

***Nonconservative LMI criteria for characterization of
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Application to stability and input-output analysis of
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Nonconservative LMI criteria for characterization of delay-independent properties of delay systems. Application to stability and input-output analysis of systems with complex parameter

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Abstract: This report focuses on delay-independent stability and delay-independent input-output properties (passivity, H_∞ performance, circle criterion, Popov criterion) of delay systems. The main results show that, as it is the case for the usual, rational, systems, the properties under study here may be characterized by solvability of some linear matrix inequalities. The latter are constructed by use of some quadratic Lyapunov-Krasovskii functionals, generalizing a well-known class. The method also applies to the analysis of systems with uncertain complex parameter, for which the results are related to the search for analytic parameter-dependent Lyapunov functions. Illustrative examples are provided.

Key-words: linear delay systems, parameter-dependent systems, delay-independent stability, quadratic Lyapunov-Krasovskii functionals, linear matrix inequalities, input-output properties, passivity, H_∞ performance, circle criterion, Popov criterion.

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Caractérisation des propriétés indépendantes du retard des systèmes à retard par des critères LMI non conservatifs.

Application à l'étude de la stabilité et des propriétés entrée-sortie des systèmes à paramètre complexe

Résumé : Ce rapport est consacré à l'étude de la stabilité indépendante du retard et de propriétés entrée-sortie indépendantes du retard (passivité, performance H_∞ , critère du cercle, critère de Popov) des systèmes à retard. Les résultats principaux montrent que, comme c'est le cas pour les systèmes rationnels usuels, les propriétés étudiées ici peuvent être caractérisées par la solvabilité d'inégalités linéaires matricielles (*linear matrix inequalities, LMI* en anglais). Ces dernières sont construites à partir de fonctionnelles quadratiques de Lyapunov-Krasovskii généralisant une classe bien connue. La méthode employée s'applique également à l'analyse des systèmes à paramètre complexe incertain, pour lesquels les résultats sont liés à la recherche de fonctions de Lyapunov dépendant de manière analytique du paramètre. Des exemples numériques illustrent les résultats.

Mots-clés : Systèmes linéaires à retard, systèmes dépendant d'un paramètre, stabilité indépendante du retard, fonctionnelles de Lyapunov-Krasovskii quadratiques, inégalités linéaires matricielles, propriétés entrée-sortie, passivité, performance H_∞ , critère du cercle, critère de Popov.

1 Introduction

1.1 Presentation of the problem and background

Our primary concern in the present paper is the study of *delay-independent stability and input-output properties* for the following class of linear delay systems

$$\dot{x} = A_0x(t) + A_1x(t-h) + B_0u(t) + B_1u(t-h), \quad y(t) = C_0x(t) + C_1x(t-h) + D_0u(t) + D_1u(t-h) \quad (S_h)$$

with input u and output y , where, for $n, p, q \in \mathbb{N}$, $(A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times p} \times \mathbb{R}^{q \times p}$. We mean by this, properties which are verified for any nonnegative value of the delay $h \geq 0$. As a matter of fact, the delays are usually imperfectly known: a way to ensure robustness with respect to this uncertainty is un fact to employ delay-independent properties. More precisely, we shall study here internal stability, passivity, \mathcal{H}_∞ performance analysis, circle criterion and Popov criterion.

Classical argument [20, 21, 15] permits to relate the analysis of delay-independent properties of S_h , to the study of properties of a class of systems with a complex parameter z , $|z| \leq 1$. This new variable replaces in the transfer the occurrences of e^{-sh} , $h \geq 0$ (s is the Laplace variable). The progress made in the domain of delay systems thus give the possibility to apply the same methods and results to the study of the following class of finite-dimensional systems, parametrized by the complex parameter $|z| \leq 1$,

$$\dot{x} = (A_0 + zA_1)x + (B_0 + zB_1)u, \quad y = (C_0 + zC_1)x + (D_0 + zD_1)u. \quad (S_z)$$

As it is usually the case in robust control, the complex parameter z may play the role of some unbounded unmodeled dynamics. When studying system S_z , one allows the matrices of the system to be complex.

The properties we are about to study are defined in terms of the transfer H of size $q \times p$ with *two* frequency domain variables, defined by

$$H(s, z) \stackrel{\text{def}}{=} (C_0 + zC_1)(sI_n - A_0 - zA_1)^{-1}(B_0 + zB_1) + D_0 + zD_1.$$

For the usual subclass of rational systems obtained for $(A_1, B_1, C_1, D_1) = (0, 0, 0, 0)$, characterization of stability and input-output properties by *linear matrix inequalities* (further abbreviated LMIs) is a classical result, see e.g. [5]. Recall the practical benefit of the use of LMIs: their resolution is a standard convex optimization problem [5], achievable in polynomial-time, and for which very efficient interior-point methods have been developed and are available as toolboxes in widely-spread control-oriented scientific softwares, such as MATLAB or SCILAB. From a methodological point of view, in order to determine the exact form of the LMI adapted to the property under study, a crucial role is played by the quadratic function $x(t)^T P x(t)$, where the positive definite matrix P is the unknown of the LMI.

In the study of delay-independent properties of linear delay systems such as S_h , it has been observed that the class of quadratic Lyapunov-Krasovskii functionals

$$x(t)^T P x(t) + \int_{t-h}^t x(\tau)^T Q x(\tau) d\tau, \quad (1)$$

parametrized by the two matrices $P = P^T > 0, Q = Q^T > 0$, instead of $x(t)^T P x(t)$ for the finite-dimensional systems, permits to derive various sufficient LMI conditions. Delay-independent stability criteria have been obtained lately [25, 14, 39], conditions for delay-independent guaranteed L^2 cost have been found [30, 10, 41, 11], as well as estimates of delay-independent H_∞ performance [27, 8, 24, 18] or delay-independent passivity conditions [31]. All these analysis results are conservative, as they provide sufficient, not necessary, conditions (performance upper bounds). Other references may be found in [32].

On the other hand, *parameter-dependent Lyapunov functions* [16, 17] have been proposed for the study of systems with parameters, especially *affine* parameter-dependent Lyapunov functions [13, 12]. These approaches too, provide numerically checkable, conservative, conditions.

The properties under study here may be easily proved to be equivalent to the existence, for any complex z with $|z| \leq 1$, of a hermitian positive definite solution $P(z)$ to a certain *parameter-dependent* LMI. Following the result by Delchamps [9] on regularity of the solutions of Riccati equation wrt coefficients, result

generalized by Kamen in [22], one deduces that $P(z)$ may be supposed *analytic in z and z^** . In other words, there exists a family of hermitian matrices $\tilde{P}_k \in \mathbb{R}^{kn \times kn}$ such that

$$P(z) = \lim_{k \rightarrow +\infty} \begin{pmatrix} I_n & & & \\ & z^* I_n & & \\ & & \dots & \\ & & & z^{(k-1)*} I_n \end{pmatrix} \tilde{P}_k \begin{pmatrix} I_n \\ z I_n \\ \dots \\ z^{k-1} I_n \end{pmatrix}, \quad (2)$$

where the limit is uniform in the complex unitary disk. So the assumption of affine dependency of the Lyapunov function wrt the parameter is generally speaking quite restrictive.

1.2 Presentation of the results

The central idea for obtention of the criteria provided in the present paper, corresponds to an extension of the class of functionals (1). For any positive integer k , define

$$\mathcal{X}_k(t) \stackrel{\text{def}}{=} \begin{pmatrix} x(t) \\ x(t-h) \\ \vdots \\ x(t-(k-1)h) \end{pmatrix}, \quad \mathcal{U}_k(t) \stackrel{\text{def}}{=} \begin{pmatrix} u(t) \\ u(t-h) \\ \vdots \\ u(t-(k-1)h) \end{pmatrix}, \quad \mathcal{Y}_k(t) \stackrel{\text{def}}{=} \begin{pmatrix} y(t) \\ y(t-h) \\ \vdots \\ y(t-(k-1)h) \end{pmatrix} \quad \text{for system } S_h, \quad (3a)$$

$$\mathcal{X}_k(t) \stackrel{\text{def}}{=} \begin{pmatrix} I_n \\ z I_n \\ \vdots \\ z^{k-1} I_n \end{pmatrix} x(t), \quad \mathcal{U}_k(t) \stackrel{\text{def}}{=} \begin{pmatrix} I_p \\ z I_p \\ \vdots \\ z^{k-1} I_p \end{pmatrix} u(t), \quad \mathcal{Y}_k(t) \stackrel{\text{def}}{=} \begin{pmatrix} I_q \\ z I_q \\ \vdots \\ z^{k-1} I_q \end{pmatrix} y(t) \quad \text{for system } S_z. \quad (3b)$$

The LMIs obtained in the present contribution corresponds to the search, in appropriate context, for quadratic functionals of the type

$$V_k(x(\cdot))(t) \stackrel{\text{def}}{=} \mathcal{X}_k(t)^T P_k \mathcal{X}_k(t) + \int_{t-h}^t \mathcal{X}_k(\tau)^T Q_k \mathcal{X}_k(\tau) d\tau, \quad (4a)$$

