event is positive by (7).

Due to stationarity, we may put $y = 0$. Consider the following representation:

$$\mathbf{P}\{\exists x_i^0 \,:\, 0 \in \operatorname{int}\big(\cap_n C_0^n(x_i^0)\big)\} = \mathbf{E} \sum_{x_i^0 \in \Pi_0} \mathbb{1}\Big(0 \in \operatorname{int}\big(\cap_n C_0^n(x_i^0)\big)\Big).$$

By the Campbell theorem and by the isotropy, the right-hand side equals

$$\int_{\mathbb{R}^d} \mathbf{P}^0\{-z \in \operatorname{int}\big(\cap_n C_0^n(0)\big)\}\, dz = \int_{\mathbb{R}^d} \mathbf{P}^0\{z \in \operatorname{int}\big(\cap_n C_0^n(0)\big) \,\big|\, b(z,\rho) \subset C^0(0)\} \times \mathbf{P}^0\{b(z,\rho) \subset C^0(0)\}\, dz \quad (15)$$

for arbitrary $\rho > 0$. By Theorem 2, the first factor under the integral in (15) is greater than $1 - a_1 \sum_{n=1}^{\infty} \big(\phi(\lambda_n)/\lambda_n\big)^{A(\rho)}$ provided that $\rho \geq c\sqrt{d}$. The second factor equals the probability that no points of Π_0 lie in the figure $\cup_{\|z-z'\|=\rho} b(z', \|z'\|)$. This figure is obtained by rotation of a cardioid around its symmetry axis; by construction, it is contained in the ball $b(z, \|z\| + 2\rho)$. Therefore,

$$\mathbf{P}^0\{b(z,\rho) \subset C^0(0)\} > \exp\{-b_d(\|z\| + 2\rho)^d\}.$$

Using this estimate we get

$$\int_{\mathbb{R}^d} \mathbf{P}^0\{b(z,\rho) \subset C^0(0)\}\, dz > \int_{\mathbb{R}^d} \exp\{-b_d(\|z\| + 2\rho)^d\}\, dz > \int_0^{\rho} r^{d-1} d b_d \exp\{-b_d(3\rho)^d\}\, dr = b_d \rho^d \exp\{-b_d(3\rho)^d\}. \quad (16)$$

Hence

$$\mathbf{P}\{\exists x_i^0 \,:\, y \in \operatorname{int}\big(\cap_n C_0^n(x_i^0)\big)\} > \sup_{\rho \geq c\sqrt{d}} \left[b_d \rho^d \exp\{-b_d(3\rho)^d\} \left(1 - a_1 \sum_{n=1}^{\infty} \Big(\frac{\phi(\lambda_n)}{\lambda_n}\Big)^{A(\rho)}\right)\right] > 0. \quad (17)$$

The following corollary gives bounds for the probability of a cell's extinction.

Corollary 5 *Under conditions of Theorem 2, for any* $1 \leq N \leq \infty$ *one has* $0 < \mathbf{P}^0\{C_0^N(0) = \emptyset\} < 1$. *The corresponding bounds for the exponential case* $\lambda_n = \lambda^n$ *are given by (19) and (20) below.*

Proof. Obviously the cell centered in the origin dies on level n if no points of Π_n fall into $b(0, R_{n-1}(0))$. Therefore,

$$\begin{aligned}\mathbf{P}^0\{C_0^N(0)=\emptyset\} &\geq \sup_{1\leq n\leq N}\mathbf{P}^0\{C_0^n(0)=\emptyset\}\\ &\geq \sup_{1\leq n\leq N}\mathbf{P}^0\{\Pi_n\big(b(0,R_{n-1}(0))\big)=0\}\\ &= \sup_{1\leq n\leq N}\mathbf{E}^0\exp\{-b_d\lambda_n R_{n-1}^d(0)\}\\ &= \sup_{1\leq n\leq N}\Big[1-\int_0^\infty db_d\lambda_n r^{d-1}\exp\{-b_d\lambda_n r^d\}\mathbf{P}^0\{R_{n-1}(0)>r\}\,dr\Big].\end{aligned}$$

By Theorem 2, for $r \geq c\sqrt{d}$,

$$\mathbf{P}^0\{R_{n-1}(0)>r\} \leq \mathbf{P}^0\{R_\infty(0)>r\} \leq a_1\sum_{n=1}^\infty\Big(\frac{\phi(\lambda_n)}{\lambda_n}\Big)^{A(r)}.$$

