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Optimal shape design of inductor coils for surface hardening

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Résumé : A shape optimization problem is considered related to the design of induction hardening facilities. The mathematical model consists of a vector potential formulation for Maxwell's equations coupled with the energy balance and an ODE to describe the solid–solid phase transition in steel during heating. Depending on the shape of the coil we control the volume fraction of the high temperature phase. The coil is modeled as a tube and is defined by a unit–speed curve. The shape optimization problem is formulated over the set of admissible curves. The existence of an optimal control is proved. To obtain the form of the shape gradient of the cost functional, the material derivative method is applied. Finally, the first order necessary optimality conditions are estabished for an optimal tube.

Mots-clé : shape optimization, surface hardening, heat equation, Maxwell's equations, optimality conditions

(Abstract: pto)

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Un problème d'optimisation de forme relatif à la modélisation d'endurcissement par induction

Abstract: Nous considérons un problème d'optimisation de forme relatif à la modélisation d'endurcissement par induction. Le modèle mathématique consiste en une formulation potentielle vectorielle pour les équations de Maxwell couplée au bilan d'énergie et à une EDO afin de décrire la phase de transition solide-solide dans l'acier pendant le chauffage. En agissant sur la forme de l'anneau, nous contrôlons la fraction de volume de la phase à haute température. L'anneau est modélisé par un tube et est défini par une courbe de vitesse unité. Le problème d'optimisation de forme est formulé pour l'ensemble des courbes admissibles. Nous prouvons l'existence d'un contrôle optimal. On utilise la méthode de la dérivée matérielle afin d'obtenir le gradient de forme de la fonctionnelle de coût. Enfin, on établit, dans le cas d'un tube optimal, les conditions nécessaires d'optimalité du premier ordre.

Key-words: optimisation de forme, équation non-linéaire de la chaleur, équations de Maxwell, conditions nécessaires d'optimalité

1 Introduction

We investigate the problem of finding the optimal design for an inductor coil in induction hardening machines. The mode of operation of these machines relies on the transformer principle. A given current density in the inductor coil induces eddy currents inside the workpiece. Because of the Joule effect these eddy currents lead to an increase in temperature in the boundary layers of the workpiece. Then the current is switched off and the workpiece is quenched by spray–water cooling. The solid–solid phase transitions during this heat treatment lead to the desired hardening effect.

For an induction hardening machine there are generally speaking two control parameters. One is the frequency of the AC–current applied. This is predefined by choosing a particular machine. Thereby, also the greatest possibe hardening depth is fixed, since it depends on frequency through the skin effect.

The second control parameter is the shape of the inductor coil. These coils are made individually for the specific workpiece from long copper tubes of quadratic or circular cross section. The design of decent coils for specific hardening purposes up to now mostly depends on experience.

However, there is a growing demand in industry for a more precise process control, mainly for two reasons. One is the general goal of weight reduction especially in automotive industry, leading to components made of thinner and thinner steel sheets. Surface hardening of these sheets is a very delicate task, since one must be careful not to harden the complete sheet, which would lead to undesirable fatigue effects. The second one is the tendency for using high quality steels with only small carbon content, which again demands for a very precise process control, now for metallurgical reasons, since the hardenability of a steel is directly related to its carbon content.

There are already numerous papers on modeling and simulation of induction heating machines, e.g. [6], [8], [10], [14]; results on the mathematical analysis can be found in [2]. Models for phase transitions in steel have been investigated in [10], [12] and [18]. Optimal control problems in the case of laser surface hardening have been considered in [1], [13], and for a 2D induction heating problen in [3].

In this paper for the first time a control problem for the 3D induction heating process including phase transitions is investigated. In Sec. 2 we derive the model, consisting of a vector potential formulation of Maxwell's equations, the balance of internal energy and an ODE to describe the phase transition during heating, and prove its well-posedness. In Sec. 3 we formulate the shape design problem for the inductor coil modeled as a tube with circular cross section. In Sec. 4, we prove the existence of an optimal design and finally in the last section first order necessary optimality conditions are derived.

2 The state equations

2.1 The vector potential formulation of Maxwell's equations

We consider the following slightly idealized geometric setting (cf. fig. 1). Let $D \subset \mathbb{R}^3$ with sufficiently smooth boundary and $\Omega \subset D$ be the coil. Its boundary $\partial\Omega$ is dissected into two parts. In Γ_1 the normal component of the current density will be prescribed, this is where in reality the coil is connected to the primary circuit of the hardening machine. Σ is the workpiece to be hardened and $G := \Omega \cup \Sigma$ the set of conductors. Moreover, we define $Q = \Sigma \times (0,T)$.

In eddy current problems we can neglect displacement currents, hence we consider the following set of Maxwell equations:

$$\operatorname{curl} H = J, \tag{2.1a}$$

$$\operatorname{curl} E = -B_t \tag{2.1b}$$

$$\operatorname{div} B = 0 \tag{2.1c}$$

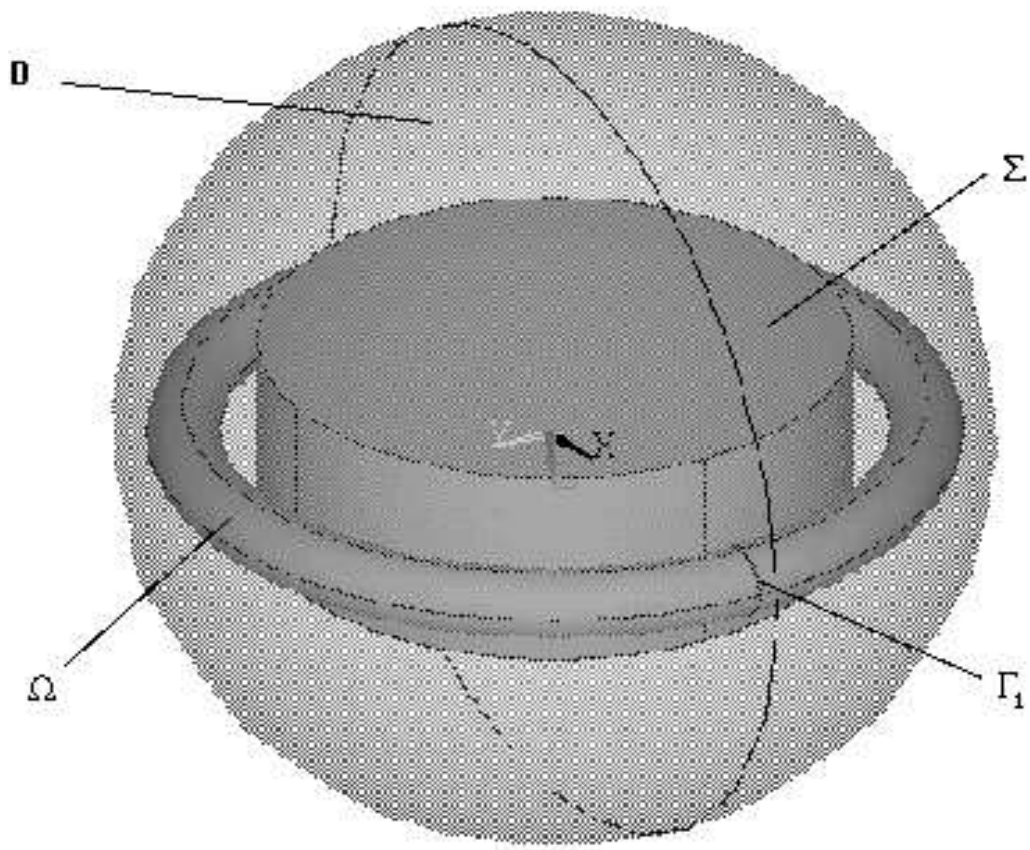

FIG. 1 – *The setting*

Here, E is the electric field, B the magnetic induction, H the magnetic field and J the current density. In addition we consider the following linear constitutive relations

$$\begin{aligned} J &= \sigma E, \quad \text{in } D, \\ B &= \mu H, \quad \text{in } D, \end{aligned}$$

with the magnetic permeability μ and the electric conductivity σ. We assume zero current density outside conductors, i.e.

$$\sigma(x) = \begin{cases} \sigma_0 > 0, & \text{in } \bar{G}, \\ 0, & \text{in } D \setminus \bar{G}, \end{cases}$$

The magnetic permeability takes different values in the coil (usually made of copper), in the workpiece and in the surrounding air. Hence, we assume

$$\mu(x) = \begin{cases} \mu_1, & \text{in } \bar{\Omega} \\ \mu_2, & \text{in } \bar{\Sigma} \\ \mu_3, & \text{in } D \setminus (\bar{\Omega} \cup \bar{\Sigma}). \end{cases} \tag{2.2}$$

Using (2.1a,b) one now introduces the magnetic vector potential A and the scalar potential ϕ such that

$$B = \text{curl } A, \tag{2.3}$$

$$E = -A_t - \text{ grad } \phi. \tag{2.4}$$

Then, Maxwell's equations (2.1a–c) can be rewritten in the following way:

$$\sigma A_t + \text{ curl } \left(\frac{1}{\mu} \text{ curl } A\right) + \sigma \text{ grad } \phi = 0, \quad \text{ in } D. \tag{2.5a}$$

The scalar potential ϕ is determined by the continuity equation div $J = 0$, i.e.

$$- \text{ div } \left(\sigma \text{ grad } \phi + \sigma A_t\right) = 0, \quad \text{ in } G. \tag{2.5b}$$

Since B is not uniquely defined by (2.3), we impose the Coulomb gauge

$$\text{div } A = 0. \tag{2.6}$$

Then (2.5b) can be reduced to the Laplace equation

$$-\Delta\phi = 0, \quad \text{ in } G \tag{2.7a}$$

with boundary condition

$$-\sigma_0 \frac{\partial \phi}{\partial n} = \begin{cases} j_g, & \text{in } \Gamma_1, \\ 0, & \text{in } \partial\Sigma \cup \Gamma_2, \end{cases} \tag{2.7b}$$

with $\Gamma_2 = \partial\Omega \setminus \Gamma_1$.

The system (2.7a,b) is a linear elliptic problem, which can be solved separately in Ω and Σ. At the workpiece boundary $\partial\Sigma$, we have homogenous boundary conditions, i.e. the solution is constant in Σ. Since only the gradient enters in (2.5a), we restrict the domain of ϕ to the coil Ω.