$P_k, Q_k \in \mathbb{R}^{kn \times kn}$ for the stability, or

$$V_k(x(\cdot), u(\cdot))(t) \stackrel{\text{def}}{=} \mathcal{X}_k(t)^T P_k \mathcal{X}_k(t) + \int_{t-h}^t \begin{pmatrix} \mathcal{X}_k(\tau) \\ \mathcal{U}_k(\tau) \end{pmatrix}^T Q_k \begin{pmatrix} \mathcal{X}_k(\tau) \\ \mathcal{U}_k(\tau) \end{pmatrix} d\tau, \quad (4b)$$

$P_k \in \mathbb{R}^{kn \times kn}$, $Q_k \in \mathbb{R}^{k(n+p) \times k(n+p)}$ for the input-output properties. For the parametrized system S_z , this corresponds to the use of quadratic functionals of the type

$$V_k(x(\cdot))(t) \stackrel{\text{def}}{=} \mathcal{X}_k(t)^* P_k \mathcal{X}_k(t) + (1 - |z|^2) \int_0^t \mathcal{X}_k(\tau)^* Q_k \mathcal{X}_k(\tau) d\tau,$$

$$V_k(x(\cdot), u(\cdot))(t) \stackrel{\text{def}}{=} \mathcal{X}_k(t)^* P_k \mathcal{X}_k(t) + (1 - |z|^2) \int_0^t \begin{pmatrix} \mathcal{X}_k(\tau) \\ \mathcal{U}_k(\tau) \end{pmatrix}^* Q_k \begin{pmatrix} \mathcal{X}_k(\tau) \\ \mathcal{U}_k(\tau) \end{pmatrix} d\tau,$$

for hermitian matrices P_k, Q_k . To each property under study will be associated a *family of LMIs* indexed by the positive integer k , and whose unknowns are the matrices P_k and Q_k . As usual, these LMIs are constructed in order that their solvability ensures decreasingness of adequate quantity, along the trajectories of the system.

The core of our contribution may be summarized as follows. For each family of LMIs, we show that solvability for positive rank k implies that the associated stability or input-output property holds. Also, solvability for rank k implies solvability for $k' \geq k$, so these sufficient conditions are more and more precise (less and less conservative). A key feature is that they are also *necessary*, in the sense that: if a certain property holds, then the corresponding LMIs are fulfilled *from a certain rank k and beyond*. Thus, the

conservatism vanishes asymptotically. The stability and input-output properties under study are hence *characterized* by solvability of some LMI problems. To our knowledge, no contribution of this nature has been previously published in the literature, for the kind of properties studied here.

The result presented above is not surprising, when interpreted in terms of parameter-dependent Lyapunov functions. To any family of matrices \tilde{P}_k defining the expansion (2) of a solution $P(z)$ for the underlying parameter-dependent LMI, one may biunivocally associate a family of square matrices P_k of size kn such that

$$P(z) = \lim_{k \rightarrow +\infty} \frac{1}{1 + |z|^2 + \dots + |z|^{2(k-1)}} (I_n \quad z^* I_n \quad \dots \quad z^{(k-1)*} I_n) P_k \begin{pmatrix} I_n \\ z I_n \\ \dots \\ z^{k-1} I_n \end{pmatrix}. \quad (5)$$

Moreover, in the situations we shall study, it may be proved that the existence of expansion (2) for $P(z)$ definite positive for any $|z| \leq 1$, is *equivalent* to the existence of an expansion of type (5) with $P_k > 0$.

It turns out that the P_k are the same than in the LMIs based on functionals (4). More precisely, for any $k \in \mathbb{N}$ and any positive definite P_k of size kn , the formula

$$\frac{1}{1 + |z|^2 + \dots + |z|^{2(k-1)}} (I_n \quad z^* I_n \quad \dots \quad z^{(k-1)*} I_n) P_k \begin{pmatrix} I_n \\ z I_n \\ \dots \\ z^{k-1} I_n \end{pmatrix}$$

defines, for any $z, |z| \leq 1$, a solution of the underlying parameter-dependent LMI, *if and only if* there exists Q_k such that (P_k, Q_k) fulfills the corresponding LMI at rank k . In conclusion, solvability of the LMIs at rank k is equivalent to existence of a parameter-dependent quadratic Lyapunov functions $x^*(t)P(z)x(t)$ for system S_z , polynomial in z, z^* of degree $k - 1$.

The organisation of the paper is as follows. Notations are given in Section 2, and some preparatory work is done. The studied properties are rapidly presented, and the associated LMIs and parameter-dependent LMIs are given in Section 3. The main results are enunciated in Section 4 (Theorems 1 and 2, related to systems S_z and S_h respectively). Three numerical examples illustrating the method are given in Section 5. The proofs of the results are provided in Section 6. Concluding remarks are given in Section 7. Discrete-time and continuous-time versions of Kalman-Yakubovich-Popov lemma being used as a central tool in the demonstrations, a well-suited version is reproduced in Appendix.

2 Notations and preparatory results

- The matrices $I_n, 0_n, 0_{n \times p}$ are resp. the $n \times n$ identity matrix and the $n \times n$ and $n \times p$ zero matrices, sometimes simply abbreviated $I, 0$. The symbol \otimes denotes Kronecker product. The conjugate and transconjugate of M , are denoted M^T and M^* . \mathbb{N} is the set of positive integers, and by definition $\overline{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cup \{\infty\}$. By $\overline{\mathbb{D}}$ is denoted the closed unit ball in \mathbb{C} . The unit circle is denoted as the boundary $\partial\mathbb{D}$. By $\overline{\mathbb{C}^+}$ is meant the closed set of complex numbers with nonnegative real part. Last, for sake of space, we shall use the abbreviations $R_{k,(12b)}$ and $\mathcal{R}_{(12a)}(z)$, to designate the matrix R_k defined in (12b) and the function $\mathcal{R}(z)$ defined in (12a), and so on ...

- Define, for any $k \in \mathbb{N}$, the matrices $J_{n,k}^u, J_{n,k}^l \in \mathbb{R}^{kn \times (k+1)n}$ (u, l as *upper, lower*), $J_{n,k} \in \mathbb{R}^{2kn \times (k+1)n}$ by

$$J_{n,k}^u \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn \times n} \end{pmatrix}, \quad J_{n,k}^l \stackrel{\text{def}}{=} \begin{pmatrix} 0_{kn \times n} & I_{kn} \end{pmatrix}, \quad J_{n,k} \stackrel{\text{def}}{=} \begin{pmatrix} J_{n,k}^u \\ J_{n,k}^l \end{pmatrix} = \begin{pmatrix} I_{kn} & 0_{kn \times n} \\ 0_{kn \times n} & I_{kn} \end{pmatrix},$$

and similarly $J_{p,k}^u, J_{p,k}^l, J_{p,k}$. Define also $J_k^u, J_k^l \in \mathbb{R}^{k(n+p) \times (k+1)(n+p)}$, $J_k \in \mathbb{R}^{2k(n+p) \times (k+1)(n+p)}$ by

$$J_k^u \stackrel{\text{def}}{=} \begin{pmatrix} J_{n,k}^u & 0_{kn \times (k+1)p} \\ 0_{kp \times (k+1)n} & J_{p,k}^u \end{pmatrix}, \quad J_k^l \stackrel{\text{def}}{=} \begin{pmatrix} J_{n,k}^l & 0_{kn \times (k+1)p} \\ 0_{kp \times (k+1)n} & J_{p,k}^l \end{pmatrix}, \quad J_k \stackrel{\text{def}}{=} \begin{pmatrix} J_{n,k} & 0_{2kn \times (k+1)p} \\ 0_{2kp \times (k+1)n} & J_{p,k} \end{pmatrix}.$$

Define at last $\tilde{J}_{n,k} \in \mathbb{R}^{3kn \times (k+2)n}$, $\tilde{J}_k \in \mathbb{R}^{3k(n+p) \times (k+2)(n+p)}$, as

$$\tilde{J}_{n,k} \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & & 0_{kn \times 2n} \\ 0_{kn \times n} & I_{kn} & 0_{kn \times n} \\ 0_{kn \times 2n} & & I_{kn} \end{pmatrix}, \quad \tilde{J}_k \stackrel{\text{def}}{=} \begin{pmatrix} \tilde{J}_{n,k} & 0_{3kn \times (k+2)p} \\ 0_{3kp \times (k+2)n} & \tilde{J}_{p,k} \end{pmatrix}.$$

- For $z \in \mathbb{C}$, let $v_k(z) \in \mathbb{C}^k$,

$$v_k(z) \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ \vdots \\ z^{k-1} \end{pmatrix}.$$

This quantity, already used in Section 1, will intervene naturally in the forthcoming calculations. Indeed,

$$\mathcal{X}_k = v_k(\nabla) \otimes x \text{ for system } S_h, \quad \mathcal{X}_k = v_k(z) \otimes x \text{ for system } S_z,$$

where the operator ∇ is defined by: $(\nabla x)(t) = x(t-h)$, and $(\nabla^k x)(t) \stackrel{\text{def}}{=} (\nabla \circ \nabla^{k-1} x)(t) = x(t-kh)$.

