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Bruno Gaujal — Emmanuel Hyon

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Optimal Routing Policy in Two Deterministic Queues

Bruno Gaujal[*] , Emmanuel Hyon[†]

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Abstract: We consider the problem of routing customers to one of two parallel queues where inter-arrival times and service times are deterministic. We provide an explicit formula for the average waiting time of the customers sent to one of the queues when the routing policy is an upper mechanical word. This formula is based on a special continued fraction decomposition of the service time in the queue. Using this formula we provide an algorithm computing the optimal routing policy for two queues. In general, this policy is an upper mechanical word with a rational ratio, and hence is periodic.

Key-words: Mechanical Words, Continued Fractions, Deterministic Queues

[*] LORIA, 615 route du jardin botanique, BP 101, F-54606 Villers-les-Nancy, France. E-mail: Bruno.Gaujal@loria.fr.
[†] LORIA, 615 route du jardin botanique, BP 101, F-54606 Villers-les-Nancy, France. E-mail: Emmanuel.Hyon@loria.fr.

Politique de routage optimale de deux files déterministes

Résumé : Dans cet article nous étudions le routage de clients dans deux files d'attente déterministes parallèles, les temps d'interarrivées étant supposés déterministes. Nous donnons une formule explicite de la moyenne du temps d'attente des clients introduits dans l'une des deux files quand la politique de routage est un mot mécanique. Cette formule repose sur une décomposition en fraction continue particulière du temps de service. A partir de cette formule nous obtenons un algorithme qui calcule la politique de routage optimale dans les deux files. Généralement cette politique est un mot mécanique périodique et la proportion de clients envoyés dans chaque file est rationnelle.

Mots-clés : Mots mécaniques, Fractions continues, Files déterministes

1 Introduction

A classic problem in the literature of the control of queues is the determination of the optimal routing policy for parallel queues. When the queues are homogeneous (*i.e.* all customer service times are identically distributed and independent) and the service discipline is FIFO, then the optimal policy is often the *join the shortest queue* policy, as shown in [5], for many cost functions. When the service times in the queues have different distributions, the optimal policy for two parallel queues with exponential service times was studied in [4] using dynamic programming. Threshold policies were proved optimal. The non-Markovian case with no state information was studied in [2] where it was shown that the optimal policy in terms of average waiting time must be a *mechanical word*, although no explicit computation of an optimal policy was actually provided in the heterogeneous case.

In the deterministic case (both the arrivals and the service times are constant deterministic variables), the optimal policy was computed in [8] when the system is fully loaded.

Here, we consider two deterministic FIFO queues with infinite buffers and arbitrary constant arrival and service times (hence the load is arbitrary). As proved in [2], the optimal policy is a mechanical word, and we are able to compute explicitly the slope of this optimal word, which is a rational number as long as the system is not fully loaded. Hence, the optimal policy is always periodic even when the parameters (service times and arrival times) are irrational numbers.

In order to obtain this result, we start in Section 2 by introducing upper and lower mechanical words. In Section 3, we define a new continued fraction decomposition which helps us for identifying special factors of upper mechanical words. In Section 4, we give an explicit formula for the average waiting time in one deterministic queue when the arrival process is an upper mechanical word. This function is continuous but not differentiable at certain rational points called *jumps* in the following. It is increasing and concave between jumps. Finally, in Section 5, we consider the case of two queues. Using the properties of the average waiting time in one queue, we can show that the optimal routing policy is given by a jump in one of the two queues, and hence is periodic, as long as the system is not fully loaded. An algorithm is provided that computes the optimal jump in finite time (and hence the optimal policy). In the last section, several examples are studied in detail in order to illustrate the strange behavior of the optimal policy in such a simple system (or so it seems).

2 Mechanical words

Let $A = \{0, 1\}$ be the alphabet. The free monoid A^* is the set of the finite *words* on A. An infinite word is an element of $A^{\mathbb{N}}$.

Definition 1 (Slope). *The slope of a finite nonempty word w is the number :*

$$\alpha(w) = \frac{|w|_1}{|w|},$$

where $|w|_1$ is the number of letters equal to one in w and $|w|$ is the length of w.

Let w_n, $\forall\, n \geq 1$ be the prefix of length n of an infinite word w. If the sequence $\alpha(w_n)$ converges when $n \to \infty$ the limit is called the slope of w.

For a real number x, we denote by $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) the largest (resp. smallest) integer smaller (resp. larger) than x.

Definition 2 (Mechanical word). *The upper mechanical word with slope α is the infinite word $\overline{m}_\alpha$ where the n^{th} letter, with $n \geq 0$, is :*

$$\overline{m}_\alpha(n) = \lceil (n+1) \times \alpha \rceil - \lceil n \times \alpha \rceil. \quad (1)$$

The lower mechanical word with slope α is the infinite word $\underline{m}_\alpha$ where the n^{th} letter, with $n \geq 0$, is :

$$\underline{m}_\alpha(n) = \lfloor (n+1) \times \alpha \rfloor - \lfloor n \times \alpha \rfloor. \quad (2)$$

The characteristic word of slope α is the infinite word c_α where the n^{th} letter , with $n \geq 0$, is :

$$c_\alpha(n) = \lfloor (n+2) \times \alpha \rfloor - \lfloor (n+1) \times \alpha \rfloor .$$

Lemma 3. *If α is a rational number ($\alpha = \frac{p}{q}$), then $\overline{m}_\alpha, \underline{m}_\alpha, c_\alpha$ are periodic of period q. If α is an irrational number, then $\overline{m}_\alpha, \underline{m}_\alpha, c_\alpha$ are all aperiodic.*

Proof. This result is proved for example in [6]. □

In the following, by a slight abuse of notation when an infinite word w is periodic we also denote by w its shortest period.

Example 4 (Graphical interpretation). *The terminology comes from the following graphical interpretation. For example let $\alpha = 3/7$, the lower mechanical word is 0010101, the upper mechanical word is 1010100 and the characteristic word is 0101010.*

Consider the straight line with equation $y = \alpha x$, and consider the points with integer coordinates just below the line : $P_n = (n, \lfloor n\alpha \rfloor)$, the ones just above the line : $P'_n = (n, \lceil n\alpha \rceil)$ and the points $P''_n = (n, \lfloor (n+1)\alpha \rfloor)$. The points P_n form a representation of the lower mechanical word in the following sense : two consecutive points are joined by either an horizontal straight line segment ,if $\lfloor n+1\alpha \rfloor - \lfloor n\alpha \rfloor = 0$, or a diagonal, if $\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor = 1$. Similarly the points P'_n are a representation of the upper mechanical word and the points P''_n are a representation of the characteristic word (see figure 1).

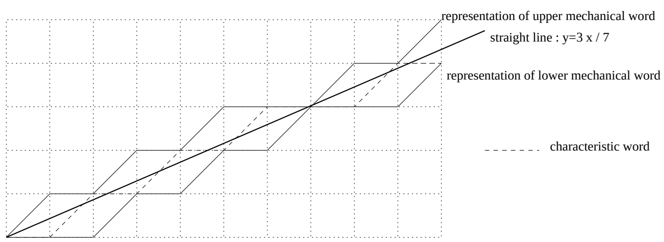

Figure 1: Mechanical words associated with the line $y = \frac{3}{7}x$.

Remark 5. *Since $\lceil n\alpha \rceil = \lfloor n\alpha \rfloor + 1$ except when $n\alpha$ is an integer, one has $\overline{m}_\alpha(n) = \underline{m}_\alpha(n)$ when $n\alpha$ and $(n+1)\alpha$ are not integer numbers. When α is an irrational number, since $\overline{m}_\alpha(0) = 1$ and $\underline{m}_\alpha(0) = 0$ then $\underline{m}_\alpha = 0c_\alpha$ and $\overline{m}_\alpha = 1c_\alpha$. When α is a rational number $\overline{m}_\alpha$ is a shift of $\underline{m}_\alpha$ (i.e. there exists a integer number k such that $\underline{m}_\alpha(n+k) = \overline{m}_\alpha(n)$ for all n).*

Lemma 6. *Let α be a real number, then the maximum number of consecutive 0 in $\overline{m}_\alpha$ is $\lceil \alpha^{-1} \rceil - 1$ and the minimum number of consecutive 0 in $\overline{m}_\alpha$ is $\lfloor \alpha^{-1} \rfloor - 1$.*

Proof. If α^{-1} is an integer number it can be checked that $\overline{m}_\alpha$ is the word composed by one followed by $\alpha^{-1} - 1$ consecutive 0.

Suppose now that α^{-1} is not an integer number. Let k such that $\lceil (n+k+1)\alpha \rceil - \lceil n\alpha \rceil = 0$ and $\lceil (n+k+2)\alpha \rceil - \lceil n\alpha \rceil = 1$ that means we have k consecutive letters of $\overline{m}_\alpha$ equal to 0. Let $s_n = \lceil \alpha n \rceil - \alpha n$. Let n such that $(1 - s_n)\alpha^{-1} \geq \lfloor \alpha^{-1} \rfloor$. Note that such a n exists since for a given ε we can find n such that $s_n \leq \varepsilon$. We have

$$\begin{aligned} \lceil (n+k+1)\alpha \rceil - \lceil n\alpha \rceil = 0 \quad &\Rightarrow \quad \lceil (k+1)\alpha + n\alpha \rceil = \lceil n\alpha \rceil \;\Rightarrow\; (k+1)\alpha \leq 1 - s_n \,, \\ \lceil (n+k+2)\alpha \rceil - \lceil n\alpha \rceil = 1 \quad &\Rightarrow \quad \lceil (k+2)\alpha + n\alpha \rceil = 1 + \lceil n\alpha \rceil \;\Rightarrow\; (k+2)\alpha > 1 - s_n \,. \end{aligned}$$

Hence $(1 - s_n)\alpha^{-1} - 1 < (k+1) \leq (1 - s_n)\alpha^{-1}$. Then

$$\lfloor \alpha^{-1} \rfloor - 1 < (k+1) \leq \lceil \alpha^{-1} \rceil .$$

□

3 Expansions in continued fractions

3.1 Classical expansion in continued fraction

In this part we will exhibit the link between expansion in continued fraction and the prefixes of characteristic words as presented in [6] and [7].

Let ρ, $0 < \rho < 1$ be a real number. The computation of its expansion in a continued fraction is given by :

$$\left\{ \begin{array}{ccc} \rho = 0 + \frac{1}{m_1+\rho_1} & ; & m_1 = \lfloor \rho^{-1} \rfloor \\ \rho_n = \frac{1}{m_{n+1}+\rho_{n+1}} & ; & m_{n+1} = \lfloor (\rho_n)^{-1} \rfloor \quad ; \forall\, n \geq 1 \end{array} \right\} . \tag{3}$$

If ρ is a rational number, the process stops after a finite number of steps when ($\rho_n = \frac{1}{m_{n+1}+0}$).

A simple continued fraction

$$0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \dots \cfrac{1}{m_n + \cdots}}},$$

is symbolically written in the form $[0, m_1, m_2, \dots, m_n, \dots]$.

Let $(m_1, m_2, \dots, m_n, \dots)$ be a sequence of integers, with $m_1 \geq 0$ and $m_n > 0$ for $n > 1$. To such a sequence we associate a sequence $((s_n)_{n \geq -1})$ of finite words defined by

$$s_{-1} = 1\,,\ s_0 = 0\,,\ s_n = s_{n-1}^{m_n} s_{n-2} \ \ (n \geq 1)\,.$$

This sequence $(s_n)_{n\geq -1}$ is called the *standard sequence* associated with the sequence $(m_n)_{n\geq 1}$.

Proposition 7. *Let $\rho = [0, m_1, m_2, \dots, m_n, \dots]$ be the continued fraction expansion of some irrational ρ with $0 < \rho < 1$, and let $(s_n)_{n\geq -1}$ be the standard sequence associated to $(m_1, m_2, \dots)$. Then every s_n is a prefix of the characteristic word c_ρ and*

$$c_\rho = \lim_{n\to\infty} s_n\,.$$

Proof. This result is proved in [6].

3.2 Mechanical expansion in continued fractions

The tight relation between the continued fraction of ρ and the characteristic word c_ρ does not extent easily to the mechanical words. In the following, we will define a special expansion in continued fraction (called mechanical continued fraction expansion). This new continued fraction allows us to find the decomposition in special factors of upper mechanical words.

The construction of the mechanical continued fraction expansion of a number α with $0 < \alpha < 1$ is given by :

$$\left\{ \begin{array}{ccccc} \alpha = \frac{1}{l_1+\alpha_1} & ; & l_1 = \lfloor \alpha^{-1} \rfloor & ; & \\ 1-\alpha_n = \frac{1}{l_{n+1}+\alpha_{n+1}} & ; & l_{n+1} = \lfloor (1-\alpha_n)^{-1} \rfloor & ; \forall\, n \geq 1 \end{array} \right\} . \tag{4}$$

When α is a rational number the construction finishes after a finite number of steps when $1 - \alpha_n = \frac{1}{l_n}$. When α is an irrational number the construction is infinite and we obtain the infinite mechanical continued fraction expansion of α.

An infinite mechanical continued fraction expansion is written in the form $\langle l_1, l_2, l_3, \dots, l_{n-1}, l_n, \dots \rangle$. A finite mechanical continued fraction is written in the form $\langle l_1, l_2, \dots, l_{n-1}, l_n \rangle$. A partial mechanical continued fraction of a number α is written in the form $\langle l_1, l_2, \dots, l_n + \alpha_n \rangle$.

