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*Cuts and flows in infinite periodic graphs;  
application to the minimization of circuit registers*

Bruno Gaujal — Jean Mairesse

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*Rapport  
de recherche*





## Cuts and flows in infinite periodic graphs; application to the minimization of circuit registers

Bruno Gaujal\* , Jean Mairesse†

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**Abstract:** In this paper, we prove that bi-infinite and causal periodic graphs admit periodic maximal flows and consecutive minimal cuts. By viewing such a graph as the unfolding of the dependences in a digital circuit, we also show how this can be used for register minimization in circuit design.

**Key-words:** Graph theory, digital circuits

\* ENSEM/ INRIA Lorraine, rue du jardin botanique, B.P. 101, 54602 Villers-les-Nancy Cedex, France. Email: gaujal@loria.fr.

† LIAFA, CNRS-Université Paris 7, Case 7014, 2 place Jussieu, 75251 Paris Cedex 05, France. Email: mairesse@liafa.jussieu.fr.

# Coupes et flots dans les graphes périodiques infinis; application à la minimisation des registres dans les circuits digitaux

**Résumé :** Dans cet article, nous prouvons qu'un graphe périodique bi-infini et causal admet un flot maximal périodique et une coupe minimale consécutive. En considérant un tel graphe comme le dépliage des dépendances dans un circuit digital, nous montrons également comment cela peut être utilisé pour minimiser le nombre de registres dans la conception de circuits digitaux.

**Mots-clés :** théorie des graphes, circuits digitaux

## 1 Introduction

One-periodic and causal bi-infinite graphs (formally defined in section 2) have been heavily studied because they can be seen as dependence graphs of Uniform Recurrence Equations or as dependence graphs of digital circuits or PRAM program loops (see [1, 3, 5, 6, 7] for examples of the use of such graphs).

Here, we prove that in such a graph, the size of a maximal flow (made of node disjoint paths) equals the size of a minimal cut. Furthermore, there exists a maximal flow which is one-periodic and a minimal cut which is consecutive.

A synchronous circuit can be viewed as a finite arc-valued graph (the valuations correspond to the registers of the circuit), whose unfolding is a one-periodic and causal bi-infinite graph. We say that a circuit  $\mathcal{R}_1$  has a larger computational power than a circuit  $\mathcal{R}_2$ , if any sequence of data which can be computed using  $\mathcal{R}_2$  can also be computed using  $\mathcal{R}_1$  (precise definitions are given in § 4). Our results on bi-infinite graphs have the following consequence. Let us consider a circuit  $\mathcal{R}$  with unfolding  $\mathcal{D}$ . The minimal number of registers of a circuit with a computational power equal or larger than the one of  $\mathcal{R}$  is equal to the size of a minimal cut of  $\mathcal{D}$ . We also propose an explicit way of constructing a circuit with minimal number of registers. This improves on some classic results by Leiserson and Saxe on the retiming of circuits [8].

The current paper is a follow-up to [5] in the sense that it extends most of the results established in [5] to the case of one-periodic bi-infinite graphs (whereas only recycled graphs were considered in [5]). The reader could refer to [5] for more detailed introductory comments and motivation purposes. Although most of the results from [5] remain true in the general case, the methods used for the proofs are completely different. In [5], an extensive use of pebble games defined on the recycled graph is made. There is no natural extension nor real need for these games in the general case, and we do not consider them at all. Here, on the other hand, we introduce and use extensively a new concept: the ‘splits’ of a bi-infinite graph.

The paper is structured as follows. Section 2 introduces the notations and some definitions. In particular one-periodic and causal bi-infinite graphs are defined as well as the notion of splits. In section 3, many properties of splits are given, the most important one being the existence of a strongly consecutive minimal border split. The ‘maxflow - mincut’ result described above is also proved in this section. Finally section 4 shows how these results from graph theory can be used in the design of digital circuits to minimize the number of registers.

## 2 Notations and Definitions

Let  $\mathcal{D} = (\mathcal{V}, \mathcal{A})$  be a bi-infinite directed graph with set of nodes  $\mathcal{V} = (V \times \mathbb{Z})$ , where  $V$  is a finite set, and set of arcs  $\mathcal{A} \subset \mathcal{V} \times \mathcal{V}$ . We write  $(i, n) \rightarrow (j, m)$  if there is an arc from  $(i, n)$  to  $(j, m)$ . The graph  $\mathcal{D}$  is assumed to be one-periodic, that is:  $(i, n) \rightarrow (j, m) \iff (i, n+1) \rightarrow (j, m+1)$ ; and we assume that all the nodes have finite in and out-degree. The  $i$ -th *line* in  $\mathcal{D}$  is the set of nodes  $\{(i, n), n \in \mathbb{Z}\}$ ; the  $n$ -th *column* in  $\mathcal{D}$  is the set of nodes  $\{(i, n), i \in V\}$ . Throughout the paper, an integer  $n$  is said to be positive (resp. negative) if  $n \geq 0$  (resp.  $n \leq 0$ ), and strictly positive (resp. strictly negative) if  $n > 0$  (resp.  $n < 0$ ). A node  $v = (i, n)$  is positive (resp. strictly positive, negative, strictly negative) if  $n$  is positive (resp. strictly positive, negative, strictly negative).

A *path* in  $\mathcal{D}$  is a subgraph of  $\mathcal{D}$  with nodes  $\{v_i, i \in I\}$  (with  $I = \{0, \dots, n\}$  or  $I = \mathbb{N}$  or  $I = \mathbb{Z}^-$  or  $I = \mathbb{Z}$ ) and arcs  $v_i \rightarrow v_{i+1}$  for all  $i$  such that  $i, i+1 \in I$ . We use the notation  $\dots \rightarrow v_i \rightarrow v_{i+1} \rightarrow \dots$ , to denote the path. A path is respectively *finite*, *positive-infinite*, *negative-infinite*, or *bi-infinite*, if  $I = \{0, \dots, n\}$ ,  $I = \mathbb{N}$ ,  $I = \mathbb{Z}^-$ , or  $I = \mathbb{Z}$ . The *length* of a finite path is the number of its arcs. For two nodes  $u$  and  $v$ , the notation  $u \xrightarrow{*} v$  means that there exists a finite path in  $\mathcal{D}$  starting in  $u$  and ending in  $v$ , the notation  $u \xrightarrow{P} v$  means that the path  $P$  starts in  $u$  and ends in  $v$ . It is often convenient to identify a path with

the set of its nodes. Below, a path in  $\mathcal{D}$  is either a subgraph of  $\mathcal{D}$  or a subset of  $\mathcal{V}$ , depending on the context.

A *flow* is a set of node-disjoint bi-infinite paths. A *cut* is a set of nodes which intersects every bi-infinite path.

For simplicity, if  $v = (i, n)$  is a node in  $\mathcal{V}$ , we use the notation  $v + k$  for the node  $(i, n + k)$ . The notation will also be used for sets of nodes, and for paths: if  $P$  is the path  $\cdots \rightarrow v_i \rightarrow v_{i+1} \rightarrow \cdots$  then  $P + k$  is the path  $\cdots \rightarrow (v_i + k) \rightarrow (v_{i+1} + k) \rightarrow \cdots$ .

The graph  $\mathcal{D}$  has the *causality property* if it contains at least one bi-infinite path and if all the positive-infinite paths contain a finite number of negative nodes. It implies in particular that the graph  $\mathcal{D}$  is acyclic. Under the causality assumption, Farkas Lemma states that it is possible to perform a renumbering of the nodes such that all the arcs point from left to right. More precisely, there exists  $c_i \in \mathbb{Z}, i \in V$ , such that for  $(i, n) \rightarrow (j, m) \in \mathcal{A}$ , we have  $(m + c_j) - (n + c_i) \geq 0$ . Let  $\mathcal{D}' = (\mathcal{V}, \mathcal{A}')$  be the graph with set of arcs defined by  $(i, n) \rightarrow (j, m) \in \mathcal{A}'$  if  $(i, n - c_i) \rightarrow (j, m - c_j) \in \mathcal{A}$ ; the arcs in  $\mathcal{D}'$  are pointing from left to right, that is  $(i, n) \rightarrow (j, m) \implies m \geq n$ . This renumbering actually amounts to a ‘retiming’ of the system, see § 4.

>From now on, we always assume that  $\mathcal{D}$  verifies the causality property, and we assume without loss of generality that all the arcs are pointing from left to right.

**Definition 2.1.** A positive (negative) split  $S$  is a subset of  $\mathcal{V}$  such that for any bi-infinite path  $P$ ,  $S \cap P$  contains a positive infinite path and no negative infinite path.

For instance the set  $\{(u, n), u \in V, n \in \mathbb{N}\}$  is a positive split and the set  $\{(u, -n), u \in V, n \in \mathbb{N}\}$  is a negative split.