One shows easily that

$$J_{n,k}(v_{k+1}(z) \otimes I_n) = \begin{pmatrix} v_k(z) \otimes I_n \\ z v_k(z) \otimes I_n \end{pmatrix}, \quad \tilde{J}_{n,k}(v_{k+2}(z) \otimes I_n) = \begin{pmatrix} v_k(z) \otimes I_n \\ z v_k(z) \otimes I_n \\ z^2 v_k(z) \otimes I_n \end{pmatrix}.$$

For system S_h , this corresponds in time-domain to the identities

$$J_{n,k} \mathcal{X}_{k+1}(t) = \begin{pmatrix} \mathcal{X}_k(t) \\ \mathcal{X}_k(t-h) \end{pmatrix}, \quad \tilde{J}_{n,k} \mathcal{X}_{k+2}(t) = \begin{pmatrix} \mathcal{X}_k(t) \\ \mathcal{X}_k(t-h) \\ \mathcal{X}_k(t-2h) \end{pmatrix}. \quad (6)$$

Similarly,

$$J_k \begin{pmatrix} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{pmatrix} = \begin{pmatrix} \mathcal{X}_k(t) \\ \mathcal{X}_k(t-h) \\ \mathcal{U}_k(t) \\ \mathcal{U}_k(t-h) \end{pmatrix}, \quad J_k^u \begin{pmatrix} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{pmatrix} = \begin{pmatrix} \mathcal{X}_k(t) \\ \mathcal{U}_k(t) \end{pmatrix}, \quad J_k^l \begin{pmatrix} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{pmatrix} = \begin{pmatrix} \mathcal{X}_k(t-h) \\ \mathcal{U}_k(t-h) \end{pmatrix}. \quad (7)$$

- Last are given some functional sets. Define the following sets of functions on $\overline{\mathbb{D}}$, with values in $\mathbb{C}^{n \times n}$:

$$\Phi_n \stackrel{\text{def}}{=} \{P \in [\mathbb{C}^{n \times n}]^{\overline{\mathbb{D}}} : \forall k \in \mathbb{N}, \exists P_k \in \mathbb{C}^{kn \times kn}, P(\cdot) = \lim_{k \rightarrow +\infty} (v_k(\cdot) \otimes I_n)^* P_k (v_k(\cdot) \otimes I_n)\},$$

$$\Phi_n^+ \stackrel{\text{def}}{=} \{P \in \Phi_n : \forall k \in \mathbb{N}, P_k = P_k^* \geq 0\},$$

$$\Psi_n \stackrel{\text{def}}{=} \{P \in [\mathbb{C}^{n \times n}]^{\overline{\mathbb{D}}} : \forall k \in \mathbb{N}, \exists P_k \in \mathbb{C}^{kn \times kn}, P_k = P_k^* > 0, P(\cdot) = \lim_{k \rightarrow +\infty} \frac{1}{\|v_k(\cdot)\|^2} (v_k(\cdot) \otimes I_n)^* P_k (v_k(\cdot) \otimes I_n)\}.$$

The limits in the previous expressions are to be understood in the sense of the uniform convergence in $\overline{\mathbb{D}}$. The set Φ_n is hence the closure of the set of complex matrices of size $n \times n$, whose coefficients are polynomial in z, z^* . The elements of Φ_n are analytic in the two variables z, z^* in every point of $\overline{\mathbb{D}}$. The coefficients themselves converge when k goes to infinity: for any element of Φ_n , one may assume without loss of generality that, for any integer $k \geq 2$,

$$P_{k-1} = \begin{pmatrix} I_{(k-1)n} & 0_{(k-1)n \times n} \end{pmatrix} P_k \begin{pmatrix} I_{(k-1)n} \\ 0_{n \times (k-1)n} \end{pmatrix}. \quad (8)$$

The fact that Φ_n^+ is stable by addition will be used.

3 Delay-independent properties of delay systems and related LMIs

In the present section, we present shortly the properties that will be studied and introduce the associated LMIs and parameter-dependent LMIs. The LMIs have the general form

$$P_k = P_k^* > 0, Q_k = Q_k^*, R_k(P_k, Q_k) < 0,$$

for different expressions of R_k .

3.1 Delay-independent stability

One considers here the asymptotic stability of system S_h with $u \equiv 0$. This property is equivalent to

$$\forall s \in \overline{\mathbb{C}^+}, \det(sI_n - A_0 - e^{-sh}A_1) \neq 0.$$

In order that this property is fulfilled for any $h \geq 0$, it is sufficient that $A_0 + zA_1$ is Hurwitz for any $z \in \overline{\mathbb{D}}$, i.e. that S_z is stable for any $z \in \overline{\mathbb{D}}$. This may be characterized by the existence, for any z in $\overline{\mathbb{D}}$, of a hermitian definite positive solution $P(z)$ to the following *parameter-dependent* LMI:

$$(A_0 + zA_1)^*P(z) + P(z)(A_0 + zA_1) < 0. \quad (9a)$$

One the other hand, it may be easily verified that, along the trajectories of S_h ,

$$\begin{aligned} \frac{d}{dt}[V_{k,(4a)}(x(\cdot))(t)] &= \begin{pmatrix} \mathcal{X}_k(t) \\ \mathcal{X}_k(t-h) \end{pmatrix}^T \left(\begin{pmatrix} P_k(I_k \otimes A_0) + (I_k \otimes A_0)^T P_k & P_k(I_k \otimes A_1) \\ (I_k \otimes A_1)^T P_k & 0_{kn} \end{pmatrix} + \begin{pmatrix} Q_k & 0_n \\ 0_n & -Q_k \end{pmatrix} \right) \begin{pmatrix} \mathcal{X}_k(t) \\ \mathcal{X}_k(t-h) \end{pmatrix} \\ &= \mathcal{X}_{k+1}(t)^T R_k \mathcal{X}_{k+1}(t), \end{aligned}$$

where, due to (6),

$$R_k(P_k, Q_k) \stackrel{\text{def}}{=} J_{n,k}^T \left[\begin{pmatrix} P_k(I_k \otimes A_0) + (I_k \otimes A_0)^T P_k & P_k(I_k \otimes A_1) \\ (I_k \otimes A_1)^T P_k & 0_{kn} \end{pmatrix} + \begin{pmatrix} Q_k & 0_n \\ 0_n & -Q_k \end{pmatrix} \right] J_{n,k}. \quad (9b)$$

Hence, it is likely that existence of positive P_k, Q_k ensuring negativeness of $R_{k,(9b)}$ is sufficient to ensure that $A_0 + zA_1$ is Hurwitz for any $z \in \overline{\mathbb{D}}$. The same considerations apply to the study of the trajectories of S_z .

3.2 Delay-independent passivity

A notion useful for stability and stabilization purpose is the *passivity* of system S_h , that is internal stability and strict positive realness of $-H(s, e^{-sh})$. Same argument than before shows that this property is fulfilled for any $h \geq 0$, provided that, $\forall z \in \overline{\mathbb{D}}$,

$$\begin{pmatrix} P(z)(A_0 + zA_1) + (A_0 + zA_1)^*P(z) & P(z)(B_0 + zB_1) + (C_0 + zC_1)^* \\ (B_0 + zB_1)^*P(z) + (C_0 + zC_1) & (D_0 + zD_1) + (D_0 + zD_1)^* \end{pmatrix} < 0. \quad (10a)$$

The correct LMI characterization of delay-independent passivity will be deduced from the fact that

$$\frac{d}{dt}[V_{k,(4b)}(x(\cdot), u(\cdot))(t)] = \begin{pmatrix} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{pmatrix}^T R_k \begin{pmatrix} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{pmatrix} + \mathcal{U}_k(t)^T \mathcal{Y}_k(t),$$

where

$$\begin{aligned}
R_k(P_k, Q_k) \stackrel{\text{def}}{=} & J_k^T \left[\left(\begin{array}{c|ccc} P_k(I_k \otimes A_0) + (I_k \otimes A_0)^* P_k & P_k(I_k \otimes A_1) & P_k(I_k \otimes B_0) & P_k(I_k \otimes B_1) \\ \hline (I_k \otimes A_1)^* P_k & & & \\ (I_k \otimes B_0)^* P_k & & & \\ (I_k \otimes B_1)^* P_k & & & \end{array} \right) \right. \\
& + \left(\begin{array}{c} 0_{2kn \times kp} \\ I_{kp} \\ 0_{kp} \end{array} \right) \left((I_k \otimes C_0) \quad (I_k \otimes C_1) \quad (I_k \otimes D_0) \quad (I_k \otimes D_1) \right) \\
& \left. + \left(\begin{array}{c} (I_k \otimes C_0)^* \\ (I_k \otimes C_1)^* \\ (I_k \otimes D_0)^* \\ (I_k \otimes D_1)^* \end{array} \right) \left(0_{kp \times 2kn} \quad I_{kp} \quad 0_{kp} \right) \right] J_k + J_k^{uT} Q_k J_k^u - J_k^{lT} Q_k J_k^l .
\end{aligned} \tag{10b}$$