Choose $r_* > c\sqrt{d}$ such that the last expression is less than 1 for all $r > r_*$. Then we obtain the following estimate

$$\begin{aligned}\mathbf{P}^0\{C_0^n(0)=\emptyset\} \geq 1 &- \int_0^{r_*} db_d r^{d-1}\lambda^n\exp\{-b_d\lambda^n r^d\}\,dr\\ &- \int_{r_*}^\infty db_d r^{d-1}\lambda^n a_1\sum_{n=1}^\infty\Big(\frac{\phi(\lambda_n)}{\lambda_n}\Big)^{A(r)}dr,\end{aligned}$$

which can be worked out in each particular case. For instance, in the exponential case considered in Corollary 1, with $r_* = c_*$ and (13) we have

$$\begin{aligned}\mathbf{P}^0\{C_0^n(0)=\emptyset\} \geq 1 &- \int_0^{c_*} db_d r^{d-1}\lambda^n\exp\{-b_d\lambda^n r^d\}\,dr\\ &- \int_{c_*}^\infty db_d r^{d-1}\lambda^n c_1\exp\{-(b_d\lambda^n+c_2)r^d\}\,dr. \quad (18)\end{aligned}$$

Calculating the integrals in (18) and summing up the results, we obtain the estimate

$$\mathbf{P}^0\{C_0^N(0)=\emptyset\} \leq \sup_{1\leq n\leq N}(2\lambda)^{-b_d\lambda^n/c_2}\left[1-\frac{2db_d^2\lambda^{n-1}}{c_2(b_d\lambda^n+c_2)}\right]. \quad (19)$$

In order to prove an upper bound, write

$$\begin{aligned}\mathbf{P}^0\{\exists n\,:\,C_0^n(0)=\emptyset\} &\leq \mathbf{P}^0\{r_\infty(0)=0\}\\ &\leq \mathbf{P}^0\{r_\infty(0)=0\,|\,r_0(0)>r\}+\mathbf{P}^0\{r_0(0)\leq r\},\end{aligned}$$

for arbitrary $r > 0$. The event $\{r_0(0) > r\}$ means that the ball $b(0, 2r)$ contains no points of Π_0, therefore its probability equals $\exp\{-b_d 2^d r^d\}$. Using the estimate (7) of Theorem 2, for $r > c\sqrt{d}$, we finally get

$$\mathbf{P}^0\{C_0^N(0) = \emptyset\} \le 1 - \sup_{r \ge c\sqrt{d}} \left[\exp\{-b_d 2^d r^d\} \left(1 - a_1 \sum_{n=1}^{\infty} \left(\frac{\phi(\lambda_n)}{\lambda_n}\right)^{A(\rho)}\right)\right],$$

and in the exponential case, with $r_* = c_*$,

$$\mathbf{P}^0\{C_0^N(0) = \emptyset\} \le 1 - \sup_{r \ge c_*} \left[\exp\{-b_d 2^d r^d\} \left(1 - c_1 \exp\{-c_2 r^d\}\right)\right]. \tag{20}$$

4 Limit tessellation

Heuristic arguments suggest that the difference between two successive aggregate cells becomes smaller and smaller if the intensities of the point processes grow sufficiently fast. One might expect, and Corollary 4 proves this, that with positive probability the family of cells $\{C_0^n(x_i^0)\}$ centered in the same nucleus x_i^0 possesses a non-empty "core" $\operatorname{int}\left(\cap_n C_0^n(0)\right)$. However, *a priori* it is unclear if the boundaries of the cells stabilize as $n \to \infty$. Therefore, it is important to know, whether there exists and in what sense a *limit* object for the process of tessellations, and if the answer is positive, whether this limit object is itself a tessellation. In this section we will address these questions.

Define the limit cell $C_0^\infty(x_i^0)$ as

$$C_0^\infty(x_i^0) = \operatorname{cl}\left(\cup_m \cap_{n \ge m} C_0^n(x_i^0)\right). \tag{21}$$

Thus $C_0^\infty(x_i^0)$ is the closure of those x that belong to $C_0^n(x_i^0)$ for all n starting from some m. Our main purpose is to show that such defined objects under a suitable condition constitute a tessellation in $\mathbb{R}^d$.

Theorem 3 *Assume that a sequence of stationary aggregate tessellations satisfies*

$$\mathbf{P}\{\exists x_i^0 \in \Pi_0 \,:\, 0 \in \operatorname{int}(\cap_{n=0}^\infty C_0^n(x_i^0))\} > 0\,. \tag{22}$$

Then the sets $\{C_0^\infty(x_i^0)\}$ a.s. constitute a tessellation of $\mathbb{R}^d$.

To prove this theorem we need the following lemma.