Assuming that the tangential component of A vanishes on ∂D, i.e.

$$n \times A = 0, \tag{2.8}$$

we introduce the spaces

$$\begin{aligned} H(\text{curl},D) &= \{v \in \mathbf{L}^2(D);\ \text{curl } v \in \mathbf{L}^2(D)\}, \\ H_0(\text{curl},D) &= \{v \in H(\text{curl},D);\ n \times v\Big|_{\partial D} = 0\}, \\ H(\text{div},D) &= \{v \in \mathbf{L}^2(D);\ \text{div } v \in \mathbf{L}^2(D)\}, \end{aligned}$$

where $\mathbf{L}$ denotes the vector–valued counterpart $\mathbf{L} = [L]^3$ for any real–valued Sobolev space L. Then, there holds (cf. [7]):

$$H_0(\text{curl},D) \cap H(\text{div},D) = \{v \in \mathbf{H}^1(D);\ n \times v\Big|_{\partial D} = 0\}.$$

Finally, we introduce the Hilbert space

$$\mathbf{X} = \{v \in H_0(\text{curl},D);\ \text{div } v = 0\},$$

which is a closed subspace of $\mathbf{H}^1(D)$, equipped with the norm

$$\|v\|_{\mathbf{X}} = \|\text{ curl } v\|_{\mathbf{L}^2(D)}$$

and recall the Green's formula

$$< n \times f, g >= \int_D \text{ curl } f \cdot g \, dx - \int_D f \cdot \text{ curl } g \, dx, \tag{2.9}$$

for all $f \in H(\text{ curl },D)$ and $g \in \mathbf{H}^1(D)$, where $< .,. >$ denotes the duality pairing between $\mathbf{H}^{-1/2}(\partial D)$ and $\mathbf{H}^{1/2}(\partial D)$.

We assume

(H1) $j_g \in H^1(0,T; H^1(\Gamma_1))$, such that $\int_{\Gamma_1} j_g \, dx = 0$, and $\int_{\Gamma_1} j_{g,t} \, dx = 0$,

(H2) $A_0 \in \mathbf{X}$.

Using (2.8) and (2.9) we obtain the following weak formulation of (2.5a), (2.7a,b):

Find (A,ϕ), such that $A(0) = A_0$,

$$\sigma_0 \int_G A_t \cdot v\, dx + \int_D \frac{1}{\mu} \text{ curl } A \cdot \text{ curl } v\, dx + \sigma_0 \int_\Omega \nabla\phi \cdot v\, dx = 0, \tag{2.10}$$

$$\sigma_0 \int_\Omega \nabla\phi \cdot \nabla u\, dx + \int_{\Gamma_1} j_g u\, dx = 0, \tag{2.11}$$

for all $v \in \mathbf{X}$ and $u \in H^1(\Omega)$.

In view of (H1), we obtain easily:

Lemma 2.1 *Assume (H1), then (2.11) has a solution. Moreover, $\nabla\phi$ is uniquely defined and satisfies*

$$\|\nabla\phi\|_{H^1(0,T;\mathbf{L}^2(\Omega))} \le C,$$

where C only depends on T and j_g.

We assume further

(H3) There exists $y_0 \in \mathbf{X}$ such that

$$\sigma_0 \int_G y_0 \cdot v\, dx + \int_D \frac{1}{\mu} \text{ curl } A_0 \cdot \text{ curl } v\, dx + \sigma_0 \int_\Omega \nabla\phi \cdot \nabla v\, dx = 0,$$

for all $v \in \mathbf{X}$, where ϕ is the unique solution to (2.11) (cf. [16]).

Then, we have

Lemma 2.2 *Assume (H1)–(H3) and let ϕ be the unique solution to (2.11). Then, (2.10) has a unique solution $A \in L^\infty(0,T;\mathbf{X})$. Moreover, there exists a constant $C > 0$ such that*

$$\|A_t\|_{L^4(0,T;L^4(G))} \le C.$$

Proof:

Uniqueness follows from standard arguments. To prove existence, we fix $M \in \mathbb{N}$ and define $h = T/M$, $A^0 = A_0$, and

$$\phi^m(x) = \frac{1}{h} \int_{(m-1)h}^{mh} \phi(x,t)dt \quad \text{for } m \in \{1, \dots M\}.$$

Then, we consider the time-discrete version of (2.10):

$$\sigma_0 \int_G \delta_h A^m \cdot v\, dx + \int_D \frac{1}{\mu} \text{ curl } A^m \cdot \text{ curl } v\, dx + \sigma_0 \int_\Omega \nabla\phi^m \cdot v\, dv = 0 \text{ for all } v \in \mathbf{X}, \tag{2.12}$$

for $m \in \{1, \dots, M\}$, with $\delta_h A^m = \frac{1}{h}(A^m - A^{m-1})$.

Owing to (H2), the first two terms in (2.12) define a coercive bilinear form on $\mathbf{X}$, hence applying the Lax–Milgram lemma, inductively we obtain a unique solution A^m to (2.12) for $m \in \{1, \dots, M\}$. Next, we insert $v = A^m - A^{m-1}$ into (2.12) and sum up for $m = 1, \dots k$:

$$\sigma_0 \sum_{m=1}^k h \int_G \left|\delta_h A^m\right|^2 dx + \sum_{m=1}^k \int_G \frac{1}{\mu} \text{ curl } A^m \cdot \Big(\text{ curl } A^m - \text{ curl } A^{m-1}\Big) dx$$
$$+ \sigma_0 \sum_{m=1}^k \int_\Omega \nabla\phi^m \cdot (A^m - A^{m-1}) dx = 0.$$

Using the inequalities of Young and Hölder, we obtain

$$\begin{aligned}
&\sigma_0 \sum_{m=1}^{k} \int_\Omega \nabla\phi^m \cdot (A^m - A^{m-1})dx \\
&\leq \frac{\sigma_0}{2} \sum_{m=1}^{k} h \int_\Omega \left|\nabla\phi^m\right|^2 dx + \frac{\sigma_0}{2} \sum_{m=1}^{k} h \int_\Omega \left|\delta_h A^m\right|^2 dx.
\end{aligned}$$

Routine calculations show

$$\begin{aligned}
&\sum_{m=1}^{k} \int_G \frac{1}{\mu} \text{ curl } A^m \cdot \Big(\text{ curl } A^m - \text{ curl } A^{m-1}\Big) dx \\
&= \sum_{m=1}^{k} \int_G \frac{1}{\mu} \Big| \text{ curl } A^m\Big|^2 dx - \sum_{m=1}^{k} \int_G \frac{1}{\mu} \text{ curl } A^m \cdot \text{ curl } A^{m-1}\, dx \\
&= \sum_{m=1}^{k} \int_G \frac{1}{2\mu} \Big| \text{ curl } A^m - \text{ curl } A^{m-1}\Big|^2 dx + \int_G \frac{1}{2\mu} \Big| \text{ curl } A^k\Big|^2 dx - \int_G \frac{1}{2\mu} \Big| \text{ curl } A^0\Big|^2 dx.
\end{aligned}$$

Hence, using Lemma 2.1, we obtain

$$\begin{aligned}
&\frac{\sigma_0}{2} \sum_{m=1}^{M} h \left\|\delta_h A^m\right\|^2_{\mathbf{L}^2(G)} + \frac{1}{2\mu_1} \max_{1\leq k\leq M} \left\| \text{ curl } A^k\right\|^2_{\mathbf{L}^2(D)} + \\
&\qquad + \frac{1}{2\mu_1} \sum_{m=1}^{M} \left\| \text{ curl } A^m - \text{ curl } A^{m-1}\right\|^2_{\mathbf{L}^2(D)} \leq c_1, \qquad (2.13)
\end{aligned}$$

with a constant c_1 independent of M.

To obtain higher regularity, we take the difference of (2.12) for m and $m-1$ divided by h,

$$\sigma_0 \int_G \delta_h^2 A^m \cdot v\, dx + \int_D \frac{1}{\mu} \text{ curl } \delta_h A^m \cdot \text{ curl } v\, dx + \sigma_0 \int_\Omega \nabla\delta_h\phi^m \cdot v\, dx = 0 \text{ for all } v \in \mathbf{X},$$

with $\delta_h^2 A^m = \frac{1}{h}(\delta_h A^m - \delta_h A^{m-1})$. In view of (H3), we define $\delta_h A^0 := y_0$. Then, the previous equation is valid for $m \in \{1,\ldots,M\}$.

Putting $v = \delta_h A^m - \delta_h A^{m-1}$, in the same manner as above we get

$$\begin{aligned}
&\frac{\sigma_0}{2} \sum_{m=1}^{M} h \left\|\delta_h^2 A^m\right\|^2_{\mathbf{L}^2(G)} + \frac{1}{2\mu_1} \max_{1\leq k\leq M} \left\| \text{ curl } \delta_h A^k\right\|^2_{\mathbf{L}^2(D)} + \\
&\qquad + \frac{1}{2\mu_1} \sum_{m=1}^{M} \left\| \text{ curl } \delta_h A^m - \text{ curl } \delta_h A^{m-1}\right\|^2_{\mathbf{L}^2(D)} \leq c_2,
\end{aligned}$$

with a constant c_2 independent of M.

Now, we define approximating functions

$$\begin{aligned}
A_h(t) &= A^m + \tfrac{t-mh}{h}(A^m - A^{m-1}) && \text{for } t \in [(m-1)h, mh], \\
\hat{A}_h &= A^m, && \text{for } t \in ((m-1)h, mh].
\end{aligned}$$

In view of (2.13), we have

$$\|A_h\|_{H^1(0,T;\mathbf{L}^2(G))\cap L^\infty(0,T;\mathbf{X})} \leq c_3, \qquad (2.14a)$$

$$\|\hat{A}_h\|_{L^\infty(0,T;\mathbf{X})} \leq c_4, \qquad (2.14b)$$

and

$$\|A_h - \hat{A}_h\|^2_{L^2(0,T;\mathbf{X})} \leq \frac{h}{3} \sum_{m=1}^{M} \int_D \Big| \text{ curl } A^m - \text{ curl } A^{m-1} \Big|^2 dx \longrightarrow 0,$$

for $h \to 0$. Thus, we have the convergences

$$\begin{array}{rcll} A_h & \longrightarrow & A, & \text{weakly–star in } L^\infty(0,T;\mathbf{X}) \cap H^1(0,T;\mathbf{L}^2(G)), \\ \hat{A}_h & \longrightarrow & \hat{A}, & \text{weakly–star in } L^\infty(0,T;\mathbf{X}). \end{array}$$

Moreover,

$$A = \hat{A} \qquad \text{a.e. in } D \times (0,T)$$

and A is a solution to (2.10).