Let  $S$  and  $T$  be two subsets of  $\mathcal{V}$ . We define:

- $\gamma^+(S) \stackrel{\text{def}}{=} \{v \in \mathcal{V} \setminus S \mid \exists u \in S, u \rightarrow v\}$ ;
- $\gamma^-(S) \stackrel{\text{def}}{=} \{v \in \mathcal{V} \setminus S \mid \exists u \in S, v \rightarrow u\}$ ;
- $\Gamma^+(S) \stackrel{\text{def}}{=} \{v \in \mathcal{V} \setminus S \mid \exists u \in S, u \overset{*}{\rightarrow} v\}$ ;
- $\Gamma^-(S) \stackrel{\text{def}}{=} \{v \in \mathcal{V} \setminus S \mid \exists u \in S, v \overset{*}{\rightarrow} u\}$ ;
- $\Gamma^{+\infty}(S) \stackrel{\text{def}}{=} \{v \in \mathcal{V} \setminus S \mid \exists u \in S, u \overset{*}{\rightarrow} v, \exists P \text{ positive-infinite path}, v \in P, P \cap S = \emptyset\}$ ;
- $\Gamma^{-\infty}(S) \stackrel{\text{def}}{=} \{v \in \mathcal{V} \setminus S \mid \exists u \in S, v \overset{*}{\rightarrow} u, \exists P \text{ negative-infinite path}, v \in P, P \cap S = \emptyset\}$ ;
- $\Gamma^-(S, T) \stackrel{\text{def}}{=} \{v \in \mathcal{V} \setminus S \mid \exists u \in S, v \overset{*}{\rightarrow} u, \exists P, v \xrightarrow{P} u, P \cap T = \emptyset\}$ .

By definition, we have  $\gamma^+(S) \subset \Gamma^+(S)$  and  $\Gamma^{+\infty}(S) \subset \Gamma^+(S)$ . We also have  $\gamma^-(S) \subset \Gamma^-(S)$  and  $\Gamma^{-\infty}(S) \subset \Gamma^-(S)$ . We check that if  $S$  is a positive (negative) split, then  $\Gamma^{+\infty}(S)$  ( $\Gamma^{-\infty}(S)$ ) is empty.

**Lemma 2.2.** Let  $S$  be a positive (negative) split, then  $\gamma^-(S)$  ( $\gamma^+(S)$ ) is a cut.

*Proof.* We consider a negative split  $S$ , the proof for positive splits is similar. Let  $P$  be a bi-infinite path. By definition,  $S \cap P$  contains a negative-infinite path. On the other hand,  $P$  is not included in  $S$  since  $S \cap P$  can not contain a positive-infinite path. Let  $u$  and  $v$  be consecutive nodes of  $P$  such that  $u \in S, v \notin S$ . Then, by definition  $v$  belongs to  $\gamma^+(S)$ .  $\square$

Since  $\mathcal{D}$  contains at least one bi-infinite path, it implies that there is no empty cut in  $\mathcal{D}$ . Also since the graph is one-periodic and since the nodes have a finite degree, it implies the existence of finite cuts. A finite cut with a minimal number of nodes is called a *minimal cut*.

**Definition 2.3.** A positive (resp. negative) Minimal Border Split (MBS) is a positive (resp. negative) split  $S$  such that  $\gamma^-(S)$  (resp.  $\gamma^+(S)$ ) is a minimal cut.

The fact that a MBS always exists is proved below in Proposition 3.2. The different notions introduced so far are illustrated in Figure 1.

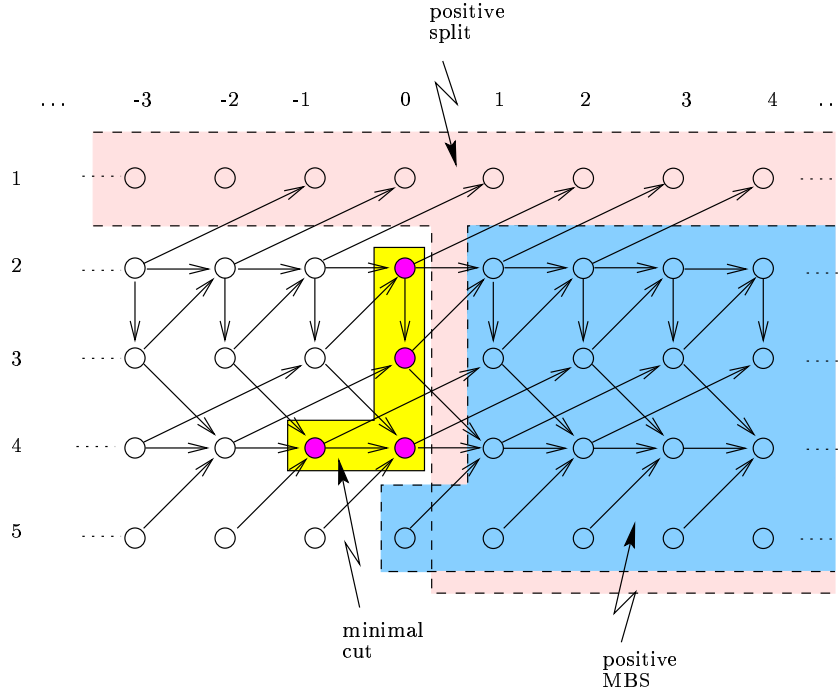


Figure 1: Split, MBS and minimal cut.

### 3 A Maxflow - Mincut Result

We denote by  $\chi(\mathcal{D})$  the cardinality of a minimal cut in  $\mathcal{D}$ . We define  $H(\mathcal{D}) = \max\{\gamma \mid \exists i, j, (i, 0) \rightarrow (j, \gamma)\}$ , the maximal ‘depth’ of an arc in  $\mathcal{D}$ . Next lemma is a converse of Lemma 2.2.

**Lemma 3.1.** Let  $C$  be a finite cut then  $\Gamma^{+\infty}(C)$  ( $\Gamma^{-\infty}(C)$ ) is a positive (negative) split.

*Proof.* Let  $P$  be a bi-infinite path. It intersects the cut  $C$ . Since  $C$  is finite, there exists a last node  $u$  at which  $P$  intersects  $C$ , and let  $v$  be the successor of  $u$  in  $P$ . The sub-path of  $P$  starting in  $v$  is by construction included in  $\Gamma^{+\infty}(C)$ . We conclude that  $\Gamma^{+\infty}(C) \cap P$  contains a positive-infinite path.

Let  $n$  be the smallest index of a column intersected by  $C$ . Since all the arcs point from left to right, a successor of a node in  $C$  belongs to a column with index larger or equal than  $n$ . In particular, there is only a finite number of negative nodes in  $\Gamma^{+\infty}(C)$ . As a consequence,  $\Gamma^{+\infty}(C) \cap P$  can not contain a negative-infinite path. □

**Proposition 3.2.** The graph  $\mathcal{D}$  contains positive and negative MBSs.

*Proof.* Let us prove the existence of positive MBSs, the negative case being similar. Let  $C$  be a minimal cut. We define

$$S \stackrel{\text{def}}{=} \Gamma^{+\infty}(C) \cup \Gamma^-(\Gamma^{+\infty}(C), C). \tag{1}$$



Let us prove first that  $S$  is a positive split. First of all, note that  $S$  does not intersect  $C$  by definition. Let  $P$  be a bi-infinite path. By the same argument as in Lemma 3.1,  $P \cap \Gamma^{+\infty}(C)$  contains a positive-infinite path. Now assume that  $P \cap S$  contains a negative-infinite path  $P^-$  and let  $a$  be a node in  $P^-$ . By definition of  $S$ , there exists a positive-infinite path  $Q$  included in  $S$  and containing  $a$ . The set  $P^- \cup Q$  contains a bi-infinite path which does not intersect  $C$ . This contradicts the fact that  $C$  is a cut. We conclude that  $S$  is a split.

Consider  $u \in \gamma^-(S)$ , we have  $u \in \gamma^-(\Gamma^{+\infty}(C)) \cup \gamma^-(\Gamma^-(\Gamma^{+\infty}(C), C))$ . If  $u$  belongs to  $\gamma^-(\Gamma^{+\infty}(C))$ , it implies that we have either  $u \in C$  or  $u \in \Gamma^-(\Gamma^{+\infty}(C), C)$ , hence it implies that  $u$  belongs to  $S$  which is a contradiction. We conclude that  $u$  belongs to  $\gamma^-(\Gamma^-(\Gamma^{+\infty}(C), C))$ , which implies that  $u$  belongs to  $C$ . Hence we have  $\gamma^-(S) \subset C$ . Now  $\gamma^-(S)$  is a cut by Lemma 2.2 and we conclude that  $\gamma^-(S) = C$  by minimality of  $C$ . Therefore  $S$  is a MBS.  $\square$

If  $S$  is a negative split, we set  $\overline{S} \stackrel{\text{def}}{=} \mathcal{V} \setminus (S \cup \gamma^+(S))$ . If  $S$  is a positive split, we set  $\overline{S} \stackrel{\text{def}}{=} \mathcal{V} \setminus (S \cup \gamma^-(S))$ .

**Proposition 3.3.** *The set  $S$  is a positive MBS if and only if  $\overline{S}$  is a negative MBS. In this case, we have  $\gamma^-(S) = \gamma^+(\overline{S})$  and we have  $\overline{\overline{S}} = S$ .*

*Proof.* Let  $S$  be a positive MBS. Let us prove first that  $\overline{S}$  is a negative split. We remark that  $\mathcal{V}$  is the disjoint union of  $S, \overline{S}$  and  $\gamma^-(S)$ . Let  $P$  be a bi-infinite path. By definition,  $S \cap P$  contains a positive-infinite path. Therefore,  $\overline{S} \cap P$  can not contain a positive-infinite path.

Let  $P^-$  be a negative-infinite path included in  $P$ , then  $S \cap P^-$  is finite. Since  $\gamma^-(S)$  is finite,  $\overline{S} \cap P^-$  contains a negative-infinite path. We conclude that  $\overline{S}$  is a negative split. It remains to be shown that  $\gamma^+(\overline{S})$  is a minimal cut.