Lemma 1 *Assume that a sequence of stationary aggregate tessellations satisfies condition (22). Then for any $y \in \mathbb{R}^d$ there exists an almost surely unique limit cell $C_0^\infty(x_i^0)$ such that $y \in \operatorname{int}\left(C_0^\infty(x_j^0)\right)$.*

Proof. Because of the stationarity, it is sufficient to prove the Lemma for $y = 0$. Let $C_m^n(x_i^m)$ be the cells of the aggregate tessellation Θ_m^n defined as

$$\Theta_m^n = \Theta^m \circ \Theta^{m+1} \circ \ldots \circ \Theta^n .$$

Denote by $x^n(0)$ the point of Π_n such that $0 \in C^n(x^n(0))$ and consider the following r.v.'s:

$$\begin{aligned}
&T_1 = \min\{n > 0 \,:\, 0 \notin \operatorname{int} \cap_{k=0}^n C_0^k\left(x^0(0)\right)\}\,,\\
&T_2 = \min\{n > T_1 \,:\, 0 \notin \operatorname{int} \cap_{k=T_1}^n C_{T_1}^k\left(x^{T_1}(0)\right)\}\,,\\
&\ldots\\
&T_n = \min\{n > T_{n-1} \,:\, 0 \notin \operatorname{int} \cap_{k=T_{n-1}}^n C_{T_{n-1}}^k\left(x^{T_{n-1}}(0)\right)\}\,,\\
&\ldots
\end{aligned}$$

Thus T_n are the levels at which the aggregate cell containing the origin changes. Clearly, if some $T_i = \infty$ then $T_j = \infty$ for all $j \geq i$. By assumption (22), we have $\mathbf{P}\{T_1 < \infty\} = p > 0$. Therefore, for all $n > 1$,

$$\mathbf{P}\{T_n < \infty\} \leq \mathbf{P}\{T_1 < \infty\} = p\,.$$

Moreover, T_n are stopping times and the distribution of T_{n+1} given $\{T_n = k\}$ depends only on $\Pi_{k+1}, \Pi_{k+2}, \ldots$. Therefore,

$$\mathbf{P}\{T_1 < \infty, \ldots, T_n < \infty\} \leq p^n\,.$$

Thus $\mathbf{P}\{\forall k,\ T_k < \infty\} = 0$ so that a.s. there exists $n \geq 0$ such that $T_n < \infty$, but $T_{n+1} = \infty$. The lemma statement now holds for a.s. unique x_i^0 such that $x^n(0) \in C_0^n(x_i^0)$.

Proof of Theorem 3. Recall the definition of a tessellation from Section 1. We start from verifying condition *(b)*. By Lemma 1, with probability 1, for every $i \neq j$,

$$\operatorname{int}\left(C_0^\infty(x_i^0)\right) \cap \operatorname{int}\left(C_0^\infty(x_j^0)\right) = \emptyset.$$

Otherwise, with positive probability for some $i \neq j$,

$$\left|\operatorname{int}\left(C_0^\infty(x_i^0)\right) \cap \operatorname{int}\left(C_0^\infty(x_j^0)\right)\right| > 0,$$

and hence with positive probability a point $y \in \mathbb{R}^d$ is covered by the intersection of the interiors of the two cells. This contradicts the uniqueness in Lemma 1. Consequently, for the boundary Γ_0^∞ of the limit tessellation one has

$$|\Gamma_0^\infty| = |\mathbb{R}^d \setminus \cup_i \operatorname{int}\left(C_0^\infty(x_i^0)\right)| = 0 \quad \text{a.s.}\,. \tag{23}$$

Let us verify *(c)*. For a Borel set $B \subset \mathbb{R}^d$ introduce a random variable

$$N(B) = \limsup_{n\to\infty} \sum_{x_i^0 \in \Pi_0} \mathbb{1}\{C_0^n(x_i^0) \cap B \neq \emptyset\}$$

Obviously, $N(B) = \infty$ if an infinite number of cells $C_0^\infty(x_i^0)$ intersects B. If σ_n denotes the sigma-algebra generated by the sequence of processes Π_k, $k = n, \dots, \infty$ then the event $\{N(B) = \infty\}$ belongs to the tail sigma-algebra $\sigma_\infty = \cap_n \sigma_n$. Since Π_k are independent, the zero-one law applies so that $\mathbf{P}\{N(B) = \infty\} = 0$ or 1. From (22) it follows that for some $\varepsilon > 0$

$$\mathbf{P}\{\exists x_i^0 \in \Pi_0 \,:\, b(0,\varepsilon) \in \operatorname{int}(\cap_{n=0}^\infty C_0^n(x_i^0))\} > 0\,,$$

and hence, $\mathbf{P}\{N(b(0,\varepsilon)) = \infty\} = 0$. Every bounded set $B \subset \mathbb{R}^d$ can be covered by a finite family $\{b(t_k,\varepsilon)\}_{k<K}$ of copies of $b(0,\varepsilon)$ shifted by t_k. Because of stationarity, $N(b(0,\varepsilon))$ and $N(b(t_k,\varepsilon))$ have the same distribution for each $t_k \in \mathbb{R}^d$, therefore

$$\mathbf{P}\{N(B) = \infty\} \le \sum_{k=1}^K \mathbf{P}\{N(b(t_k,\varepsilon)) = \infty\} = 0.$$