Now, let $y_m = \delta_h A^m$ and

$$\begin{array}{rcll} A_h^{(1)}(t) & = & y^m + \frac{t-mh}{h}(y^m - y^{m-1}) & \text{for } t \in [(m-1)h,mh], \\ \hat{A}_h^{(1)} & = & y^m, & \text{for } t \in ((m-1)h,mh]. \end{array}$$

As before, we get

$$\begin{array}{rcll} A_h^{(1)} & \longrightarrow & y, & \text{weakly–star in } L^\infty(0,T;\mathbf{X}) \cap H^1(0,T;\mathbf{L}^2(G)), \\ & & & \text{strongly in } L^2(0,T;\mathbf{L}^2(G)) \\ \hat{A}_h^{(1)} & \longrightarrow & \hat{y}, & \text{weakly–star in } L^\infty(0,T;\mathbf{X}), \end{array}$$

and $y = \hat{y}$ a.e. in $D \times (0,T)$. On the other hand, we have

$$\hat{A}_h^{(1)} = A_{h,t} \longrightarrow A_t, \quad \text{weakly in } L^2(0,T;\mathbf{L}^2(G)).$$

Hence, we conclude

$$y = A_t \qquad \text{a.e. in } G \times (0,T).$$

Using Sobolev embedding theorem we finally obtain

$$\|A_t\|_{L^4(0,T;\mathbf{L}^4(G))} \leq \|y\|_{L^\infty(0,T;\mathbf{X})} \leq c_4. \tag{2.15}$$

□

2.2 Energy balance and the formation of austenite

A good measure for the hardness penetration depth in the workpiece is the formation of austenite during heating, which can be described by the following initial–value problem derived by Leblond & Deveaux [15] (for details, we refer to [10]):

$$z(0) = 0, \tag{2.16a}$$

$$z_t(t) = \frac{1}{a(\theta)} \Big[b(\theta) - z \Big]^+, \tag{2.16b}$$

where z is the volume fraction of austenite and θ the temperature. To avoid technical difficulties, we assume for the positive part function $[.]^+$ and the temperature dependent coefficients a,b:

(H4) $[x]^+ = x \cdot \mathcal{H}(x)$, where $\mathcal{H}$ is a regularized, smooth Heaviside function,

(H5) $b \in C^{1,1}(\mathbb{R})$, $b(x) \in [0,1]$ for all $x \in \mathbb{R}$,

(H6) $a \in C^{1,1}(\mathbb{R})$, $m \leq a(x) \leq M$ for all $x \in \mathbb{R}$, and constants $0 < m < M$.

We consider the following semi–linear energy balance equation:

$$\begin{aligned} \rho c_p \theta_t - \text{ div } (k \text{ grad } \theta) &= -\rho L z_t + \sigma_0 \left| A_t \right|^2, \quad \text{in } \Sigma \times (0,T) && (2.17a) \\ \frac{\partial \theta}{\partial n} &= 0, \quad \text{in } \partial\Sigma \times (0,T) && (2.17b) \\ \theta(.,0) &= \theta_0 \quad \text{in } \Sigma. && (2.17c) \end{aligned}$$

ρ, c_p, k, L are density, specific heat at constant pressure, heat conductivity and latent heat, assumed to be constant. The first term on the right–hand side of (2.17a) measures the latent heat inside the workpiece Σ, which is consumed during the formation of austenite. The second one describes the Joule heating $\sigma_0 |E|^2$, cf. (2.4). Note that $\nabla\phi \equiv 0$ in Σ.

Using the results of [12], we can easily prove

Lemma 2.3 *Assume (H4)–(H6) and let $\theta_0 \in H^1(\Sigma)$. Then (2.16a,b), (2.17a–c) has a unique solution $(z,\theta) \in W^{1,\infty}(0,T; L^\infty(\Sigma)) \times H^{2,1}(Q)$.*

Here, we have used the abbreviation $Q = \Sigma \times (0,T)$.

3 The shape design problem

The technological aim is to obtain a certain, possibly uniform penetration depth of austenite inside the workpiece. The most important control parameter to achieve this goal is the shape of the coil Ω. Hence, we consider the following cost functional:

$$J(\Omega) = \int_\Sigma \Big(z(x,T) - \bar{z}(x) \Big)^2 \, dx, \tag{3.1}$$

where $\bar{z}$ is a given distribution of austenite.

Note that the cost functional depends on Ω only implicitely, through the solution to the Maxwell equation (2.10).

Inductor coils are manufactured from copper tubes with approximately quadratic cross–section. For convenience, we will consider tubes with circular cross-section. These tubes can easily be generated from curves in the following way (cf. Gray, [11]).

Let $\gamma : [0,l] \longrightarrow \mathbb{R}^3$ be a unit speed curve, i.e.

$$|T(s)| = 1, \qquad s \in [0,l],$$

where $T(s) = \gamma'(s)$ is the unit tangential vector. Since the arc–length of a unit–speed curve is given by

$$\int_0^t |\gamma'(s)| ds = t,$$

unit–speed curves are said to be parametrized by arc length. If the curvature $\kappa(s) = |\gamma''(s)|$ does not vanish, we can define the unit normal and binormal vector fields N and B, respectively, by

$$\begin{aligned} N(s) &= \frac{1}{\kappa(s)} T'(s), \\ B(s) &= T(s) \times N(s). \end{aligned}$$

If $\kappa(s) = 0$, one can easily choose two vectors N, B to form an orthogonal system with $T(s)$.

Then, the tube with circular cross–section $R > 0$ corresponding to γ is given by

$$\Omega = \Omega(\gamma) = \{\omega_\gamma(s,r,\vartheta) \,|\, 0 \le s \le l, 0 \le r \le R, 0 \le \vartheta \le 2\pi\},$$

with

$$\omega_\gamma(s,r,\vartheta) = \gamma(s) + r\cos\vartheta N(s) + r\sin\vartheta B(s). \tag{3.2}$$

The faces of the tube, i.e. the parts, where the inductor is connected to the hardening machine (cf. (2.7b)) are defined by

$$\Gamma_1 = \{\omega_\gamma(0,r,\vartheta) \,|\, 0 \le r \le R, 0 \le \vartheta \le 2\pi\} \cup \{\omega_\gamma(l,r,\vartheta) \,|\, 0 \le r \le R, 0 \le \vartheta \le 2\pi\}.$$

The lateral boundary is parametrized by

$$\Gamma_2 = \{\omega_\gamma(s,R,\vartheta) \,|\, 0 \le s \le l, 0 \le \vartheta \le 2\pi\}.$$

Therefore, $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

Even simply connected curves may generate tubes with intersecting parts. For obvious technical reasons this case has to be excluded. To this end we introduce the notion of *reach* (cf. Federer [9]).

Definition 3.1 *(Federer, 1959)*

Let $\boldsymbol{\Gamma}_\gamma = \{\gamma(s)\,, s \in [0,l]\}$ be the trace of γ. We call Unp $(\boldsymbol{\Gamma}_\gamma)$ *the set of all points $x \in \mathbb{R}^3$, for which there exists a unique projection onto $\boldsymbol{\Gamma}_\gamma$. For $y \in \boldsymbol{\Gamma}_\gamma$ we define*

$$\begin{aligned} \text{reach } (\boldsymbol{\Gamma}_\gamma,y) &= \sup\Big\{r\,, \{x \in \mathbb{R}^3\,, |x-y| < r\} \subset \text{ Unp } (\boldsymbol{\Gamma}_\gamma)\Big\}, \qquad \textit{and} \\ \text{reach } (\boldsymbol{\Gamma}_\gamma) &= \inf\Big\{\text{reach } (\boldsymbol{\Gamma}_\gamma,y)\,, y \in \boldsymbol{\Gamma}\Big\}. \end{aligned}$$

In other words, reach of a subset $B \subset \mathbb{R}^n$ is the largest ε such that for all x in an ε–surrounding of B, there exists a unique projection onto B. If B is convex, then reach $(B) = \infty$. On the other hand, if B is concave with a reentrant corner, e.g. an L-shaped domain, then reach $(B) = 0$. In our situation, to avoid too narrow twists of the curve γ, we demand

$$\text{reach } (\boldsymbol{\Gamma}_\gamma) \ge R + \delta, \tag{3.3}$$

where $\delta > 0$ is a given positive parameter and R is the tube radius.

Now we can introduce the set of admissible curves

$$\begin{aligned} U_{ad} = \{\gamma : [0,l] \to \mathbb{R}^3 \setminus U_{R+\delta}(\Sigma)\,, |\gamma'| = 1 \text{ in } [0,l], \text{reach } (\boldsymbol{\Gamma}_\gamma) \ge R+\delta, \\ 0 < L_1 \le l \le L_2, \|\gamma\|_{H^4[0,l]} \le M, \gamma(0) = \gamma(l) = 0, \\ \gamma'(0) = \gamma'(l), \gamma''(0) = \gamma''(l), \gamma'''(0) = \gamma'''(l)\}, \end{aligned}$$

the corresponding set of admissible domains

$$\mathcal{U}_{ad} = \{\Omega(\gamma)\,, \gamma \in U_{ad}\},$$

and give a precise definition of our control problem, which reads

(CP) Minimize $J(\Omega)$, given by (3.1)
subject to
$\Omega \in \mathcal{U}_{ad}$
and the state equations (2.10), (2.11), (2.16), (2.17).

Note that (CP) is a non–convex optimization (control) problem due to the non–convexity of $\mathcal{U}_{ad}$ and of the cost functional.

For the derivation of optimality conditions it will be convenient to introduce perturbations of the admissible curves, hence we introduce the further notation

$$I(\gamma) = J(\Omega(\gamma)). \tag{3.4}$$

4 The existence of an optimal domain Ω^*

Theorem 4.1
Assume (H1)–(H6), then (CP) admits a solution $\Omega^* \in \mathcal{U}_{ad}$.

For the proof we take a minimizing sequence $\{\Omega_n\} \subset \mathcal{U}_{ad}$ for (3.1). We have $\Omega_n = \Omega(\gamma_n)$ and $\gamma_n : [0,l_n] \to \mathbb{R}^n \setminus U_\delta(\Sigma)$. We extend γ_n by defining

$$\gamma_n(s) = \sum_{k=0}^{3} \gamma_n^{(k)}(l_n)(s-l_n)^k, \quad \text{for } s \in [l_n,L_2].$$

Hence $\{\gamma_n\}$ is bounded in $H^4(0,L_2)$ and there exists a subsequence (still indicated by n) satisfying

$$\gamma_n \to \gamma^* \quad \text{in } C^3(0,L_2).$$

Extracting possibly a further subsequence we also have $l_n \to l^* \in [L_1,L_2]$. Obviously, γ is a unit–speed curve satisfying also (3.4). Let $P = 0$ be the prescribed endpoint of the curves defined in $\mathcal{U}_{ad}$, then

$$|\gamma_n(l^*) - P| \le |\gamma_n(l_n) - P| + |\gamma_n(l^*) - \gamma_n(l_n)| \le c|l^* - l_n| \to 0, \quad \text{for } n \to \infty.$$

In the same manner, we obtain $\gamma^{*'}(0) = \gamma^{*'}(l^*)$, $\gamma^{*''}(0) = \gamma^{*''}(l^*)$ and $\gamma^{*'''}(0) = \gamma^{*'''}(l^*)$, and thus, we have $\Omega^* = \Omega(\gamma^*) \in \mathcal{U}_{ad}$.

