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THÈME 2



*Rapport
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On the number of cylindrical shells

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Thème 2 — Génie logiciel
et calcul symbolique
Projets Prisme

Rapport de recherche n° 4234 — Juillet 2001 — 8 pages

Abstract: Given a set P of n points in three dimensions, a cylindrical shell or zone cylinder is formed by two cylindrical cylinders with the same axis such that all points of P are between the two cylinders. We prove that the number of cylindrical shells enclosing P passing through combinatorially different subsets of P has size $\Omega(n^3)$ and $O(n^4)$ (previous known bound was $O(n^5)$).

Key-words: Computational geometry, metrology, convex hull

Sur le dénombrement des paires de cylindres coaxiaux

Résumé : Étant donné un ensemble P de n points en dimension trois, on cherche à dénombrer le nombre de paires de cylindres coaxiaux telle que tous les points de P soient compris entre les deux cylindres. Nous démontrons que le nombre de paires de cylindres coaxiaux contenant P est borné par $\Omega(n^3)$ et $O(n^4)$ (la meilleure borne connue était $O(n^5)$).

Mots-clés : Géométrie algorithmique, métrologie, enveloppe convexe

1 Introduction

Given a set P of n points in three dimensions. A cylindrical shell is formed by the space between two coaxial circular cylinders in 3D. The shell is said to enclose P , if the set P is between the two cylinders. The difference between the radii of the two cylinders is the width of the shell.

Finding the minimum width shell enclosing P is an important metrology problem, this width can be viewed as a measure of the quality of cylindrical object. Devillers and Preparata reduce the problem to linear programming in the special case of small width and find a provably good approximation [DP00]. Agarwal, Aronov and Sharir compute in quadratic time a constant factor approximation of the width [AAS00]. Har-Peled and Varadarajan propose to compute an $1 + \epsilon$ approximation in time linear in n and exponential in $\frac{1}{\epsilon}$ [HPV01].

There is few works about exact algorithms to compute the width, In general position, 6 points defined a constant number of shells (less than 150) having these 6 points on their boundary [DMPT01], thus a naive algorithm will have $O(n^7)$ complexity. Agarwal, Aronov and Sharir [AAS00] have proposed a reduction to a convex hull in 10 dimensions which yields to a $O(n^5)$ complexity algorithm.

In this paper, we will prove upper and lower bounds for the number of combinatorially different enclosing cylindrical shells for a set of n points in three dimensional space. These bounds yields to a $O(n^4)$ complexity for the exact algorithm of Agarwal, Aronov and Sharir computing the smallest width enclosing shell.

2 Lower bound example

Theorem 1 *Given n points in 3 dimensions, the number of combinatorially different enclosing shells is $\Omega(n^3)$.*

Proof: We consider three sets of n points:

$$\begin{aligned} p_i &= \left(\frac{i}{n}, 0, 0\right), 1 \leq i \leq n \\ q_i &= \left(\frac{i}{n}, 0, \zeta\right), 1 \leq i \leq n \\ r_i &= \left(x_i, \frac{i}{n}, 0\right), 1 \leq i \leq n \end{aligned}$$

such that the r_i belongs to the circle \mathcal{C} in plane $z = 0$ of center $c = (3, \frac{1}{2}, 0)$ and radius 6 and ζ is some positive parameter to be chosen later (see Figure 1).

Then for any triple (i, j, k) the aim is to define a shell such that the internal cylinder is tangent to line $y = z = 0$ at p_i and to line $y = \zeta - z = 0$ at q_j and such that the external cylinder is tangent (outside) to Circle \mathcal{C} at r_k .

The two first tangencies are granted provided that the axis of the shell is directed by vector $p_i q_j$ and passes through a point of line $p_i y$, that point is chosen at the intersection