Let us denote by $\overline{M}$ the set of all upper mechanical words. Denote by $\overline{M_k}$ the set of upper mechanical words with slope α such that $(k+1)^{-1} < \alpha \leq k^{-1}$.

Let us now introduce the morphism $\varphi_k : \overline{M_k} \to A^* \cup A^{\mathbb{N}}$.

$$\varphi_k \;:\; \begin{array}{ll} 1\overbrace{0\ldots0}^{k-1} & \mapsto 1 \\ 1\overbrace{0\ldots0}^{k} & \mapsto 0 \end{array} .$$

If $\alpha = k^{-1}$ then the word $\varphi_k(\overline{m}_\alpha)$ is reduced to the letter one.

Introduce the morphism

$$\Phi : \begin{array}{ccc} \overline{M} & \to & A^* \cup A^{\mathbb{N}} \\ \overline{m}_\alpha & \mapsto & \varphi_{\lfloor \alpha^{-1} \rfloor}(\overline{m}_\alpha) \end{array} .$$

We now show that Φ has its values in $\overline{M}$.

Lemma 8. *Let $\overline{m}_\alpha$ be the upper mechanical word of slope α, then $\Phi(\overline{m}_\alpha)$ is the upper mechanical word of slope $1 - (\alpha^{-1} - \lfloor \alpha^{-1} \rfloor)$.*

Proof. If α^{-1} is an integer number $\Phi(\overline{m}_\alpha)$ is reduced to the word “1”, then we have the result since $1 - (\alpha^{-1} - \lfloor \alpha^{-1} \rfloor) = 1$.

Recall that $\overline{m}_\alpha(n)$ is defined by Formula (1). An integer a_k is the index of the k^{th} occurrence of the letter one in $\overline{m}_\alpha$ if $\overline{m}_\alpha(0)\ldots\overline{m}_\alpha(a_k)$ contains k letters one and $\overline{m}_\alpha(0)\ldots\overline{m}_\alpha(a_k - 1)$ contains $k-1$ letters one. This means

$$\lceil \alpha(a_k + 1) \rceil = k + 1\,,\; \lceil \alpha a_k \rceil = k\,.$$

These equalities imply $a_k = \lfloor k\alpha^{-1} \rfloor$.

Define now the function ψ_k, $\psi_k : \overline{M_k} \to A^* \cup A^{\mathbb{N}}$.

$$\psi_k \;:\; \begin{array}{ll} 1\overbrace{0\ldots0}^{k-1} & \mapsto 0 \\ 1\overbrace{0\ldots0}^{k} & \mapsto 1\,. \end{array}$$

Let $l = \lfloor \alpha^{-1} \rfloor$ and let $w = \psi_l(\overline{m}_\alpha)$. The sequence $(w(k))_{k\geq 0}$ of letters of w can be computed by $w(k) = a_{k+1} - a_k - l$. Then $w(k) = \lfloor (k+1)\alpha^{-1} - l(k+1) \rfloor - \lfloor k\alpha^{-1} - kl \rfloor$, hence w is the lower mechanical word of slope $\alpha^{-1} - l$

Define now the function γ such that

$$\gamma \;:\; \begin{array}{ccc} 1 & \mapsto & 0 \\ 0 & \mapsto & 1\,. \end{array}$$

The function γ transforms a lower mechanical word of slope α into an upper mechanical word of slope $1 - \alpha$ and conversely. Let $w' = \gamma(w)$, w' is the upper mechanical word of slope $1 - (\alpha^{-1} - l)$.

It can be checked that $\Phi = \gamma \circ \psi$, hence $\Phi(\overline{m}_\alpha)$ is the upper mechanical word of slope $1 - (\alpha^{-1} - \lfloor \alpha^{-1} \rfloor)$. □

The following corollary shows the relation between $\Phi(\overline{m}_\alpha)$ and the mechanical continued fraction of $\overline{m}_\alpha$.

Corollary 9. *Let α , $0 < \alpha < 1$ be any given real number such that $\alpha = \langle l_1, l_2, \ldots, l_{n-1}, l_n + \alpha_n \rangle$, with $n \geq 1$. Let $w = \Phi(\overline{m}_\alpha)$. Then $\alpha(w) = \langle l_2, l_3, \ldots, l_{n-1}, l_n + \alpha_n \rangle$.*

Proof. By Equations (4), we have $\alpha = (l_1 + \alpha_1)^{-1}$ and $1 - \alpha_1 = \langle l_2, l_3, \ldots, l_{n-1}, l_n + \alpha_n \rangle$. By Lemma 8, we obtain

$$\alpha(w) = 1 - (\alpha^{-1} - \lfloor \alpha^{-1} \rfloor) = 1 - (\alpha^{-1} - l_1) = 1 - \alpha_1\,.$$

Hence $\alpha(w) = \langle l_2, l_3, \ldots, l_{n-1}, l_n + \alpha_n \rangle$. □

This will allow us to use induction on the number of terms in the mechanical continued fraction in the following.

Theorem 10 ((x,y)-factor decomposition). *Let α, $0 < \alpha < 1$ be any given real number with $\alpha = \langle l_1, l_2, \dots, l_n, \dots \rangle$. Define two sequences, $\{x_i(\alpha)\}_{i\geq 0}$ and $\{y_i(\alpha)\}_{i\geq 0}$, computed by :*

$$x_0(\alpha) = 1,\ y_0(\alpha) = 0,\ x_i(\alpha) = x_{i-1}(\alpha)(y_{i-1}(\alpha))^{l_i - 1},\ y_i(\alpha) = x_{i-1}(\alpha)(y_{i-1}(\alpha))^{l_i},\ \textit{with } i \geq 1.$$

Then the upper mechanical word $\overline{m}_\alpha$ can be factorized only using the two factors $x_i(\alpha)$ and $y_i(\alpha)$.

These two sequences are called (x-y)-factor decomposition sequences associated with the mechanical expansion of α.

Proof. We will study finite sequences $\{x_i(\alpha)\}_{0\leq i\leq n}$, $\{y_i(\alpha)\}_{0\leq i\leq n}$ associated with the partial mechanical continued fraction expansion of α and prove the result by induction.

Step 1. We have $\alpha = \langle l_1 + \alpha_1 \rangle$. By Lemma 6, it is checked that $\overline{m}_\alpha$ can be factorized only using x_1 and y_1.

Step n. We have $\alpha = \langle l_1, l_2, \dots, l_n +\ alpha_n \rangle$.

Let $w = \Phi(\overline{m}_\alpha)$, by Corollary 9, $\alpha(w) = \langle l_2, l_3, \dots, l_{n-1}, l_n + \alpha_n \rangle$. Let $\{x_i(\alpha(w))\}_{0\leq i\leq n-1}$ and $\{y_i(\alpha(w))\}_{0\leq i\leq n-1}$ be the two (x-y)-factor decomposition sequences associated with the partial mechanical expansion of $\alpha(w)$. By induction hypothesis on the number of terms in the partial expansion w can be factorized only using $x_i(\alpha(w))$ and $y_i(\alpha(w))$. Introduce now the sequences x'_i and y'_i such that

$$x'_0 = 1\,,\ y'_0 = 0\,,\ \varphi_{l_1}(x'_i) = x_{i-1}(\alpha(w))\,,\ \varphi_{l_1}(y'_i) = y_{i-1}(\alpha(w))\,,\ \forall i \geq 1\,.$$

Then $w = \Phi(\overline{m}_\alpha)$ can be factorized only using $\varphi_{l_1}(x'_i)$ and $\varphi_{l_1}(y'_i)$, $\forall i \geq 0$. Therefore $\overline{m}_\alpha$ can be factorized only using x'_i and y'_i, since $\Phi(\overline{m}_\alpha) = \varphi_{l_1}(\overline{m}_\alpha)$. Note now that the sequences x'_i, y'_i are the (x-y)-factor decomposition sequences associated with $\langle l_1, l_2, \dots, l_n \rangle$. □

Lemma 11. *Let α, $0 < \alpha < 1$ be any given rational number with $\alpha = \langle l_1, l_2, \dots, l_{n-1}, l_n \rangle$. Then $x_n(\alpha) = \overline{m}_\alpha$.*

Proof. We also use here an induction on the number of terms in the partial mechanical expansion.

Step 1. Considering that $\alpha = \langle l_1 \rangle$ then α is a rational number with $\alpha^{-1} = l_1$. Hence $\overline{m}_\alpha = 10\dots0$, where the number of consecutive 0 is $\alpha^{-1} - 1 = l_1 - 1$.

Step n. We have $\alpha = \langle l_1, l_2, \dots, l_{n-1}, l_n \rangle$, therefore $w = \Phi(\overline{m}_\alpha)$ has a slope $\alpha(w) = \langle l_2, \dots, l_n \rangle$. Using the induction hypothesis we obtain $w = x_{n-1}(\alpha(w))$. Since $x_{n-1}(\alpha(w)) = \varphi_{l_1}(x_n(\alpha))$, then $\overline{m}_\alpha = x_n(\alpha)$. □

Remark 12. *Since $\Phi^{(n)}(\overline{m}_\alpha) = \varphi_{l_n} \circ \dots \circ \varphi_{l_2} \circ \varphi_{l_1}(\overline{m}_\alpha)$, then $x_n(\alpha)$ and $y_n(\alpha)$ are the only factors of $\overline{m}_\alpha$, such that $\Phi^{(n)}(x_n(\alpha)) = 1$ and $\Phi^{(n)}(y_n(\alpha)) = 0$.*

Remark 13. *Since $\forall\, n \geq 0$, $\Phi^{(n)}(\overline{m}_\alpha)$ is an upper mechanical word and since all upper mechanical words begin with one, then the sequence $x_i(\alpha)$ is a sequence of prefixes of $\overline{m}_\alpha$ verifying*

$$\lim_{i\to\infty} x_i(\alpha) = \overline{m}_\alpha\,.$$

Example 14. *Here $\alpha = \frac{8}{19}$. The mechanical expansion of α using Equations* (4) *is : $\langle 2, 1, 2, 2 \rangle$.*

$$\overline{m}_{8/19} = 1010100101010010100\,.$$

$\Phi(\overline{m}_{8/19}) = 11011010$, $\Phi^{(2)}(\overline{m}_{8/19}) = 10100$ *and* $\Phi^{(3)}(\overline{m}_{8/19}) = 10$.

Using the (x-y)-factor decomposition we obtain $x_1 = 10$ and $y_1 = 100$; $x_2 = x_1$ and $y_2 = x_1 y_1$; $x_3 = x_2 y_2$ and $y_3 = x_2(y_2)^2$ with finally $x_4 = x_3 y_3$.

We can check Theorem 10 and Lemma 11.

$$\overline{m}_{8/19} = \underbrace{\overbrace{\underbrace{\overbrace{10}^{x_1}}_{x_2}\underbrace{\overbrace{10}^{x_1}\overbrace{100}^{y_1}}_{y_2}}^{x_3}\overbrace{\underbrace{\overbrace{10}^{x_1}}_{x_2}\underbrace{\overbrace{10}^{x_1}\overbrace{100}^{y_1}}_{y_2}\underbrace{\overbrace{10}^{x_1}\overbrace{100}^{y_1}}_{y_2}}^{y_3}}_{x_4}\,.$$

Theorem 10 and Lemma 11 underline the connection between the construction of factors of an upper mechanical word and the mechanical continued fraction expansion of its slope. This connection could be seen like an equivalent for upper mechanical words of the relation between continued fraction expansion and characteristic words presented in Proposition (7).

In the following, the i^{th} factors of the word $\overline{m}_\alpha$ will be denoted x_i and y_i when no confusion is possible.

4 Average waiting time in a single queue

The aim of this section is to compute and to study the average waiting time of customers, $W_S(w)$ in a ./D/1/∞/FIFO queue with a constant service time, S, and with constant inter-arrival times. Using an appropriate time scale, the inter-arrival times can be chosen to be equal to one. The input sequence w is such that $w(n)$ is 0 if no customer enter the queue at slot n and $w(n)$ is 1 if one customer enters the queue at slot n. In the following, we will only consider an input sequence which is the upper mechanical word of slope α, $\overline{m}_\alpha$.

Let α, $0 \leq \alpha \leq 1$ be the ratio of customers sent in the ./D/1/∞/FIFO queue. The stability condition of the queue is

$$\alpha \leq \frac{1}{S}. \tag{5}$$

4.1 Jumps

We now show that the computation of the average waiting time $W_S(\overline{m}_\alpha)$ tightly depends on the factorization of $\overline{m}_\alpha$ in (x-y)-factors of S^{-1}. For that we will exhibit special rational numbers.

Lemma 15. *Let $p = \langle l_1, l_2, \ldots, l_{n-1}, l_n, l_{n+1}\rangle$ be a rational number.*

-i) For all real number α which partial mechanical expansion is $\langle l_1, l_2, \ldots, l_{n-1}, l_n + \alpha_n\rangle$, such that l_{n+1} is the smallest integer satisfying

$$\alpha_n \leq \frac{l_{n+1}-1}{l_{n+1}}\,.$$

Then $x_{n+1}(\alpha) = x_n(p)(y_n(p))^{l_{n+1}-2}$ and $y_{n+1}(\alpha) = x_{n+1}(p)$.

-ii) For all real number β which partial mechanical expansion is $\langle l_1, l_2, \ldots, l_{n-1}, l_n + \beta_n\rangle$ with

$$\frac{l_{n+1}-1}{l_{n+1}} \leq \beta_n < 1$$

then $x_{n+1}(\beta) = x_n(p)(y_n(p))^k$ and $y_{n+1}(\beta) = x_n(p)(y_n(p))^{k+1}$ with $k \geq l_{n+1}-1$.