For an arbitrary curve γ the lateral boundary Γ_2 of the corresponding tube is parametrized by the function $\tilde{\omega}_\gamma(s,\vartheta) = \omega_\gamma(s,R,\vartheta)$ defined in (3.2). The normal and binormal vectors $N(s)$,$B(s)$ depend on the second derivative of γ, therefore $\tilde{\omega}_\gamma(s,\vartheta)$ is a C^1 function and the lateral boundary Γ_2 is uniformly Lipschitz continuous, with a Lipschitz constant depending on M in the definition of the family $\mathcal{U}_{ad}$. Therefore, for the sequence $\{\Omega_n\}$ corresponding to our subsequence of curves $\{\gamma_n\}$ selected above, we have the following properties:

$$D \setminus \overline{\Omega_n} \longrightarrow D \setminus \overline{\Omega^*} \quad \text{in the sense of Hausdorf metric,} \tag{4.1a}$$

$$\chi_n = \chi_{\Omega_n} \longrightarrow \chi = \chi_{\Omega^*} \quad \text{in } L^p(D) \text{ for all } p < \infty, \tag{4.1b}$$

$$\text{for all } K \subset\subset \Omega^* \text{ there exists } N \text{ such that} K \subset\subset \Omega_n \text{ for all } n > N. \tag{4.1c}$$

We refer the reader to [5] for the compactness results in the class of uniformly Lipschitz domains. Using these properties, in the following Lemmas we are going to pass to the limit for $n \to \infty$ in the state equations and show the convergence of solutions, which implies $J(\Omega_n) \to J(\Omega^*)$ and, therefore the existence of an optimal domain $\Omega^* = \Omega(\gamma^*)$.

We begin with the equation for the scalar potential (2.11). Denoting

$$\widetilde{\nabla\phi_n}(x) = \begin{cases} \nabla\phi_n(x) & ,x \in \Omega_n, \\ 0 & ,x \in D \setminus \Omega_n. \end{cases}$$

we have

$$\sigma_0 \int_D \chi_n \widetilde{\nabla\phi_n} \cdot \nabla u \, dx + \int_{\Gamma_1} j_g u \, dx = 0, \qquad \text{for all } u \in H^1(D). \tag{4.2}$$

and obtain

Lemma 4.1 *There exists a subsequence satisfying*

$$\widetilde{\nabla\phi_n} \longrightarrow \widetilde{\nabla\phi} \qquad \textit{strongly in } H^1(0,T;\mathbf{L}^2(D)). \tag{4.3}$$

Proof:

Since $\{\widetilde{\nabla\phi_n}\}$ is bounded in $L^2(0,T;\mathbf{L}^2(D))$, (4.3) holds weakly in $L^2(0,T;\mathbf{L}^2(D))$. Moreover, taking $u = \widetilde{\nabla\phi_n}$ in (4.2) we have

$$\int_D |\widetilde{\nabla\phi_n}|^2\,dx = -\frac{1}{\sigma_0}\int_{\Gamma_1} j_g\phi_n\,dx \longrightarrow -\frac{1}{\sigma_0}\int_{\Gamma_1} j_g\phi\,dx = \int_D |\widetilde{\nabla\phi}|^2\,dx$$

and thus, strong convergence in $L^2(0,T;\mathbf{L}^2(D))$. Differentiating (4.2) formally with respect to t and reasoning as above completes the proof. □

Now, we consider the equation for the magnetic vector potential (2.10). We denote

$$\mu_n(x) = \begin{cases} \mu_1 & \text{, in } \overline{\Omega_n}, \\ \mu_2 & \text{, in } \overline{\Sigma}, \\ \mu_3 & \text{, in } D\setminus(\overline{\Omega_n}\cup\overline{\Sigma}) = D\setminus\overline{G_n}, \end{cases}$$

which can be expressed using characteristic functions as

$$\mu_n = \mu_1\chi_n + \mu_2\chi_\Sigma + \mu_3(1-\chi_n-\chi_\Sigma).$$

According to (4.1b), we have

$$\mu_n \longrightarrow \mu \text{ in } L^p(D) \text{ for all } p\in[1,\infty).$$

We rewrite (2.10) as

$$\sigma_0\int_{G_n} A_t^n\cdot v\,dx + \int_D \frac{1}{\mu_n}\,\mathrm{curl}\,A^n\cdot\,\mathrm{curl}\,v\,dx + \sigma_0\int_D \widetilde{\nabla\phi_n}\cdot v\,dx = 0.$$

Making a priori estimates analogous to the discrete ones of Lemma 2.2 (cf. (2.14), (2.15)), $G_n = \Omega_n\cup\Sigma$, we obtain

Lemma 4.2 *There exists a subsequence $\{A^n\}$ satisfying*

$$\begin{aligned} A^n &\longrightarrow A, && \textit{weakly-star in } L^\infty(0,T;\mathbf{X}) \\ \chi_{G_n}A_t^n &\longrightarrow \chi_G A_t, && \textit{weakly in } L^2(0,T;L^2(D)), \\ A^n_{t\,|\Sigma} &\longrightarrow A_{t\,|\Sigma}, && \textit{strongly in } C(0,T;L^4(\Sigma)). \end{aligned}$$

The equations for temperature and phase transition (2.16),(2.17) depend only implicitely on the shape of Ω_n, namely through A^n. Standard a priori estimates for this system yield

$$\|z\|_{W^{1,\infty}(0,T;L^\infty(\Sigma))} + \|\theta\|_{H^{2,1}(Q)} \le c_1,$$

with a constant c_1 independent of n. Hence, using Lebesgue's convergence theorem we obtain

Lemma 4.3 *There exist subsequences $\{\theta_n\}, \{z_n\}$ satisfying*

$$\begin{aligned} \theta_n &\longrightarrow \theta, && \textit{strongly in } C([0,T];H^1(\Sigma)), \\ z_n &\longrightarrow z, && \textit{strongly in } C([0,T];L^2(\Sigma)). \end{aligned}$$

In view of Lemmas 4.1 – 4.3, we can pass to the limit in the state equations (2.10), (2.11), (2.16), (2.17) and in the cost functional (3.1), which concludes the proof of Theorem 4.1.

5 Necessary optimality conditions

5.1 Introduction and main results

We propose the following procedure in order to derive the first order optimality conditions.

Let $\Omega = \Omega(\gamma)$ be an admissible domain. First, we investigate $J(\Omega)$ using the speed method [17]. Given an admissible vector field V with $\sup V \cap \partial\Omega \neq \emptyset$ and the associated mapping $T_\tau(V) : \mathbb{R}^3 \to \mathbb{R}^3$, we show that there exists the limit

$$dJ(\Omega; V) = \lim_{\tau \to 0} (J(\Omega_\tau) - J(\Omega)), \tag{5.1}$$

where $\Omega_\tau = T_\tau(V)(\Omega)$. Furthermore, the Eulerian derivative $dJ(\Omega; V)$ is linear and continuous with respect to to V. Therefore, the shape gradient $g_{\partial\Omega}$ is supported on $\partial\Omega$ and

$$dJ(\Omega; V) =< g_{\partial\Omega}, V \cdot \nu > ,$$

where ν is the outer unit normal vector on the lateral boundary of the tube Ω.

Next step is to relate the perturbations of $\partial\Omega$ by means of $T_\tau(V)$ with perturbations of the curve γ in the form γ_ε, where γ_ε is a unit–speed parametrization of $\tilde{\gamma}_\varepsilon = \gamma + \varepsilon\beta$. We associate with $\Omega_\varepsilon = \Omega(\gamma_\varepsilon)$ the vector field $V(\beta)$ for $\varepsilon = 0$. To this end, for a given parametrization X_ε of $\partial\Omega_\varepsilon$ for $\varepsilon \geq 0$ we just have to evaluate

$$V(\beta) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(X_\varepsilon - X).$$

The field is defined on $\partial\Omega$ and we have

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(J(\Omega_\varepsilon) - J(\Omega)) = dJ(\Omega; V(\beta)) =< g_{\partial\Omega}, V(\beta) \cdot \nu > .$$

Therefore, we can use the shape derivative $dJ(\Omega; V)$ in order to evaluate

$$dI(\gamma; \beta) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(I(\gamma_\varepsilon) - I(\gamma)) =< g_{\partial\Omega}, V(\beta) \cdot \nu > .$$

This means that knowing the form of the shape gradient $g_{\partial\Omega}$ for $J(\Omega)$ and of the speed vector field $V(\beta)$ on $\partial\Omega$ associated with the deformations $\Omega_\varepsilon = \Omega(\gamma_\varepsilon)$ of the tube $\Omega = \Omega(\gamma)$, we can evaluate the directional derivative $dI(\gamma; \beta)$ and derive the optimality conditions. In the same way, we can obtain the second order derivative of $I(\gamma)$ (for example for Newton's method).

Our main result is

Theorem 5.1 *Assume (H1)–(H6) then there exists an optimal curve γ^* and an optimal domain $\Omega^* = \Omega(\gamma^*)$, such that the following optimality system is satisfied:*

(1) The state equations (2.10), (2.11), (2.16), (2.17) written with $\Omega^* = \Omega(\gamma^*)$.

(2) The adjoint state equations:

Find (η, B, p, r) such that for all $\varphi \in H^1(\Omega^*)$, $\psi \in \mathbf{X}$ and $\xi \in H^1(\Sigma)$,

$$\int_{\Omega^*} \nabla\eta \cdot \nabla\varphi \, dx + \sigma_0 \int_{\Omega^*} B \cdot \nabla\varphi \, dx = 0, \quad a.e. \text{ in } (0,T) \tag{5.2a}$$

$$B(T) = 0, \quad \text{in } \Sigma \tag{5.2b}$$

$$-\sigma_0 \int_{G^*} B_t \cdot \psi \, dx + \int_D \frac{1}{\mu} \, curl\, B \cdot \, curl\, \psi \, dx$$

$$-2\sigma_0 \int_\Sigma \left(pA_t\right)_t \cdot \psi = 0, \quad a.e. \text{ in } (0,T) \tag{5.2c}$$

$$
\begin{aligned}
p(T) &= 0, \quad \text{in } \Sigma && (5.2d)\\
-\rho c_p \int_\Sigma p_t \xi \, dx + k \int_\Sigma \nabla p \cdot \nabla \xi \, dx - \int_\Sigma \frac{\partial f}{\partial \theta} r \xi &= 0, \quad \text{a.e. in } (0,T) && (5.2e)\\
r(T,x) + 2(z(T,x) - \bar{z}(x)) &= 0, \quad \text{in } \Sigma && (5.2f)\\
-r_t - \rho L p_t - \frac{\partial f}{\partial z} r &= 0, \quad \text{in } \Sigma \times (0,T). && (5.2g)
\end{aligned}
$$