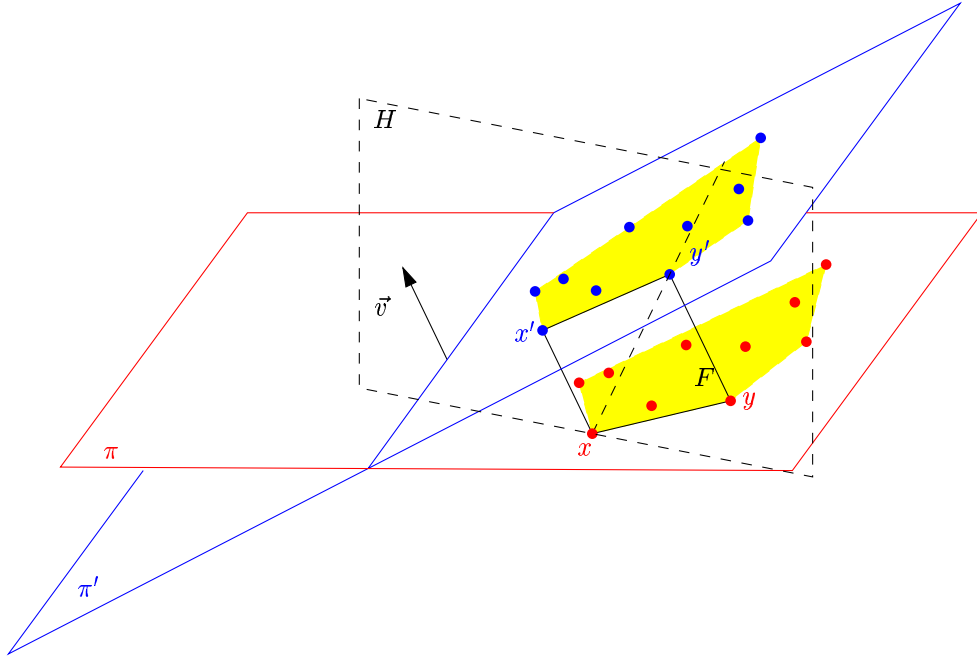


Figure 2: For the proof of Theorem 2.

Theorem 2 Let π, π' be two hyperplanes in \mathbb{R}^d and \vec{v} be a vector of \mathbb{R}^d which does not belong to the directions of π or π' . Let π^+ be one of the half hyperplane of π delimited by π' . Let $\mathcal{U} \subset \pi^+$ be a finite set of n points and $\mathcal{U}' \subset \pi'$ be the projection of \mathcal{U} on π' along direction \vec{v} . Let $\mathcal{V} = \mathcal{U} \cup \mathcal{U}'$. The convex hull $CH(\mathcal{V})$ of \mathcal{V} has the same asymptotic complexity as $CH(\mathcal{U})$, that is $O(n^{\lfloor \frac{d-1}{2} \rfloor})$.

Proof: The space is divided in four quadrants by the hyperplanes π and π' (see Figure 2). Three of these quadrants are obviously separated from \mathcal{V} by π or π' and thus cannot intersect $CH(\mathcal{V})$. Consider a face F of $CH(\mathcal{V})$ in the fourth quadrant, we first claim that \vec{v} belongs to the direction of F . Indeed, if we denote H an hyperplane supporting F and n_H the normal to H , x a vertex of F in π and x' its projection on π' , y' a vertex of F in π' and y its projection on π . Since H is a supporting hyperplane we must have x' and y on the same side of H that is $\text{sign}(x\vec{x}' \cdot n_H) = \text{sign}(y'\vec{y} \cdot n_H)$, but since $x\vec{x}'$ and $y'\vec{y}$ are collinear to \vec{v} and in opposite directions, we get that \vec{v} belongs to H .

In fact we have that the faces of $CH(\mathcal{V})$ are either faces of $CH(\mathcal{U})$, faces of $CH(\mathcal{U}')$ or faces linking a face of $CH(\mathcal{U})$ to its projection in $CH(\mathcal{U}')$. Thus the total number of faces of $CH(\mathcal{V})$ is three times the number of faces of $CH(\mathcal{U})$. ■

Unfortunately, we have not found any suitable generalization of Theorem 2 where π and π' are k -flats with $k \leq d - 2$. For example if $\mathcal{U} = \{p_i = (i, i^2, 0, 0); 1 \leq i \leq n\}$ and $\mathcal{U}' = \{p'_i = (0, 0, i, i^2); 1 \leq i \leq n\}$ then the $O(n^2)$ simplices $(p_i, p_{i+1}, p_j, p_{j+1})$ are convex hull faces.

4 Minimum width cylindrical shell algorithm

Given a set \mathcal{S} of n 3D points Agarwal, Aronov and Sharir [AAS00, page 511 col.2] exhibit a transformation mapping a 3D point $p = (x, y, z)$ in two half spaces P^* and P^\dagger in 10 dimensions :

$$\begin{aligned} P^* : \quad \varphi_9 &\leq (x^2 + y^2) + 2x\varphi_1 + 2y\varphi_2 + 2z\varphi_3 - 2xy\varphi_4 - 2xz\varphi_5 - 2yz\varphi_6 \\ &\quad + (y^2 + z^2)\varphi_7 + (x^2 + z^2)\varphi_8 \\ P^\dagger : \quad \varphi_{10} &\geq (x^2 + y^2) + 2x\varphi_1 + 2y\varphi_2 + 2z\varphi_3 - 2xy\varphi_4 - 2xz\varphi_5 - 2yz\varphi_6 \\ &\quad + (y^2 + z^2)\varphi_7 + (x^2 + z^2)\varphi_8 \end{aligned}$$

where $\varphi_1, \dots, \varphi_{10}$ are the coordinates in 10 dimensions.

Agarwal, Aronov and Sharir proves that the complexity of the intersection of the P^* and P^\dagger bounds the complexity of the enclosing shells, and since we have an intersection of $2n$ half spaces in 10 dimensions get an $O(n^5)$ bound.