In order to prove *(a)*, we need to show that the set

$$\Phi = \mathbb{R}^d \setminus \cup_i C_0^\infty(x_i^0)$$

is a.s. empty. Observe that $\Phi \subset \Gamma_0^\infty$, and therefore Φ contains only boundary points if non-empty. For such a point $y \in \Phi$, there exists a sequence

$$\{y_k\} \subset \cup_i C_0^\infty(x_i^0)$$

converging to y. Being itself a bounded set, this sequence visits only a finite number of limit cells $C_0^\infty(x_i^0)$. At least one of these cells contains an infinite subsequence $\{y_{k_n}\}$, and hence it contains y because cells are closed sets. We come to a contradiction with the non-emptiness of Φ.

Corollary 6 *Assume that the intensities $\{\lambda_n\}$ for* **PVAT** *satisfy condition (5). Then the family $\{C_0^\infty(x_i^0)\}$ constitutes a tessellation of $\mathbb{R}^d$.*

Proof. In view of Corollary 4, the condition (22) is satisfied and thus Theorem 3 applies.

Remark 2 Examining the proof of Lemma 1 one may see that a stronger result is proved there: for all $y \in \mathbb{R}^d$

$$\mathbf{P}\{\exists x_i^0 \in \Pi_0 \,:\, y \in \cup_m \operatorname{int} \cap_{m\ge n} C_0^n(x_i^0)\} = 1\,.$$

One always has

$$\cup_m \operatorname{int} \cap_{m\geq n} C_0^n(x_i^0) \subseteq \operatorname{int} \cup_m \cap_{m\geq n} C_0^n(x_i^0) \subseteq \operatorname{int} C_0^\infty(x_i^0)\,.$$

Replacing $\operatorname{int}\big(C_0^\infty(x_i^0)\big)$ with $\cup_m \operatorname{int} \cap_{m\geq n} C_0^n(x_i^0)$ in the proof of part *(b)* of Theorem 3 yields

$$|\mathbb{R}^d \setminus \cup_i \big(\cup_m \operatorname{int} \cap_{m\geq n} C_0^n(x_i^0)\big)| = 0$$

instead of (23). So, in fact, one has almost surely

$$\cup_m \operatorname{int} \cap_{m\geq n} C_0^n(x_i^0) = \operatorname{int} \cup_m \cap_{m\geq n} C_0^n(x_i^0) = \operatorname{int} C_0^\infty(x_i^0)\,. \tag{24}$$

It is possible to define the limit tessellations in various other ways. Let us show that the most natural ones are essentially equivalent in that they define the same tessellation.

Let $\{A_n\}$ be a sequence of closed subsets of $\mathbb{R}^d$. Recall the definitions of the lower and upper set limits:

$$\begin{aligned}
\liminf A_n &= \{x \,:\, \exists\{x_n\} \text{ such that } x_n \in A_n \text{ and } x = \lim_n x_n\}\,,\\
\limsup A_n &= \{x \,:\, \exists\{x_{n_k}\} \text{ such that } x_{n_k} \in A_{n_k} \text{ and } x = \lim_k x_{n_k}\}\,.
\end{aligned}$$

In words, a point belongs to $\liminf A_n$ if and only if any its neighborhood intersects with all sets A_n starting from some m; a point belongs to $\limsup A_n$ if and only if any its neighborhood intersects infinitely many sets A_n. Both limits are closed sets (see, e. g., [8, Prop. 1.2.3]). Define the sets

$$D_0^\infty(x_i^0) = \operatorname{cl}\big(\cap_m \cup_{n\geq m} C_0^n(x_i^0)\big)\,, \tag{25}$$

$$\underline{E}(x_i^0) = \liminf_n C_0^n(x_i^0)\,, \tag{26}$$

$$\overline{E}(x_i^0) = \limsup_n C_0^n(x_i^0)\,. \tag{27}$$

Corollary 7 *Assume that a sequence of stationary aggregate tessellations satisfies condition (22). Then the sets $C_0^\infty(x_i)$, $D_0^\infty(x_i^0)$, $\underline{E}(x_i^0)$, and $\overline{E}(x_i^0)$ defined by (21), (25-27) have almost surely the same interior, may differ only on the set of a null measure and thus define the same tessellation.*

Proof. Obviously, $C_0^\infty(x_i^0) \subseteq D_0^\infty(x_i^0) \subseteq \overline{E}(x_i^0)$ and $C_0^\infty(x_i^0) \subseteq \underline{E}(x_i^0) \subseteq \overline{E}(x_i^0)$ Moreover, $\overline{E}(x_i^0) \cap \operatorname{int}\big(C_0^\infty(x_j^0)\big) = \emptyset$ for any $j \neq i$. Therefore,

$$\begin{aligned}
D_0^\infty(x_i^0) \setminus C_0^\infty(x_i^0) \subseteq \overline{E}(x_i^0) \setminus C_0^\infty(x_i^0) &\subseteq \overline{E}(x_i^0) \setminus \operatorname{int} C_0^\infty(x_i^0)\\
&= \overline{E}(x_i^0) \setminus \cup_i \operatorname{int} C_0^\infty(x_i^0) \subseteq \mathbb{R}^d \setminus \cup_i \operatorname{int} C_0^\infty(x_i^0) = \Gamma_0^\infty\,.
\end{aligned}$$