Assume that  $\gamma^+(\overline{S})$  is not included in  $\gamma^-(S)$  and let  $a$  belong to  $\gamma^+(\overline{S}) \setminus \gamma^-(S)$ . We have  $a \in S$  (since  $S \cup \overline{S} \cup \gamma^-(S) = \mathcal{V}$ ), but then,  $\gamma^-(\{a\}) \subset \gamma^-(S)$  intersects  $\overline{S}$ , which is impossible by definition of  $\overline{S}$ .

We conclude that we have  $\gamma^+(\overline{S}) \subset \gamma^-(S)$ . Since  $\gamma^-(S)$  is a minimal cut and since  $\gamma^+(\overline{S})$  is a cut by Lemma 2.2, it implies that  $\gamma^+(\overline{S}) = \gamma^-(S)$ . It follows that  $\overline{\overline{S}} = S$ .  $\square$

We remark that if  $S$  is only a split, then  $\overline{S}$  need not be a split.

**Proposition 3.4.** *The intersection of two positive (negative) splits is a positive (negative) split. The intersection of two positive (negative) MBSs is a positive (negative) MBS.*

*Proof.* For splits, the property is a direct consequence of the definition. We focus on negative MBSs, the positive case being similar. Let  $S_1$  and  $S_2$  be two negative MBSs. We have

$$\gamma^+(S_1 \cap S_2) \subset (S_1 \cap \gamma^+(S_2)) \cup (\gamma^+(S_1) \cap S_2) \cup (\gamma^+(S_1) \cap \gamma^+(S_2)). \quad (2)$$

Similarly, we have

$$\gamma^-(\overline{S_1} \cap \overline{S_2}) \subset (\overline{S_1} \cap \gamma^-(\overline{S_2})) \cup (\gamma^-(\overline{S_1}) \cap \overline{S_2}) \cup (\gamma^-(\overline{S_1}) \cap \gamma^-(\overline{S_2})). \quad (3)$$

By Proposition 3.3, we have  $\gamma^-(\overline{S_i}) = \gamma^+(S_i)$ ,  $i = 1, 2$ . By replacing in Equation (3), we get

$$\gamma^-(\overline{S_1} \cap \overline{S_2}) \subset (\overline{S_1} \cap \gamma^+(S_2)) \cup (\gamma^+(S_1) \cap \overline{S_2}) \cup (\gamma^+(S_1) \cap \gamma^+(S_2)). \quad (4)$$

We add the two inequalities obtained by taking the cardinals in Equations (2) and (4). Using that  $(S_i, \gamma^+(S_i), \overline{S_i})$  is a partition of the nodes in  $\mathcal{D}$ , and that  $S_1$  and  $S_2$  are negative MBSs, we obtain

$$|\gamma^+(S_1 \cap S_2)| + |\gamma^-(\overline{S_1} \cap \overline{S_2})| \leq |\gamma^+(S_1)| + |\gamma^+(S_2)| = 2\chi(\mathcal{D}). \quad (5)$$

We know that  $\overline{S_1} \cap \overline{S_2}$  is a positive split and that  $S_1 \cap S_2$  is a negative split. Therefore,  $\gamma^-(\overline{S_1} \cap \overline{S_2})$  and  $\gamma^+(S_1 \cap S_2)$  are cuts which implies that  $|\gamma^-(\overline{S_1} \cap \overline{S_2})| \geq \chi(\mathcal{D})$  and  $|\gamma^+(S_1 \cap S_2)| \geq \chi(\mathcal{D})$ . By replacing  $|\gamma^-(\overline{S_1} \cap \overline{S_2})|$  in the inequality (5), we obtain that  $|\gamma^+(S_1 \cap S_2)| \leq \chi(\mathcal{D})$ . We deduce that  $|\gamma^+(S_1 \cap S_2)| = \chi(\mathcal{D})$ , which completes the proof.  $\square$

**Definition 3.5.** A set of nodes  $S$  in  $\mathcal{D}$  is consecutive if  $(i, n) \in S$  and  $(i, n+k) \in S$  implies  $(i, n+h) \in S$ ,  $\forall 0 \leq h \leq k$ . The set  $S$  is strongly consecutive if we have either a)  $\forall i, (i, n) \in S \Rightarrow (i, n+h) \in S, \forall h \geq 0$ , or b)  $\forall i, (i, n) \in S \Rightarrow (i, n-h) \in S, \forall h \geq 0$ .

**Proposition 3.6.** There exist positive (negative) MBSs which are strongly consecutive.

*Proof.* We only treat the positive case. Let  $C$  be a minimal cut. We define the positive MBS  $S = \Gamma^{+\infty}(C) \cup \Gamma^-(\Gamma^{+\infty}(C), C)$  as in (1). There is no reason why  $S$  should be strongly consecutive. Below, we are going to transform  $S$  to get a strongly consecutive MBS.

Since  $C$  is finite,  $C$  spans over a finite number of columns in  $\mathcal{D}$ , say  $0, 1, \dots, M$ . We define  $K = |V|H(\mathcal{D})$ , that is the number of lines times the maximal ‘depth’ of an arc in  $\mathcal{D}$ .

First, let us make sure that  $S$  does not contain any node on a column with index strictly smaller than  $-K$ . Since  $C$  contains only positive nodes,  $\Gamma^{+\infty}(C)$  also contains only positive nodes. However  $\Gamma^-(\Gamma^{+\infty}(C), C)$  may contain negative nodes. Let us assume that it contains a negative node  $v$  on a column with index strictly smaller than  $-K$ . By definition, there is a finite path  $P$  from  $v$  to a node  $w$  in  $\Gamma^{+\infty}(C)$  and which does not intersect  $C$ . Since  $K = |V|H(\mathcal{D})$ , the negative part of  $P$  must contain at least two nodes on the same line, say  $(i, n)$  and  $(i, m)$ ,  $n < m < 0$ . Since  $\mathcal{D}$  is one-periodic, by repeating the sub-path between  $(i, n)$  and  $(i, m)$ , we can construct a negative-infinite path  $P^-$  that ends in  $(i, m)$ . Since this path contains only strictly negative nodes, it does not intersect  $C$ . By definition of  $\Gamma^{+\infty}(C)$ , there exists a positive-infinite path  $P^+$  starting in  $w$  and which does not intersect  $C$ . Now by concatenating  $P^-$ , the sub-path of  $P$  going from  $(i, m)$  to  $w$  and  $P^+$ , we obtain a bi-infinite path which does not intersect  $C$ . This is a contradiction.

Let us now consider a node  $(i, n) \in S$  for  $n \geq M$ . We want to prove that  $(i, n+1)$  also belongs to  $S$ . Assume that  $(i, n)$  belongs to  $\Gamma^{+\infty}(C)$ . By definition, there exists  $u \in C$ , a finite path  $P_1$  between  $u$  and  $(i, n)$  and a positive-infinite path  $P^+$  starting in  $(i, n)$ . By minimality of the cut  $C$ , there exists a bi-infinite path containing  $u$ , hence a negative-infinite path  $P^-$  ending in  $u$ . By concatenating  $P^-, P_1$  and  $P^+$ , we obtain a bi-infinite path  $P$  containing  $u$ . By one-periodicity of  $\mathcal{D}$ , the node  $(i, n+1)$  is contained in the bi-infinite path  $P+1$ . Since  $C$  is a cut, the path  $P+1$  must intersect  $C$ . Hence there exists  $v \in C$  such that  $v \xrightarrow{*} (i, n+1)$ . Furthermore, the positive-infinite path included in  $P+1$  and starting with  $(i, n+1)$  does not intersect  $C$  since we have  $n+1 > M$  and since all the arcs are pointing from left to right in  $\mathcal{D}$ . We conclude that  $(i, n+1)$  belongs to  $\Gamma^{+\infty}(C)$ .

Now assume that  $(i, n)$  belongs to  $\Gamma^-(\Gamma^{+\infty}(C), C)$ , then there exists  $(j, m) \in \Gamma^{+\infty}(C)$ ,  $(m \geq n)$ , and a finite path  $P$  such that  $(i, n) \xrightarrow{P} (j, m)$  and  $P \cap C = \emptyset$ . We have  $(i, n+1) \xrightarrow{P+1} (j, m+1)$  and  $(P+1) \cap C = \emptyset$  (since  $n+1 > M$ ). Furthermore, the node  $(j, m+1)$  belongs to  $\Gamma^{+\infty}(C)$  (see above). We conclude that  $(i, n+1)$  belongs to  $\Gamma^-(\Gamma^{+\infty}(C), C)$ . To summarize, we have proved that  $\forall n \geq M$ ,  $(i, n) \in S \Rightarrow (i, n+1) \in S$ .

Now, on any given line  $i$ ,  $S$  has two possible shapes:

- (A)  $\forall n < -K$ ,  $(i, n) \notin S$  and  $\exists M_i, \forall n \geq M_i$ ,  $(i, n) \in S$ ;
- (B)  $\forall n < -K$ ,  $(i, n) \notin S$  and  $\forall n \geq M$ ,  $(i, n) \notin S$ .