As a matter of fact, it ensues from the previous identity and the formulas in (7), that, whenever $R_{k,(10b)} < 0$ and the integral converges,

$$\int_0^{+\infty} u(t)^T y(t) dt = \frac{1}{k} \int_0^{+\infty} \mathcal{U}_k(t)^T \mathcal{Y}_k(t) dt \geq -V_{k,(4b)}(x(\cdot), u(\cdot))(0) .$$

3.3 Delay-independent \mathcal{H}_∞ performance

When S_h is internally stable and the input u is seen as a perturbation, a possible way of evaluating its influence on the output y is to compute the norm of the corresponding input-output operator. Considering the latter as a $L^2 \rightarrow L^2$ map, yields the \mathcal{H}_∞ -norm

$$\|H(s, e^{-sh})\|_\infty \stackrel{\text{def}}{=} \sup_{\omega \in \mathbb{R}} \|H(j\omega, e^{-j\omega h})\| ,$$

where the matrix norm is the largest singular value. The appropriate parameter-dependent LMI and LMI are here as follows.

$$\left(\begin{array}{cc} P(z)(A_0 + zA_1) + (A_0 + zA_1)^* P(z) & P(z)(B_0 + zB_1) \\ (B_0 + zB_1)^* P(z) & -I_p \end{array} \right) + \left(\begin{array}{c} (C_0 + zC_1)^* \\ (D_0 + zD_1)^* \end{array} \right) \left(\begin{array}{cc} (C_0 + zC_1) & (D_0 + zD_1) \end{array} \right) < 0 . \tag{11a}$$

$$\begin{aligned}
R_k(P_k, Q_k) \stackrel{\text{def}}{=} & J_k^T \left[\left(\begin{array}{c|ccc} P_k(I_k \otimes A_0) + (I_k \otimes A_0)^* P_k & P_k(I_k \otimes A_1) & P_k(I_k \otimes B_0) & P_k(I_k \otimes B_1) \\ \hline (I_k \otimes A_1)^* P_k & & & \\ (I_k \otimes B_0)^* P_k & & & \\ (I_k \otimes B_1)^* P_k & & & \end{array} \right) \right. \\
& + \left(\begin{array}{c} (I_k \otimes C_0)^* \\ (I_k \otimes C_1)^* \\ (I_k \otimes D_0)^* \\ (I_k \otimes D_1)^* \end{array} \right) \left((I_k \otimes C_0) \quad (I_k \otimes C_1) \quad (I_k \otimes D_0) \quad (I_k \otimes D_1) \right) - \left(\begin{array}{c|cc} 0_{2kn} & 0_{2kn \times 2kp} \\ \hline 0_{2kp \times 2kn} & I_{kp} \quad 0_{kp} \\ & 0_{kp} \quad 0_{kp} \end{array} \right) \\
& \left. + J_k^{uT} Q_k J_k^u - J_k^{lT} Q_k J_k^l \right] J_k .
\end{aligned} \tag{11b}$$

One has here

$$\frac{d}{dt} [V_{k,(4b)}(x(\cdot), u(\cdot))(t)] = \left(\begin{array}{c} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{array} \right)^T R_{k,(11b)} \left(\begin{array}{c} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{array} \right) + \mathcal{U}_k(t)^T \mathcal{U}_k(t) - \mathcal{Y}_k(t)^T \mathcal{Y}_k(t) ,$$

in such a way that, when $R_{k,(11b)} < 0$ and u is square summable,

$$\int_0^{+\infty} (u(t)^T u(t) - y(t)^T y(t)) dt = \frac{1}{k} \int_0^{+\infty} (\mathcal{U}_k(t)^T \mathcal{U}_k(t) - \mathcal{Y}_k(t)^T \mathcal{Y}_k(t)) dt \geq -V_{k,(4b)}(x(\cdot), u(\cdot))(0) .$$

3.4 Delay-independent circle criterion

When system S_h is closed by a nonlinear element $u = f(t, y)$ fulfilling the following sector condition, defined by the matrix $K \in \mathbb{R}^{p \times q}$:

$$\forall t \geq 0, \forall y \in \mathbb{R}^q, f(t, y)^T (Ky - f(t, y)) \geq 0 ,$$

a way to ensure asymptotic stability of the closed-loop is to verify *circle criterion* [35, 42], that is internal stability plus strict positive realness of $I_p - KH(s, e^{-sh})$. This case will be treated with the following notations.

$$\mathcal{R}(z) \stackrel{\text{def}}{=} \begin{pmatrix} (A_0 + zA_1)^* P(z) + P(z)(A_0 + zA_1) & (C_0 + zC_1)^* K^T + P(z)(B_0 + zB_1) \\ K(C_0 + zC_1) + (B_0 + zB_1)^* P(z) & -2I_p + K(D_0 + zD_1) + (D_0 + zD_1)^* K^T \end{pmatrix} < 0 . \quad (12a)$$

$$\begin{aligned} R_k(P_k, Q_k) &\stackrel{\text{def}}{=} J_k^T \left[\left(\begin{array}{c|ccc} P_k(I_k \otimes A_0) + (I_k \otimes A_0)^* P_k & P_k(I_k \otimes A_1) & P_k(I_k \otimes B_0) & P_k(I_k \otimes B_1) \\ \hline (I_k \otimes A_1)^* P_k & & & \\ (I_k \otimes B_0)^* P_k & & & \\ (I_k \otimes B_1)^* P_k & & & \end{array} \right) \right. \\ &+ \begin{pmatrix} 0_{2kn \times kp} \\ I_{kp} \\ 0_{kp} \end{pmatrix} \left((I_k \otimes KC_0) \quad (I_k \otimes KC_1) \quad (I_k \otimes KD_0) \quad (I_k \otimes KD_1) \right) \\ &+ \left((I_k \otimes KC_0) \quad (I_k \otimes KC_1) \quad (I_k \otimes KD_0) \quad (I_k \otimes KD_1) \right)^* \begin{pmatrix} 0_{kp \times 2kn} & I_{kp} & 0_{kp} \end{pmatrix} \\ &\left. - 2 \begin{pmatrix} 0_{2kn \times kp} \\ I_{kp} \\ 0_{kp} \end{pmatrix} \begin{pmatrix} 0_{kp \times 2kn} & I_{kp} & 0_{kp} \end{pmatrix} \right] J_k + J_k^{uT} Q_k J_k^u - J_k^{lT} Q_k J_k^l . \end{aligned} \quad (12b)$$

When $R_{k,(12b)} < 0$, one has

$$\frac{d}{dt} [V_{k,(4b)}(x(\cdot), u(\cdot))(t)] = \begin{pmatrix} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{pmatrix}^T R_{k,(12b)} \begin{pmatrix} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{pmatrix} - \mathcal{U}_k(t)^T ((I_k \otimes K) \mathcal{Y}_k(t) - \mathcal{U}_k(t)) ,$$

which is nonpositive, provided that f fulfills the sector condition.

3.5 Uniform delay-independent Popov criterion

When system S_h is square ($p = q$) and closed by a nonlinear *time-invariant, decentralized* [23] element $u = f(y)$ fulfilling sector condition

$$\forall y \in \mathbb{R}^p, f(y)^T (Ky - f(y)) \geq 0 ,$$

when moreover $D_0 = D_1 = 0$, the results from Section 3.4 may be improved: a way to ensure asymptotic stability of the closed-loop is to verify *Popov criterion* [33, 29], i.e. internal stability and strict positive realness of $I_p - (I_p + \eta s)KH(s, e^{-sh})$, for a certain (non necessarily nonnegative [1]) diagonal matrix $\eta \in \mathbb{R}^{p \times p}$. We provide here necessary and sufficient condition for a delay-independent version of Popov criterion where the slope η *does not depend upon* h . We call this *uniform* delay-independent Popov criterion.