-iii) For any number γ which partial mechanical expansion is $\langle l_1, l_2, \ldots, l_{n-1}, l_n + \gamma_n\rangle$, then $1-\gamma_n$ is the proportion of $x_n(\beta) = x_n(\alpha) = x_n(p)$ in $\overline{m}_\gamma$ (i.e. the number of $x_n(\beta)$ divided by the number of $x_n(\beta)$ added with the number of $y_n(\beta)$ in $\overline{m}_\alpha$).

Proof. Note that $1 - 1/l_{n+1} = (l_{n+1}-1)/l_{n+1}$. Hence the Equations (4) lead to $\beta \leq p \leq \alpha$.

-i) For all i, $0 \leq i \leq n$, we have $x_i(p) = x_i(\alpha)$ and $y_i(p) = y_i(\alpha)$, since the mechanical expansions are equal until the n^{th} coefficient. Compute l_{n+1} the smallest integer such that

$$\alpha_n \leq \frac{l_{n+1}-1}{l_{n+1}}\,,$$

that implies

$$l_{n+1} \geq \frac{1}{1-\alpha_n} \;\Rightarrow\; l_{n+1}-1 = \lfloor \frac{1}{1-\alpha_n} \rfloor.$$

Hence the mechanical expansion of α is $\langle l_1, l_2, \ldots, l_n, (l_{n+1}-1)+\alpha_{n+1}\rangle$. Therefore $x_{n+1}(\alpha) = x_n(p)(y_n(p))^{l_{n+1}-2}$ and $y_{n+1}(\alpha) = x_n(p)(y_n(p))^{l_{n+1}-1} = x_{n+1}(p)$.

-ii) $\frac{l_{n+1}-1}{l_{n+1}} \leq \beta_n$ implies

$$\lfloor \frac{1}{1-\beta_n} \rfloor \geq l_{n+1},$$

therefore the number of consecutive $y_n(\beta)$ in $x_{n+1}(\beta)$ is larger than $l_{n+1}-1$. Since $x_n(p) = x_n(\beta)$ and $y_n(p) = y_n(\beta)$ then we proved the result.

-iii) By Remark 12, $x_n(\gamma)$ and $y_n(\gamma)$ are the only factors of $\overline{m}_\gamma$, such that $\Phi^{(n)}(x_n(\gamma)) = 1$ and $\Phi^{(n)}(y_n(\gamma)) = 0$. Hence the composition in $x_n(\gamma)$, $y_n(\gamma)$ of $\overline{m}_\gamma$ is the composition in 1 and 0 of $\Phi^{(n)}(\overline{m}_\gamma)$. By Lemma 8 and Corollary 9, $\Phi^{(n)}(\overline{m}_\gamma)$ is an upper mechanical word of slope $1-\gamma_n$. Therefore using Definition 1, $1-\gamma_n$ is the proportion of x_n in $\overline{m}_\gamma$. □

The previous Lemma says that in all mechanical words with slope $\alpha \leq p = \langle l_1, l_2, \ldots, l_{n-1}, l_n, l_{n+1}\rangle$, the sequence $x_n(p)y_n(p)^{l_{n+1}-2}$ never appears. But as soon as the slope is greater than p then the sequence $x_n(p)y_n(p)^{l_{n+1}-2}$ appears. Intuitively, when you increase the slope the number of ones increases and the number of zero decreases, hence the minimum number of consecutive y_n decreases until $x_n(p)y_n(p)^{l_{n+1}-2}$ appears.

Definition 16 (jumps). *Denote the mechanical continued fraction expansion of S^{-1} by $\langle l_1, \ldots, l_n, \ldots\rangle$. Define the rational number $r_i(S^{-1})$ by $r_i(S^{-1}) = \langle l_1, l_2, \ldots, l_{i-1}, l_i + 1\rangle$. The number $r_i(S^{-1})$ is called the i^{th} jump of W_S. When S is a rational number, then $S^{-1} = \langle l_1, l_2, \ldots, l_N\rangle$ and we define the last jump $r_N(S^{-1})$ by $r_N(S^{-1}) = \langle l_1, l_2, \ldots, l_N\rangle$.*

Note that the jumps form a sequence of rational numbers increasing towards S^{-1}. When S^{-1} is rational, the number of jumps is finite and the last jump equals S^{-1}.

Let us define n_i and q_i to be the two relatively prime integers such that $r_i(S^{-1}) = n_i/q_i$. Therefore by Definition 1, $n_i = |\overline{m}_{r_i(S^{-1})}|_1$ and $q_i = |\overline{m}_{r_i(S^{-1})}|$. For example, if $r_2 = \langle l_1, l_2 + 1\rangle$, then it can be checked that $n_2 = l_2 + 1$ and $q_2 = l_1(l_2 + 1) + l_2$. From now on $r_i(S^{-1})$ will be denoted by r_i when no confusion is possible.

Lemma 17. *The number r_{i+1} is the rational number with the smallest denominator in $]r_i, S^{-1}]$.*

Proof : Let $\alpha = \langle l_1, l_2, \ldots, l_i + \alpha_i\rangle$ be any given rational number in $]r_i, S^{-1}]$. By Equations (4) we have $\lfloor (1-\alpha_i)^{-1} \rfloor \geq l_{i+1}$. Hence by Lemma 15, $x_i(y_i)^{l_{i+1}}$ is a factor of $\overline{m}_\alpha$. That means $|\overline{m}_\alpha| \geq |x_i(y_i)^{l_{i+1}}|$. Since $x_i(y_i)^{l_{i+1}}$ is the mechanical word $\overline{m}_{r_{i+1}}$ then $|x_i(y_i)^{l_{i+1}}| = q_{i+1}$. Considering that $|\overline{m}_\alpha|$ is the denominator of α ends the proof. □

Remark 18 (Connection with classical continued fractions). *The rational number c_k, obtained by keeping only k terms in a classical expansion in continued fractions is called the k^{th} convergent. The even convergents c_{2n} of a simple continued fraction form an increasing sequence. Hence if $[m_0, m_1, m_2, \ldots]$ denote the classical continued fraction of S^{-1}, then the even convergents satisfy $c_{2n} = [m_0, m_1, \ldots, m_{2n}] \leq S^{-1}$. If $c_{2n} = [m_0, m_1, \ldots, m_{2n}]$ is an even convergent of S^{-1}, then the rational numbers $[m_0, m_1, \ldots, m_{2n-1}, \rho_{2n}]$ with $\rho_{2n} \in \{1, 2, \ldots, m_{2n}\}$ are called intermidiate convergents.*

Note now that the sequence of intermediate convergents and even convergents is the sequence of jumps $r_i(S^{-1})$, since this sequence increases towards S^{-1} and each term of this sequence is the rational with the smallest denominator in the interval formed by the preceeding intermediate convergent and S^{-1} and since $[c_1 + 1] = r_1$.

However it seems hard to use the intermediate convergents to compute the average waiting time as it is shown further.

4.2 Formula of the average waiting time

We will see that the factors x_i tend to increase the load in the queue whereas the factors y_i tend to decrease the load. The decomposition of $\overline{m}_{r_i}$ in (x-y) factors of S^{-1} allows us to compute recursively the average waiting time of the input sequence m_{r_i} : $W_S(\overline{m}_{r_i})$. From that we can compute $W_S(\overline{m}_\alpha)$ for any rational number α, $r_i < \alpha < r_{i+1}$ and finally we extend it to any irrational number α.

In the following, the workload denotes the amount of service (in time units) remaining to be done by the server. Let d_i^1 denote the increase of the workload after an input sequence equal to x_i. Let d_i^2 denote the maximal possible decrease of the workload after an input sequence equal to y_i.

Lemma 19. *If the initial workload is equal to zero then using x_i as input sequence the workload during x_i is never null and remains non negative at the end of the sequence. If the initial workload is equal to zero then using y_i as input sequence the workload is never null during y_i until its last letter (which is always a 0) and is null at the end of the sequence.*

Proof. The proof holds by induction on i.

Step 1. Assume S^{-1} is not an integer, we have $l_1 < S < l_1 + 1$. Therefore $S - l_1 - 1 < 0 < S - l_1$ and the result is proved for step one since the workload after x_1 is equal to $S - l_1$ and the workload after y_1 is $\max(S - l_1 - 1, 0) = 0$. Moreover Equations (4) yield $d_1^1 = \alpha_1$ and $d_1^2 = \alpha_1 - 1$. When S^{-1} is an integer, we have $S - l_1 = 0$.

Step 2. By Theorem 10, x_2 is composed by x_1 followed by $(l_2 - 1)$ consecutive y_1 and y_2 is composed by x_1 followed by l_2 consecutive y_1. Hence to prove the result we have to show that $\forall\, j,\ 1 \le j \le \max(l_2 - 1, 1)$:

$$\begin{aligned} \alpha_1 + (l_2 - j)(\alpha_1 - 1) &\geq 0\,, \\ \alpha_1 + (l_2)(\alpha_1 - 1) + 1 &\geq 0\,, \\ \alpha_1 + (l_2)(\alpha_1 - 1) &\leq 0\,, \end{aligned}$$

this last inequality implying the nullity of the workload. By Lemma 15, l_2 is the smallest integer satisfying $\alpha_1 \le l_2/(l_2 + 1)$, then for all j such that $1 \le j \le \max(l_2 - 1, 1)$ we have

$$\frac{l_2}{l_2 + 1} \geq \alpha_1 > \frac{l_2 - j}{l_2 + 1 - j}\,.$$

Since $(l_2 - 1)/(l_2) \geq (l_2 - 1)/(l_2 + 1)$, the three inequalities are satisfied.

It can be checked that $\alpha_1 + (l_2 - 1)(\alpha_1 - 1) = (1 - \alpha_1)\alpha_2$ and $\alpha_1 + (l_2)(\alpha_1 - 1) = (1 - \alpha_1)(\alpha_2 - 1)$. Therefore $d_2^1 = (1 - \alpha_1)\alpha_2$ and $d_2^2 = (1 - \alpha_1)(\alpha_2 - 1)$.

Step i+1. Suppose that $d_i^2 = (1 - \alpha_1)\dots(1 - \alpha_{i-1})(\alpha_i - 1)$ and $d_i^1 = (1 - \alpha_1)\dots(1 - \alpha_{i-1})\alpha_i$. We have to show that $\forall\, j,\ 1 \le j \le \max(l_{i+1} - 1, 1)$:

$$(1 - \alpha_1)\dots(1 - \alpha_{i-1})\,[\alpha_i + (l_{i+1} - j)(\alpha_i - 1)] > 0\,, \tag{6}$$

$$(1 - \alpha_1)\dots(1 - \alpha_{i-1})\,[\alpha_i + (l_{i+1}(\alpha_i - 1)] \leq 0\,. \tag{7}$$

These inequalities are satisfied indeed, since l_{i+1} is the smallest integer such that $\alpha_i \le l_{i+1}/(l_{i+1} + 1)$. Moreover, by the induction hypothesis, during the sequence y_i the workload is positive until the last 0, when the initial workload is null. Consider now the last factor y_i in y_{i+1}, its initial workload is positive by Inequality (6). If the workload is never null during an input sequence with an initial workload equal to zero, then the same result holds with a positive initial workload. Hence the workload is never null during the last y_i and also during y_{i+1}, up to its last letter.

Using $\alpha_i + (l_{i+1} - 1)(\alpha_i - 1) = (1 - \alpha_i)\alpha_{i+1}$ and $\alpha_i + (l_{i+1})(\alpha_i - 1) = (1 - \alpha_i)(\alpha_{i+1} - 1)$ we obtain

$$d_i^2 = (1 - \alpha_1)\dots(1 - \alpha_{i-1})(\alpha_i - 1) \ \text{ and } \ d_i^1 = (1 - \alpha_1)\dots(1 - \alpha_{i-1})\alpha_i\,. \tag{8}$$

That finishes the proof. □

Remark 20. *When $S = \langle l_1, l_2, \dots, l_N\rangle$ is a rational number, $1 - \alpha_{N-1} = 1/l_N$ which is equivalent to $\alpha_{N-1} = (l_N - 1)/l_N$. Then $\alpha_{N-1} + (l_N - 1)(\alpha_{N-1} - 1) = 0$. This yields $d_N^1 = 0$ and $d_N^2 = 0$. Hence the workload during x_N is positive until the last epoch of the admission sequence $x_N(S)^{-1} = x_N(r_N)$, and null at the end.*

We give now a direct consequence of Lemma 19. The workload d_i^1 and the maximal decreases of the workload d_i^2 can be recursively computed by :

$$\forall\, i \geq 2,\; d_i^1 = d_{i-1}^1 + (l_i - 1)d_{i-1}^2,\; d_i^2 = d_{i-1}^1 + (l_i)d_{i-1}^2, \text{ with } d_1^1 = S - l_1\,,\; d_1^2 = S - l_1 - 1\,. \qquad (9)$$

Example 21 (Average waiting time with S = 51/20 and $\overline{m}_{5/13}$). *Let $S = 51/20 = 2.55$. The partial mechanical expansion of order three of $20/51$ is $\langle 2, 2, 1 + 2/7\rangle$. The (x,y)-factor decomposition gives $x_1 = 10$, $y_1 = 100$, $x_2 = 10100$, $y_2 = 10100100$, $x_3 = x_2$ and $y_3 = 1010010100100$. Note that $r_3(20/51) = 5/13$ and that $\overline{m}_{5/13} = y_3$.*

The Equation (9) *leads to*

$$\begin{aligned} d_1^1 = 0.55 \quad &; \quad d_1^2 = -0.45\,, \\ d_2^1 = 0.55 - 0.45 = 0.1 \quad &; \quad d_2^2 = 0.55 - 0.90 = -0.35\,, \\ d_3^1 = d_2^1 = 0.1 \quad &; \quad d_3^2 = 0.1 - 0.35 = -0.25\,. \end{aligned}$$

The figure 2 represents the workload during a sequence y_3. The quantity d_1^1 is represented by a, d_1^2 by b, d_2^1 by c, d_2^2 by d and d_3^2 by e.