What happens in between the columns  $-K$  and  $M - 1$  (or  $M_i - 1$ ) is a-priori not known. We now construct a strongly consecutive MBS, starting from  $S$ . Let us define  $\tilde{M} = \max(M, \max_i M_i)$  where the second maximum is taken over the indices  $i$  such that  $M_i$  is defined. Consider

$$S' \stackrel{\text{def}}{=} \bigcap_{n=0}^{K+\tilde{M}} (S+n). \quad (6)$$

The set  $S+n$  is a MBS for all  $n$  by one-periodicity of  $\mathcal{D}$ . By using Proposition 3.4, we conclude that  $S'$  is also a MBS. Let us now consider line  $i$ . If we are in case (A), it is easy to check that  $(i, n)$  belongs to  $S'$  if and only if we have  $n > h_i + K + \tilde{M}$ , where  $h_i$  is the largest integer such that  $(i, h_i) \notin S$  ( $h_i < \tilde{M}$ ). If we are in case (B), it is immediate that  $S'$  does not contain any node on line  $i$ . This shows that  $S'$  is strongly consecutive.  $\square$

**Proposition 3.7.** *A one-periodic and causal bi-infinite graph  $\mathcal{D}$  with finite degree has a minimal cut which is consecutive. If  $S$  is a positive (negative) strongly consecutive MBS then  $\gamma^-(S)$  ( $\gamma^+(S)$ ) is a consecutive minimal cut.*

*Proof.* Let  $S$  be a positive and strongly consecutive MBS, the existence of which follows from Proposition 3.6. If  $(i, n) \in \gamma^-(S)$ , using that  $S$  is strongly consecutive, we have either  $(i, n+1) \in \gamma^-(S)$  or  $(i, n+1) \in S$ . In the last case, we have  $(i, n+h) \in S, \forall h > 0$ . We have proved that  $\gamma^-(S)$  is strongly consecutive and by definition  $\gamma^-(S)$  is a minimal cut, which completes the proof.  $\square$

Clearly, a flow is constituted by a *finite* number of bi-infinite paths. The cardinality of a flow is the number of its paths. A *maximal flow* is a flow with maximal cardinality. A flow  $\mathcal{F}$  is *one-periodic* if for all path  $P$  in  $\mathcal{F}$ , the path  $P+1$  belongs to  $\mathcal{F}$ .

We now prove a ‘maxflow - mincut’ (see [4]) type of result for periodic bi-infinite graphs. We need to introduce some additional notations

Given the bi-infinite graph  $\mathcal{D} = (V \times \mathbb{Z}, \mathcal{A})$ , and  $k, l \in \mathbb{Z}, k \leq l$ , we define the finite directed graph  $\mathcal{D}[k, l]$  as follows. Let  $N = \{(i, n), i \in V, k \leq n \leq l\}$ . The set of nodes of  $\mathcal{D}[k, l]$  is  $N \cup \{L, R\}$  and the set of arcs is

$$\begin{aligned} \{(u, v) \in \mathcal{A} \mid u, v \in N\} \cup \{(L, v) \mid v \in N, \exists u \in (V \times \mathbb{Z}) \setminus N, (u, v) \in \mathcal{A}\} \\ \cup \{(u, R) \mid u \in N, \exists v \in (V \times \mathbb{Z}) \setminus N, (u, v) \in \mathcal{A}\}. \end{aligned} \quad (7)$$

Paths in  $\mathcal{D}[k, l]$  are defined as usual. A *cut* in  $\mathcal{D}[k, l]$  is a subset of the nodes which intersects any path from  $L$  to  $R$ . A *flow* in  $\mathcal{D}[k, l]$  is a set of paths from  $L$  to  $R$  which are node-disjoint except for the extremities  $L$  and  $R$ . Minimal cuts and maximal flows in  $\mathcal{D}[k, l]$  are defined accordingly.

**Theorem 3.8.** *Let us consider a one-periodic and causal bi-infinite graph  $\mathcal{D}$  with finite degree. The maximal cardinality of a flow is equal to the minimal cardinality of a cut. Furthermore, there exists a one-periodic maximal flow.*

*Proof.* Given a flow, two different paths in that flow intersect a given cut at different sets of nodes. Applying the argument to a minimal cut, we get that the cardinality  $k$  of a maximal flow verifies  $k \leq \chi(\mathcal{D})$ . Let  $S$  be a strongly consecutive positive MBS and let  $C = \gamma^-(S)$  be the corresponding consecutive minimal cut. The cut  $C$  spans over the columns say  $0, 1, \dots, M$ . We define  $K = |V|H(\mathcal{D})$  as in Proposition 3.6.

Using an argument similar to the one in Proposition 3.6, we can prove the following: let  $P_1$  be a path from a node on column  $-K$  to a node on column  $M + 2K$ , then there exists a bi-infinite path  $P$  which coincides with  $P_1$  between the columns  $0$  and  $M + K$ . Using this, given a path  $P$  from  $L$  to  $R$  in

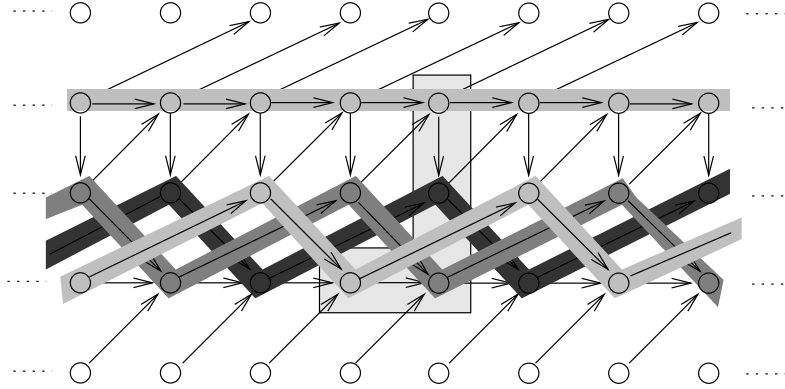


Figure 2: Illustration of Theorem 3.8.

$\mathcal{D}[-K, M + 2K]$ , we can associate with it a bi-infinite path in  $\mathcal{D}$  which coincides with  $P$  on the nodes  $\{(i, n), i \in V, 0 \leq n \leq M + K\}$ . The other way round, given a bi-infinite path in  $\mathcal{D}$ , we can associate with it a path from  $L$  to  $R$  in  $\mathcal{D}[-K, M + 2K]$ , such that the two paths coincide on the set of nodes  $\{(i, n), i \in V, -K \leq n \leq M + 2K\}$ . It follows clearly that  $C$  is also a minimal cut in  $\mathcal{D}[-K, M + 2K]$ .

In the finite graph  $\mathcal{D}[-K, M + 2K]$ , the max flow-min cut theorem [4] can be used: there exists a maximal flow  $F$  in  $\mathcal{D}[-K, M + 2K]$  of cardinality  $\chi(\mathcal{D})$  and each path in this flow intersects  $C$  in exactly one node. Let us first introduce a convention. If  $P$  is the path  $L \rightarrow u_1 \rightarrow \dots \rightarrow u_k$  in  $\mathcal{D}[-K, M + 2K]$ , then  $P - 1$  is the path  $L \rightarrow u_i - 1 \rightarrow \dots \rightarrow u_k - 1$ , where  $u_i$  is the first of the nodes  $(u_1, \dots, u_k)$  not belonging to the column  $-K$  ( $P - 1$  is the path restricted to the node  $L$  if  $u_i$  does not exist). If  $P$  is the path  $u_1 \rightarrow \dots \rightarrow u_k \rightarrow R$ , then  $P + 1$  is defined similarly.

In the flow  $F$ , we consider only some paths. Let us consider  $a$  in  $C \setminus (C + 1)$  and let  $P_a$  be the path in  $F$  intersecting  $C$  in  $a$ .

We can decompose  $P_a$  as  $P_a^- \rightarrow a \rightarrow P_a^+$ . Now let us view  $P_a := P_a \setminus \{L, R\}$  as belonging to  $\mathcal{D}$ . We want to prove that  $P_a^- \cap S = \emptyset$  and that  $P_a^+ \subset S$ . If  $P_a^-$  intersects  $S$ , then it also intersects  $\gamma^-(S) = C$  (unless it is the first node of  $P_a^-$  which belongs to  $S$  but this is not possible since  $S$  spans over columns with index larger than or equal to  $M$ ), but this is a contradiction since  $P_a$  intersects  $C$  only once (in  $a$ ). Now consider  $P_a^+$ . First, we construct a bi-infinite path  $P_a^\infty$  which coincides with  $P_a$  on the nodes  $\{(i, n), i \in V, 0 \leq n \leq M + K\}$ . By definition,  $P_a^\infty \cap S$  contains a positive-infinite path  $\Pi$  and no negative-infinite path. We consider the longest possible path  $\Pi$ , that is if  $u$  is the first node of  $\Pi$  then the predecessor  $v$  of  $u$  in  $P_a^\infty$  does not belong to  $S$ . We deduce that  $v$  belongs to  $\gamma^-(S) = C$ . Since  $P_a^\infty$  and  $P_a$  coincide on the columns  $0, \dots, M$ , we get that  $v$  also belongs to  $P_a^+$ . Therefore,  $v = a$ , and  $P_a^+$  coincides with  $\Pi$  on the nodes  $\{(u, n), u \in V, 0 \leq n \leq M + K\}$ . Now by definition of  $K$ , we have  $\{(u, n), u \in V, n > M + K\} \subset S$ . We conclude that  $P_a^+$  is included in  $S$ .

Note that  $P_a$  contains a node  $a'$  in  $(C + 1)$  (if not, then  $P_a - 1$  would not intersect  $C$ ). Since  $P_a$  intersects  $C$  only once, it implies that  $a' \in (C + 1) \setminus C$ . We decompose  $P_a$  even further as  $P_a^- \rightarrow a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_n \rightarrow P_a^{++}$  with  $a_i \in \mathcal{V}$ ,  $a_0 = a$  and  $a_n = a'$ . We denote by  $Q_a$  the sub-path  $a_0 \rightarrow \dots \rightarrow a_n$ .