Using the quantities already defined in Section 3.4, we fix here:

$$\mathcal{R}_{(12a)}(z) + \frac{k-1}{k} \begin{pmatrix} 0_n & (\eta K(C_0 + zC_1)(A_0 + zA_1))^* \\ \eta K(C_0 + zC_1)(A_0 + zA_1) & \eta K(C_0 + zC_1)(B_0 + zB_1) + (\eta K(C_0 + zC_1)(B_0 + zB_1))^* \end{pmatrix} < 0. \quad (13a)$$

$$\begin{aligned} R_k(P_k, Q_k, \eta) &\stackrel{\text{def}}{=} R_{k,(12b)} + \tilde{J}_{k-1}^T \begin{bmatrix} \begin{pmatrix} (I_{k-1} \otimes \eta K C_0 A_0)^* \\ (I_{k-1} \otimes \eta K C_0 A_1)^* \\ 0_{(k-1)n \times (k-1)p} \\ (I_{k-1} \otimes \eta K C_0 B_0)^* \\ (I_{k-1} \otimes \eta K C_0 B_1)^* \\ 0_{(k-1)p} \end{pmatrix} & (0_{(k-1)p \times 3(k-1)n} & I_{(k-1)p} & 0_{(k-1)p \times 2(k-1)p}) \end{bmatrix} \\ &+ \begin{pmatrix} 0_{3(k-1)n \times (k-1)p} \\ I_{(k-1)p} \\ 0_{2(k-1)p \times (k-1)p} \end{pmatrix} \begin{pmatrix} (I_{k-1} \otimes \eta K C_0 A_0) & (I_{k-1} \otimes \eta K C_0 A_1) & 0_{(k-1)p \times (k-1)n} & (I_{k-1} \otimes \eta K C_0 B_0) & (I_{k-1} \otimes \eta K C_0 B_1) & 0_{(k-1)p} \\ 0_{(k-1)p \times (k-1)n} & (I_{k-1} \otimes \eta K C_1 A_0) & (I_{k-1} \otimes \eta K C_1 A_1) & 0_{(k-1)p} & (I_{k-1} \otimes \eta K C_1 B_0) & (I_{k-1} \otimes \eta K C_1 B_1) \end{pmatrix} \\ &+ \begin{pmatrix} 0_{(k-1)n \times (k-1)p} \\ (I_{k-1} \otimes \eta K C_1 A_0)^* \\ (I_{k-1} \otimes \eta K C_1 A_1)^* \\ 0_{(k-1)p} \\ (I_{k-1} \otimes \eta K C_1 B_0)^* \\ (I_{k-1} \otimes \eta K C_1 B_1)^* \end{pmatrix} \begin{pmatrix} 0_{(k-1)p \times 3(k-1)n} & I_{(k-1)p} & 0_{(k-1)p \times 2(k-1)p} \end{pmatrix} \tilde{J}_{k-1}. \end{aligned} \quad (13b)$$

From the computations in Section 3.4 and (7), one gets

$$\begin{aligned} \frac{d}{dt} [V_{k,(4b)}(x(\cdot), u(\cdot))(t)] &= \begin{pmatrix} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{pmatrix}^T R_{k,(13b)} \begin{pmatrix} \mathcal{X}_{k+1}(t) \\ \mathcal{U}_{k+1}(t) \end{pmatrix} - \mathcal{U}_k(t)^T ((I_k \otimes K) \mathcal{Y}_k(t) - \mathcal{U}_k(t)) \\ &\quad - 2 \frac{d}{dt} \left[\sum_{k'=0}^{k-2} \sum_{i=1}^p \eta_i K_i \int_0^{y_i(t-k'h)} f_i(v) dv \right], \end{aligned}$$

which is nonpositive, provided that $R_{k,(13b)} < 0$ and f fulfills the sector condition. In the last term may be recognized the derivative of a Lur'e term, equal to $-2(I_{k-1} \otimes \eta K) \mathcal{Y}_{k-1}(t)^T \mathcal{U}_{k-1}(t)$.

4 Main results

The results are first expressed in terms of properties of the class of systems S_z , $z \in \overline{\mathbb{D}}$.

Theorem 1. (LMI characterization of properties of the parametrized system S_z) Consider the system S_z with complex coefficients.

- System S_z is asymptotically stable for any $z \in \overline{\mathbb{D}}$, if and only if there exist $k \in \mathbb{N}$ and hermitian matrices $P_k \in \mathbb{C}^{kn \times kn}$, $Q_k \in \mathbb{C}^{kn \times kn}$ such that P_k and $-R_{k,(9b)}(P_k, Q_k) \in \mathbb{C}^{(k+1)n \times (k+1)n}$ are hermitian definite positive.

- System S_z is passive for any $z \in \overline{\mathbb{D}}$, if and only if there exist $k \in \mathbb{N}$ and hermitian matrices $P_k \in \mathbb{C}^{kn \times kn}$, $Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}$ such that P_k and $-R_{k,(10b)}(P_k, Q_k) \in \mathbb{C}^{(k+1)(n+p) \times (k+1)(n+p)}$ are hermitian definite positive.

- System S_z is internally stable with \mathcal{H}_∞ -norm $\|H(s, z)\|_\infty$ less than 1 for any $z \in \overline{\mathbb{D}}$, if and only if there exist $k \in \mathbb{N}$ and hermitian matrices $P_k \in \mathbb{C}^{kn \times kn}$, $Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}$ such that P_k and $-R_{k,(11b)}(P_k, Q_k) \in \mathbb{C}^{(k+1)(n+p) \times (k+1)(n+p)}$ are hermitian definite positive.

- System S_z is internally stable with $I_p - KH(s, z)$ strictly positive real for any $z \in \overline{\mathbb{D}}$, if and only if there exist $k \in \mathbb{N}$ and hermitian matrices $P_k \in \mathbb{C}^{kn \times kn}$, $Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}$ such that P_k and $-R_{k,(12b)}(P_k, Q_k) \in \mathbb{C}^{(k+1)(n+p) \times (k+1)(n+p)}$ are hermitian definite positive.

• Let $\eta \in \mathbb{R}^{p \times p}$ be a diagonal matrix. If $p = q$ and $D_0 = D_1 = 0$, then system S_z is internally stable with $I_p - (I_p + \eta s)KH(s, z)$ strictly positive real for any $z \in \overline{\mathbb{D}}$, if and only if there exist $k \in \mathbb{N}$ and hermitian matrices $P_k \in \mathbb{C}^{kn \times kn}$, $Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}$ such that P_k and $-R_{k, (13b)}(P_k, Q_k, \frac{k}{k-1}\eta) \in \mathbb{C}^{(k+1)(n+p) \times (k+1)(n+p)}$ are hermitian definite positive.

• In every case, if the LMI associated to the problem is feasible for $k \in \mathbb{N}$, then the LMIs corresponding to $k' \geq k$ are also solvable. ■

Proof of Theorem 1 is given in Section 6.

The previous results may be recast in terms of *scaled small-gain conditions*, involving *structured singular values* [43]. As an example, when $p = q$, $A_0 + zA_1$ is Hurwitz and $\|H(s, z)\|_\infty < 1$ for all $z \in \overline{\mathbb{D}}$, if and only if

$$\sup_{s \in j\mathbb{R}} \mu_\Delta \left(\begin{pmatrix} (sI_n - A_0)^{-1}A_1 & (sI_n - A_0)^{-1} & 0_{n \times q} & (sI_n - A_0)^{-1}B_0 \\ 0_n & 0_n & 0_{n \times q} & B_1 \\ 0_{q \times n} & 0_{q \times n} & 0_q & D_1 \\ C_0(sI_n - A_0)^{-1}A_1 + C_1 & C_0(sI_n - A_0)^{-1} & I_q & C_0(sI_n - A_0)^{-1}B_0 + D_0 \end{pmatrix} \right) < 1, \quad (14)$$

where the uncertainty set Δ is equal to $\{\text{diag}\{zI_{2n+q}; Z\} : z \in \mathbb{C}, Z \in \mathbb{C}^{q \times q}\}$. To prove this, express $H(s, z)$ as the linear fractional transformation

$$C_0(sI_n - A_0)^{-1}B_0 + D_0 + z \begin{pmatrix} C_0(sI_n - A_0)^{-1}A_1 + C_1 & C_0(sI_n - A_0)^{-1} & I_q \end{pmatrix} \left(I_{2n+q} - \begin{pmatrix} (sI_n - A_0)^{-1}A_1 & (sI_n - A_0)^{-1} & 0_{n \times q} \\ 0_n & 0_n & 0_{n \times q} \\ 0_{q \times n} & 0_{q \times n} & 0_q \end{pmatrix} \right)^{-1} \begin{pmatrix} (sI_n - A_0)^{-1}B_0 \\ B_1 \\ D_1 \end{pmatrix}$$

and use the Main loop theorem [43, Theorem 11.7]. The usual method for numerical verification of condition (14) involves time-consuming frequency sweeping and use of upper bound of μ_Δ , which is here equal to μ_Δ [43].

Sweeping may be avoided, using the fact that (14) is equivalent to

$$\mu_\Delta \left(\begin{pmatrix} (I_n - A_0)^{-1}(I_n + A_0) & \sqrt{2}(I_n - A_0)^{-1}A_1 & \sqrt{2}(I_n - A_0)^{-1} & 0_{n \times q} & \sqrt{2}(I_n - A_0)^{-1}B_0 \\ \sqrt{2}(I_n - A_0)^{-1} & (I_n - A_0)^{-1}A_1 & (I_n - A_0)^{-1} & 0_{n \times q} & (I_n - A_0)^{-1}B_0 \\ 0_n & 0_n & 0_n & 0_{n \times q} & B_1 \\ 0_q & 0_{q \times n} & 0_{q \times n} & 0_q & D_1 \\ \sqrt{2}C_0(I_n - A_0)^{-1}A_1 + C_1 & C_0(I_n - A_0)^{-1} & I_q & C_0(I_n - A_0)^{-1}B_0 + D_0 & \end{pmatrix} \right) < 1, \quad (15)$$

with here $\Delta = \{\text{diag}\{z_s I_n; z I_{2n+q}; Z\} : z_s, z \in \mathbb{C}, Z \in \mathbb{C}^{q \times q}\}$. However, up to our knowledge, there does not exist method of exact evaluation for this structured singular value.