Replacing $D_0^\infty(x_i^0)$ by $\underline{E}(x_i^0)$, we obtain that the same relation is also true for $\underline{E}(x_i^0)$. the assertion now follows from (23).

The following complementary result asserts the convergence of the boundaries

$$\Gamma_0^n = \mathbb{R}^d \setminus \cup_i \operatorname{int}\left(C_0^n(x_i^0)\right).$$

of the aggregate tessellations Θ_0^n to the the boundary Γ_0^∞ of the limit one. This limit boundary can be a fractal as it is shown in Section 6 for the case of **PVAT** .

Corollary 8 *Assume that a sequence of stationary aggregate tessellations satisfies condition (22). Then*

$$\liminf \Gamma_0^n = \limsup \Gamma_0^n = \Gamma_0^\infty \quad a.s. \tag{28}$$

Also, the distances

$$d(x, \Gamma_0^n) = \inf\{\|x - y\|,\ y \in \Gamma_0^n\} \tag{29}$$

converge a.s. to $d(x, \Gamma_0^\infty)$ *for every* $x \in \mathbb{R}^d$.

Proof. According to [8, Prop. 1.2.5]), the two definitions (28) and (29) of convergence are equivalent in $\mathbb{R}^d$. We will first show that $\Gamma_0^\infty \subset \liminf \Gamma_0^n$. If $x \in \Gamma_0^\infty$, then by (23), $x \in C_0^\infty(x_i^0) \cap C_0^\infty(x_j^0)$ for some $i \neq j$. Therefore, a neighborhood of x hits $\cap_{n\geq m} C_0^n(x_i^0)$ and $\cap_{n\geq m} C_0^n(x_j^0)$ starting from some m. Then it must also hit Γ_0^n for all $n \geq m$.

Let us verify that $\limsup \Gamma_0^n \subset \Gamma_0^\infty$. Suppose that $x_{n_k} \in \Gamma_0^{n_k}$ and $x = \lim_k x_{n_k}$. If $x \notin \Gamma_0^\infty$, then $x \in \operatorname{int}\left(C_0^\infty(x_i^0)\right)$ for some i. By (24), $x \in \operatorname{int} \cap_{n\geq m} C_0^n(x_i^0)$ for some m, and the sequence of $x_{n_k} \in \Gamma_0^{n_k}$ cannot converge to x. From this contradiction it follows that $x \in \Gamma_0^\infty$.

Corollary 9 *Under condition (22), for each* $y \in \mathbb{R}^d$,

$$\lim_n \mathbf{P}^0\{y \in C_0^n(0)\} = \mathbf{P}^0\{y \in C_0^\infty(0)\}.$$

Proof. By Corollary 7,

$$\mathbf{P}^0\{y \in C_0^\infty(0)\} = \mathbf{P}^0\{y \in D_0^\infty(0)\}.$$

Using the continuity property of the probability measures, we obtain

$$\mathbf{P}^0\{y \in C_0^\infty(0)\} = \lim_m \mathbf{P}^0\{y \in \cup_{n\geq m} C_0^n(0)\} \geq \lim_n \mathbf{P}^0\{y \in C_0^n(0)\},$$
$$\mathbf{P}^0\{y \in C_0^\infty(0)\} = \lim_m \mathbf{P}^0\{y \in \cap_{n\geq m} C_0^n(0)\} \leq \lim_n \mathbf{P}^0\{y \in C_0^n(0)\}.$$

Remark 3 Replacing in the last proof $\mathbf{P}^0\{y \in \cdot\}$ by $|\cdot|$, we get

$$\lim_n |C_0^n(0)| = |C_0^\infty(0)| \quad \text{a.s.}$$

Also,

$$\mathbf{E}^0 |C_0^\infty(0)| = 1$$

due to the stationarity of the tessellation.

5 Spherical contact distribution function of the boundary

From now on we confine ourselves to **PVAT** with exponentially growing intensities. As we have shown in the previous section, if (5) holds, the limit Poisson-Voronoi tessellations exist and the boundary of the limit cells Γ_0^∞ is a random closed set defined by (23). One of its important characteristics is the *spherical contact distribution function* $H(r)$, defined as

$$H(r) = \mathbf{P}\{\Gamma_0^\infty \cap b(0,r) \neq \emptyset \,\big|\, 0 \notin \Gamma_0^\infty\}, \qquad r \geq 0.$$

Here, as Lemma 1 shows, the probability of the condition is one, thus $H(0) = 0$. Some information on the degree of variability of the cell boundary can be derived from the rate at which $H(r)$ decreases as r tends to zero. The aim of this section is to prove the following result.