The sub-paths  $Q_a$ , defined for each  $a \in C \setminus (C + 1)$ , form a matching between  $C \setminus (C + 1)$  and  $(C + 1) \setminus C$ . Let  $Q$  be the subgraph of  $\mathcal{D}$  defined by  $Q = \cup_{k \in \mathbb{Z}} \cup_{a \in C \setminus (C + 1)} (Q_a + k)$ . Let us set  $Q'_a = a_1 \rightarrow \dots \rightarrow a_{n-1}$ , where the sub-paths  $Q'_a$  are possibly empty. The set of lines spanned by  $Q'_a, a \in C \setminus (C + 1)$ , does not intersect the set of lines spanned by  $C$ . Otherwise, it would contradict the fact that the paths  $P_a$  intersect the consecutive cut  $C$  in only one node. Clearly,  $Q$  is a node-disjoint union of bi-infinite paths if and only if the sub-paths  $Q'_a, a \in C \setminus (C + 1)$ , span over disjoint sets of lines.

We now prove that this last point holds. We work in the finite graph  $\mathcal{D}[-K, M + 2K]$ . Let us assume there exist  $a, b \in C \setminus (C + 1)$  and two different nodes  $a_k \in Q'_a, b_h \in Q'_b$  belonging to the same line, for instance  $a_k = (i, t)$  and  $b_h = (i, t + e)$ ,  $e \geq 1$ . By one-periodicity, we can construct a path

$$\begin{aligned} P_b^- - 1 \rightarrow b_0 - 1 \rightarrow \dots \rightarrow b_h - 1 (= a_k + e - 1) \\ \rightarrow a_{k+1} + e - 1 \rightarrow \dots \rightarrow a_n + e - 1 \rightarrow P_a^{++} + e - 1. \end{aligned}$$

The sub-path  $P_b^- - 1 \rightarrow b_0 - 1 \rightarrow \dots \rightarrow b_{h-1} - 1 \rightarrow b_h - 1$  does not intersect  $C$  since  $b' - 1$  is the only node of  $P_b - 1$  belonging to  $C$  and  $b' = b_m$  with  $m > h$ . The sub-path  $a_{k+1} + e - 1 \rightarrow \dots \rightarrow P_a^{++} + e - 1$  does not intersect  $C$  since  $a_{k+1} \rightarrow \dots \rightarrow P_a^{++}$  is included in  $S$  (which is strongly consecutive) which implies that  $a_{k+1} + e - 1 \rightarrow \dots \rightarrow P_a^{++} + e - 1$  is also included in  $S$  and hence does not intersect  $C$ . To summarize, we have built a path from  $L$  to  $R$  in  $\mathcal{D}[-K, M + 2K]$  not intersecting  $C$ , which is impossible. Therefore the paths of the form  $Q'_a, a \in C \setminus (C + 1)$ , intersect distinct sets of lines. It implies that  $\cup_{k \in \mathbb{Z}} \cup_a (Q_a + k)$  defines a flow which is one-periodic and consists of exactly  $\chi(\mathcal{D})$  paths which concludes the proof.  $\square$

The above result is to be compared with a result proved in [2] using a rather similar approach. In [2], a finite graph left invariant by a symmetry is studied, and it is proved that the minimal cut is left invariant by the symmetry, but not the maximal flow. Here, our graph is left invariant by translations, and we proved that the maximal flow is also invariant by translations, but obviously not the minimal cut.

A bi-infinite graph  $\mathcal{D} = (\mathcal{V}, \mathcal{A})$  is  $k$ -periodic, if:  $(i, n) \rightarrow (j, m) \in \mathcal{A} \iff (i, n + k) \rightarrow (j, m + k) \in \mathcal{A}$ . A  $k$ -periodic flow is defined in an analog way.

**Corollary 3.9.** *A  $k$ -periodic and causal bi-infinite graph  $\mathcal{D}$  with finite degree has a  $k$ -periodic maximal flow.*

*Proof.* From  $\mathcal{D}$ , we construct a one-periodic graph  $\mathcal{D}_1$  by merging  $k$  consecutive columns into one. There exists a one-periodic maximal flow in  $\mathcal{D}_1$  (Proposition 3.7). This flow when considered in  $\mathcal{D}$  is  $k$ -periodic.  $\square$

This simple construction shows that all the results proved here for one-periodic graphs extend easily to  $k$ -periodic graphs for arbitrary  $k$ .

We say that a positive split  $S$  is *complete* if it is of the form  $\{(i, n), i \in V, n \geq n_i\}$  for  $n_i \in \mathbb{Z}, i \in V$  (with an analog definition for negative splits). In particular, a complete split is strongly consecutive. The next proposition will be useful in the applications presented in section 4.

**Proposition 3.10.** *There exist complete positive (negative) MBSs.*

*Proof.* Let  $S$  be any positive and strongly consecutive MBS of  $\mathcal{D}$ . We denote by  $\text{lin}(S)$  the set of the lines intersected by  $S$ . We assume that there is at least one node on column 0 in each line in  $\text{lin}(S)$ . We define  $K = |V|H(\mathcal{D})$ . Let  $U = \{(j, m) \text{ s.t. } j \notin \text{lin}(S) \text{ and } m > K\}$  and let  $T = \Gamma^-(U, S) \cup U \cup S$ . We are going to prove that  $T$  is complete and that  $\gamma^-(T) = \gamma^-(S)$ .

The proof goes along the following lines. First, we show that  $\Gamma^-(U, S)$  is a finite set. This implies that  $\Gamma^-(U, S) \cup U \cup S$  is a positive split. In the last part of the proof we show that  $\gamma^-(T) = \gamma^-(S)$  which implies that  $T$  is a MBS.

Let us consider  $x \in \Gamma^-(U, S)$ . By definition, there exists a path  $P$  from  $x$  to some node  $y$  in  $U$ , and such that  $P \cap S = \emptyset$ . Let us assume that  $P$  is of length larger than  $|V|$ . Assume also that  $P$  (except for the ending node  $y$ ) ranges over lines that do not belong to  $\text{lin}(S)$ . Then  $P$  must contain at least two nodes on the same line. By one-periodicity, we can associate with  $P$  a bi-infinite path that does not intersect  $S$ , which is impossible.

We conclude that  $P$  is either of length strictly less than  $|V|$  or that it intersects the lines of  $S$ . Let us assume that the latter holds. Let  $z$  be the last node of  $P$  belonging to a line in  $\text{lin}(S)$ . The path is of the form  $P = x \xrightarrow{*} z \rightarrow w \xrightarrow{*} y$  with  $z = (j, m) \in \text{lin}(S)$ ,  $w = (k, o) \in \text{lin}(U)$  and  $y = (l, p) \in U$  (with possibly  $w = y$ ). By definition of  $S$ , we have  $m < 0$  and thus  $o < H(\mathcal{D})$ . On the other hand, by definition of  $U$ , we have  $p \geq K$ . Therefore, the sub-path  $Q = w \xrightarrow{*} y$  is contained in  $\text{lin}(U)$  and its length is strictly larger than  $(|V| - 1)H(\mathcal{D})$ . It implies that  $Q$  hits at least twice the same line. Hence, we can associate with it a bi-infinite path that does not intersect  $S$ . This is again impossible.

We conclude that the path  $P$  is of length strictly less than  $|V|$ . Therefore,  $\Gamma^-(U, S)$  is finite and  $T = \Gamma^-(U, S) \cup U \cup S$  is a positive split.

To finish the proof, we show that  $\gamma^-(T) = \gamma^-(S)$ . First, we have

$$\gamma^-(T) = (\gamma^-(\Gamma^-(U, S)) \cup \gamma^-(U) \cup \gamma^-(S)) \setminus (\Gamma^-(U, S) \cup U \cup S) . \quad (8)$$

By definition of  $\Gamma^-(U, S)$ , we have  $\gamma^-(\Gamma^-(U, S)) \subset S$ . Furthermore, we have  $\gamma^-(U) \subset (S \cup \Gamma^-(U, S))$ . By replacing in Equation (8), we obtain

$$\gamma^-(T) = \gamma^-(S) \setminus (\Gamma^-(U, S) \cup U) \subset \gamma^-(S) .$$

Since  $T$  is a split,  $\gamma^-(T)$  is a cut. Since  $\gamma^-(S)$  is a minimal cut, we conclude that  $\gamma^-(T) = \gamma^-(S)$ . It follows that  $\Gamma^-(U, S) \cup U \cup S$  is a MBS. It is not a priori clear that this MBS should be strongly consecutive. However, the same transformation as in Proposition 3.6 can be applied to it to get a strongly consecutive MBS. It follows from the construction that the new MBS is complete.  $\square$

## 4 Application to Register Minimization in Circuits

In this section, we also consider directed graphs with several arcs between a given couple of nodes. In this case, given an arc  $e$  we denote by  $i(e)$  and  $t(e)$  the initial and terminal node of the arc.

Let  $\mathcal{D} = (V \times \mathbb{Z}, \mathcal{A})$  be a one-periodic bi-infinite graph with finite degree as defined in § 2. Such a graph can be folded into a finite graph. The *reduced graph associated with  $\mathcal{D}$*  is the finite arc-valued directed graph  $\mathcal{R} = (V, E, \Delta)$  where  $V$  is the set of nodes,  $E$  the set of arcs and  $\Delta : E \rightarrow \mathbb{Z}$  the valuation, with  $E$  and  $\Delta$  defined as follows: there is an arc  $e \in E$  with  $i(e) = u$ ,  $t(e) = v$  and  $\Delta(e) = \delta$  if  $(u, 0) \rightarrow (v, \delta) \in \mathcal{A}$ . According to this definition, there might be several arcs with different values between two given nodes. We call  $\Delta(e)$  the *delay* of the arc  $e$ .