The results of Theorem 1 offer an attractive numerical alternative, as resolution of LMIs is a convex problem, solvable in polynomial time. Computation of the structured singular values is known to be a NP-hard problem (see [6, 38], and specifically [37] for problems related to delay systems). The results in Theorem 1 hence furnishes approximation of some NP problems, of arbitrary precision, by families of P problems. Their main weakness, in the present state, is *non-decidability*: no information is known on the size of the least k , if any, for which the LMIs are solvable (in other words, of the largest k which is necessary to test numerically to get an exact conclusion on the veracity of the property under study). This is an important complexity issue, both from theoretical and practical point of view. On the other hand, some numerical experiments (see below Section 5) indicate that small values of k (typically smaller than the dimension n of the state variable of the system) often yield correct answers.

We now express the results adapted to the issue of characterization of delay-independent properties of system S_h . It turns out that almost no generality is lost when replacing in the transfer the terms e^{-sh} , $h \geq 0$, by z , $|z| \leq 1$. Indeed, the set of the systems fulfilling a property of the parametrized systems S_z , constitutes the *interior* of the set of the systems fulfilling the corresponding delay-independent property of S_h . The topology which is meant, is the topology induced by the norm of the largest singular value on the product space $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times p} \times \mathbb{R}^{q \times p}$ of the coefficients $(A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1)$.

In particular, the LMI conditions we obtain characterize “robustified versions” (with respect to infinitesimal uncertainties on the matrices defining S_h) of the delay-independent properties.

Theorem 2. (LMI characterization of delay-independent properties of the delay system S_h)
Consider the system S_h with real coefficients.

- *All the systems in a neighborhood of S_h are asymptotically stable for any $h \geq 0$, if and only if there exist $k \in \mathbb{N}$ and symmetric matrices $P_k \in \mathbb{R}^{kn \times kn}$, $Q_k \in \mathbb{R}^{kn \times kn}$ such that P_k and $-R_{k,(9b)}(P_k, Q_k) \in \mathbb{R}^{(k+1)n \times (k+1)n}$ are symmetric definite positive.*

- *All the systems in a neighborhood of S_h are passive for any $h \geq 0$, if and only if there exist $k \in \mathbb{N}$ and symmetric matrices $P_k \in \mathbb{R}^{kn \times kn}$, $Q_k \in \mathbb{R}^{k(n+p) \times k(n+p)}$ such that P_k and $-R_{k,(10b)}(P_k, Q_k) \in \mathbb{R}^{(k+1)(n+p) \times (k+1)(n+p)}$ are symmetric definite positive.*

- *All the systems in a neighborhood of S_h are internally stable with \mathcal{H}_∞ -norm $\|H(s, e^{-sh})\|_\infty$ less than 1 for any $h \geq 0$, if and only if there exist $k \in \mathbb{N}$ and symmetric matrices $P_k \in \mathbb{R}^{kn \times kn}$, $Q_k \in \mathbb{R}^{k(n+p) \times k(n+p)}$ such that P_k and $-R_{k,(11b)}(P_k, Q_k) \in \mathbb{R}^{(k+1)(n+p) \times (k+1)(n+p)}$ are symmetric definite positive.*

- *All the systems in a neighborhood of S_h verify circle criterion for any $h \geq 0$, if and only if there exist $k \in \mathbb{N}$ and symmetric matrices $P_k \in \mathbb{R}^{kn \times kn}$, $Q_k \in \mathbb{R}^{k(n+p) \times k(n+p)}$ such that P_k and $-R_{k,(12b)}(P_k, Q_k) \in \mathbb{R}^{(k+1)(n+p) \times (k+1)(n+p)}$ are symmetric definite positive.*

- *Let $\eta \in \mathbb{R}^{p \times p}$ be a diagonal matrix. If $p = q$ and $D_0 = D_1 = 0$, then all the systems in a neighborhood of S_h fulfill Popov criterion with slope η for any $h \geq 0$, if and only if there exist $k \in \mathbb{N}$ and symmetric matrices $P_k \in \mathbb{R}^{kn \times kn}$, $Q_k \in \mathbb{R}^{k(n+p) \times k(n+p)}$ such that P_k and $-R_{k,(13b)}(P_k, Q_k, \frac{k}{k-1}\eta) \in \mathbb{R}^{(k+1)(n+p) \times (k+1)(n+p)}$ are symmetric definite positive.*

- *In every case, if the LMI associated to the problem is feasible for $k \in \mathbb{N}$, then the LMIs corresponding to $k' \geq k$ are also solvable. ■*

Proof of Theorem 2 is given in Section 6. The result on Popov theorem may be adapted to treat the case of systems with time-varying nonlinearities with time derivative fulfilling a sector condition [2].

Remark 3. It may be proved that if the LMI condition (9b) admits a solution (P_k, Q_k) with P_k positive, there is no loss in assuming that Q_k is positive definite too. Whether this is the case for the other criteria is an open question. It may conjectured that there is no loss of generality in assuming that $Q_k > 0$ too. Indeed, both options have been tested in the numerical examples presented in Section 5 below (with and without the assumption $Q_k > 0$), showing no difference. See as an example Table 1, where the computation times are compared (generally speaking, it seems that the latter are increased when the constraint $Q_k > 0$ is added).

However, it must be emphasized that the functionals (4) play a central role in the derivation of the LMI criteria, as demonstrated in Section 3, although they are positive definite only if P_k and Q_k are themselves positive definite in the various LMIs. □

5 Numerical examples

We present here three academical examples, in order to test the algorithms, to check the coherency of their results and to provide an indication on the computation times necessitated by the resolution of the LMIs.

The computations presented here have been achieved using the SCILAB package `lmitool`¹. The indication of the CPU time necessary for the resolution of the LMIs has been measured on a computer equipped with a Pentium III 800MHz.

¹Scilab is a free software developed by INRIA, which is distributed with all its source code. For the distribution and details, see Scilab’s homepage on the web at the address <http://www-rocq.inria.fr/scilab/>

5.1 Example 1 – Delay-independent \mathcal{H}_∞ performance

System S_h is considered, with $n = 2, p = 1, q = 2$ and

$$\begin{aligned} A_0 &= \begin{pmatrix} -8 & 19 \\ -4 & 3 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 \\ -2 & 3 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \\ C_0 &= \begin{pmatrix} 3 & 4 \\ 4 & -2 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -2 & 5 \\ -4 & -1 \end{pmatrix}, \quad D_0 = \begin{pmatrix} 5 \\ 9 \end{pmatrix}, \quad D_1 = \begin{pmatrix} -3 \\ -4 \end{pmatrix}. \end{aligned}$$

One wishes to estimate the supremum over $h \geq 0$ of the norms $\|H(s, e^{-sh})\|_\infty$.

The properties of the system are numerically evaluated as follows. One maximizes, for different values of the integer index k , the value of the parameter $\lambda > 0$ such that there exist P_k, Q_k, R_k positive, such that $R_{(11b)}(P_k, Q_k)$ is negative, with C_0 , resp. C_1, D_0, D_1 , replaced by λC_0 , resp. $\lambda C_1, \lambda D_0, \lambda D_1$. The corresponding maximal value is denoted λ_k . Then, from Theorem 2,

$$\lambda_k^{-1/2} \searrow \lim_{\varepsilon \searrow 0} \sup_{H_\varepsilon \in B(H; \varepsilon)} \sup_{h \geq 0} \|H_\varepsilon(s, e^{-sh})\|_\infty.$$

Here, $B(H; \varepsilon)$ represents the ball of center H and radius $\varepsilon > 0$ in the product metric space given before Theorem 2.

Independently, the value of $\|H(s, e^{-sh})\|_\infty$ is also computed directly for different values of $h \geq 0$. The effective value, evaluated by gridding, is found to be

$$\sup_{h \geq 0} \|H(s, e^{-sh})\|_\infty = 64.3.$$

The results are given in Table 1. Addition of the constraint $Q_k > 0$ when solving the LMI (see Remark 3) does not change the results, but increases the computation times.

The value of $\|H(s, e^{-sh})\|_\infty$ as functions of h are plotted in Figure 1. The successive estimates of the norms are also indicated. The curve $\omega \mapsto \|H(j\omega, e^{-j\omega h})\|$ is provided in Figure 2 for $h = 3$ as an example, showing already brutal variations.

As an example, one may conclude from what precedes that the feedback interconnection of $H(s, e^{-sh})$ with any stable proper transfer $G(s)$ of norm $\|G\|_\infty < \rho$, $\rho \geq 0$ fixed, is well-posed and internally stable [43] for any $h \geq 0$, *if and only if*

$$\rho \leq 1/64.3.$$

5.2 Example 2 – Delay-independent absolute stability

One now consider the system S_h , with $n = 2, p = 1, q = 1$, given by

$$\begin{aligned} A_0 &= \begin{pmatrix} -8 & 19 \\ -4 & 3 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 \\ -2 & 3 \end{pmatrix}, \quad B_0 = \begin{pmatrix} -4 \\ 12 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \\ C_0 &= \begin{pmatrix} 3 & -1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -2 & 5 \end{pmatrix}, \quad D_0 = 0, \quad D_1 = 0. \end{aligned} \tag{16}$$

One wishes to determine the largest sector $[0, K]$ for which absolute stability is ensured, for every nonnegative value of h .