Theorem 4 *For* **PVAT** *with exponentially growing intensities:* $\lambda_n = \lambda^n$ *for some* $\lambda > 1$, *there exist constants* $K > 0$ *and* $q \in (0,1)$ *such that for all* $r \geq 0$,

$$H(r) = \mathbf{P}\{b(0,r) \cap \Gamma_0^\infty \neq \emptyset\} \leq K r^{dq}.$$

The values of q *and* K *are given in* (37).

Proof. Consider the cells of the limit tessellation $\{C_n^\infty(x_i^n)\}$ defined in the same way as in (21). Let $X(n,r)$ be the nuclei of those cells whose boundary crosses the ball $b(0,r)$, that is

$$X(n,r) = \{x_i^n \in \Pi_n \,:\, \partial C_n^\infty(x_i^n) \cap b(0,r) \neq \emptyset\}.$$

We will first prove the estimate: for each $n \geq 1$ and for each $s > 0$,

$$\mathbf{P}\{b(0,r) \cap \Gamma_0^\infty \neq \emptyset\} < f(r,n,s)^n, \tag{30}$$

where

$$f(r,n,s) = 1 - \mathbf{P}\{X(n,r) \subset b(0,s)\}\mathbf{P}\{b(0,s) \subset C^n(x^{n-1}(0))\}.$$

Recall that $x^{n-1}(0)$ denotes the closest to 0 point of Π_{n-1}. Consider the events

$$E(m,r) = \{b(0,r) \cap \Gamma_m^\infty \neq \emptyset\},$$

where Γ_m^∞ is the boundary of the tessellation $\{C_m^\infty(x_i^m)\}$. Note that $E(0,r)$ is the event in the left-hand side of (30). Since $\Gamma_m^\infty \subset \Gamma_{m+1}^\infty$, we have $E(m,r) \subset E(m+1,r)$ and therefore,

$$\begin{aligned}\mathbf{P}\{E(0,r)\} &= \mathbf{P}\{E(1,r)\}\mathbf{P}\{E(0,r) \,|\, E(1,r)\} \\ &= \mathbf{P}\{E(n,r)\} \prod_{m=1}^{n} \mathbf{P}\{E(m-1,r) \,|\, E(m,r)\} \\ &< \prod_{m=1}^{n} \mathbf{P}\{E(m-1,r) \,|\, E(m,r)\}. \quad (31)\end{aligned}$$

For every m and $s_m > 0$,

$$\mathbf{P}\{E(m-1,r) \,|\, E(m,r)\} \leq f(r,m,s_m), \quad (32)$$

Indeed, if $b(0,r) \cap \Gamma_m^\infty \neq \emptyset$ and

$$X(m,r) \subset b(0,s_m) \subset C^{m-1}(x^{m-1}(0))$$

then the cells $C_m^\infty(x_i^m)$ for which $x_i^m \in X(m,r)$ join in $C_{m-1}^\infty(x^{m-1}(0))$ so that $b(0,r) \cap \Gamma_{m-1}^\infty = \emptyset$.

Since the intensity of each Π_n equals λ^n, the distributions of

$$(\Pi_n, \Pi_{n+1}, \dots) \quad \text{and} \quad (\lambda^{1/d}\Pi_{n+1}, \lambda^{1/d}\Pi_{n+2}, \dots)$$

coincide. Consequently, the sets $X(m+1,r)$ and $\lambda^{-1/d}X(m,\lambda^{1/d}r)$ have the same distribution, and

$$\begin{aligned}\mathbf{P}\{X(m+1,r) \subset b(0,s_m)\} &= \mathbf{P}\{\lambda^{-1/d}X(m,\lambda^{1/d}r) \subset b(0,s_m)\} \\ &= \mathbf{P}\{X(m,\lambda^{1/d}r) \subset b(0,\lambda^{1/d}s_m)\} \\ &< \mathbf{P}\{X(m,r) \subset b(0,\lambda^{1/d}s_m)\},\end{aligned}$$

as $X(m,r) \subseteq X(m,\lambda^{1/d}r)$. Also

$$\mathbf{P}\{b(0,s_m) \subset C^m(x^m(0)\} = \mathbf{P}\{b(0,\lambda^{1/d}s_m) \subset C^{m-1}(x^{m-1}(0))\}$$

so that $f(r,m-1,\lambda^{1/d}s_m) < f(r,m,s_m)$. Alternatively, $f(r,m,s_m) < f(r,m+1,\lambda^{-1/d}s_m) < f(r,n,\lambda^{-(n-m)/d}s_m)$ by induction. Thus, taking $s_m = \lambda^{(n-m)/d}s$ in (32), by (31) we obtain (30).