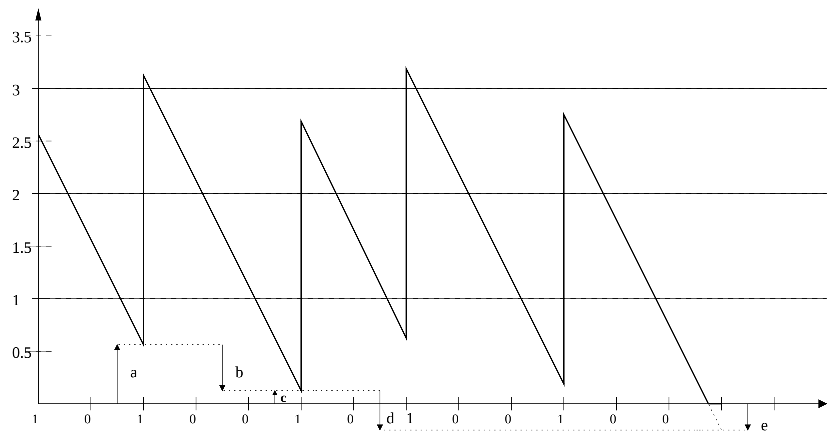

Figure 2: Representation of the workload

We have $|\overline{m}_{r_i}|_1 = |x_{i-1}|_1 + l_i|y_{i-1}|_1$, ($|\overline{m}_{r_N}|_1 = |x_{N-1}|_1 + (l_N - 1)|y_{N-1}|_1$ when $S^{-1} = \langle l_1, l_2, l_3, \dots, l_N\rangle$) and $|x_{i-1}|_1 = |y_{i-1}|_1 - |y_{i-2}|_1$. The recursive computation of $n_i = |\overline{m}_{r_i}|_1$ is given by :

$$\begin{aligned} n_0 &= 0, & (10)\\ n_1 &= 1, & (11)\\ n_i &= (l_i + 1)n_{i-1} - n_{i-2} \;\; \forall i \geq 2, & (12)\\ n_N &= l_N n_{N-1} - n_{N-2} \;\; \text{if } S^{-1} = \langle l_1, \dots, l_N\rangle. & (13) \end{aligned}$$

This allows us to compute the average waiting time of $W_S(\overline{m}_{r_i})$ recursively.

The sum of the waiting times of customers admitted during the sequence w when the first customer of the sequence waits for a time t is denoted by $K^t(w)$. We also denote the sum of the waiting times of $\overline{m}_{r_i}$ over one period by K_i^t and K_i^0 by K_i. Note first, that we can focus on one period since the workload after $\overline{m}_{r_i}$ is null.

Lemma 22 (Formula of $W_S(\overline{m}_{r_i})$). *The average waiting time of $\overline{m}_{r_i}$ is :*

$$W_S(\overline{m}_{r_i}) = \frac{K_i}{n_i}\,,$$

with $K_1 = 0$,

$$K_2 = l_2 d_1^1 + (l_2-1)(l_2/2)d_1^2, \tag{14}$$

or if $S^{-1} = \langle l_1, l_2 \rangle$

$$K_2 = (l_2-1)d_1^1 + (l_2-2)(l_2-1/2)d_1^2, \tag{15}$$

and

$$K_i = (l_i+1)K_{i-1} - K_{i-2} + (n_{i-1}l_i - n_{i-2})\, d_{i-1}^1 + n_{i-1}(l_i-1)\frac{l_i}{2}d_{i-1}^2 \;\forall\, i \geq\, 2, \tag{16}$$

and if $S^{-1} = \langle l_1, \ldots, l_N \rangle$

$$K_N = l_N K_{N-1} - K_{N-2} + (n_{N-1}(l_N-1) - n_{N-2})\, d_{N-1}^1 + n_{N-1}(l_N-2)\frac{l_N-1}{2}d_{N-1}^2. \tag{17}$$

Proof. The minimum number of consecutive zeros in $\overline{m}_{r_1}$ is equal to l_1. This means that the inter-arrival time in the queue is equal to l_1+1 and since $S < l_1+1$ none of the customer has to wait, therefore $K_1 = 0$.

Recall that we have by Definition 16 $\overline{m}_{r_2} = x_1(y_1)^{l_2}$. The waiting time of a customer is the workload at the epoch of its arrival. By Lemma 19 the first customer admitted does not wait, on the other hand the next l_2 customers will have to wait. Moreover the workload during $\overline{m}_{r_2}$ is positive until the last 0 hence the waiting time of the second customer is d_1^1, the one of the third customer is $d_1^1 + d_1^2$, the one of the j^{th} is $d_1^1 + (j-2)d_1^2$ and the one of the last customer, which is the $(l_2+1)^{th}$, is $d_1^1 + (l_2-1)d_1^2$. Summing we obtain $K_2 = l_2 d_1^1 + (l_2-1)(l_2/2)d_1^2$.

Let w be an admission sequence during which the workload is never null if the initial workload is equal to zero. When the first customer of w waits for a time t, the waiting time of a customer admitted during w is its waiting time when the first customer does not wait, increased by t. Therefore $K^t(w) = (|w|_1 \times t) + K^0(w)$.

We have $\overline{m}_{r_i} = x_{i-1}(y_{i-1})^{l_i}$. Let us compute the waiting times of the first customers of the l_i consecutive y_{i-1}. The waiting time of the first customer of the first y_{i-1} is d_{i-1}^1. The waiting time of the first customer of the second y_{i-1} is $d_{i-1}^1 + d_{i-1}^2$. The waiting time of the first customer of the j^{th} y_{i-1} is $d_{i-1}^1 + (j-1)d_{i-1}^2$. The waiting time of the first customer of the last y_{i-1} (the l_i^{th}) is $d_{i-1}^1 + (l_i-1)d_{i-1}^2$.

We decompose K_i.

$$\begin{aligned}
K_i &= K^0(x_{i-1}) + K_{i-1}^{d_{i-1}^1} + K_{i-1}^{d_{i-1}^1+d_{i-1}^2} + \ldots + K_{i-1}^{d_{i-1}^1+(j-1)d_{i-1}^2} + \ldots + K_{i-1}^{d_{i-1}^1+(l_i-1)d_{i-1}^2}\,, \\
&= K^0(x_{i-1}) + n_{i-1}d_{i-1}^1 + K_{i-1} + n_{i-1}(d_{i-1}^1 + d_{i-1}^2) + K_{i-1} + \ldots \\
&\qquad\qquad + n_{i-1}(d_{i-1}^1 + (l_i-1)d_{i-1}^2) + K_{i-1}\,, \\
&= K^0(x_{i-1}) + l_i K_{i-1} + n_{i-1}(l_i d_{i-1}^1 + (l_i-1)\frac{l_i}{2}d_{i-1}^2)\,,
\end{aligned}$$

since $y_{i-1} = x_{i-1}y_{i-2}$,

$$\begin{aligned}
K_i &= K_{i-1} - K_{i-2}^{d_{i-1}^1} + l_i K_{i-1} + n_{i-1}(l_i d_{i-1}^1 + (l_i-1)\frac{l_i}{2}d_{i-1}^2)\,, \\
&= K_{i-1} - (n_{i-2}d_{i-1}^1 + K_{i-2}) + l_i K_{i-1} + n_{i-1}(l_i d_{i-1}^1 + (l_i-1)\frac{l_i}{2}d_{i-1}^2)\,.
\end{aligned}$$

When $S^{-1} = \langle l_1, l_2, \ldots, l_N \rangle$, the N-th jump is a secial case since $\overline{m}_{r_N(S^{-1})} = \overline{m}_{S^{-1}}$ is x_N and the number of consecutive y_{N-1} following x_{N-1} is $l_N - 1$. □

Theorem 23 (Formula of $W_S(\overline{m}_\alpha)$ for rational ratio). *Let r_i be the i^{th} jump of $W_S(\overline{m}_\alpha)$, and r_{i+1} the $i+1^{th}$ jump of $W_S(\overline{m}_\alpha)$. Let $\alpha = \langle l_1, \ldots, l_i + \alpha_i \rangle$ be a rational number such that $\alpha \in (r_i, r_{i+1})$. Then the average waiting time of $\overline{m}_\alpha$ is*

$$W_S(\overline{m}_\alpha) = \frac{(1-\alpha_i)K_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,K_i}{(1-\alpha_i)n_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,n_i}\,. \tag{18}$$

If α is a rational number such that $\alpha \leq r_1$, then $W_S(\overline{m}_\alpha) = 0$.

Proof. By Theorem 10, $\overline{m}_\alpha$ can be decomposed only using $x_{i+1}(\alpha)$ and $y_{i+1}(\alpha)$. Since by Equations (4)

$$\frac{l_{i+1}}{l_{i+1}+1} \leq \alpha_i \leq 1\,,$$

then we obtain the composition of $x_{i+1}(\alpha)$ and $y_{i+1}(\alpha)$ with Lemma 15:

$$x_{i+1}(\alpha) = x_i(r_{i+1})(y_i(r_{i+1}))^k \text{ and } y_{i+1}(\alpha) = x_i(r_{i+1})(y_i(r_{i+1}))^{k+1} \text{ with } k \geq (l_{i+1}+1)-1.$$

Let us rewrite these two equalities into :

$$x_{i+1}(\alpha) = x_i(r_{i+1})(y_i(r_{i+1}))^{l_{i+1}}(y_i(r_{i+1}))^{k'} \text{ and } y_{i+1}(\alpha) = x_i(r_{i+1})(y_i(r_{i+1}))^{l_{i+1}}(y_i(r_{i+1}))^{k'+1},$$

with $k' \geq 0$. We have $x_{i+1}(\alpha) = \overline{m}_{r_{i+1}}(\overline{m}_{r_i})^{k'}$ and $y_{i+1}(\alpha) = \overline{m}_{r_{i+1}}(\overline{m}_{r_i})^{k'+1}$.

Since by Lemma 19 and Remark 20 the workload after $\overline{m}_{r_i}$ is always null, then the workload after an admission sequence $x_{i+1}(\alpha)$ or $y_{i+1}(\alpha)$ is also null. Therefore the workload after one period of $\overline{m}_\alpha$ is null and we simply have to focus on one period of $\overline{m}_\alpha$. Computing $K^0(x_{i+1}(\alpha))$ and $K^0(y_{i+1}(\alpha))$ gives

$$K^0(x_{i+1}(\alpha)) = K_{i+1} + k'K_i \text{ and } K^0(y_{i+1}(\alpha)) = K_{i+1} + (k'+1)K_i\,.$$

We write $\alpha_i = n/q$, α_i being rational. The number of $\overline{m}_{r_{i+1}}$ in $\overline{m}_\alpha$ is the number of $x_i(S^{-1}) = x_i(r_{i+1})$ in $\overline{m}_\alpha$. By Lemma 15 the total number of $x_i(S^{-1})$ in $\overline{m}_\alpha$ is equal to the total number of ones in $\Phi^{(i)}(\overline{m}_\alpha)$. Since $1-n/q$ is the slope of $\Phi^{(i)}(\overline{m}_\alpha)$ and since $\overline{m}_\alpha$ and $\Phi^{(i)}(\overline{m}_\alpha)$ are finite words. Then by Definition 1 the total number of $x_i(S^{-1})$ in $\overline{m}_\alpha$ is $(1-\frac{n}{q})q$. Using a similar argument the total number of $\overline{m}_{r_i}$ in $\overline{m}_\alpha$ is $(\frac{n}{q})q$. The number of $\overline{m}_{r_i}$ which do not belong to $\overline{m}_{r_{i+1}}$ is equal to $q(\frac{n}{q}) - l_{i+1}(1-\frac{n}{q})q$. Therefore

$$K(\overline{m}_\alpha) = q(1-\frac{n}{q})K_{i+1} + q[(\frac{n}{q}) - l_{i+1}(1-\frac{n}{q})]K_i\,.$$

Compute now n_α the total number of ones in $\overline{m}_\alpha$. This number is the number of $\overline{m}_{r_{i+1}}$ multiplied by the number of ones in $\overline{m}_{r_{i+1}}$ added with the number of $\overline{m}_{r_i}$ which do not belong to $\overline{m}_{r_{i+1}}$ multiplied by the number of ones in $\overline{m}_{r_i}$:

$$n_{\alpha_i} = q(1-\frac{n}{q})n_{i+1} + q[(\frac{n}{q}) - l_{i+1}(1-\frac{n}{q})]n_i\,.$$

We obtain

$$W_\alpha(S) = \frac{(1-\frac{n}{q})K_{i+1} + [\frac{n}{q} - (l_{i+1}(1-\frac{n}{q}))]K_i}{(1-\frac{n}{q})n_{i+1} + [\frac{n}{q} - (l_{i+1}(1-\frac{n}{q}))]n_i}\,.$$

When $\alpha \leq r_1$, by Lemma 6 the minimum number of consecutive 0 in $\overline{m}_\alpha$ is greater than l_1. Therefore the inter-arrival times of $\overline{m}_\alpha$ are greater than l_1+1, hence none of the customers has to wait and $W_S(\overline{m}_\alpha) = 0$. ☐

We will now extend this result to irrational numbers. First, we will define the average waiting time of an infinite word w. Let w_n be the prefix of length n of w, then

$$W_S(w) = \lim_{n\to\infty} \frac{K^0(w_n)}{|w_n|_1}\,,$$

if the limit exists.