(3) The optimality condition

$$
\begin{aligned}
\int_0^T \int_{\partial\Omega^*} \nabla\varphi \cdot \nabla\eta V(h(\beta)) \cdot \nu \, dx\, dt \; + \sigma_0 \int_0^T \int_{\partial\Omega^*} A_t \cdot BV(h(\beta)) \cdot \nu \, dxdt \, + \\
+ \int_0^T \int_{\partial\Omega^*} \nabla\varphi \cdot BV(h(\beta)) \cdot \nu \, dxdt \, + \\
+ \int_0^T \int_{\partial\Omega^*} \Big[\frac{1}{\mu} \; curl \; A\Big]_{\partial\Omega^*} \cdot \; curl \; BV(h(\beta)) \cdot \nu \, dxdt \quad \geq 0, \qquad (5.3)
\end{aligned}
$$

for all $\beta \in T_{\mathcal{U}_{ad}}(\gamma^*)$, where $V(h)$ is the speed of $\partial\Omega_\varepsilon$ at $\varepsilon = 0$ in the direction $h = h(\beta)$ given by

$$
h(\beta)(s) = \beta(s) - (\gamma^*)'(s) \int_0^s (\gamma^*)' \cdot \beta' d\tilde{s}.
$$

β is an admissible tangent direction, which defines the unit–speed perturbation γ_ε of γ^* by putting

$$
\tilde{\gamma}_\epsilon(s) = \gamma^*(s) + \varepsilon\beta(s), \quad 0 \leq s \leq l^*
$$

and

$$
\gamma_\varepsilon(s) = \tilde{\gamma}_\varepsilon(\tau_\varepsilon(s)),
$$

where the parameter transformation τ_ε is characterized by

$$
\begin{cases}
|\tau_\varepsilon'(s)| &= |\gamma'(\tau_\varepsilon(s)) + \varepsilon\beta'(\tau_\varepsilon(s))|^{-1}\\
\tau_\varepsilon(0) &= 0.
\end{cases}
$$

Admissible $\beta \in T_{\mathcal{U}_{ad}}(\gamma^*)$ satisfy the following conditions

(i) if $l^* = L_1$, then β satisfies

$$
\frac{\partial \tau_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}(L_1) = -\int_0^{L_1} \gamma' \cdot \beta' ds \geq 0,
$$

if $l^* = L_2$, then β satisfies

$$
\frac{\partial \tau_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}(L_2) = -\int_0^{L_2} \gamma' \cdot \beta' ds \leq 0;
$$

(ii) if $\|\gamma^*\|_{H^4(0,l^*)} = M$, then β satisfies

$$
(h(\beta),\gamma)_{H^4(0,l^*)} \leq 0.
$$

Remark 5.1 *Note that γ_ε is admissible only, if* reach $(\gamma_\varepsilon) \geq R + \delta$ *for $\varepsilon > 0$, small enough.*

The proof of theorem 5.1 uses the following results on the shape differentiability of the shape functional $J(\Omega)$.

Theorem 5.2 *Assume (H1)–H(6), then the shape fuctional $J(\Omega)$ is shape differentiable at any domain $\Omega \in \mathcal{U}_{ad}$.*

A direct consequence of Theorem 5.2 is that we can apply the *structure theorem* (cf. Theorem 2.27 in [17]) and obtain

Corollary 5.1 *There exists a distribution $g_{\partial\Omega}$ supported on the lateral boundary $\Gamma_2 \subset \partial\Omega$ such that*

$$dJ(\Omega; V) = < g_{\partial\Omega}, V \cdot \nu > .$$

5.2 Proof of Theorem 5.2

For a given admissible domain $\Omega \in \mathcal{U}_{ad}$ and a speed vector field

$$V \in C^1(-\tau_1, \tau_1; C^2(\mathbb{R}^3; \mathbb{R}^3)),$$

such that supp $V \supset \bar{\Omega}$ and supp $V \cap \Sigma = \emptyset$, we denote

$$\Omega_\tau = T_\tau(V)(\Omega), \qquad \tau \in [0, \tau_1]$$

the family of domains associated with the flow of the field V, in particular $\Omega_0 = \Omega$. All equations defined in Ω_τ can be transported to the fixed domain Ω using the transformation $T_\tau^{-1} : \Omega_\tau \to \Omega$.

In the sequel we indicate functions on Ω_τ with subscript τ and functions transported to the fixed domain Ω with superscript τ, i.e. $f^\tau = f_\tau \circ T_\tau$.

The following lemma describes the transport of div and grad to the fixed domain. The proof can be found in [17], Sec. 2.

Lemma 5.1 *Let $B_1(\tau)^{-1} = \det(DT_\tau) DT_\tau^{-1}$, then we have*

(1)

$$(\ grad\ \varphi) \circ T_\tau = {}^*DT_\tau^{-1} \cdot\ grad\ (\varphi \circ T_\tau) \quad \text{for all } \varphi \in H^1(\mathbb{R}^N),$$

(2)

$$(\ div\ \psi) \circ T_\tau = \frac{1}{\det(DT_\tau)}\ div\ \Big(B_1(\tau)^{-1}(\psi \circ T_\tau)\Big), \quad \text{for all } \psi \in \mathbf{H}^1(\Omega).$$

Using Lemma 5.1, we obtain for (2.11) and $u \in H^1(\Omega_\tau)$

$$\begin{aligned} -\int_{\Gamma_1} j_g u\, dx &= \sigma_0 \int_{\Omega_\tau} \nabla\phi_\tau \cdot \nabla u\, dx \\ &= \sigma_0 \int_{\Omega} \det(DT_\tau)\Big(\nabla\phi_\tau \cdot \nabla u\Big) \circ T_\tau\, dx \\ &= \sigma_0 \int_{\Omega} B_2(\tau) \nabla\phi^\tau \nabla(u \circ T_\tau)\, dx \end{aligned}$$

with

$$B_2(\tau) = \beta(\tau) DT_\tau^{-1}\, {}^*DT_\tau^{-1} \quad \text{and} \quad \beta(\tau) = \det(DT_\tau).$$

Hence, (2.11) is replaced with

$$-\int_{\Gamma_1} j_g u\, dx = \alpha_0(\tau,\phi^\tau,u), \qquad \text{for all } u \in H^1(\Omega), \tag{5.4}$$

and

$$\alpha_0(\tau,\phi^\tau,u) = \sigma_0 \int_\Omega B_2(\tau)\nabla\phi^\tau \cdot \nabla u\, dx.$$

For the Maxwell equation (2.10) the situation is more complicated, since functions A_τ that are divergence free on Ω_τ generally loose this property when transported to the fixed domain. Therefore we introduce an auxiliary unknown function

$$\eta^\tau = B_1(\tau)^{-1}A^\tau, \tag{5.5}$$

for which we have (cf. Lemma 5.1(2))

$$\text{div } A_\tau = 0 \quad \text{in } \Omega_\tau \quad \Longleftrightarrow \quad \text{div } \eta^\tau = 0 \quad \text{in } \Omega.$$

In the same manner we replace the test functions v in Ω_τ with

$$w = B_1(\tau)^{-1} v \circ T_\tau \quad \text{in } \Omega. \tag{5.6}$$

Next we transport the curl operater to the fixed domain. Let e_i be the unit vector in x_i direction, then we may write

$$\text{curl } A = \Big(\text{div } (A \times e_1), \text{div } (A \times e_2), \text{div } (A \times e_3)\Big)^T$$

and using Lemma 5.1, we obtain

$$\begin{aligned}(\text{ curl } A_\tau) \circ T_\tau = \frac{1}{\det(DT_\tau)}\Big(&\text{div } [B_1(\tau)^{-1}(A^\tau \times e_1)], \\ &\text{div } [B_1(\tau)^{-1}(A^\tau \times e_2)], \text{div } [B_1(\tau)^{-1}(A^\tau \times e_3)]\Big)^T.\end{aligned} \tag{5.7}$$

Invoking (5.6) and (5.7) we obtain for (2.10):

$$\begin{aligned}\sigma_0 \int_{G_\tau} \frac{\partial A_\tau}{\partial t} v\, dx &= \sigma_0 \int_G \det(DT_\tau) A_t^\tau \cdot v \circ T_\tau\, dx \\ &= \sigma_0 \int_G B_3(\tau)\, \eta_t^\tau \cdot w\, dx = \alpha_1(\tau,\eta_t^\tau,w),\end{aligned}$$

with

$$B_3(\tau) = \det(DT_\tau)\, {}^*B_1(\tau) B_1(\tau) = \frac{1}{\det(DT_\tau)}\, {}^*DT_\tau DT_\tau.$$

Using Einstein's summation convention, we get for the next term

$$\begin{aligned}\int_D \frac{1}{\mu} \text{ curl } A_\tau \cdot \text{ curl } v\, dx &= \int_D \frac{\det(DT_\tau)}{\mu}\Big(\text{div } (A_\tau \times e_i) \text{ div } v \times e_i)\Big) \circ T_\tau\, dx \\ &= \int_D \frac{1}{\beta(\tau)\mu} \text{ div } [B_1(\tau)^{-1}(A^\tau \times e_i)] \text{ div } [B_1(\tau)^{-1}(v \circ T_\tau \times e_i)]\, dx \\ &= \int_D \frac{1}{\beta(\tau)\mu} \text{ div } [B_1(\tau)^{-1}(B_1(\tau)\eta^\tau \times e_i)] \text{ div } [B_1(\tau)^{-1}(B_1(\tau) w \times e_i)]\, dx \\ &= \alpha_2(\tau,\eta^\tau,w).\end{aligned}$$

For the last term in (2.10) we have

$$\begin{aligned}\int_{\Omega_\tau} \nabla\phi_\tau \cdot v\,dx &= \int_\Omega \det(DT_\tau)\Big(\nabla\phi_\tau \cdot v\Big) \circ T_\tau\,dx \\ &= \int_\Omega \det(DT_\tau)\,{}^*DT_\tau^{-1}\nabla\phi^\tau \cdot B_1(\tau)w\,dx \\ &= \int_\Omega \nabla\phi^\tau \cdot w\,dx = F(\nabla\phi^\tau,w).\end{aligned}$$

Altogether, we have replaced (2.10) with

$$\alpha_1(\tau,\eta_t^\tau,w) + \alpha_2(\tau,\eta^\tau,w) + F(\phi^\tau,w) = 0, \qquad \text{for all } w \in X, \tag{5.8a}$$

$$\eta_0^\tau = B_1(\tau)^{-1}A_0 \circ T_\tau. \tag{5.8b}$$

With these preparations we can derive the material derivatives. We begin with

Lemma 5.2 *B_1, B_2, B_3, β are differentiable. The derivatives at $\tau = 0$ are given by*

$$\begin{aligned}\beta'(0) &= \mathit{div}\ V(0), \\ B_1'(0) &= -\ \mathit{div}\ V(0)I + DV(0), \\ B_2'(0) &= \mathit{div}\ V(0)I - 2\varepsilon(V(0)), \\ B_3'(0) &= -\ \mathit{div}\ V(0)I + 2\varepsilon(V(0)).\end{aligned}$$

Here, $\varepsilon(V(0))$ is the symmetrized part of $DV(0)$, i.e. $\varepsilon(V(0)) = \frac{1}{2}(DV(0) + {}^*DV(0))$. For the proof we refer again to [17], Sec. 2.13.