An intersection of half spaces problem can be transformed in a convex hull of points problem through duality if we know a point inside the intersection. Here we can remark that for some α large enough (larger than the square distance between the origin and its farthest neighbor in \mathcal{S}), point $(0, 0, 0, 0, 0, 0, 0, 0, \alpha)$ is certainly inside the half spaces. By a translation of that vector, the hyperplane

$$0 = \psi_0 + \sum_{i=1}^{10} \psi_i \varphi_i$$

is first transform in the hyperplane

$$\begin{aligned} 0 &= \psi_0 + \sum_{i=1}^9 \psi_i \varphi_i + \psi_{10}(\varphi_{10} - \alpha + \alpha) \\ 0 &= (\psi_0 + \psi_{10}\alpha) + \sum_{i=1}^{10} \psi_i \phi_i \end{aligned}$$

where ϕ_i are the coordinates in the new frame. This equation is normalized by dividing it by ψ_8 and by duality we get the point

$$\left(\frac{\psi_0 + \psi_{10}\alpha}{\psi_8}, \frac{\psi_1}{\psi_8}, \frac{\psi_2}{\psi_8}, \frac{\psi_3}{\psi_8}, \frac{\psi_4}{\psi_8}, \frac{\psi_5}{\psi_8}, \frac{\psi_6}{\psi_8}, \frac{\psi_7}{\psi_8}, \frac{\psi_9}{\psi_8}, \frac{\psi_{10}}{\psi_8} \right)$$

Through this process, we get to points p^* and p^\dagger

$$\begin{aligned}
p^* &= \left(\frac{x^2 + y^2}{x^2 + z^2}, \frac{2x}{x^2 + z^2}, \frac{2y}{x^2 + z^2}, \frac{2z}{x^2 + z^2}, \right. \\
&\quad \left. \frac{-2xy}{x^2 + z^2}, \frac{-2xz}{x^2 + z^2}, \frac{-2yz}{x^2 + z^2}, \frac{y^2 + z^2}{x^2 + z^2}, \frac{1}{x^2 + z^2}, 0 \right) \\
p^\dagger &= \left(\frac{x^2 + y^2}{x^2 + z^2} + \frac{\alpha}{x^2 + z^2}, \frac{2x}{x^2 + z^2}, \frac{2y}{x^2 + z^2}, \frac{2z}{x^2 + z^2}, \right. \\
&\quad \left. \frac{-2xy}{x^2 + z^2}, \frac{-2xz}{x^2 + z^2}, \frac{-2yz}{x^2 + z^2}, \frac{y^2 + z^2}{x^2 + z^2}, 0, \frac{1}{x^2 + z^2} \right) \\
&= p^* + \frac{1}{x^2 + z^2} (\alpha, 0, 0, 0, 0, 0, 0, 0, -1, 1)
\end{aligned}$$

We remark that p^* belongs to the half-hyperplane $\pi^+ : \phi_{10} = 0, \phi_9 \geq 0$ p^\dagger is the projection of p^* on $\pi' : \phi_9 = 0$ along direction $\vec{v} = (\alpha, 0, 0, 0, 0, 0, 0, 0, -1, 1)$. The hypotheses for Theorem 2 are verified and we conclude that the complexity of the intersections of the P^* and P^\dagger ($p \in \mathcal{S}$) can be reduced to $O(n^4)$. We can state the following theorem, improving the previous result by a factor of n :

Theorem 3 *Given a set \mathcal{S} of n points in \mathbb{R}^3 , a minimum-width cylindrical shell containing \mathcal{S} can be computed in $O(n^4)$ time.*

5 Conclusion and open problems

We have proven that the number of combinatorially different cylindrical shells enclosing a set of n points is $\Omega(n^3)$. This result apply to the number of all shells. Of course in general position, the minimum width shell is unique, we may ask what is the maximal number of minimum width shells in any position and what is the maximal number of shells whose width is locally minimal in general position.

We have also proven that the number of combinatorially different shells is $O(n^4)$ which reduces the complexity of Agarwal, Aronov and Sharir algorithm to that complexity. A clear open problem is to reduce the gap between these lower and upper bounds.

References

- [AAS00] Pankaj K. Agarwal, Boris Aronov, and Micha Sharir. Exact and approximation algorithms for minimum-width cylindrical shells. In *Proc. 11th ACM-SIAM Sympos. Discrete Algorithms*, pages 510–517, 2000 (to appear DCG).

- [DMPT01] Olivier Devillers, Bernard Mourrain, Franco Preparata, and Philippe Trebuchet. On circular cylinders by four or five points in space. Rapport de recherche 4195, INRIA, 2001.
- [DP00] Olivier Devillers and Franco P. Preparata. Evaluating the cylindricity of a nominally cylindrical point set. In *Proc. 11th ACM-SIAM Sympos. Discrete Algorithms*, pages 518–527, 2000.
- [HPV01] Sariel Har-Peled and Kasturi R. Varadarajan. Geometric shape approximation via linearization, 2001. manuscript.



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