Theorem 24 (Formula of $W_S(\overline{m}_\alpha)$ for irrational ratio). *Let $\alpha = \langle l_1, \ldots, l_i + \alpha_i\rangle$ be an irrational number such that $\alpha \in (r_i, r_{i+1})$, where r_i is the i^{th} jump of $W_S(\overline{m}_\alpha)$, and r_{i+1} is the $i+1^{th}$ jump of $W_S(\overline{m}_\alpha)$. The average waiting time of $\overline{m}_\alpha$ is also given by the Formula* (18) *that is :*

$$W_S(\overline{m}_\alpha) = \frac{(1-\alpha_i)K_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,K_i}{(1-\alpha_i)n_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,n_i}\,.$$

Proof. Let n be an integer. Let $(\overline{m}_\alpha)_n$ be the prefix of length n of $\overline{m}_\alpha$. Let $(\overline{m}_\alpha)_{s_n}$ be the greatest prefix of $\overline{m}_\alpha$ uniquely composed by factors $x_{i+1}(\alpha)$ and $y_{i+1}(\alpha)$ which length is smaller than n, and let $(\overline{m}_\alpha)_{g_n}$ be the smallest prefix uniquely composed by factors $x_{i+1}(\alpha)$ and $y_{i+1}(\alpha)$ which length is greater than n. We obtain

$$\frac{K^0((\overline{m}_\alpha)_{s_n})}{|(\overline{m}_\alpha)_{s_n}|_1} \leq \frac{K^0((\overline{m}_\alpha)_n)}{|(\overline{m}_\alpha)_n|_1} \leq \frac{K^0((\overline{m}_\alpha)_{g_n})}{|(\overline{m}_\alpha)_{g_n}|_1}.$$

The prefix $(\overline{m}_\alpha)_{g_n}$ is composed by $(\overline{m}_\alpha)_{s_n}$ followed by either $x_{i+1}(\alpha)$ or $y_{i+1}(\alpha)$. As shown above, the workload after any sequence $x_{i+1}(\alpha)$ and $y_{i+1}(\alpha)$ is null. Hence the workload after $(\overline{m}_\alpha)_{g_n}$ and after $(\overline{m}_\alpha)_{s_n}$ is also null. Since the factor $y_{i+1}(\alpha)$ is longer than the factor $x_{i+1}(\alpha)$, then $|x_{i+1}(\alpha)|_1 \leq |y_{i+1}(\alpha)|_1$ and $K^0(x_{i+1}(\alpha)) \leq K^0(y_{i+1}(\alpha))$. That yields $K^0((\overline{m}_\alpha)_{g_n}) \leq K^0((\overline{m}_\alpha)_{s_n}) + K^0(y_{i+1}(\alpha))$ and $|(\overline{m}_\alpha)_{g_n}|_1 \leq |(\overline{m}_\alpha)_{s_n})|_1 + |y_{i+1}(\alpha)|_1$. Therefore

$$\frac{K^0((\overline{m}_\alpha)_{s_n})}{|(\overline{m}_\alpha)_{s_n}|_1} \cdot \frac{|(\overline{m}_\alpha)_{s_n}|_1}{|(\overline{m}_\alpha)_{s_n}|_1 + |y_{i+1}(\alpha)|_1} \leq \frac{K^0((\overline{m}_\alpha)_n)}{|(\overline{m}_\alpha)_n|_1} \leq \frac{K^0((\overline{m}_\alpha)_{s_n})}{|(\overline{m}_\alpha)_{s_n}|_1} + \frac{K^0(y_{i+1}(\alpha))}{|(\overline{m}_\alpha)_{s_n}|_1}. \tag{19}$$

Let us focus on $K^0((\overline{m}_\alpha)_{s_n})$. For all $n \geq n_{i+1}$, this sum is strictly positive. Moreover this sum, as shown in the previous proof, only depends on the number of $y_{i+1}(r_{i+1})$ and "*remaining*" $y_i(r_{i+1})$. The number of $y_{i+1}(r_{i+1})$ and "*remaining*" $y_i(r_{i+1})$ is given by the slope of $\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))$. Hence

$$\begin{aligned} K^0((\overline{m}_\alpha)_{s_n}) = {} & \left(|\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))| \cdot \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right) K_{i+1} + \\ & \left(|\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))|\right) \left[\left(1 - \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right) - l_{i+1} \cdot \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right] K_i \,. \end{aligned}$$

and

$$\begin{aligned} |(\overline{m}_\alpha)_{s_n}|_1 = {} & \left(|\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))| \cdot \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right) n_{i+1} + \\ & \left(|\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))|\right) \left[\left(1 - \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right) - l_{i+1} \cdot \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right] n_i \,. \end{aligned}$$

Finally we have

$$\begin{aligned} \frac{K^0((\overline{m}_\alpha)_{s_n})}{|(\overline{m}_\alpha)_{s_n}|_1} = {} & \\ & \frac{\alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))K_{i+1} + \left[\left(1 - \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right) - l_{i+1} \cdot \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right] K_i}{\alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))n_{i+1} + \left[\left(1 - \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right) - l_{i+1} \cdot \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n}))\right] n_i}. \end{aligned}$$

Note now that the number $|(\overline{m}_\alpha)_{s_n}|$ depending on n, when $n \to \infty$, $|(\overline{m}_\alpha)_{s_n}| \to \infty$ and also $|(\overline{m}_\alpha)_{s_n}|_1 \to \infty$. Definition 1 and Lemma 15 give

$$\lim_{n\to\infty} \alpha(\Phi^{(i)}((\overline{m}_\alpha)_{s_n})) = \lim_{n\to\infty} \alpha(\Phi^{(i)}((\overline{m}_\alpha)_n)) = 1 - \alpha_i \,.$$

This implies

$$\lim_{n\to\infty} \frac{K^0((\overline{m}_\alpha)_{s_n})}{|(\overline{m}_\alpha)_{s_n}|_1} = \frac{(1-\alpha_i)K_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,K_i}{(1-\alpha_i)n_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,n_i}. \tag{20}$$

Since $y_{i+1}(r_{i+1})$ is a finite factor then $|y_{i+1}(r_{i+1})|_1$ and $K(y_{i+1}(r_{i+1}))$ are bounded. It comes

$$\begin{aligned} \lim_{n\to\infty} \frac{K^0((\overline{m}_\alpha)_{s_n})}{|(\overline{m}_\alpha)_{s_n}|_1} \cdot \frac{|(\overline{m}_\alpha)_{s_n}|_1}{|(\overline{m}_\alpha)_{s_n}|_1 + |y_{i+1}(\alpha)|_1} &= \lim_{n\to\infty} \frac{K^0((\overline{m}_\alpha)_{s_n})}{|(\overline{m}_\alpha)_{s_n}|_1} \\ \lim_{n\to\infty} \frac{K^0((\overline{m}_\alpha)_{s_n})}{|(\overline{m}_\alpha)_{s_n}|_1} + \frac{K^0(y_{i+1}(\alpha))}{|(\overline{m}_\alpha)_{s_n}|_1} &= \lim_{n\to\infty} \frac{K^0((\overline{m}_\alpha)_{s_n})}{|(\overline{m}_\alpha)_{s_n}|_1}. \end{aligned}$$

Using Equation (19) and Equation (20) we obtain

$$\frac{(1-\alpha_i)K_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,K_i}{(1-\alpha_i)n_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,n_i} \leq \lim_{n\to\infty} \frac{K^0((\overline{m}_\alpha)_n)}{|(\overline{m}_\alpha)_n|_1}$$
$$\leq \frac{(1-\alpha_i)K_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,K_i}{(1-\alpha_i)n_{i+1} + [\alpha_i - l_{i+1}\,(1-\alpha_i)]\,n_i}\,.$$

□

4.3 Properties

Here, we will give some properties of the function $W_S(\overline{m}_\alpha)$. Most of them are required to compute the optimal ratio in the case of two queues.

Lemma 25 (Continuity). *The function $\alpha \to W_S(\overline{m}_\alpha)$ is continuous.*

Proof. Let $\alpha = \langle l_1, \dots, l_n + \alpha_n\rangle$ and $\beta = \langle l_1, \dots, l_n + \beta_n\rangle$ be two points in the interval . Note that α and β are both in the interval $[r_n, r_{n+1}]$. If $|\alpha - \beta|$ goes to zero, then $|\alpha_n - \beta_n|$ also goes to zero. By Equation (18), $|W_S(\overline{m}_\alpha) - W_S(\overline{m}_\beta)|$ goes to zero. □

Using similar arguments, one can show that the function $\alpha \to W_S(\overline{m}_\alpha)$ is infinitely differentiable on each interval $]r_n, r_{n+1}[$, but it is not differentiable at the jumps.

Lemma 26. *The function $i \to \frac{K_i}{n_i}$ is strictly increasing.*

Proof. We have

$$\frac{K_{i+1}}{n_{i+1}} - \frac{K_i}{n_i} = \frac{n_i \times K_{i+1} - n_{i+1} \times K_i}{n_i \times n_{i+1}}\,.$$

Since $\forall\, i \geq 1$, $n_i \times n_{i+1} > 0$ we have to focus on the sign of $n_i K_{i+1} - n_{i+1} K_i$. Using Equations (12) and (16) yields

$$\begin{aligned}
&n_i K_{i+1} - n_{i+1} K_i \\
&\quad= n_i \left[(l_{i+1}+1)K_i + (n_i l_{i+1} - n_{i-1})\, d_i^1 + n_i(l_{i+1}-1)\frac{l_{i+1}}{2} d_{i+1}^2 - K_{i-1}\right] - n_{i+1}K_i \\
&\quad= n_i \left[(l_{i+1}+1)K_i - K_{i-1} + (n_i l_{i+1} - n_{i-1})\, d_i^1 + n_i(l_{i+1}-1)\frac{l_{i+1}}{2} d_i^2\right] \\
&\qquad\quad + (n_{i-1})K_i - n_i(l_i+1)K_i \\
&\quad= (n_{i-1}K_i - n_i K_{i-1}) + n_i \left[(n_i l_{i+1} - n_{i-1})\, d_i^1 + n_i(l_{i+1}-1)\frac{l_{i+1}}{2} d_i^2\right]\,,
\end{aligned}$$

reordering we obtain

$$n_i K_{i+1} - n_{i+1} K_i = (n_{i-1}K_i - n_i K_{i-1}) + n_i^2 (l_{i+1}-1)(d_i^1 + \frac{l_{i+1}}{2} d_i^2) + n_i(n_i - n_{i-1}) d_i^1\,. \qquad (21)$$

On the first hand, by Lemma 19, we know that d_i^1 is strictly positive and d_{i+1}^1 is non negative and that d_i^2 and d_{i+1}^2 are non positive. On the other hand Equation (9) gives us $d_i^1 + (l_{i+1}-1)d_i^2 = d_{i+1}^1$ and $d_i^1 + l_{i+1} d_i^2 = d_{i+1}^2$. This implies $\forall\, l_{i+1} \geq 2$,

$$d_{i+1}^1 + \frac{l_{i+1}}{2} d_{i+1}^2 > 0\,.$$

That means $\forall\, l_{i+1} \geq 1$,

$$(l_{i+1}-1)(d_i^1 + \frac{l_{i+1}}{2} d_{i+1}^2) \geq 0.$$

Since $K_2 - n_2K_1 = K_2$ ($K_1 = 0$) and since $K_2 = l_2\left(d_1^1 + \frac{(l_2-1)}{2}d_1^2\right)$ and since $d_1^1 + \frac{(l_2-1)}{2}d_1^2 > d_2^1 \geq 0$, then $K_2 > 0$, and by induction we have $\forall i \geq 0$:

$$n_iK_{i+1} - n_{i+1}K_i > 0\,. \tag{22}$$

Therefore $\forall i \geq 0$,

$$\frac{K_{i+1}}{n_{i+1}} - \frac{K_i}{n_i} > 0.$$

□

Remark 27. *As proved in [8] if S is an irrational number then the average waiting time $W_{S^{-1}}(\overline{m}_{S^{-1}}) = 1/2$. Hence the sequence $\frac{K_i}{n_i}$ goes to $1/2$ when i goes to infinity.*

Lemma 28 (Monotonicity). *If $\alpha \in]r_i, r_{i+1}[$, then the function $W_S(\overline{m}_\alpha)$ is strictly increasing in α.*

Proof. First we will show by induction on i that the derivative function $\frac{\partial \alpha_i}{\partial \alpha}$ is non positive.