Corollary 5.2 *For $\tau > 0$ small enough, we have*

$$\begin{aligned}\beta(\tau) &= 1 + \tau\beta'(0) + o(\tau) \\ B_i(\tau) &= I + \tau B_i'(0) + o(\tau), \qquad i = 1,2,3.\end{aligned}$$

A particular consequence of Corollary 5.1 is

Corollary 5.3 *Let $\tau > 0$ be small enough and g_i be real–valued functions satisfying $g_i(\tau) = o(\tau)$, $i = 0,1,2$.*

(1) For all $u_1,u_2 \in H^1(\Omega)$ we have

$$\begin{aligned}\alpha_0(\tau,u_1,u_2) &= \alpha_0(0,u_1,u_2) + \tau\alpha_{0,\tau}(0,u_1,u_2) + \tilde{\alpha}_0(\tau,u_1,u_2), \\ \alpha_{0,\tau}(0,u_1,u_2) &= \int_\Omega B_2'(0)\nabla u_1\nabla u_2\,dx, \\ \left|\tilde{\alpha}_0(\tau,u_1,u_2)\right| &\le g_0(\tau)\|\nabla u_1\|_{\mathbf{L}^2(\Omega)}\|\nabla u_2\|_{\mathbf{L}^2(\Omega)};\end{aligned}$$

(2) For all $w_1,w_2 \in \mathbf{L}^2(D)$, we have

$$\begin{aligned}\alpha_1(\tau,w_1,w_2) &= \alpha_1(0,w_1,w_2) + \tau\alpha_{1,\tau}(0,w_1,w_2) + \tilde{\alpha}_1(\tau,w_1,w_2), \\ \alpha_{1,\tau}(0,w_1,w_2) &= \int_G B_3'(0)w_1 \cdot w_2\,dx, \\ \left|\tilde{\alpha}_1(\tau,w_1,w_2)\right| &\le g_1(\tau)\|w_1\|_{\mathbf{L}^2(\Omega)}\|w_2\|_{\mathbf{L}^2(\Omega)},\end{aligned}$$

(3) For all $w_1,w_2 \in \mathbf{X}$, we have

$$\begin{aligned}
\alpha_2(\tau,w_1,w_2) &= \alpha_2(0,w_1,w_2) + \tau\alpha_{2,\tau}(0,w_1,w_2) + \tilde{\alpha}_2(\tau,w_1,w_2),\\
\alpha_{2,\tau}(0,w_1,w_2) &= -\int_D \frac{\beta'(0)}{\mu}\ curl\ w_1 \cdot\ curl\ w_2\,dx + \int_D \frac{1}{\mu}\ div\ [(B_1^{-1})'(0)(w_1\times e_i)]\,(\ curl\ w_2)_i\,dx\\
&\quad + \int_D \frac{1}{\mu}\ div\ [B_1'(0)w_1\times e_i]\,(\ curl\ w_2)_i\,dx\\
&\quad + \int_D \frac{1}{\mu}\,(\ curl\ w_1)_i\ div\ [(B_1^{-1})'(0)(w_2\times e_i)]\,dx\\
&\quad + \int_D \frac{1}{\mu}\,(\ curl\ w_1)_i\ div\ [B_1'(0)w_1\times e_i]\,dx,\\
\left|\tilde{\alpha}_2(\tau,w_1,w_2)\right| &\le g_2(\tau)\|w_1\|_{\mathbf{X}}\|w_2\|_{\mathbf{X}}.
\end{aligned}$$

Using Corollary 5.1, we can prove

Lemma 5.3 *(Stability)*

Assume (H1)–(H6), then there exists a constant $C>0$ such that

(1) $\|\nabla\phi^\tau - \phi\|_{H^1(0,T;\mathbf{L}^2(\Omega))} \le C\cdot|\tau|$,

(2) $\|\eta^\tau - A\|_{L^2(0,T;\mathbf{X})} + \|\eta_t^\tau - A_t\|_{L^4(0,T;\mathbf{L}^4(G))} \le C\cdot|\tau|$,

(3) $\|z^\tau - z\|_{H^1(0,T;L^2(\Sigma))\cap L^\infty(0,T;L^4(\Sigma))} \le C\cdot|\tau|$,

(4) $\|\theta^\tau - \theta\|_{H^{2,1}(Q)} \le C\cdot|\tau|$.

Remark 5.2 *(z^τ,θ^τ) is the solution to (2.16),(2.17) where A_t in (2.17a) has been replaced with η_t^τ. In view of (5.5) we have $\eta_t^\tau = A_t^\tau = A_{\tau,t}$ on Σ.*

Proof: According to Lemma 5.1 we have $B_2(\tau) = I + \tau B_2'(\xi)$ for τ small enough and $\xi\in(0,\tau)$. Using (H1) this gives immediately

$$\|\nabla\phi^\tau\|_{H^1(0,T;\mathbf{L}^2(\Omega))} \le c_1$$

independent of τ. Moreover, we have

$$\begin{aligned}
0 &= \alpha_0(\tau,\phi^\tau,u) - \alpha_0(0,\phi,u)\\
&= \alpha_0(0,\phi^\tau-\phi,u) + \tau\int_\Omega B_2'(\xi)\phi^\tau\nabla\phi^\tau\cdot\nabla u\,dx.
\end{aligned}$$

Inserting $u = \phi^\tau - \phi$ and using Young's inequality we obtain

$$\|\nabla\phi^\tau - \nabla\phi\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \le c_2|\tau|.$$

Since the same estimate holds true for $\phi_t^\tau - \phi_t$, assertion (1) is proved. In the same way, writing $B_1(\tau) = I + \tau B_1'(\xi)$, $\quad B_3(\tau) = I + \tau B_3'(\xi)$, (note that $(B_1^{-1})' = -B_1'$), we obtain

$$\|\eta^\tau\|_{L^2(0,T;\mathbf{X})} \le c_3,$$

independent of τ, and, defining

$$\eta_{0,t}^\tau = B_1(\tau)^{-1}y\circ T_\tau \tag{5.9}$$

(cf. (H3) and (5.8b)), we get

$$\|\eta_t^\tau\|_{L^4(0,T;\mathbf{L}^4(G))} \le c_4.$$

Now we take the difference of (5.8) for η^τ and A and obtain

$$\begin{aligned} 0 &= \alpha_1(\tau,\eta_t^\tau,w) + \alpha_2(\tau,\eta^\tau,w) + F(\phi^\tau,w) - \alpha_1(0,A_t,w) - \alpha_2(0,A,w) - F(\phi,w) \\ &= \alpha_1(0,\eta_t^\tau - A_t,w) + \alpha_2(0,\eta^\tau - A,w) + F(\phi^\tau - \phi,w) + \tau \int_G B_3'(\xi)\eta_t^\tau w dx + \tau G(\eta^\tau,w), \end{aligned}$$

with a function G that satisfies

$$|G(w_1,w_2)| \le c_5 \|w_1\|_{\mathbf{X}} \cdot \|w_2\|_{\mathbf{X}}.$$

Putting $w = \eta^\tau - A$ and integrating in time leads to

$$\begin{aligned} &\frac{\sigma_0}{2} \int_G |\eta^\tau(t) - A(t)|^2 \, dx + \int_0^t \int_D \frac{1}{\mu} |\operatorname{curl}(\eta^\tau - A)|^2 \, dx \, dt \\ &\le \int_0^t \int_D |\nabla(\phi^\tau - \phi) \cdot (\eta^\tau - A)| \, dx \, dt + \frac{\sigma_0}{2} \int_G |\eta_0^\tau - A_0|^2 \, dx \\ &+ |\tau| c_5 \int_0^t \|\eta^\tau\|_{\mathbf{X}} \cdot \|\eta^\tau - A\|_{\mathbf{X}} + |\tau| c_6 \int_0^t \|\eta_t^\tau\|_{\mathbf{L}^2(G)} \cdot \|\eta^\tau - A\|_{\mathbf{L}^2(G)}. \end{aligned}$$

Applying the inequalities of Young and Gronwall and using (5.8b) the first part of assertion (2) is proved. Differentiating (5.8a) formally with respect to time, making the same computations as before but using (5.9) finishes the proof of assertion (2).

Now, we test the difference of (2.16) for z^τ and z with $\bar{z}_t = z_t^\tau - z_t$. Using (H4)–(H6) and Young's inequality, defining $\bar{\theta} = \theta^\tau - \theta$, we obtain

$$\begin{aligned} \frac{1}{2} \int_0^t \int_\Sigma \bar{z}_s^2 \, dx &\le \frac{1}{2} \int_0^t \int_\Sigma \Big(\frac{[b(\theta^\tau) - z^\tau]^+}{a(\theta^\tau)} - \frac{[b(\theta) - z]^+}{a(\theta)} \Big)^2 \, dx ds \\ &\le c_7 \int_0^t \int_\Sigma \bar{\theta}^2 \, dx ds + c_8 \int_0^t \int_\Sigma \bar{z}^2 \, dx ds \end{aligned}$$

Invoking Gronwall's inequality leads to

$$\|\bar{z}\|_{H^1(0,T;L^2(\Sigma))} \le c_9 \|\bar{\theta}\|_{L^2(Q)}. \tag{5.10}$$

Next, we test the difference of (2.17) for θ^τ and θ with $\bar{\theta}_t$ and apply the inequalities of Hölder and Young to get:

$$\begin{aligned} &\rho c_p \int_0^t \int_\Sigma \bar{\theta}_s^2 \, dx ds + \frac{k}{2} \int_\Sigma \left| \nabla \bar{\theta}(t) \right|^2 dx ds \\ &\le \rho L \int_0^t \int_\Sigma \bar{z}_s \bar{\theta}_s \, dx ds + \sigma_0 \int_0^t \int_\Sigma \left| (\eta_s^\tau - A_s) \cdot (\eta_s^\tau + A_s) \right|^2 dx ds \\ &\le \frac{\rho c_p}{2} \int_0^t \int_\Sigma \bar{\theta}_s^2 \, dx ds + c_{10} \int_0^t \|\bar{\theta}\|_{L^2(\Sigma)}^2 + c_{11} \int_0^t \|\eta_s^\tau - A_s\|_{L^4(\Sigma)}^2 \|\eta_s^\tau + A_s\|_{L^4(\Sigma)}^2. \end{aligned}$$

Using (5.11), assertion (2) and Gronwall's inequality, we get

$$\|\bar{\theta}\|_{L^\infty(0,T;H^1(\Sigma)) \cap H^1(0,T;L^2(\Sigma))} \le c_{12} |\tau|.$$

In view of the last inequality we can test the difference of (2.16) for z^τ and z with $\bar{z}^3$, apply the inequalities of Young and Gronwall and obtain

$$\|\bar{z}\|_{L^\infty(0,T;L^4(\Sigma))} \leq c_{13}|\tau|.$$

Testing (2.17) with $-\Delta\bar{\theta}$ making the same computations as above finishes the proof. □

Remark 5.3 *All the unknowns depend on the shape of Ω_τ, either explicitly as A^τ and ϕ^τ or implicitly as θ^τ and z^τ. For all these quantities, we call*

$$\dot{f} = \lim_{\tau\to 0} \frac{f^\tau - f}{\tau}$$

the strong material derivative of f, whenever the limit exists in the strong sense.