By extension, a finite arc-valued directed graph is called a *reduced graph*. Let  $\mathcal{R} = (V, E, \Delta)$  be a reduced graph, its *unfolding* is the one-periodic bi-infinite graph  $\mathcal{D} = (V \times \mathbb{Z}, \mathcal{A})$  with  $(u, n) \rightarrow (v, m) \in \mathcal{A} \iff \exists e \in E, i(e) = u, t(e) = v, \Delta(e) = m - n$ . A reduced graph is *causal* if its unfolding is causal. Clearly, a reduced graph is causal if and only if any circuit  $(e_0, e_1, \dots, e_n), e_i \in E, t(e_n) = i(e_0)$ , is such that  $\Delta(e_0) + \dots + \Delta(e_n) > 0$ . From now on, we consider only causal reduced graphs.

A *digital circuit* is made of *gates* computing data according to boolean logical functions, *wires* connecting the gates and *memory registers* on the wires which are storing the data between two computation cycles. With a digital circuit, we associate a (causal) reduced graph  $\mathcal{R} = (V, E, \Delta)$ , whose nodes, arcs and delays correspond respectively to the gates, wires and number of registers on the wires of the circuit. Since the number of registers has to be positive, a specificity of the reduced graph associated with a digital circuit is that it has only positive delays. So for our purpose, a digital circuit is simply a reduced graph with positive delays. All the definitions and notations introduced for reduced graphs will also be used for digital circuits. The unfolding of the reduced graph can be viewed as the graph of the dependences between the computations performed by the circuit (the node  $(i, n)$  corresponds to the  $n$ -th computation at gate  $i$ ). Our goal is to minimize the number of registers used in a circuit in a sense to be made

precise below (problem **Min-Register**). A more thorough discussion of the relation between circuits and reduced graphs is proposed in [8, 5].

#### 4.1 Computations in digital circuits

Let  $\mathcal{R} = (V, E, \Delta)$  be a digital circuit. Let  $\mathcal{Q}$  be a finite set (corresponding to all the different values that one register can store) and let  $\mathcal{F}$  be the set of functions from  $\mathcal{Q}^k$  to  $\mathcal{Q}$  for all  $k \in \mathbb{N}$ .

A *specialization*  $\sigma$  of the circuit consists in mapping one function of  $\mathcal{F}$  to each gate of  $\mathcal{R}$ :

$$\begin{aligned} \sigma &: V \rightarrow \mathcal{F} \\ u &\mapsto F_u \end{aligned}$$

The function  $F_u$  attached to gate  $u$  must have as many arguments as  $u$  has input arcs. In particular, if  $u$  is a node with no predecessor, then  $F_u$  is a constant (since  $F_u$  is a function from  $\mathcal{Q}^0$  to  $\mathcal{Q}$ ).

On a wire  $e$  with  $\Delta(e) > 0$ , we denote the registers by  $(e, 1), \dots, (e, \Delta(e))$  where the ranking is performed according to the physical order of the registers on the wire. Let  $M = \{(e, n), e \in E, 1 \leq n \leq \Delta(e)\}$  be the set of the registers. An *initial condition*  $I$  assigns an initial value to each register of the circuit:

$$\begin{aligned} I &: M \rightarrow \mathcal{Q} \\ m &\mapsto I(m) \end{aligned}$$

The *computation* of  $(\mathcal{R}, \sigma, I)$  is the sequence  $(x(u, n))_{u \in V, n \in \mathbb{N}}$  where  $x(u, n) \in \mathcal{Q}$  is the  $n$ -th value computed at gate  $u$  if the values stored initially in the registers have been set by  $I$  and if the functions computed at each gate are those given by  $\sigma$ . More formally, we have

$$x(u, n) = F_u[x(i(e_1), n - \Delta(e_1)), \dots, x(i(e_k), n - \Delta(e_k))], \quad n \in \mathbb{N}, u \in V, \quad (9)$$

where  $e_1, \dots, e_k$ , are the arcs with terminal node  $u$  (listed according to some total order on  $E$ ), and where, if  $n - \Delta(e_j) < 0$ ,

$$x(i(e_j), n - \Delta(e_j)) = I((e_j, \Delta(e_j) - n)).$$

The *computational power* of a digital circuit  $\mathcal{R}$  is defined as follows. The sequence  $(x(u, n))_{u \in V, n \in \mathbb{N}}$  of elements of  $\mathcal{Q}$  is *computable* by  $\mathcal{R}$  if there exists  $\sigma$  and  $I$ , such that the sequence is computed by  $(\mathcal{R}, \sigma, I)$ . We say that a circuit  $\mathcal{R}_2$  (nodes  $V_2$ ) has a *larger computational power* than a circuit  $\mathcal{R}_1$  (nodes  $V_1$ ) if for each specialization  $\sigma_1$  of  $\mathcal{R}_1$ , there exists a specialization  $\sigma_2$  of  $\mathcal{R}_2$  and an injective mapping

$$\theta : V_1 \times \mathbb{N} \rightarrow V_2 \times \mathbb{N}, \quad (10)$$

such that for any initial condition  $I_1$  for  $\mathcal{R}_1$  there exists an initial condition  $I_2$  for  $\mathcal{R}_2$  such that the computation  $(x(u, n))_{u \in V_1, n \in \mathbb{N}}$  of  $(\mathcal{R}_1, \sigma_1, I_1)$  and the computation  $(y(u, n))_{u \in V_2, n \in \mathbb{N}}$  of  $(\mathcal{R}_2, \sigma_2, I_2)$  satisfy

$$\forall (u, n) \in V_1 \times \mathbb{N}, \quad x(u, n) = y(\theta(u, n)).$$

Roughly speaking, this means that everything which can be computed by  $\mathcal{R}_1$  can also be retrieved from a computation carried over  $\mathcal{R}_2$ . Obviously, this retrieval is effective only if  $\theta$  is simple enough.

#### 4.2 Forward splitting and duplication

Given a reduced graph (resp. circuit)  $\mathcal{R} = (V, E, \Delta)$ , we define the *total number of delays* (resp. *total number of registers*) of  $\mathcal{R}$  as follows

$$\Delta_A(\mathcal{R}) = \sum_{e \in E} \Delta(e). \quad (11)$$

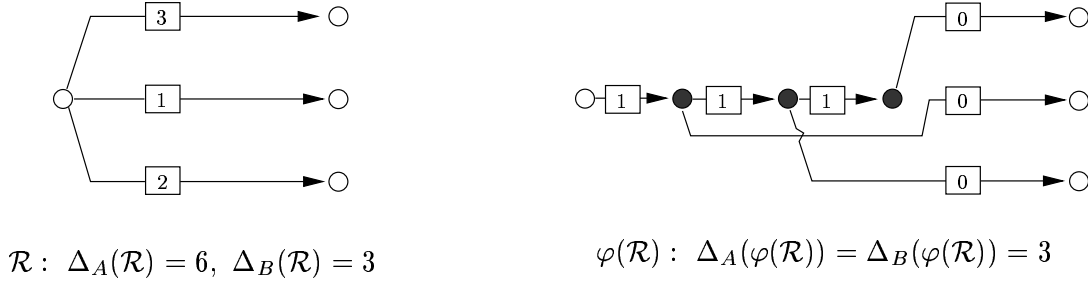


Figure 3: The forward splitting transformation.

Another quantity of interest is the following one

$$\Delta_B(\mathcal{R}) = \sum_{u \in V} \max_{e \in E, i(e)=u} \Delta(e). \quad (12)$$

Since  $\Delta_A(\mathcal{R}) = \sum_{u \in V} \sum_{e \in E, i(e)=u} \Delta(e)$ , we have  $\Delta_B(\mathcal{R}) \leq \Delta_A(\mathcal{R})$ . Introducing the quantity  $\Delta_B$  is relevant because of the following result. There exists a reduced graph denoted  $\varphi(\mathcal{R})$  (set of nodes  $\varphi(V)$ ), with positive delays, which has a larger computational power than  $\mathcal{R}$ , and such that  $\Delta_A(\varphi(\mathcal{R})) = \Delta_B(\mathcal{R})$ . The graph  $\varphi(\mathcal{R})$  is obtained from  $\mathcal{R}$  by *duplication* and *forward splitting*, see [8, 5]. (In [5], *duplication* is studied in the context of recycled graphs, but it is easy to see that the construction extends to the general case). In Figure 3, we give an example of the forward splitting transformation. The duplication transformation is designed to get rid of negative delays.

### 4.3 Solution of the Min-Register problem

We want to solve the **Min-Register** problem which is defined as follows:

*Given a circuit  $\mathcal{R}$ , find another circuit with at least the same computational power and with as few registers as possible. This number will be denoted by  $\text{Min-Reg}(\mathcal{R})$ .*

To do so, we first introduce the classical notion of retiming [8]. A *retiming* is a function  $r : V \rightarrow \mathbb{Z}$ . Given a reduced graph  $\mathcal{R}$ , it specifies a new reduced graph  $\mathcal{R}_r$  and a new unfolded graph  $\mathcal{D}_r$  as follows:

- $\mathcal{R}_r = (V, E, \Delta_r)$  with, for  $e \in E$ ,  $\Delta_r(e) = \Delta(e) + r(i(e)) - r(t(e))$ ;
- $\mathcal{D}_r = (V \times \mathbb{Z}, \mathcal{A}_r)$  is the unfolding of  $\mathcal{R}_r$ ; that is  $(i, n) \rightarrow (j, m) \in \mathcal{A}_r \iff (i, n + r(i)) \rightarrow (j, m + r(j)) \in \mathcal{A}$ .