Step 1. Recall that by Equations (4) we have : $\alpha = (l_1 + \alpha_1)^{-1}$ this means $\alpha_1 = \alpha^{-1} - l_1$. Hence

$$\frac{\partial \alpha_1}{\partial \alpha} < 0\,.$$

Step i. By Equations (4), $\alpha_{i-1} = 1 - (l_i + \alpha_i)^{-1}$, then $\alpha_i = (1 - \alpha_{i-1})^{-1} - l_i$. Therefore

$$\frac{\partial \alpha_i}{\partial \alpha_{i-1}} > 0\,, \tag{23}$$

and using the induction hypothesis

$$\frac{\partial \alpha_i}{\partial \alpha} = \frac{\partial \alpha_i}{\partial \alpha_{i-1}}\left(\frac{\partial \alpha_{i-1}}{\partial \alpha}\right) < 0\,. \tag{24}$$

Second, using Equations (18) and (22), we have

$$\frac{\partial W_S(\overline{m}_\alpha)}{\partial \alpha_i} = \frac{-\left(n_iK_{i+1} - n_{i+1}K_i\right)}{\left[\left(1 - \alpha_i\right)n_{i+1} + \left(\alpha_i - l_{i+1}\left(1 - \alpha_i\right)\right)n_i\right]^2} < 0\,. \tag{25}$$

Since

$$\frac{\partial W_S(\overline{m}_\alpha)}{\partial \alpha} = \frac{\partial W_S(\overline{m}_\alpha)}{\partial \alpha_i} \cdot \frac{\partial \alpha_i}{\partial \alpha}\,,$$

then

$$\frac{\partial W_S(\overline{m}_\alpha)}{\partial \alpha} > 0\,.$$

□

Lemma 29 (Concavity). *If $\alpha \in]r_i, r_{i+1}[$, then the function $\alpha \to W_S(\overline{m}_\alpha)$ is strictly concave.*

Proof. First, we prove by induction on i that the second derivative function $\frac{\partial^2 \alpha_i}{\partial \alpha^2}$ is strictly positive.

Step 1. It can be checked that

$$\frac{\partial^2\left(\alpha^{-1} - l_1\right)}{\partial \alpha^2} > 0\,.$$

Step i. We obtain :

$$\frac{\partial^2 \alpha_i}{\partial \alpha^2} = \frac{\partial \alpha_i}{\partial \alpha_{i-1}} \cdot \left(\frac{\partial \alpha_{i-1}}{\partial \alpha}\right)^2 + \frac{\partial \alpha_i}{\partial \alpha_{i-1}} \cdot \frac{\partial^2 \alpha_{i-1}}{\partial \alpha^2}.$$

The Formula (23) and the induction hypothesis can be written respectively as :

$$\frac{\partial \alpha_i}{\partial \alpha_{i-1}} \; > \; 0 \;\; \text{and} \;\; \frac{\partial^2 \alpha_{i-1}}{\partial \alpha^2} \; > \; 0\,,$$

hence

$$\frac{\partial^2 \alpha_i}{\partial \alpha^2} > 0\,. \tag{26}$$

Using formulas (18) and (25) it comes

$$\frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial \alpha_i^2} = \frac{2\,(n_i K_{i+1} - n_{i+1} K_i)\,(-n_{i+1} + n_i(1 - l_{i+1}))}{\left[(1-\alpha_i)\,n_{i+1} + (\alpha_i - l_{i+1}\,(1-\alpha_i))\,n_i\right]^3}\,.$$

Equations (4) and (12) give $l_{i+1} \geq 1$ and $n_{i+1} > 0$. This implies, with Equation (22), $(n_i K_{i+1} - n_{i+1} K_i) \times (-n_{i+1} + n_i(1 - l_{i+1})) < 0$. Let us also note that $\alpha_i - l_{i+1}(1-\alpha_i)$=$(1-\alpha_i)\alpha_{i+1}$ which yields $(1-\alpha_i)\,n_{i+1} + (\alpha_i - l_{i+1}\,(1-\alpha_i))\,n_i > 0$. Therefore

$$\frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial \alpha_i^2} < 0\,. \tag{27}$$

We have

$$\frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial \alpha^2} = \frac{\partial^2 \alpha_i}{\partial \alpha^2} \cdot \frac{\partial W_S(\overline{m}_\alpha)}{\partial \alpha_i} + \frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial \alpha_i^2} \cdot \left(\frac{\partial \alpha_i}{\partial \alpha}\right)^2 .$$

By inequalities 25, 26 and 27 we know

$$\frac{\partial^2 \alpha_i}{\partial \alpha^2} \; > \; 0, \;\; \frac{\partial W_S(\overline{m}_\alpha)}{\partial \alpha_i} \; < \; 0 \;\; \text{and} \;\; \frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial \alpha_i^2} \; < 0 \;\; .$$

Then

$$\frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial \alpha^2} < 0.$$

□

Lemma 30 (Concavity, II). *If $\alpha \in]r_i, r_{i+1}[$, then the function $\alpha \to \alpha W_S(\alpha)$ is concave.*

Proof. First we show by induction on i that the function $2\frac{\partial \alpha_i}{\partial \alpha} + \alpha \frac{\partial^2 \alpha_i}{\partial \alpha^2}$ is nonnegative.

Step 1. We check that

$$2\frac{\partial \alpha_1}{\partial \alpha} + \alpha \frac{\partial^2 \alpha_1}{\partial \alpha^2} = \frac{\partial^2 (\alpha \cdot \alpha_1)}{\partial \alpha^2} = \frac{\partial^2 (1 - l_1 \alpha)}{\partial \alpha^2} = 0\,.$$

Step i. We obtain

$$\begin{aligned} 2\frac{\partial \alpha_i}{\partial \alpha} + \alpha \frac{\partial^2 \alpha_i}{\partial \alpha^2} &= 2\frac{\partial \alpha_i}{\partial \alpha_{i-1}} \cdot \frac{\partial \alpha_{i-1}}{\partial \alpha} + \alpha \left[\frac{\partial \alpha_i}{\partial \alpha_{i-1}} \cdot \frac{\partial^2 \alpha_{i-1}}{\partial \alpha^2} + \left(\frac{\partial \alpha_i}{\partial \alpha_{i-1}}\right)^2 \cdot \frac{\partial^2 \alpha_i}{\partial \alpha_{i-1}^2}\right] \\ &= \alpha \left(\frac{\partial \alpha_i}{\partial \alpha_{i-1}}\right)^2 \frac{\partial^2 \alpha_i}{\partial \alpha_{i-1}^2} + \frac{\partial \alpha_i}{\partial \alpha_{i-1}} \left(2\frac{\partial \alpha_{i-1}}{\partial \alpha} + \frac{\partial^2 \alpha_{i-1}}{\partial \alpha^2}\right)\,. \end{aligned}$$

Using Equations (4), the Inequality (23) and the induction hypothesis we get :

$$\frac{\partial \alpha_i}{\partial \alpha_{i-1}} \; > \; 0\,, \; \frac{\partial^2 \alpha_i}{\partial \alpha_{i-1}^2} = \frac{\partial^2 (1-\alpha_i)^{-1}}{\partial \alpha_{i-1}^2} \; > \; 0 \;\; \text{and} \;\; 2 \cdot \frac{\partial \alpha_{i-1}}{\partial \alpha} + \frac{\partial^2 \alpha_{i-1}}{\partial \alpha^2} \; \geq \; 0\,,$$

therefore

$$2\frac{\partial\alpha_i}{\partial\alpha}+\alpha\frac{\partial^2\alpha_i}{\partial\alpha^2}\geq 0\,. \tag{28}$$

Therefore, we have

$$\begin{aligned}\frac{\partial^2(\alpha W_S(\overline{m}_\alpha))}{\partial\alpha} &= 2\frac{\partial W_S(\overline{m}_\alpha)}{\partial\alpha}+\alpha\frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial\alpha}\\ &= 2\frac{\partial W_S(\overline{m}_\alpha)}{\partial\alpha_i}\left(\frac{\partial\alpha_i}{\partial\alpha}\right)+\alpha\left[\left(\frac{\partial\alpha_i}{\partial\alpha}\right)^2\frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial\alpha_i^2}+\frac{\partial^2\alpha_i}{\partial\alpha^2}\frac{\partial W_S(\overline{m}_\alpha)}{\partial\alpha_i}\right]\\ &= \frac{\partial W_S(\overline{m}_\alpha)}{\partial\alpha_i}\left[2\frac{\partial\alpha_i}{\partial\alpha}+\alpha\frac{\partial^2\alpha_i}{\partial\alpha^2}\right]+\alpha\left(\frac{\partial\alpha_i}{\partial\alpha}\right)^2\frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial\alpha_i^2}\,.\end{aligned}$$

Inequalities (25), (27) and (28) give respectively

$$\frac{\partial W_S(\overline{m}_\alpha)}{\partial\alpha_i}<0\,,\ \frac{\partial^2 W_S(\overline{m}_\alpha)}{\partial\alpha_i^2}<0\ \text{ and }\ 2\frac{\partial\alpha_i}{\partial\alpha}+\alpha\frac{\partial^2\alpha_i}{\partial\alpha^2}\geq 0\,,$$

then

$$\frac{\partial^2(\alpha W_S(\overline{m}_\alpha))}{\partial\alpha}\leq 0\,.$$

□

To illustrate the results obtained in this section let us take an example.

Example 31 (Curve of $W_{5/4}(\alpha)$). *The mechanical expansion of* $4/5$ *is* $\langle 1,1,1,2\rangle$ *and* $\overline{m}_{4/5}$ *is equal to* 11110*. The factor decomposition is :* $x_1=1$ *and* $y_1=10$*, then* $x_2=x_1=1$ *and* $y_2=x_1y_1=110$*, then* $x_3=x_2=1$ *and* $y_3=x_2y_2=1110$*, then* $x_4=x_3y_3=11110=\overline{m}_{4/5}$*.*

We have $r_1(4/5)=\langle 2\rangle=1/2$*,* $r_2(4/5)=\langle 1,2\rangle=2/3$*,* $r_3(4/5)=\langle 1,1,2\rangle=3/4$ *and* $r_4(4/5)=\langle 1,1,1,2\rangle=4/5$*, therefore* $n_1=1$*,* $n_2=2$*,* $n_3=3$ *and* $n_4=4$*.*

Let us now compute K_i*, we obtain* $K_1=0$*,* $K_2=1/4$*,* $K_3=3/4$ *and* $K_4=3/2$*. Hence we have :*

$$\frac{K_1}{n_1}=0\,,\ \frac{K_2}{n_2}=\frac{1}{8}\,,\ \frac{K_3}{n_3}=\frac{1}{4}\,,\ \frac{K_4}{n_4}=\frac{3}{8}\,.$$

Recall that $\alpha=\langle l_1+\alpha_1\rangle=\langle l_1,l_2+\alpha_2\rangle=\langle l_1,l_2,l_3+\alpha_3\rangle$*. Formula* (18) *leads to*

$$\begin{aligned} W_S(\overline{m}_\alpha) &= 0 & &;\ \textit{for}\ \alpha\leq\frac{1}{2}\\ W_S(\overline{m}_\alpha) &= \frac{1-\alpha_1}{4} & &;\ \textit{for}\ \frac{1}{2}\leq\alpha\leq\frac{2}{3}\ \Leftrightarrow\ \frac{1}{2}\leq\alpha_1\leq 1\,.\\ W_S(\overline{m}_\alpha) &= \frac{2-\alpha_2}{4\,(\alpha_2+1)} & &;\ \textit{for}\ \frac{2}{3}\leq\alpha\leq\frac{3}{4}\ \Leftrightarrow\ \frac{1}{2}\leq\alpha_2\leq 1\,.\\ W_S(\overline{m}_\alpha) &= \frac{3}{4(2\alpha_3+1)} & &;\ \textit{for}\ \frac{3}{4}\leq\alpha\leq\frac{4}{5}\ \Leftrightarrow\ \frac{1}{2}\leq\alpha_3\leq 1\,.\end{aligned}$$

The function $W_S(\overline{m}_\alpha)$ *for all* $0\leq\alpha\leq 4/5$ *, is shown in figure 3.*

5 Average waiting time for two queues

5.1 Presentation of the Model

From now on as presented in figure 4, we will study a system made of two ./D/1/∞/FIFO queues, with constant service times S_1 in queue 1 and S_2 in queue 2. It is assumed for convenience in the following that

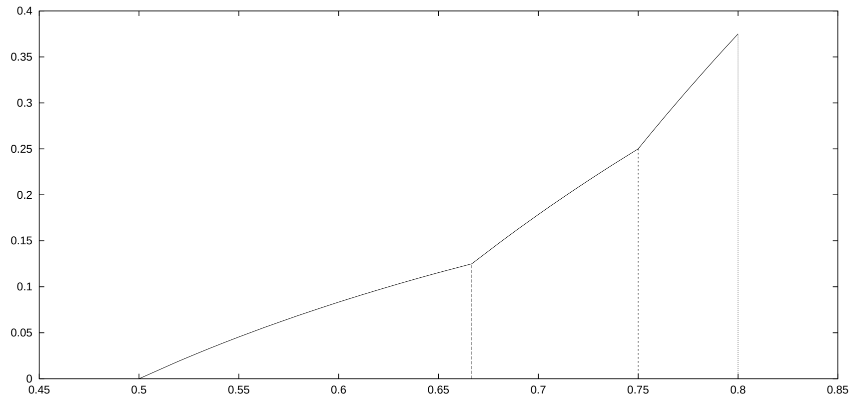

Figure 3: Curve of $W_{5/4}(\overline{m}_\alpha)$

S_2 is the smallest service time. The time unit is chosen such that inter-arrival time slots are constant and equal to one. When they arrive, the customers are routed to a queue where they wait to be treated in a FIFO order. The ratio of customers sent in queue 1 is α and therefore, the ratio of customers sent in queue 2 is $1-\alpha$ (*i.e.* no customer is lost). Our aim is to find a policy which minimizes the average waiting time of all the customers (which also minimizes the average response time of customers). The problem consists in finding an optimal allocation pattern and to find the optimal ratio (α,$1-\alpha$). The optimal allocation pattern was found in [1, 3, 2] but no way to compute the optimal ratio is provided there.