Our main result in this subsection is

Lemma 5.4 *(Strong material derivatives)*

Assume (H1)–(H6), then the following are valid:

(1) The strong material derivative

$\nabla\dot{\phi}$ exists in $H^1(0,T;\mathbf{L}^2(\Omega))$,

$\dot{A}$ exists in $L^\infty(0,T;X)$ and $W^{1,4}(0,T;L^4(G))$,

$\dot{z}$ exists in $H^1(0,T;L^2(\Sigma))$,

$\dot{\theta}$ exists in $H^{2,1}(Q)$.

(2) Moreover, $(\dot{\phi},\dot{A},\dot{z},\dot{\theta})$ satisfy the linearized state equations

$$\begin{aligned}
\alpha_0(0,\dot{\phi},u) + \alpha_{0,\tau}(0,\phi,u) &= 0, \quad \textit{for all } u \in H^1(\Omega), && (5.11a)\\
\alpha_1(0,\dot{A}_t,w) + \alpha_2(0,\dot{A},w) + F(\dot{\phi},w) & && (5.11b)\\
+\alpha_{1,\tau}(0,A_t,w) + \alpha_{2,\tau}(0,A,w) &= 0, \quad \textit{for all } w \in \mathbf{X}, && (5.11c)\\
\dot{A}(0) - DA_0V(0) + B_1'(0)A_0 &= 0, \quad \textit{in } D, && (5.11d)\\
\dot{z}_t - \frac{\partial f}{\partial\theta}\dot{\theta} - \frac{\partial f}{\partial z}\dot{z} &= 0, \quad \textit{in } Q, && (5.11e)\\
\dot{z}(0) &= 0, \quad \textit{in } \Sigma, && (5.11f)\\
\rho c_p\dot{\theta}_t - k\Delta\dot{\theta} + \rho L\dot{z}_t - 2\sigma_0 A_t\cdot\dot{A}_t &= 0, \quad \textit{in } Q, && (5.11g)\\
\frac{\partial\dot{\theta}}{\partial\nu} &= 0, \quad \textit{in } \partial\Sigma\times(0,T), && (5.11h)\\
\dot{\theta}(0) &= 0, \quad \textit{in } \Sigma, && (5.11i)
\end{aligned}$$

where f is the right-hand side of (2.16b).

Proof:

Similar to the proofs of Lemma 2.1–2.3 one can show that (5.11a–i) has a solution and that $(\nabla\dot{\phi},\dot{A},\dot{z},\dot{\theta})$ are uniquely defined. It remains to show that these solutions are the strong material derivatives. To this end let

$$\psi^\tau = \frac{1}{\tau}(\phi^\tau - \phi) - \dot{\phi}, \tag{5.12}$$

then according to Corollary 5.3(1), (5.4) and (5.11a), ψ^τ satisfies

$$\begin{aligned}
\alpha_0(0,\psi^\tau,u) &= -\frac{1}{\tau}\Big(\alpha_0(\tau,\phi^\tau,u) - \alpha_0(0,\phi^\tau,u)\Big) - \alpha_0(0,\dot{\phi},u)\\
&= \alpha_{0,\tau}(0,\phi-\phi^\tau,u) - \frac{1}{\tau}\tilde{\alpha}_0(0,\phi^\tau,u)
\end{aligned}$$

Integrating in time, inserting $u = \psi^\tau$ and using Corollary 5.3(1) once again we obtain

$$\|\nabla\psi^\tau\|_{L^2(0,T;\mathbf{L}^2(\Omega))} = O(\tau).$$

Since the same computations hold for $\nabla\phi_t$, the first part of assertions (1) and (2) is proved.

Next, defining

$$p^\tau = \frac{1}{\tau}(\eta^\tau - A) - \dot{A},$$

and using (5.12) and Corollary 5.3, we see that p^τ satisfies

$$\begin{aligned}
\alpha_1(0,q_t^\tau,w) + \alpha_2(0,p^\tau,w) = & -\frac{1}{\tau}\Big(F(\phi^\tau,w) - F(\phi,w)\Big) \\
& -\frac{1}{\tau}\Big(\alpha_1(\tau,\eta_t^\tau,w) - \alpha_1(0,\eta_t^\tau,w)\Big) - \frac{1}{\tau}\Big(\alpha_2(\tau,\eta^\tau,w) - \alpha_2(0,\eta^\tau,w)\Big) \\
& +F(\dot{\phi},w) + \alpha_{1,\tau}(0,A_t,w) + \alpha_{2,\tau}(0,A,w) \\
= & -F(\psi^\tau,w) - \alpha_{1,\tau}(0,\eta_t^\tau - A_t,w) - \alpha_{2,\tau}(0,\eta^\tau - A),w) \\
& -\frac{1}{\tau}\tilde{\alpha}_1(\tau,\eta_t^\tau,w) - \frac{1}{\tau}\tilde{\alpha}_2(\tau,\eta^\tau,w).
\end{aligned}$$

We take $w = p^\tau$ and integrate in time to obtain

$$\begin{aligned}
& \frac{\sigma_0}{2}\int_G |p^\tau|^2\,dx - \frac{\sigma_0}{2}\int_G |p_0^\tau|^2\,dx + \int_0^t\int_D \frac{1}{\mu}|\operatorname{curl} p^\tau|^2\,dxds \\
\leq\ & c_1\int_0^t\int_G |p^\tau|^2\,dx + \delta\int_0^t\int_D \frac{1}{\mu}|\operatorname{curl} p^\tau|^2\,dxds + c_2\int_0^t\int_G |\eta_t^\tau - A_t|^2\,dxds \\
& +\frac{1}{4\delta}\int_0^t\int_D \frac{1}{\mu}|\operatorname{curl}\eta^\tau - \operatorname{curl} A|^2\,dxds + c_3\int_0^t\int_G |\nabla\psi^\tau|^2\,dxds + O(\tau^2).
\end{aligned} \tag{5.13}$$

Using (5.8b), the second term in (5.13) gives

$$\int_G |p_0^\tau|^2\,dx = \int_G |\frac{1}{\tau}(B_1(\tau)^{-1}A_0\circ T_\tau - A_0) - \dot{A}_0|^2 dx.$$

According to [17], Sec. 2.14, $\tau \mapsto A_0\circ T_\tau$ is differentiable with

$$\frac{d}{d\tau}(A_0\circ T_\tau)\Big|_{\tau=0} = DA_0V(0),$$

hence

$$A_0\circ T_\tau = A_0 + \tau DA_0V(0) + o(\tau).$$

Moreover, Corollary 5.2 implies

$$B_1(\tau)^{-1} = I - \tau B_1'(0) + O(\tau^2).$$

Altogether, we obtain

$$\int_G |p_0^\tau|^2\,dx = O(\tau^2).$$

Taking δ small enough and invoking Gronwall's Lemma, we obtain from (5.13)

$$\|p^\tau\|^2_{L^\infty(0,T;\mathbf{L}^2(G))} + \int_0^T \|p^\tau\|^2_{\mathbf{X}} = O(\tau^2).$$

Now, we differentiate (5.8) formally with respect to time, using (5.9) as initial condition and make the same estimates as above. Reasoning similar to the end of the proof of Lemma 2.2, we obtain

$$\|p_t^\tau\|_{L^4(0,T;\mathbf{L}^4(G))} = O(\tau). \tag{5.14}$$

To prove the differentiability of θ^τ and z^τ, we first remark that there exists a constant $C > 0$, such that

$$\|\dot\theta\|_{H^{2,1}(Q)} + \|\dot z\|_{H^1(0,T;L^2(\Sigma))\cap L^\infty(0,T;L^4(\Sigma))} \le C.$$

We define

$$\begin{aligned} q^\tau &= \frac{1}{\tau}(\theta^\tau - \theta) - \dot\theta \\ r^\tau &= \frac{1}{\tau}(z^\tau - z) - \dot z, \end{aligned}$$

then, (q^τ,r^τ) solve

$$\rho c_p q^\tau - k\Delta q^\tau = -\rho L r^\tau + \sigma_0\tau|\dot A_t|^2 + \sigma_0 p_t^\tau \cdot \Big(2A_t + 2\tau\dot a_t + \tau p_t^\tau\Big) \tag{5.15a}$$

$$r^\tau = \frac{1}{\tau}\Big(f(\theta^\tau,z^\tau) + f(\theta,z)\Big) - \frac{\partial f}{\partial\theta}(\theta,z)\dot\theta - \frac{\partial f}{\partial z}(\theta,z)\dot z \tag{5.15b}$$

$$=: G(\tau) \tag{5.15c}$$

$$\frac{\partial q^\tau}{\partial\nu} = 0, \qquad q^\tau = 0, \quad r^\tau(0) = 0. \tag{5.15d}$$

Owing to (H4)–(H6), we can apply Taylor's formula to develop $G(\tau)$ and obtain (with a constant $\xi \in [0,1]$)

$$\begin{aligned} |G(\tau)| &= \Big|\frac{1}{\tau}\Big(f(\theta + \tau(q^\tau + \dot\theta),z + \tau(r^\tau + \dot z)) - f(\theta,z)\Big) - \frac{\partial f}{\partial\theta}(\theta,z)\dot\theta - \frac{\partial f}{\partial z}(\theta,z)\dot z\Big| \\ &= \Big|(q^\tau + \dot\theta)\frac{\partial f}{\partial\theta}(\theta + \xi\tau(q^\tau + \dot\theta),z + \xi\tau(r^\tau + \dot z)) \\ &\quad + (r^\tau + \dot z)\frac{\partial f}{\partial z}(\theta + \xi\tau(q^\tau + \dot\theta),z + \xi\tau(r^\tau + \dot z)) - \frac{\partial f}{\partial\theta}(\theta,z)\dot\theta - \frac{\partial f}{\partial z}(\theta,z)\dot z\Big| \\ &\le c_4|q^\tau| + c_5|r^\tau| + |\dot\theta|\Big|\frac{\partial f}{\partial\theta}(\theta + \xi\tau(q^\tau + \dot\theta),z + \xi\tau(r^\tau + \dot z)) - \frac{\partial f}{\partial\theta}(\theta,z)\Big| \\ &\quad + |\dot z|\Big|\frac{\partial f}{\partial z}(\theta + \xi\tau(q^\tau + \dot\theta),z + \xi\tau(r^\tau + \dot z))\frac{\partial f}{\partial z}(\theta,z)\Big| \\ &\le \Big(c_4 + c_6|\tau||\dot\theta| + c_8|\tau||\dot z|\Big)|q^\tau| + |\tau|\Big(c_6\dot\theta^2 + c_8|\dot\theta||\dot z|\Big) + \Big(c_5 + c_7|\tau||\dot\theta|\Big)|r^\tau| + c_9|\dot z||z^\tau - z|. \end{aligned}$$