Next, we find a bound for $f(r,n,s)$. We have

$$\begin{aligned}
1-\mathbf{P}\{X(n,r)\subset b(0,s)\} &= 1-\mathbf{P}\{X(0,r)\subset b(0,\lambda^{n/d}s)\}\\
&< \mathbf{P}\{\exists x_i^0 \,:\, x_i^0\notin b(0,\lambda^{n/d}s) \text{ and } C_0^\infty(x_i^0)\cap b(0,\lambda^{n/d}r)\neq\emptyset\}\\
&< \mathbf{P}\{\exists x_i^0 \,:\, x_i^0\notin b(0,\lambda^{n/d}s) \text{ and } R_\infty(x_i^0)>\|x_i^0\|-\lambda^{n/d}r\}\\
&< \mathbf{E}\sum_{x_i^0\in\Pi_0}\mathbb{1}\big(\|x_i^0\|>\lambda^{n/d}s;\ R_\infty(x_i^0)>\|x_i^0\|-\lambda^{n/d}r\big).
\end{aligned}$$

By the Campbell theorem, the last expectation equals

$$\begin{aligned}
&\int_{\|z\|>\lambda^{n/d}s}\mathbf{P}^0\{R_\infty(0)>\|z\|-\lambda^{n/d}r\}\,dz\\
&\qquad=\lambda^n\int_{\|z\|>s}\mathbf{P}^0\{R_\infty(0)>\lambda^{n/d}(\|z\|-r)\}\,dz \quad (33)
\end{aligned}$$

Assume that s and n satisfy the inequality

$$\lambda^{n/d}(s-r)\geq c_*, \tag{34}$$

with c_* as in Corollary 1. Then by (13), the right-hand side of (33) is smaller than

$$c_1\lambda^n\int_{\|z\|>s}e^{-c_2\lambda^n(\|z\|-r)^d}\,dz = c_1\lambda^n\int_s^\infty db_d\rho^{d-1}e^{-c_2\lambda^n(\rho-r)^d}\,d\rho \leq (c_1/c_2)e^{-c_2\lambda^n(s-r)^d}$$

and therefore,

$$\mathbf{P}\{X(n,r)\subset b(0,s)\}>1-(c_1/c_2)e^{-c_2\lambda^n(s-r)^d}. \tag{35}$$

Also, since there is only one Voronoi cell that may contain a ball, we may write

$$\begin{aligned}
\mathbf{P}\{b(0,s)\subset C^{n-1}(x^{n-1}(0))\} &= \mathbf{P}\{b(0,\lambda^{(n-1)/d}s)\subset C^0(x^0(0))\}\\
&= \mathbf{E}\sum_{x_i^0\in\Pi_0}\mathbb{1}\{b(0,\lambda^{(n-1)/d}s)\subset C^0(x_i^0)\}\\
&= \int\mathbf{P}^0\{b(z,\lambda^{(n-1)/d}s)\subset C^0(0)\}\,dz\,.
\end{aligned}$$

Using (16), we obtain

$$\mathbf{P}\{b(0,s)\subset C^{n-1}(x^{n-1}(0))\}>b_d\lambda^{n-1}s^de^{-b_d\lambda^{n-1}(3s)^d}. \tag{36}$$

Combining (35) and (36), for such s and n that (34) holds, we get

$$\mathbf{P}\{b(0,r)\cap\Gamma_0^\infty\neq\emptyset\}<\left[1-\left(1-\frac{c_1}{c_2}\exp\{-c_2\lambda^n(s-r)^d\}\right)\right.$$
$$\left.\times\, b_d\lambda^{n-1}s^d\exp\{-b_d\lambda^{n-1}(3s)^d\}\right]^n.$$

One of the possible choices of the constants s and n is

$$s=2r\qquad\text{and}\qquad n=d\log_\lambda(c_*/r).$$

Then

$$\mathbf{P}\{b(0,r)\cap\Gamma_0^\infty\neq\emptyset\}<Kr^{dq},$$

where

$$q=-\log_\lambda\left[1-\left(1-\frac{c_1}{c_2}e^{-c_2c_*^d}\right)\frac{b_d(2c_*)^d}{\lambda}e^{-b_d(6c_*)^d/\lambda}\right]$$
$$K=c_*^{-dq}. \tag{37}$$

Since $\lambda>1$, we have $0<q<1$. The theorem is proved.

6 Fractal boundary of the limit cells

In the case of exponentially growing intensities, the distributions of Γ_0^∞ and Γ_n^∞ scaled by $\lambda^{n/d}$ coincide, i.e., the boundary of the limit tessellation is statistically self-similar. This property is rather different from geometrical self-similarity in the sense of Γ_0^∞ being a union of scaled copies of self. However, by construction, Γ_{n-1}^∞ consists of parts of Γ_n^∞, and therefore, Γ_0^∞ has a similar structure at any scale of observation, which allows us to call it a fractal.