Let us first use circle criterion. One evaluates, for different values of k , the largest K_k for which there exist $P_k = P_k^T > 0, Q_k = Q_k^T$ such that the corresponding value $R_{k,(12b)}$ is negative. The corresponding problem is a LMI in the three unknowns P_k, Q_k, K . The results are summarized in Table 2.

When the nonlinearity does not depend upon time, the uniform delay-independent Popov criterion applies. It leads to the resolution of a problem affine in the unknowns P_k, Q_k, K plus a new scalar variable replacing ηK . Dividing the value found for this variable by the value of $kK/(k-1)$ gives value for the Popov slope η . The successive estimates K_k, η_k are given in Table 3.

Figure 3 shows the Popov curves obtained for different values of h . The three Popov loci do not intersect neither the vertical line corresponding to the largest gain K found by delay-independent circle criterion, nor

the Popov line found by uniform delay-independent Popov criterion. The tangency to these lines show the optimality of the estimate of K_k for $k \geq 2$.

One concludes from this analysis that:

- the system S_h defined by (16) and closed by a nonlinearity $u = f(t, y)$ in the sector $[0, K]$ fulfills circle criterion for any $h \geq 0$, *if and only if* $K \leq 0.03147$. This ensures asymptotic stability for any $h \geq 0$.
- the system S_h defined by (16) and closed by a time-invariant nonlinearity $u = f(y)$ in the sector $[0, K]$ fulfills Popov criterion for any $h \geq 0$, with a Popov line independent of h , *if and only if* $K \leq 0.05233$. This ensures asymptotic stability for any $h \geq 0$.

5.3 Example 3 – Asymptotic stability of some real systems with two real parameters.

Let us consider the following real system, affine in the two real parameters z_r, z_i .

$$\begin{pmatrix} \dot{x}_r \\ \dot{x}_i \end{pmatrix} = \begin{pmatrix} A_{0r} + z_r A_{1r} - z_i A_{1i} & -A_{0i} - z_r A_{1i} - z_i A_{1r} \\ A_{0i} + z_r A_{1i} + z_i A_{1r} & A_{0r} + z_r A_{1r} - z_i A_{1i} \end{pmatrix} \begin{pmatrix} x_r \\ x_i \end{pmatrix}. \quad (17)$$

One wishes to determine the largest $\rho > 0$ such that system (17) is asymptotically stable, *for any* z_r, z_i such that $z_r^2 + z_i^2 \leq \rho^2$.

Putting $x \stackrel{\text{def}}{=} x_r + ix_i$, $z \stackrel{\text{def}}{=} z_r + iz_i$, $A_0 \stackrel{\text{def}}{=} A_{0r} + iA_{0i}$, $A_1 \stackrel{\text{def}}{=} A_{1r} + iA_{1i}$, system (17) writes as the complex system

$$\dot{x} = (A_0 + zA_1)x.$$

Take here $n = 4$,

$$A_0 \stackrel{\text{def}}{=} \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}, \quad A_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The successive estimates ρ_k of the largest ρ ensuring robust asymptotic stability of (17) are given in Table 4. They are compared in Figure 4 to the exact value of ρ , equal to [7]

$$\left(\sup_{\omega \in \mathbb{R}} \rho((j\omega I_n - A_0)^{-1} A_1) \right)^{-1}.$$

The CPU time given is the duration of the test for the value of ρ_k indicated.

From the achieved analysis, it turns out that system (17) is asymptotically stable for any z_r, z_i such that $z_r^2 + z_i^2 \leq \rho^2$, *if and only if*

$$\rho \leq 0.6504.$$

6 Proofs

The general organisation of the proofs is as follows. On the one hand, the stability and input-output properties, expressed in frequency domain with the two variables s, z , are shown to be equivalent to solvability of some LMIs parametrized by the variable z (see inequalities (9a), (10a), (11a), (12a), (13a) above). This comes from an usual application of (continuous-time) Kalman-Yakubovich-Popov lemma. On the other hand, one shows that solvability of the LMIs with unknowns P_k, Q_k given in the statement of Theorems 1 and 2 (and associated to formulas (9b), (10b), (11b), (12b), (13b) above) is equivalent to the existence of solution to the same parameter-dependent LMI, polynomial in the variables z and z^* , of degree k . This

ensues from discrete-time version of Kalman-Yakubovich-Popov lemma. These assertions are exposed in Section 6.2 (Lemma 4).

The result permitting to link the stability and input-output properties to the LMIs, is then a result on *analyticity of solutions of LMIs with respect to their coefficients*. It is stated and proved in Section 6.3 (Lemma 5). The two previous ingredients are put together to prove Theorem 1 (Section 6.4), and then Theorem 2 (Section 6.5). Some additional notations are first given in Section 6.1.

6.1 Additional notations and technical results

We begin by some notations necessary to the proofs.

- Let

$$F_k \stackrel{\text{def}}{=} \begin{pmatrix} 0_{1 \times (k-1)} & 0 \\ I_{k-1} & 0_{(k-1) \times 1} \end{pmatrix}, f_k \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 0_{(k-1) \times 1} \end{pmatrix}.$$

The size of the previous matrices is $f_k: k \times 1$, $F_k: k \times k$, and the spectrum of F_k is $\{0\}$. These matrices permit some useful manipulations of the matrices J previously defined in Section 2, which will now be exposed.

The matrix $J_{n,k}$ admits the following block decomposition:

$$J_{n,k} = \begin{pmatrix} I_{kn} & 0_{kn \times n} \\ 0_{kn \times n} & I_{kn} \end{pmatrix} = \begin{pmatrix} f_k \otimes I_n & F_k \otimes I_n \\ 0_{kn \times n} & I_{kn} \end{pmatrix}. \quad (18)$$

Define

$$\mathcal{Z}_{n,k}(z) \stackrel{\text{def}}{=} (I_{kn} - z(F_k \otimes I_n))^{-1}, \mathcal{Z}_k(z) \stackrel{\text{def}}{=} \left(I_{k(n+p)} - z \begin{pmatrix} F_k \otimes I_n & 0_{kn \times kp} \\ 0_{kp \times kn} & F_k \otimes I_p \end{pmatrix} \right)^{-1} = \begin{pmatrix} \mathcal{Z}_{n,k}(z) & 0_{kn \times kp} \\ 0_{kp \times kn} & \mathcal{Z}_{p,k}(z) \end{pmatrix}.$$

Simple computations show recursively that

$$\mathcal{Z}_{n,k}(z)(f_k \otimes I_n) = v_k(z) \otimes I_n, \mathcal{Z}_k(z) \begin{pmatrix} f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{pmatrix} = \begin{pmatrix} v_k(z) \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & v_k(z) \otimes I_p \end{pmatrix}.$$

- Define the matrix $N_k \in \mathbb{R}^{(k+1)(n+p) \times (k+1)(n+p)}$ by

$$N_k \stackrel{\text{def}}{=} \begin{pmatrix} I_n & 0_{n \times p} & 0_{n \times kn} & 0_{n \times kp} \\ 0_{kn \times n} & 0_{kn \times p} & I_{kn} & 0_{kn \times kp} \\ 0_{p \times n} & I_p & 0_{p \times kn} & 0_{p \times kp} \\ 0_{kp \times n} & 0_{kp \times p} & 0_{kp \times kn} & I_{kp} \end{pmatrix}.$$

N_k is a permutation matrix, so $N_k^{-1} = N_k^T$. Using the properties established before, one shows easily that

$$\begin{aligned} J_k N_k \left(z \mathcal{Z}_k(z) \begin{pmatrix} I_{n+p} & \\ f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{pmatrix} \right) &= J_k N_k \left(z \begin{pmatrix} I_{n+p} & \\ v_k(z) \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & v_k(z) \otimes I_p \end{pmatrix} \right) \\ &= J_k \begin{pmatrix} v_{k+1}(z) \otimes I_n & 0_{(k+1)n \times p} \\ 0_{(k+1)p \times n} & v_{k+1}(z) \otimes I_p \end{pmatrix} = \begin{pmatrix} v_k(z) \otimes I_n & 0_{kn \times p} \\ z v_k(z) \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & v_k(z) \otimes I_p \\ 0_{kp \times n} & z v_k(z) \otimes I_p \end{pmatrix}, \end{aligned} \quad (19a)$$

and similarly

$$\begin{aligned} \tilde{J}_{k-1} N_k \left(z Z_k(z) \begin{pmatrix} I_{n+p} & \\ f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{pmatrix} \right) &= \tilde{J}_{k-1} \begin{pmatrix} v_{k+1}(z) \otimes I_n & 0_{(k+1)n \times p} \\ 0_{(k+1)p \times n} & v_{k+1}(z) \otimes I_p \end{pmatrix} \\ &= \begin{pmatrix} v_{k-1}(z) \otimes I_n & 0_{(k-1)n \times p} \\ z v_{k-1}(z) \otimes I_n & 0_{(k-1)n \times p} \\ z^2 v_{k-1}(z) \otimes I_n & 0_{(k-1)n \times p} \\ 0_{(k-1)p \times n} & v_{k-1}(z) \otimes I_p \\ 0_{(k-1)p \times n} & z v_{k-1}(z) \otimes I_p \\ 0_{(k-1)p \times n} & z^2 v_{k-1}(z) \otimes I_p \end{pmatrix}. \end{aligned} \quad (19b)$$