Invoking Hölder's inequality, Sobolev's embedding theorem and Lemma 5.3(3), we can conclude

$$\begin{aligned} \int_0^t \|G(\tau)\|^2_{L^2(\Sigma)}ds &\le O(\tau^2) + \int_0^t \Big\|(c_4 + c_6|\tau||\dot\theta| + c_8|\tau||\dot z|)\Big\|^2_{L^4(\Sigma)}\|q^\tau\|^2_{H^1(\Sigma)}\,ds \\ &\quad + \int_0^t \Big(c_5 + c_7|\tau|\|\dot\theta\|_{H^2(\Sigma)}\Big)\|r^\tau\|^2_{L^2(\Sigma)}\,ds. \end{aligned}$$

Now testing (5.15b) with r_t^τ and applying Gronwall's and Young's inequality leads to

$$\frac{1}{2}\int_0^t\int_\Sigma r_s^2\,dxds \le O(\tau^2) + c_{1)}\int_0^t \Big\|(c_4 + c_6|\tau||\dot\theta| + c_8|\tau||\dot z|)\Big\|^2_{L^4(\Sigma)}\|q^\tau\|^2_{H^1(\Sigma)}\,ds.$$

The proof is concluded by testing (5.15a) successively with q_t^T and $-\Delta q^T$, taking into account the previous estimate as well as (5.14) and Gronwall's inequality. □

Using Lemma 5.4, we see that the Eulerian derivative (5.1) exists and that it is linear and continuous with respect to V. This concludes the proof of Theorem 5.2.

5.3 Proof of Theorem 5.1

Let γ be an optimal solution and $\gamma_\varepsilon = \gamma + \varepsilon h + o(\varepsilon)$ in C^3 be an admissible perturbation of the curve γ (cf. Appendix). Then,

$$I(\gamma_\varepsilon) \geq I(\gamma), \tag{5.16}$$

where I is defined in (3.4). In view of Theorem 5.1, we have

$$J(\Omega(\gamma_\varepsilon)) = J(\Omega(\gamma)) + \varepsilon\, dJ(\Omega(\gamma); V(h)) + o(\varepsilon).$$

Hence, from (5.16) it follows that

$$dJ(\Omega(\gamma); V(h)) \geq 0$$

for all admissible directions $h \in C^3$; and in particular

$$dI(\gamma; h) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}[I(\gamma_\varepsilon) - I(\gamma)] = dJ(\Omega(\gamma); V(h)).$$

Applying the structure theorem (cf. Corollary 5.1) we obtain

$$dI(\gamma; h) =< g_{\partial\Omega}; V(h) \cdot \nu > .$$

In general, $g_{\partial\Omega}$ is a distribution. Assuming that the density $g_{\partial\Omega}$ is a function, it can be identified in the following way. Utilizing a general strategy to derive the first order optimality system described eg. by Cea in [4], we introduce the Lagrangean

$$\begin{aligned}
\mathcal{L}_\varepsilon(\phi,\eta; A,B; \theta,p,z,r) &= J(\Omega_\varepsilon) \\
&+ \int_0^T \int_{\Omega_\varepsilon} \nabla\phi \cdot \nabla\eta \, dxdt - \int_0^T \int_{\Gamma_1} j\eta \, dxdt \\
&+ \int_0^T \int_D \Big[\sigma_0^\varepsilon A_t \cdot B + \frac{1}{\mu_\varepsilon} \text{ curl } A \cdot \text{ curl } B\Big] \, dxdt + \sigma_0 \int_0^T \int_{\Omega_\varepsilon} \nabla\phi \cdot B \, dxdt \\
&+ \int_0^T \int_\Sigma \Big[\rho c_p \theta_t p + k\nabla\theta \cdot \nabla p + \rho L z_t p - \sigma_0 |A_t|^2 p\Big] \, dxdt \\
&+ \int_0^T \int_\Sigma \Big[z_t - f(\theta,z)\Big] r \, dxdt.
\end{aligned}$$

Here, $\Omega_\varepsilon = \Omega(\gamma_\varepsilon)$ and $\sigma_0^\varepsilon = \sigma_0$ in Ω_ε and 0 otherwise.

The adjoint state equations are obtained for $\varepsilon = 0$ by differentiation of the Langrangean with respect to the state variables, i.e. $< \frac{\partial \mathcal{L}_0}{\partial \phi}; \varphi >= 0$ gives (5.2a), $< \frac{\partial \mathcal{L}_0}{\partial A}; \psi >= 0$ gives (5.2b,c), $< \frac{\partial \mathcal{L}_0}{\partial \theta}; \xi >= 0$ gives (5.2d,e), $< \frac{\partial \mathcal{L}_0}{\partial z}; w >= 0$ gives (5.2f,g).

It is easy to see that the linear system of adjoint equations admits a unique solution. First of all, one proves that (5.2d–g) admits a unique strong solution (p,r) by a contraction mapping argument in

$H^1(0,T;L^2(\Sigma))$. Then, (5.2b,c) is solved using Lemma 2.2 and finally standard elliptic theory shows the solvability of (5.2a).

The gradient of the cost functional is given by the derivative

$$dI(\gamma;h) = dJ(\Omega(\gamma);V(h)) = \frac{\partial \mathcal{L}_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0},$$

assuming that we have enough regularity for the solution to the state and the adjoint equations. To derive the gradient, we make use of the following

Lemma 5.5 *(cf. Proposition 2.46 in [17])*
Let $f \in W^{1,1}(\mathbb{R}^3)$ and $\Omega \subset \mathbb{R}^3$ with boundary of class $C^k, k \geq 1$, then

$$\Big(\frac{\partial}{\partial \varepsilon}\int\limits_{\Omega_\varepsilon} f\,dx\Big)\Big|_{\varepsilon=0} = \int_{\partial\Omega} f\,V\cdot\nu\,dx,$$

where ν is the outer unit normal on $\partial\Omega$.

Using this lemma and taking into account the different values for permeability in the tube and in the air, we directly obtain (5.3), which concludes the proof of Theorem 5.1.

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A Calculation of the speed vector field

In the appendix we explain, how the speed vector field $V(h(\beta))$ of $\partial\Omega(\gamma_\varepsilon)$ at $\varepsilon = 0$ can be computed from a perturbation of the curve γ. To this end let $\tilde{\gamma_\varepsilon} = \gamma + \varepsilon\beta$, then γ_ε is the unit–speed curve

$$\gamma_\varepsilon(s) = \gamma(\tau_\varepsilon(s)) + \varepsilon\beta(\tau_\varepsilon(s)), \quad 0 \le s \le l_\varepsilon, \varepsilon \ge 0, \tag{A.1}$$

where $\tau_\varepsilon : [0,l_\varepsilon] \to [0,l]$ is a reparametrization satisfying $\tau_\varepsilon(0) = 0$, $\tau_\varepsilon(l_\varepsilon) = l$ and $\tau'_\varepsilon(s) > 0$ for all $s \in [0,l_\varepsilon]$.

We recall that the lateral tube boundary Γ_2 is parametrized by

$$\Gamma_2 = \{x \in \mathbb{R}^3 \mid X(u,v) = \gamma(u) + R\cos v\, N(u) + R\sin v\, B(u), 0 \le u \le l, 0 \le v < 2\pi\}.$$

Using (A.1) and the unit-speed property of γ_ε, we have

$$\gamma_\varepsilon = \gamma + \varepsilon h(\beta) + o(\varepsilon) \quad \text{in } C^3[0,l_\varepsilon],$$

where

$$h(\beta)(s) = \beta(s) - \gamma'(s)\int_0^s \gamma'(\xi)\cdot\beta'(\xi)d\xi.$$

If $l_\varepsilon > l$, we of course have to extend γ in the same way than in the proof of Theorem 4.1.

We assume that the normal and binormal vector fields $N_\varepsilon(u), B_\varepsilon(u)$ take the form

$$\begin{aligned} N_\varepsilon &= \mathbf{N} + \varepsilon N_1(h) + o(\varepsilon), \quad \text{in } C^1[0,l_\varepsilon], \\ B_\varepsilon &= \mathbf{B} + \varepsilon B_1(h) + o(\varepsilon), \quad \text{in } C^1[0,l_\varepsilon]. \end{aligned}$$

Remark A.1 *If $|\gamma''(u)| > 0$ for all $u \in [0,l]$, then we have the Frenet Formulas*

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N, \end{aligned}$$

where τ is the torsion, and the curve γ_ε defines in a unique way the fields $N_\varepsilon(u), B_\varepsilon(u)$.

The lateral boundary of $\partial\Omega_\varepsilon = \partial\Omega(\gamma_\varepsilon)$ is defined by the parametrization

$$\begin{aligned} X^\varepsilon(u,v) &= \gamma_\varepsilon(u) + R\cos v\, N_\varepsilon(u) + R\sin v\, B_\varepsilon(u) \\ &= X(u,v) + \varepsilon\Big\{h(u) + R\cos v\Big[\frac{h''(u)}{\kappa(u)} - \frac{\gamma''(u)\cdot h''(u)}{|\gamma''(u)|^2}N(u)\Big] \\ &\qquad + R\sin v\Big[-\frac{\gamma''(u)\cdot h''(u)}{\kappa^2(u)}B(u) + h'(u)\times N(u) + \frac{1}{\kappa(u)}T(u)\times h''(u)\Big]\Big\} + o(\varepsilon) \end{aligned}$$

in the space of continuous functions C, provided $\gamma,\beta \in C^3$, here $0 \le u \le l, 0 \le v < 2\pi$.

Therefore, the speed of $\partial\Omega_\varepsilon$ with respect to the parameter ε at $\varepsilon = 0$ is given by

$$\begin{aligned} V(h)(u,v) &= \frac{\partial X^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0}(u,v) \\ &= h(u) + R\cos v\Big[\frac{h''(u)}{\kappa(u)} - \frac{\gamma''(u)\cdot h''(u)}{|\gamma''(u)|^2}N(u)\Big] \\ &\qquad + R\sin v\Big[-\frac{\gamma''(u)\cdot h''(u)}{\kappa^2(u)}B(u) + h'(u)\times N(u) + \frac{1}{\kappa(u)}T(u)\times h''(u)\Big]. \end{aligned}$$

Remark A.2

(1) In order to have $V(u,v)$ to be C^1, we need $\gamma \in C^4$, since h depends on γ'.

(2) If the condition $|\gamma''(u)| \neq 0$ is not satisfied, we cannot use the Frenet formulas and should directly construct the field V from the given parametrization of $\partial\Omega_\varepsilon$.

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