The primary characteristic of a fractal is its dimension, which can be defined in several ways. We will be interested in the *Hausdorff dimension* of Γ_0^∞ (see, e. g., [5, p. 20–23] for definitions of different dimensions that we use here).

Theorem 5 *Let q be the constant defined in (37). Then for* **PVAT** *with exponentially growing intensities: $\lambda_n=\lambda^n$ for some $\lambda>1$, one has*

$$\dim_H\Gamma_0^\infty=\mathbf{E}\dim_H\Gamma_0^\infty\qquad a.s.;$$
$$\dim_H\Gamma_0^\infty<d(1-q)\qquad a.s.$$

Proof. Consider the collection of mesh cubes of size M in $\mathbb{R}^d$ and let $\{\theta_\alpha\}$ be the family of shifts translating the cube at the origin $[0,M)^d$ by the vector $M\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$. Introduce also

$$\begin{aligned} \Lambda_N &= \{\alpha : |\alpha_i| \le N,\ i = 1, \ldots, d\} \\ \Gamma_0^\infty(M) &= \Gamma_0^\infty \cap [0,M)^d \\ \theta_\alpha \Gamma_0^\infty(M) &= \Gamma_0^\infty \cap \theta_\alpha [0,M)^d. \end{aligned}$$

Since $\theta_\alpha \Gamma_0^\infty(M) \subset \Gamma_0^\infty$, with probability 1 for all α, we have

$$\dim_H \Gamma_0^\infty \ge \dim_H \theta_\alpha \Gamma_0^\infty(M),$$

and hence

$$\dim_H \Gamma_0^\infty \ge \sup_{\alpha \in \Lambda_N} \dim_H \theta_\alpha \Gamma_0^\infty(M).$$

Now by the ergodic theorem,

$$\begin{aligned} \dim_H \Gamma_0^\infty &\ge \lim_{N\to\infty} \sup_{\alpha \in \Lambda_N} \dim_H \theta_\alpha \Gamma_0^\infty(M) \\ &\ge \lim_{N\to\infty} \frac{1}{(2N)^d} \sum_{\alpha \in \Lambda_N} \dim_H \theta_\alpha \Gamma_0^\infty(M) \\ &= \mathbf{E} \dim_H \Gamma_0^\infty(M). \end{aligned}$$

Letting $M \to \infty$ in this inequality and using the property of monotonicity of the Hausdorff dimension, we get

$$\dim_H \Gamma_0^\infty \ge \mathbf{E} \dim_H \Gamma_0^\infty,$$

which implies the first equality of the theorem.

To prove the second inequality, we make use of the estimate of the Hausdorff dimension of a set by its upper box dimension (see, e. g., [5, p. 24]). Let $N_\varepsilon(B)$ be the smallest number of closed balls of radius ε that cover B. Then

$$\dim_H \Gamma_0^\infty(M) \le \limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon(\Gamma_0^\infty(M))}{-\log \varepsilon}.$$

Take expectations at both sides of this inequality. It can be easily verified that $[0,M)^d$, and hence $\Gamma_0^\infty(M)$, can be covered by a family $\{b_i\}$ of less than $\left(M\sqrt{d}/2\varepsilon\right)^d$ balls of radius ε. Thus the function in the right-hand side under the limit is bounded by a constant not depending on ε, and therefore we can exchange the limit and the expectation. Moreover, the function $\log(\cdot)$ is concave, hence

$$\mathbf{E} \log(\cdot) \le \log \mathbf{E}(\cdot).$$

Therefore,

$$\mathbf{E}\dim_H \Gamma_0^\infty(M) \le \limsup_{\varepsilon\to 0} \frac{\log \mathbf{E}\, N_\varepsilon(\Gamma_0^\infty(M))}{-\log\varepsilon}.$$

Recalling the definition of the contact distribution $H(r)$ from the previous section, we get

$$\mathbf{E}\, N_\varepsilon(\Gamma_0^\infty(M)) \le \mathbf{E}\sum_i \mathbb{1}(b_i \cap \Gamma_0^\infty(M)) \ne \emptyset) \le \left(\frac{M\sqrt{d}}{2\varepsilon}\right)^d H(\varepsilon).$$

From Theorem 4 it follows that

$$\mathbf{E}\dim_H \Gamma_0^\infty(M) \le \limsup_{\varepsilon\to 0} \frac{d\log(M\sqrt{d}/2) - d\log\varepsilon + \log K + dq\log\varepsilon}{-\log\varepsilon} = d(1-q)\,,$$

and it remains to let $M\to\infty$ to obtain the second statement of the theorem.

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