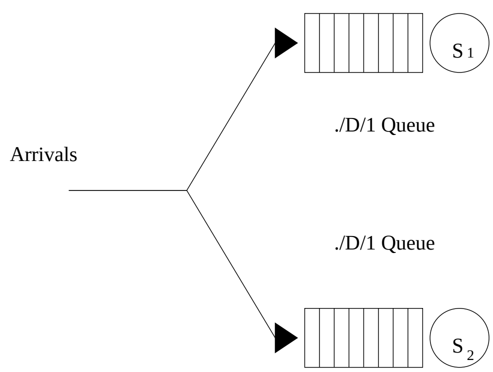

Figure 4: Admission Control in two ./D/1 queues

The input sequence w is given under the form of a binary sequence w such that when $w(n) = 1$ the n^{th} customer is sent in the first queue and when $w(n) = 0$ the n^{th} customer is sent in the second queue. The average waiting time of all the customers for an input sequence w is denoted $W_{S_1,S_2}(w)$.

5.2 Optimal Policies

To find an optimal allocation pattern we will use results shown in [2].

Theorem 32. *The average waiting time $W_{S_1,S_2}(w)$ is minimized for an lower mechanical input sequence $\underline{m}_{\alpha_{opt}}$, with some slope α_{opt}.*

Proof. This is a direct consequence of the multi-criteria optimization found in [2]. □

Note that the approach proposed in [2] is more general than the system considered here and works for any stationary services times and stationary inter-arrival times. However, it does not provide any mean to compute α_{opt}, which is what we are going to do here for deterministic queues.

Remark 33. *It can be shown that the average waiting time in a deterministic queue under an arrival process of the form $\underline{m}_\alpha$ is the same as the average waiting time when the arrival process is of the form $\overline{m}_\alpha$. Therefore, Theorem 33 applies with lower mechanical as well as upper mechanical sequences.*

This remark is essential here since when the input sequence in the first queue is an upper mechanical word of slope α ($\overline{m}_\alpha$), then the input sequence in the second queue is a lower mechanical word of slope $1-\alpha$ ($\underline{m}_{1-\alpha}$).

Using the considerations of Remark 34, and conditioning on the queue chosen for each customer the relation between $W_{S_1,S_2}(\overline{m}_\alpha)$, $W_{S_1}(\overline{m}_\alpha)$ and $W_{S_2}(\overline{m}_{1-\alpha})$ is

$$W_{S_1,S_2}(\overline{m}_\alpha) = \alpha \cdot W_{S_1}(\overline{m}_\alpha) + (1-\alpha) \cdot W_{S_2}(\overline{m}_{1-\alpha}) \, . \tag{29}$$

5.3 Optimal Ratio

We will now compute the optimal ratio α_{opt} over all possible stable ratios.

5.3.1 Stability Condition

Consider the system of the two ./D/1 queues above, the stability condition of such systems is $\rho \leq 1$, that is

$$\frac{1}{S_2} + \frac{1}{S_1} \geq 1 \, . \tag{30}$$

But the stability of the two queues individually is also necessary. Therefore by Equation (5) $\alpha \leq S_1^{-1}$ and $1-\alpha \leq S_2^{-1}$. Hence for any given number α in $\left[1-\frac{1}{S_2}\, , \, \frac{1}{S_1}\right]$ the input sequence $\overline{m}_\alpha$ is stable. The interval $\left[1-\frac{1}{S_2}\, , \, \frac{1}{S_1}\right]$ is called the interval of stability in the following.

5.3.2 Special Cases

This part considers special degenerated cases where the theory developed in the previous section is not necessary.

Lemma 34. *In the case $2 < S_2 \leq S_1$, no real optimal policy exists.*

Proof. The system is never stable since $S_1^{-1} \leq S_2^{-1} < 1/2$, implying

$$\left(\frac{1}{S_1} + \frac{1}{S_2}\right) < 1 \, .$$

□

Lemma 35. *In the case $S_2 \leq 1$, an optimal policy consists in sending all the customers in the second queue. In this case $W_{S_1,S_2}(\overline{m}_0) = 0$.*

Proof. When $S_2 \leq 1$, the service time is smaller than the inter-arrival time. Therefore each time a customer arrives the second queue is empty and we have $W_{S_2}(1) = 0$. Since $W_{S_1}(0) = 0$ then by Equation (29), $W_{S_1,S_2}(\overline{m}_0) = 0$. □

Lemma 36. *If $1 < S_2 \leq S_1 < 2$ then the round robin policy is an optimal policy.*

Proof. Lemma (22) implies $W_{S_1}(\overline{m}_{1/2}) = 0$ and $W_{S_2}(\overline{m}_{1/2}) = 0$. Therefore a possible optimal ratio is $\alpha = 1/2$ and $\overline{m}_{1/2} = 10$ is an optimal policy. □

5.3.3 Case $1 < S_2 < 2 < S_1$

This can be considered as the general case.

Characterization of optimal ratio when $\rho < 1$ We are interested to find the optimal ratio α_{opt} given by:

$$\alpha_{opt} = \min_{\alpha \in \left[1 - S_2^{-1}, S_1^{-1}\right]} W_{S_1,S_2}(\overline{m}_\alpha) \, .$$

Let r_i^1 and r_j^2 denote the jumps of W_{S_1} and W_{S_2} respectively.

Theorem 37. *For any real service time S_1 and S_2 the optimal ratio α_{opt} is a jump of W_{S_1} or W_{S_2}. Hence, it is a rational number and the optimal routing policy is periodic.*

Proof. By Lemma 31, the function $\alpha W_{S_1}(\overline{m}_\alpha)$ is concave for α in $]r_i^1, r_{i+1}^1[$ and the function $(1 - \alpha)W_{S_2}(\overline{m}_{1-\alpha})$ is also concave for α in $]1 - r_j^2, 1 - r_{j-1}^2[$. Therefore by Equation (29) the function W_{S_1,S_2} is concave for α in $]r_k^*, r_{k+1}^*[$. Where $\{r_k^*\} = \left(\{r_i^1\} \cup \{r_j^2\}\right) \cap [1 - 1/S_2, 1/S_1]$ is the set of jumps of W_{S_1,S_2}.

Hence the set of possible minima of W_{S_1,S_2} is all the jumps r_k^* or S_1^{-1} or S_2^{-1} when they are irrational. Theorem 41 says that the growing rates of the function $i \to \frac{K_i}{n_i}$ converges to infinity. This allows us to exclude the points S_1^{-1} and S_2^{-1} of the set of possible minima.

Therefore the minimum of W_{S_1,S_2} is a jump and since all the jumps are rational the optimal ratio α_{opt} is rational. This means that the optimal routing policy is periodic. □

Lemma 38. *Let p be the rational number with the smallest denominator in $\left[1 - S_2^{-1}, S_1^{-1}\right]$. Then p is the smallest jump of W_{S_1} in $[1 - 1/S_2, 1/S_1]$ and the greatest jump of W_{S_2} in $[1 - 1/S_2, 1/S_1]$. Since p is the unique common jump of W_{S_1} and W_{S_2}, we call p the double jump.*

Proof. Let r_i^1 be the greatest jump of $W_{S_1}(\overline{m}_\alpha)$ such that $r_i^1 < 1 - 1/S_2$. This implies $1 - 1/S_2 \le r_{i+1}^1 \le 1/S_1$. By Lemma 17 r_{i+1}^1 is the rational number with the smallest denominator in $]r_i^1, 1/S_1]$, then r_{i+1}^1 is the rational number with the smallest denominator in $[1 - 1/S_2, 1/S_1]$. Let r_j^2 be the greatest jump of $W_{S_2}(1 - \alpha)$ such that $1 - r_j^2 > 1/S_1$. This implies $1 - r_{j+1}^2$ is the greatest jump of $W_{S_2}(1 - \alpha)$ in $[1 - 1/S_2, 1/S_1]$ and the rational number with the smallest denominator in $[1 - 1/S_2, 1/S_1]$. Then $r_i^1 = 1 - r_{j+1}^2$. □

The double jump is not always the optimal ratio, as we will see in the example below.

Example 39 (Double jump is not optimal). *Let $S_1 = 21/5$ and $S_2 = 6/5$. The mechanical expansion of $5/21$ is $\langle 4, 1, 1, 1, 2\rangle$ and the one of $5/6$ is $\langle 1, 1, 1, 1, 2\rangle$. We have $r_1^1 = 1/5$, $r_2^1 = 2/9$, $r_3^1 = 3/13$, $r_4^1 = 4/17$, $r_5^1 = 1/S_1 = 5/21$ and $1 - r_1^2 = 1/2$, $1 - r_2^2 = 1/3$, $1 - r_3^2 = 1/4$, $1 - r_4^2 = 1/5$, $1 - r_5^2 = 1 - 1/S_2 = 1/6$. The double jump is $1/5$.*

The following numerical values have been obtained using exact computations provided by the program presented later (Section 5.4 and 6).

$$W_{S_1}(\overline{m}_{1/5}) = 0, \quad W_{S_2}(\overline{m}_{4/5}) = \frac{3}{10}, \qquad W_{S_1,S_2}(\overline{m}_{1/5}) = \frac{3}{10} * \frac{4}{5} = \frac{6}{25}.$$

$$W_{S_1}(\overline{m}_{2/9}) = \frac{1}{10}, \quad W_{S_2}(\overline{m}_{7/9}) = \frac{9}{35}, \quad W_{S_1,S_2}(\overline{m}_{2/9}) = \frac{1}{10} * \frac{2}{9} + \frac{9}{35} * \frac{7}{9} = \frac{2}{9}.$$

Since $\frac{2}{9} < \frac{6}{25}$, the optimal ratio is not the double jump. It can also be shown that the optimal ratio in this case is $\alpha_{opt} = \frac{2}{9}$.

Figure 5 displays the functions $W_{S_1}(\overline{m}_\alpha)$ (curve b*), $W_{S_2}(\overline{m}_{1-\alpha})$ (curve* c*) and $W_{S_1,S_2}((\overline{m}_\alpha)$ (curve* a*), as well as some jumps of those three curves (the double jump 1/5 is marked by the dotted line* d*, and the optimal jump 2/9 by the dotted line* e*). The vertical lines* f *show the interval of stability of the system, $W_{S_1,S_2}(\overline{m}_\alpha)$ is infinite outside this interval.*

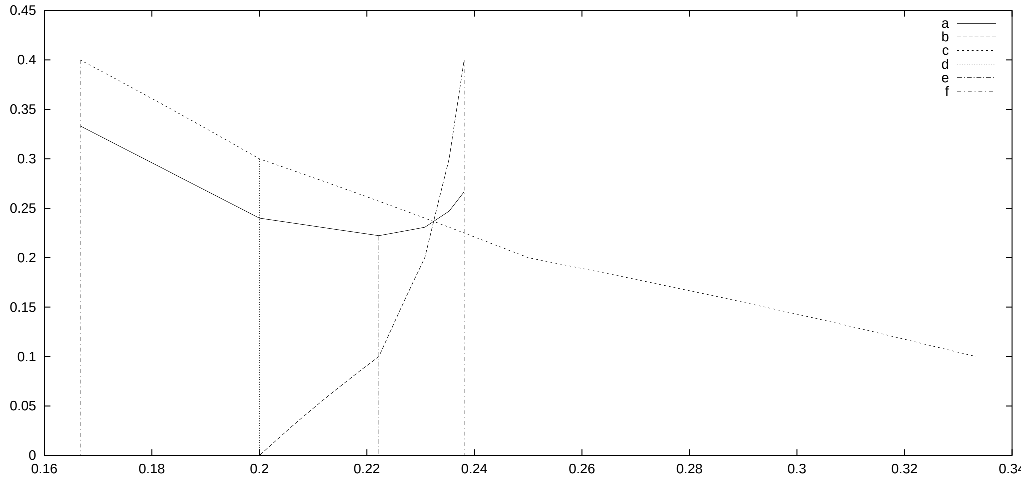

Figure 5: Curves of W_{S_1}, W_{S_2} and W_{S_1,S_2}.

Characterization of optimal ratio when $\rho = 1$ In this part, we will assume that the system is fully loaded. When $\rho = 1$ then the stability interval is reduced to a single point. There is just one ratio α satisfying the stability condition, $\alpha = S_1^{-1}$. Therefore the upper mechanical word with slope α is optimal. This is the only case where the optimal policy may be aperiodic.

5.4 Algorithm and computational issues

We present in Figure 6 an algorithm to compute the optimal ratio α_{opt}, when $\rho \leq 1$.

> Find double jump p
> current-jump := p
> Compute the next-jump-right of p
> **while** $W_{S_1,S_2}(\overline{m}_{\text{current-jump}}) > W_{S_1,S_2}(\overline{m}_{\text{next-jump-right}})$ **do**
>   current-jump:= next-jump-right
>   Compute the next-jump-right of current-jump
> **endwhile**
> Compute the next-jump-left of p
> **while** $W_{S_1,S_2}(\overline{m}_{\text{current-jump}}) > W_{S_1,S_2}(\overline{m}_{\text{next-jump-left}})$ **do**
>   current-jump:= next-jump-left
>   compute the next-jump-left of current-jump
> **endwhile**
> **return** current-jump

Figure 6: Algorithm computing α_{opt}

5.4.1 Correctness of the algorithm

We will now show that the algorithm is correct and converges to the result in a finite number of steps. For that we have to prove two preliminary lemmas.