In the example of Figure 4, the new graphs  $\mathcal{R}_r$  and  $\mathcal{D}_r$  correspond to the retiming  $r$  defined by  $r(1) = 1, r(2) = 1$  and  $r(3) = 0$ .

Let  $\mathcal{R}$  be a circuit. It should be clear that for any retiming  $r$ ,  $\mathcal{R}_r$  has the same computational power as  $\mathcal{R}$ . It follows from section 4.2 that  $\varphi(\mathcal{R}_r)$  has a larger computational power than  $\mathcal{R}$ . In particular it implies that  $\text{Min-Reg}(\mathcal{R}) \leq \min_r \Delta_B(\mathcal{R}_r) = \min_r \Delta_A(\varphi(\mathcal{R}_r))$ , where the minimum is taken over all possible retimings. Next Theorem states that there is in fact equality.

**Theorem 4.1.** *Let  $\mathcal{R}$  be a digital circuit and let  $\mathcal{D}$  be its unfolding. Then the minimal number of registers  $\text{Min-Reg}(\mathcal{R})$  is equal to  $\chi(\mathcal{D})$ , the cardinality of a minimal cut of  $\mathcal{D}$ . Furthermore, let  $S$  be a positive and complete MBS of  $\mathcal{D}$ . For  $i \in V$ , let  $n_i$  be such that  $(i, n_i - 1) \notin S, (i, n_i) \in S$ . Let  $r$  be the retiming defined by  $r(i) = n_i$ . Then the digital circuit  $\varphi(\mathcal{R}_r)$  is a solution to the **Min-Register** problem.*



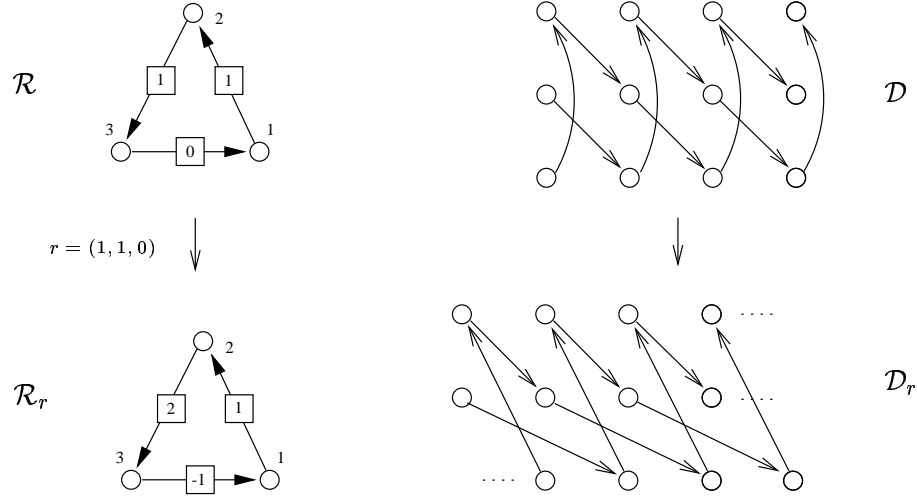


Figure 4: Retimed reduced graph and its unfolding.

*Proof.* The proof is very similar to the one of Theorem 7.4 in [5]. We first prove that the number of registers of  $\varphi(\mathcal{R}_r)$  is  $\chi(\mathcal{D})$ . Let  $f_r$  be the map from  $V \times \mathbb{Z}$  into itself defined by  $f_r((u, n)) = (u, n - r(u))$ . Obviously,  $f_r(S)$  is a positive and complete MBS of  $\mathcal{D}_r$ . Furthermore, by definition of  $r$ , we have  $f_r(S) = \{(u, n), u \in V, n \in \mathbb{N}\}$ . Now, the size of the cut  $\gamma^-(S)$  in  $\mathcal{D}$  is the same as the size of the cut  $\gamma^-(f_r(S))$  in  $\mathcal{D}_r$ . Let us consider a node  $u \in V$ . Let  $M = \max_{e \in E, i(e)=u} \Delta_r(e)$  and let  $v$  be such that there exists  $e \in E$  with  $i(e) = u, t(e) = v, \Delta(e) = M$ . There is an arc in  $\mathcal{D}_r$  from  $(u, -M)$  to  $(v, 0)$  and no arc from a node  $(u, k), k < -M$  to a positive node. Hence, by definition,  $\gamma^-(f_r(S))$  contains exactly the nodes  $(u, -M), \dots, (u, -1)$  on line  $u$ . The same argument repeated on each line shows that  $\Delta_B(\mathcal{R}_r) = |\gamma^-(S)| = \chi(\mathcal{D})$ . As recalled above, the graph  $\varphi(\mathcal{R}_r)$  has positive delays, a larger computational power than  $\mathcal{R}$  and verifies  $\Delta_A(\varphi(\mathcal{R}_r)) = \Delta_B(\mathcal{R}_r) = \chi(\mathcal{D})$ .

In the second part of the proof we show that there exists no circuit with at least the same computational power as  $\mathcal{R}$  and with fewer registers than  $\chi(\mathcal{D})$ . Let  $\mathcal{R}' = (V', E', \Delta')$  be a circuit with at least the same computational power as  $\mathcal{R}$  and let  $\mathcal{D}'$  be the unfolded graph associated with  $\mathcal{R}'$ .

Let  $C$  be a minimal cut of  $\mathcal{D}$ . According to Theorem 3.8, there exists in  $\mathcal{D}$  a one-periodic flow of cardinality  $|C|$  which defines a bijective mapping from the nodes of  $C$  to the ones of  $C + L$ , for any positive integer  $L$ . Let us choose a specialization  $\sigma : u \mapsto F_u$ , in the following way. Given  $u \in V$ , let  $e_1, \dots, e_k$ , be all the arcs with terminal node  $u$  (listed according to some total order on  $E$ ), and let  $e_l$  be the only arc belonging to the flow. Then we define  $F_u : \mathbb{Q}^k \rightarrow \mathbb{Q}$  by  $F_u(x_1, \dots, x_k) = x_l$ . By composing the functions  $F_u$ , we get an application from the nodes of  $C$  to the ones of  $C + L$  of the form  $F : \mathbb{Q}^{|C|} \rightarrow \mathbb{Q}^{|C|}$  which is a permutation of the coordinates. In particular  $F$  is bijective. Also note that when we let the initial condition  $I$  vary over all the possible values in  $\mathbb{Q}^{\Delta_A(\mathcal{R})}$  then the values of the computation of  $(\mathcal{R}, \sigma, I)$  in the cut  $C$ , namely  $(x(u, n))_{(u, n) \in C}$  cover all the values in  $\mathbb{Q}^{|C|}$ .

Since  $\mathcal{R}'$  has a larger computational power than  $\mathcal{R}$ , there exists a specialization  $\sigma'$  of  $\mathcal{R}'$  and an injective mapping  $\theta : V \times \mathbb{Z} \rightarrow V' \times \mathbb{Z}$  such that each sequence  $(x(u, n))_{u \in V, n \in \mathbb{N}}$  computed by  $(\mathcal{R}, \sigma, I)$  for some initial condition  $I$ , is related to a sequence  $(y(u, n))_{u \in V', n \in \mathbb{N}}$  computed by  $(\mathcal{R}', \sigma', I')$  for an adequate initial condition  $I'$  by  $x(u, n) = y(\theta(u, n))$ .

Let  $S'$  be a positive and complete MBS of  $\mathcal{D}'$ . The minimal cut  $\gamma^-(S')$  is denoted  $C'$  in the following. Let  $\theta(C)$  be the image by  $\theta$  of the cut  $C$  in  $\mathcal{D}'$ . Since  $\theta$  is injective, we can assume by translating  $S'$  and by choosing  $L$  large enough, that  $\theta(C) \cap (S' \cup C') = \emptyset$ , and  $\theta(C + L) \subset S'$ . This means that  $\theta(C)$  is on the “left” of  $C'$  and  $\theta(C + L)$  is on the “right” of  $C'$ . Let  $U'$  be the subset of  $V'$  consisting of the nodes

with no predecessor. We recall that given a specialization  $\sigma'$  of  $\mathcal{R}'$  we have

$$\forall u \in U', \exists c_u \in \mathcal{Q}, \forall I', \forall n \in \mathbb{N}, y(u, n) = c_u.$$

where  $(y(u, n))_{u \in U', n \in \mathbb{N}}$  are the values computed by  $(\mathcal{R}', \sigma', I')$ . We consider all the paths of  $\mathcal{D}'$  ending in  $\theta(C + L)$ . Any such path intersects the cut  $C'$  or is finite and starts in a node of the type  $(u, n), u \in U'$ . Hence for any node  $(u, n) \in \theta(C + L)$ , we have  $y(u, n) = G(y(v, k), (v, k) \in C'; c_u, u \in U')$  for some function  $G$  depending only on  $\sigma'$ . We conclude that for fixed values  $c_u, u \in U'$ , the variables  $(y(u, n))_{(u, n) \in \theta(C + L)}$  can take at most  $|\mathcal{Q}|^{|C'|}$  different values when the initial condition  $I'$  varies. Since  $F$  is bijective from  $\mathcal{Q}^{|C'|}$  into itself, it implies that  $|C| \leq |C'|$ .  $\square$

This result deserves several comments.

- First, given a digital circuit  $\mathcal{R}$  with unfolding  $\mathcal{D}$ , the quantity  $\chi(\mathcal{D})$  can be seen as the intrinsic quantity of memory needed to carry any computations which could be wired by  $\mathcal{R}$ .
- To minimize the number of registers in a digital circuit, using retiming is not always optimal but using retiming and duplication is. However if the reduced graph  $\mathcal{R}_r$  obtained after optimal retiming has only positive delays, then duplication is not necessary. This completes the results given in [8], section 8.
- As proved above, the circuit  $\mathcal{R}$  (nodes  $V$ ) can be replaced by the circuit  $\varphi(\mathcal{R}_r)$  (nodes  $\varphi(V)$ ) without loss of computational power. It is however necessary to ask, following the remark at the end of section 4.1, if the mappings  $\theta : V \times \mathbb{N} \rightarrow \varphi(V) \times \mathbb{N}$ , defined as in (10), are simple enough. Here it follows from [5] that the mappings  $\theta$  are one-periodic, that is:  $\theta(u, n) = (v_u, n + c_u)$  where  $v_u \in \varphi(V)$  and  $c_u \in \mathbb{N}$  do not depend on  $n$ .

#### 4.4 Complexity issues

The constructions presented in sections 3 and 4 can be performed using algorithms with polynomial complexity, as shown in the following.

**Lemma 4.2.** *Consider a one-periodic and causal bi-infinite graph  $\mathcal{D} = (V \times \mathbb{Z}, \mathcal{A})$  with finite degree. There exists a minimal consecutive cut  $C$  such that*

$$\max\{n : \exists u \in V, (u, n) \in C\} - \min\{n : \exists u \in V, (u, n) \in C\} \leq (2|V| + 1)H(\mathcal{D}), \quad (13)$$

where  $H(\mathcal{D}) = \max\{\gamma : \exists i, j, (i, 0) \rightarrow (j, \gamma)\}$ .

The quantity on the right-hand side of (13) is called the *span* of  $C$ .

*Proof.* Let  $\mathcal{R} = (V, E, \Delta)$  be the reduced graph associated with  $\mathcal{D}$ . As in section 2, we assume without loss of generality that  $\mathcal{R}$  has positive delays (equivalently that the arcs of  $\mathcal{D}$  are pointing from left to right). Let  $S$  be a positive and complete MBS of  $\mathcal{D}$ . According to Proposition 3.7, the set  $\gamma^{-1}(S)$  is a consecutive minimal cut.

Let us assume first that the graph  $\mathcal{R}$  is strongly connected. Let  $u$  and  $v$  be two distinct nodes of  $V$  and let  $e_1, \dots, e_l, e_i \in E$ , be a path from  $u$  to  $v$ , that is:  $i(e_1) = u, t(e_k) = i(e_{k+1}), t(e_l) = v$ . We assume that the path contains no circuit:  $\forall j \neq k, t(e_j) \neq t(e_k)$ . Let  $r : V \rightarrow \mathbb{Z}$  be such that  $\forall u \in V, (u, r(u) - 1) \notin S, (u, r(u)) \in S$ . We have

$$\begin{aligned} r(u) - r(v) &= \sum_{k=1}^l r(i(e_k)) - r(t(e_k)) \\ &\leq \sum_{k=1}^l \max_{e \in E, i(e)=i(e_k)} \Delta(e) + r(i(e)) - r(t(e)) \leq \Delta_B(\mathcal{R}_r) = \chi(\mathcal{D}), \end{aligned}$$

where the last equality is a consequence of Theorem 4.1. We conclude that the span of  $\gamma^{-1}(S)$  is less or equal than  $\chi(\mathcal{D}) + H(\mathcal{D}) - 1$ .

Assume now that  $\mathcal{R}$  is connected but not strongly connected and let  $\mathcal{R}_1, \dots, \mathcal{R}_l$ , be the maximal strongly connected subgraphs of  $\mathcal{R}$ . Let  $\mathcal{D}_1, \dots, \mathcal{D}_l$ , be the corresponding unfoldings, and let  $S_1, \dots, S_l$ , be positive and complete MBS of  $\mathcal{D}_1, \dots, \mathcal{D}_l$ . For  $n \in \mathbb{Z}$ , the shifted set  $S_i + n$  is still a positive and complete MBS of  $\mathcal{D}_i$ . In what follows, the sets  $S_i$  are supposed to be defined up to a shift. For any  $\mathcal{R}_j$  and  $\mathcal{R}_k$ ,  $j \neq k$ , such that  $\exists e \in E$ ,  $i(e) \in \mathcal{R}_j, t(e) \in \mathcal{R}_k$ , we choose  $S_j$  and  $S_k$  to verify  $\min\{n : \exists u, (u, n) \in S_k\} = \max\{n : \exists u, (u, n) \in S_j\} + H(\mathcal{D})$ . This choice ensures that  $\gamma^{-1}(\bigcup_i S_i) = \bigcup_i \gamma^{-1}(S_i)$ . It follows easily that  $C = \gamma^{-1}(\bigcup_i S_i)$  is a consecutive minimal cut of  $\mathcal{D}$ . Let  $T = \{(u, n), u \in V, n \in \mathbb{N}\}$ , we have  $\chi(\mathcal{D}) \leq |\gamma^{-1}(T)| \leq |V|H(\mathcal{D})$ . By construction and using the previous inequality, we obtain that the span of  $C$  is upper bounded by  $(2|V| + 1)H(\mathcal{D})$ .

Assume now that the graph  $\mathcal{R}$  contains several connected components  $\mathcal{R}_1, \dots, \mathcal{R}_l$ , with unfoldings  $\mathcal{D}_1, \dots, \mathcal{D}_l$ . Let  $S_1, \dots, S_l$ , be positive and complete MBS of  $\mathcal{D}_1, \dots, \mathcal{D}_l$ , chosen according to the above procedure. It is immediate that  $\bigcup_i S_i$  is a positive and complete MBS of  $\mathcal{D}$ . We shift the sets  $S_i$  in order to ensure that the span of  $C = \gamma^{-1}(\bigcup_i S_i)$  is equal to the maximum of the spans of  $\gamma^{-1}(S_i)$ . Hence the span of  $C$  is upper bounded by  $(2|V| + 1)H(\mathcal{D})$ .  $\square$

A consequence of the above Lemma is that a minimal cut of the finite graph  $\mathcal{D}[1, (2|V| + 1)H(\mathcal{D})]$  (defined as in (7)) is also a minimal cut of  $\mathcal{D}$ . This is the main ingredient in proving next Proposition.

**Proposition 4.3.** *Consider a one-periodic and causal bi-infinite graph with finite degree. A minimal consecutive cut and an associated one-periodic flow can be computed in polynomial time (in  $|V|$  and  $H(\mathcal{D})$ ). Consider a digital circuit, another circuit with a larger computational power and a minimal number of registers can be built in polynomial time.*

*Proof.* The construction goes along the following lines. Let  $\mathcal{D}$  be the bi-infinite graph and consider the finite graph  $\mathcal{D}[1, (2|V| + 1)H(\mathcal{D})]$ . Using the augmenting path algorithm of Ford and Fulkerson [4], we get a minimal cut in  $\mathcal{D}[1, (2|V| + 1)H(\mathcal{D})]$ . The time complexity of the algorithm is  $O(\chi(\mathcal{D})K)$ , where  $K$  is the number of arcs in  $\mathcal{D}[1, |V|H(\mathcal{D})]$ . Hence, using the inequality  $\chi(\mathcal{D}) \leq |V|H(\mathcal{D})$ , we obtain a time complexity  $O(|E||V|^2H(\mathcal{D})^2)$ , where  $E$  is the set of arcs of the reduced graph associated with  $\mathcal{D}$  (note that we have  $|E| \leq |V|^2H(\mathcal{D})$ ). The minimal cut that we obtained has no reason to be consecutive. We construct the MBS associated with the cut as in (1), then we construct a strongly consecutive MBS  $S$  using (6). By Proposition 3.7, the set  $\gamma^{-1}(S)$  is a consecutive minimal cut of  $\mathcal{D}$ . Using the method described in the proof of Theorem 3.8, we then get a one-periodic maximal flow. All these steps have a time complexity which is lower than  $O(|E||V|^2H(\mathcal{D})^2)$ .

Assume now that the initial bi-infinite graph  $\mathcal{D}$  was the unfolding of a digital circuit  $\mathcal{R}$ . We apply the same first steps as above. Starting from  $S$ , we build a complete MBS  $S'$  as in Proposition 3.10. Let  $r : V \times \mathbb{Z}$  be such that  $(i, r(i) - 1) \notin S'$ ,  $(i, r(i)) \in S'$ . According to Theorem 4.1, the circuit  $\varphi(\mathcal{R}_r)$  provides a solution to the minimal register problem. The circuit  $\varphi(\mathcal{R}_r)$  is obtained from  $\mathcal{R}$  by applying the *duplication* and *forward splitting* algorithms as described in [5], sections 6 and 7. The time complexity of these algorithms is  $O(\Delta_A(\mathcal{R})^2|V|^2 + |V|^3)$  (Corollary 7.5 in [5]). Using the upper bound  $\Delta_A(\mathcal{R}) \leq |E|H(\mathcal{D})$ , we conclude that the total complexity is  $O(|E|^2|V|^2H(\mathcal{D})^2 + |V|^3)$ .  $\square$

In particular, the algorithm to design circuits with a minimal number of registers has been implemented [9].

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