Lemma 40. *The growing rate of the function $i \to \frac{K_i}{n_i}$ converges to infinity.*

Proof. We show first by induction that $\forall i \geq 1$,

$$n_{i+1}q_i - n_i q_{i+1} = 1 \,. \tag{31}$$

At step one : $n_2 q_1 - n_1 q_2 = (l_2+1)(l_1+1) - (l_1(l_2+1)+l_2) = 1$.
At step $i+1$: $n_{i+1}q_i - n_i q_{i+1} = (l_{i+1}+1)q_i n_i - q_i n_{i-1} - ((l_{i+1}+1)q_i n_i - n_i q_{i-1}) = n_i q_{i-1} - n_{i-1} q_i$.
Hence the growing rate

$$\begin{aligned}\left(\frac{K_{i+1}}{n_{i+1}} - \frac{K_i}{n_i}\right)\left(\frac{n_{i+1}}{q_{i+1}} - \frac{n_i}{q_i}\right)^{-1} &= \frac{(n_i K_{i+1} - n_{i+1}K_i)(q_{i+1}q_i)}{n_{i+1}n_i} \\ &\geq (n_i K_{i+1} - n_{i+1}K_i) \\ &\geq (n_{i-1}K_i - n_i K_{i-1}) + n_i(n_i - n_{i-1})d_i^1 \\ &\geq \Sigma_{1\leq k\leq i}\left(n_k(n_k - n_{k-1})d_k^1\right) .\end{aligned}$$

Let us prove now that $\Sigma_{1\leq k}\left(n_k(n_k - n_{k-1})d_k^1\right) = +\infty$. We use Equations (8) that yield

$$n_k(n_k - n_{k-1})d_k^1 = n_k(n_k - n_{k-1})\left[(1-\alpha_1)(1-\alpha_2)\dots(1-\alpha_{k-1})\right]\alpha_k .$$

By Equation (12) we obtain

$$\begin{aligned}n_i(n_i - n_{i-1}) &= \left[(l_i+1)n_{i-1} + (n_{i-1} - n_{i-2})\right]\left[(l_i)n_{i-1} + (n_{i-1} - n_{i-2})\right] \\ &\geq \left[l_i + 1 + l_i + l_i(l_i+1)\right] n_{i-1}(n_{i-1} - n_{i-2}) .\end{aligned}$$

Hence

$$\begin{aligned}n_k(n_k - n_{k-1})d_k^1 &\geq \frac{\left((l_k+1)^2 + l_k\right)\left((l_{k-1}+1)^2 + l_{k-1}\right)\dots\left((l_2+1)^2 + l_2\right)\alpha_k}{l_2 l_3 \dots l_{k-1}} \\ &\geq \left((l_k+3)\right)\left((l_{k-1}+3)\right)\dots\left((l_2+3)\right)\alpha_k \geq 4^{k-1}\alpha_k .\end{aligned}$$

Two cases may occur, either the sequence $\{\alpha_k\}_{k\geq 0}$ does not converge to zero or it converges to zero. In the first case the sequence $n_k(n_k - n_{k-1})d_k^1$ does not converges to zero and the series $\Sigma_{k\leq 0} n_k(n_k - n_{k-1})d_k^1$ goes to infinity. In the other case there exists a number N such that $\forall k \geq N$, $\alpha_k < 1/2$. Since $\alpha_k < 1/2$ implies $l_{k+1} = 1$ and $(1-\alpha_k)\alpha_{k+1} = \alpha_k$,

$$\frac{n_{k+1}(n_{k+1} - n_k)d_{k+1}^1}{n_k(n_k - n_{k-1})d_k^1} = \frac{n_{k+1}(n_{k+1} - n_k)}{n_k(n_k - n_{k-1})} > 1$$

Which implies that the sequence is strictly increasing and that the series diverges to infinity. □

Lemma 41. *The function $i \to \frac{K_i}{q_i}$ is convex.*

Proof. The recursive computation of $q_i = |\overline{m}_{r_i}|$ is given by :

$$q_0 = 0\,,\; q_1 = l_1\,,\; \forall i \geq 1\,,\; q_i = (l_i+1)q_{i-1} - q_{i-2} .$$

We have by Equations (16) and (31)

$$\begin{aligned}&\left(\frac{K_{i+1}}{q_{i+1}} - \frac{K_i}{q_i}\right)\left(\frac{n_{i+1}}{q_{i+1}} - \frac{n_i}{q_i}\right)^{-1} - \left(\frac{K_i}{q_i} - \frac{K_{i-1}}{q_{i-1}}\right)\left(\frac{n_i}{q_i} - \frac{n_{i-1}}{q_{i-1}}\right)^{-1} \\ &\qquad = \left(\frac{K_{i+1}}{q_{i+1}} - \frac{K_i}{q_i}\right)(q_{i+1}q_i) - \left(\frac{K_i}{q_i} - \frac{K_{i-1}}{q_{i-1}}\right)(q_i q_{i-1}) \\ &\qquad = q_i K_{i+1} - q_{i+1}K_i - q_{i-1}K_i + q_i K_{i-1} .\end{aligned}$$

Using Equation (16) leads to

$$\begin{aligned}q_i K_{i+1} - q_{i+1}K_i &= q_i\left((l_{i+1}+1)K_i - K_{i-1}\right) - q_{i+1}K_i \\ &\qquad + q_i\left(n_i(l_{i+1}-1)(d_i^1 + \frac{l_{i+1}}{2}d_i^2) + (n_i - n_{i-1})d_i^1\right) \\ &= q_i\left((l_{i+1}+1)K_i - K_{i-1}\right) - (l_{i+1}q_i - q_{i-1})\,K_i \\ &\qquad + q_i\left(n_i(l_{i+1}-1)(d_i^1 + \frac{l_{i+1}}{2}d_i^2) + (n_i - n_{i-1})d_i^1\right) \\ &= q_{i-1}K_i - q_i K_{i-1} + q_i\left(n_i(l_{i+1}-1)(d_i^1 + \frac{l_{i+1}}{2}d_i^2) + (n_i - n_{i-1})d_i^1\right) .\end{aligned}$$

Hence

$$q_iK_{i+1} - q_{i+1}K_i - \left(q_{i-1}K_i - q_iK_{i-1}\right) = q_i\Big(n_i(l_{i+1}-1)(d_i^1 + \frac{l_{i+1}}{2}d_i^2) + (n_i - n_{i-1})d_i^1\Big)\,.$$

As shown in proof of Lemma 27,

$$n_i(l_{i+1}-1)(d_i^1 + \frac{l_{i+1}}{2}d_i^2) + (n_i - n_{i-1})d_i^1 \geq 0\,. \tag{32}$$

This proves the convexity of the function. □

Lemma 42. *The algorithm is correct and converges in a finite number of steps*

Proof. Correctness. Since by Lemma 42 the function $i \to \frac{K_i}{q_i}$ is convex, then by Equation (29) the function $k \to W_{S_1,S_2}(\overline{m}_{r_k^*})$ is also convex. Considering that $\alpha_{opt} \in \{r_k^*\}$ shows the correctness of the algorithm.

Finiteness. Let k, i, j be the integers such that $p = r_k = r_i^1 = 1 - r_j^2$. Let us note that $1 - r_j^2 \leq S_1 < 1 - r_{j-1}^2$. Since $\forall n \geq 1,\ r_{k+n}^* = r_{i+n}^1$, and since by Lemma 41 the growing rate of the function $i \to W_{S_1}(\overline{m}_{r_i^1})$ goes to infinity then by Equation (29) $\forall n \geq 1$ the growing rate of the function $k+n \to W_{S_1,S_2}(\overline{m}_{r_{k+n}^*})$ converges to infinity. Therefore the integer $k_0 = k + n_0$ such that $\forall n \geq n_0$

$$W_{S_1,S_2}(\overline{m}_{r_{k+n}^*}) - W_{S_1,S_2}(\overline{m}_{r_{k+n-1}^*}) > 0\,,$$

is finite. Using similar arguments for the jumps smaller than p shows that the algorithm converges in a finite number of steps. □

6 Numerical experiments

The algorithm presented above has been implemented in Maple in order to keep exact values for all the rational numbers involved in the computations. This section is dedicated to the presentation of several runs of the program in order to shown how the optimal policy (or equivalently the ratio of the optimal policy) behaves with respect to the parameters of the system, namely S_1, the service time in queue 1, S_2, the service time in queue 2 as well as the inter-arrival time (fixed to one previously, but which can be modified by scaling the time units).

In the first series of computations, we fix $S_1 = 22/5$ and $S_2 = 6/5$. Therefore, the fastest server, S_2 is c times faster than S_1, with $c = 11/3$. One could expect that the optimal routing policy sends c times more customers in the second queue than in the first queue. This policy has a ratio $\alpha = \frac{1}{c+1}$, namely $\alpha = 3/14$. However, as the experiments in Figure 7 show, the optimal ratio is $\alpha_{opt} = 1/5$ when the inter-arrival time is one.

In Figure 7, with $S_1 = 22/5$ and $S_2 = 6/5$, we let the inter-arrival time vary so that the total load ρ goes from 0 to 1. All the results presented in the figure are exact computations and do not suffer from any numerical errors. When ρ is smaller than $11/14$ (dotted line on the left), many ratios are optimal. For instance, for all $\alpha < 1/4$, $W_{S_1,S_2}(\overline{m}_\alpha) = 0$. This part is not shown in the figure. The main interest lies within the bounds $11/14 \leq \rho \leq 1$ where the optimal ratio is unique. The optimal ratio α_{opt} takes several rational values, ranging from $1/4$ to $1/5$ and ending with the intuitive ratio of the service times, $3/14$. When $\rho = 1$, then $\alpha = 3/14$ is the only point in the stability region. However, for lower values of the load, it is somewhat surprising that the optimal ratio changes with the load and deviates from the ratio of the service times, $3/14$. The changes of the optimal ratio occur according to several reasons. For example, α_{opt} moves away from $1/4$ only when $1/4$ gets out of the stability region (this occurs at $\rho = 6/7$). The same phenomenon occurs when α_{opt} jumps away from $1/5$, which occurs when $\rho = 55/56$. On the other hand, the change from 2/9 to 1/5 occurs while 2/9 is still a jump in the stability region. In general, the sudden changes of the optimal ratio when the load increases remain mysterious and deserve further studies.

The second set of computations is presented in Figure 8. Here, the inter-arrival time is fixed to one, and the service time in the first queue is fixed to $S_1 = 51/25 > 2$. We let S_2 vary from 1 to 51/26 and compute the optimal ratio.

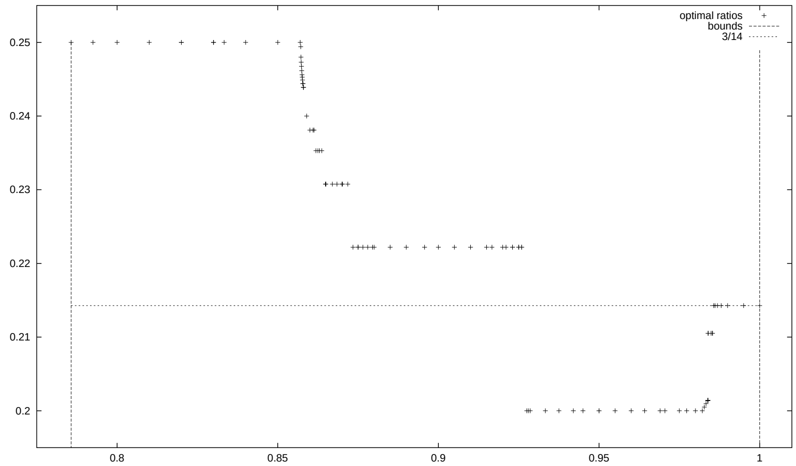

Figure 7: Optimal ratio when the total load varies

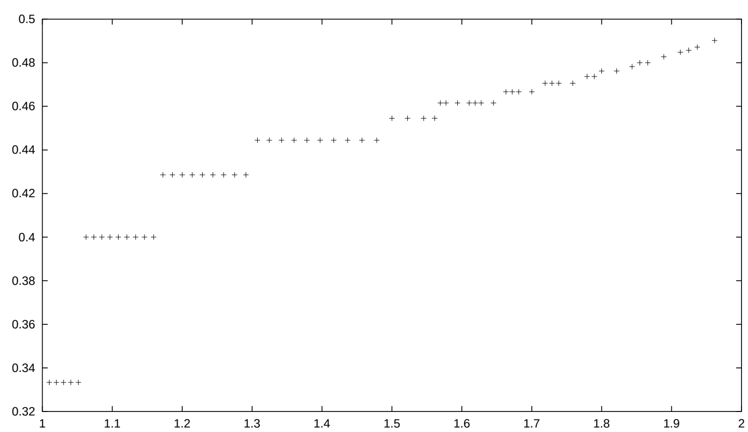

Figure 8: Optimal ratio when the service time S_2 varies

The Figure shows that α_{opt} increases to 1/2 while S_2 approaches S_1, which is natural. It also illustrates the high non-linearity of the behavior of the optimal ratio with respect to the parameters of the system.

In total, the Maple program computing α_{opt} was run hundreds of times, with many rational and irrational parameters. The maximal number of steps before finding the optimal jump never went over 15, which confirms experimentally the intuition that the optimal jump should be in the neighborhood of the double jump in general. However, we believe that one can come up with well chosen parameters for which the optimal jump is arbitrarily far (in number of jumps) from the double jump.

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Unité de recherche INRIA Lorraine
LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)

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