6.2 Properties of LMIs and parameter-dependent LMIs

Lemma 4. • For any $z \in \overline{\mathbb{D}}$, $A_0 + zA_1$ is Hurwitz if and only if (9a) admits a hermitian positive definite solution $P(z)$. For any $k \in \mathbb{N}$ and any hermitian positive definite matrix $P_k \in \mathbb{C}^{kn \times kn}$, $P(z)$ given by

$$\frac{1}{\|v_k(z)\|^2} (v_k(z) \otimes I_n)^* P_k (v_k(z) \otimes I_n) \quad (20)$$

is a hermitian positive definite solution of (9a) for any $z \in \partial\mathbb{D}$, if and only if there exists a hermitian matrix $Q_k \in \mathbb{C}^{kn \times kn}$ such that $R_{k,(9b)}(P_k, Q_k) \in \mathbb{C}^{(k+1)n \times (k+1)n}$ is hermitian definite negative.

• For any $z \in \overline{\mathbb{D}}$, $A_0 + zA_1$ is Hurwitz and $H(s, z)$ is strictly positive real if and only if (10a) admits a hermitian positive definite solution $P(z)$. For any $k \in \mathbb{N}$ and any hermitian positive definite matrix $P_k \in \mathbb{C}^{kn \times kn}$, $P(z)$ given by (20) is a hermitian positive definite solution of (10a) for any $z \in \partial\mathbb{D}$, if and only if there exists a hermitian matrix $Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}$ such that $R_{k,(10b)}(P_k, Q_k) \in \mathbb{C}^{(k+1)(n+p) \times (k+1)(n+p)}$ is hermitian definite negative,

• For any $z \in \overline{\mathbb{D}}$, $A_0 + zA_1$ is Hurwitz and $\|H(s, z)\|_\infty < 1$ if and only if (11a) admits a hermitian positive definite solution $P(z)$. For any $k \in \mathbb{N}$ and any hermitian positive definite matrix $P_k \in \mathbb{C}^{kn \times kn}$, $P(z)$ given by (20) is a hermitian positive definite solution of (11a) for any $z \in \partial\mathbb{D}$, if and only if there exists a hermitian matrix $Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}$ such that $R_{k,(11b)}(P_k, Q_k) \in \mathbb{C}^{(k+1)(n+p) \times (k+1)(n+p)}$ is hermitian definite negative.

• For any $z \in \overline{\mathbb{D}}$, $A_0 + zA_1$ is Hurwitz and $I_p - KH(s, z)$ is strictly positive real if and only if (12a) admits a hermitian positive definite solution $P(z)$. For any $k \in \mathbb{N}$ and any hermitian positive definite matrix $P_k \in \mathbb{C}^{kn \times kn}$, $P(z)$ given by (20) is a hermitian positive definite solution of (12a) for any $z \in \partial\mathbb{D}$, if and only if there exists a hermitian matrix $Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}$ such that $R_{k,(12b)}(P_k, Q_k) \in \mathbb{C}^{(k+1)(n+p) \times (k+1)(n+p)}$ is hermitian definite negative.

• Suppose $p = q$, $D_0 = D_1 = 0$, and let $\eta \in \mathbb{R}^{p \times p}$ be a diagonal matrix. For any $z \in \overline{\mathbb{D}}$, for any $k \in \mathbb{N}$, $A_0 + zA_1$ is Hurwitz and $I_p - (I_p + \frac{k-1}{k}\eta s)KH(s, z)$ is strictly positive real if and only if (13a) admits a hermitian positive definite solution $P(z)$. For any $k \in \mathbb{N}$ and any positive definite matrix $P_k \in \mathbb{C}^{kn \times kn}$, $P(z)$ given by (20) is a hermitian positive definite solution of (13a) for any $z \in \partial\mathbb{D}$, if and only if there exists a hermitian matrix $Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}$ such that $R_{k,(13b)}(P_k, Q_k, \eta) \in \mathbb{C}^{(k+1)(n+p) \times (k+1)(n+p)}$ is hermitian definite negative.

• When the system under study is real, one may replace everywhere “hermitian” (resp. \mathbb{C}) by “symmetric” (resp. \mathbb{R}) in the previous statements. ■

Proof. • The first part of the statements, establishing an equivalence between internal stability plus strict positive realness of ad hoc transfers with parameter $z \in \overline{\mathbb{D}}$, and existence of positive definite solution for (9a), (10a), (11a), (12a), (13a), is obtained by classical invocation of the continuous-time version of Kalman-Yakubovich-Popov lemma (see Lemma A.2 and the subsequent Remark A.3).

• Let us show now the second part of the statements, e.g. for the circle criterion issue. Consider the two terms containing Q_k in formulas (12b). Using the block decomposition (18), one shows that $J_k^{uT} Q_k J_k^u - J_k^{lT} Q_k J_k^l$

is equal to

$$N_k \left(\begin{array}{cc} \left(\begin{array}{cc} f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{array} \right)^T Q_k \left(\begin{array}{cc} f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{array} \right) & \left(\begin{array}{cc} f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{array} \right)^T Q_k \left(\begin{array}{cc} F_k \otimes I_n & 0_{kn \times kp} \\ 0_{kp \times kn} & F_k \otimes I_p \end{array} \right) \\ \left(\begin{array}{cc} F_k \otimes I_n & 0_{kn \times kp} \\ 0_{kp \times kn} & F_k \otimes I_p \end{array} \right)^T Q_k \left(\begin{array}{cc} f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{array} \right) & \left(\begin{array}{cc} F_k \otimes I_n & 0_{kn \times kp} \\ 0_{kp \times kn} & F_k \otimes I_p \end{array} \right)^T Q_k \left(\begin{array}{cc} F_k \otimes I_n & 0_{kn \times kp} \\ 0_{kp \times kn} & F_k \otimes I_p \end{array} \right) - Q_k \end{array} \right) N_k^T .$$

The spectrum of

$$\begin{pmatrix} F_k \otimes I_n & 0_{kn \times kp} \\ 0_{kp \times kn} & F_k \otimes I_p \end{pmatrix}$$

is $\{0\}$. One may hence apply the discrete time version of Kalman-Yakubovich-Popov lemma (see Lemma A.1). One gets that there exists $Q_k = Q_k^*$ such that $R_{k,(12b)}(P_k, Q_k) < 0$ if and only if: $\forall z \in \partial\mathbb{D}$,

$$\left(\begin{array}{cc} I_{n+p} & \\ z \mathcal{Z}_k(z) \left(\begin{array}{cc} f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{array} \right) \end{array} \right)^* N_k^T R_{k,(12b)}(P_k, 0_{k(n+p)}) N_k \left(\begin{array}{cc} I_{n+p} & \\ z \mathcal{Z}_k(z) \left(\begin{array}{cc} f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{array} \right) \end{array} \right) < 0 .$$

Using formula (19a) and developing, the previous expression is shown to be equal to

$$\begin{aligned} & \left(\begin{array}{cc} v_k(z) \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & v_k(z) \otimes I_p \end{array} \right)^* \left[\begin{array}{cc} \left(P_k(I_k \otimes (A_0 + zA_1)) + (I_k \otimes (A_0 + zA_1))^* P_k & P_k(I_k \otimes (B_0 + zB_1)) \right) \\ & (I_k \otimes (B_0 + zB_1))^* P_k & 0_{kp} \end{array} \right) \\ & + \left(\begin{array}{cc} 0_{kn} & 0_{kn \times kp} \\ (I_k \otimes K(C_0 + zC_1)) & (I_k \otimes K(D_0 + zD_1)) \end{array} \right) + \left(\begin{array}{cc} 0_{kn} & (I_k \otimes K(C_0 + zC_1))^* \\ 0_{kp \times kn} & (I_k \otimes K(D_0 + zD_1))^* \end{array} \right) \\ & - 2 \left(\begin{array}{cc} 0_{kn} & 0_{kn \times kp} \\ 0_{kp \times kn} & I_{kp} \end{array} \right) \left[\begin{array}{cc} v_k(z) \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & v_k(z) \otimes I_p \end{array} \right) , \end{aligned} \quad (21)$$

that is

$$\|v_k(z)\|^2 \begin{pmatrix} (A_0 + zA_1)^* P(z) + P(z)(A_0 + zA_1) & (C_0 + zC_1)^* K^T + P(z)(B_0 + zB_1) \\ K(C_0 + zC_1) + (B_0 + zB_1)^* P(z) & -2I_p + K(D_0 + zD_1) + (D_0 + zD_1)^* K^T \end{pmatrix} , \quad (22)$$

where $P(z)$ is defined by (20). Facts such as

$$(I_k \otimes (A_0 + zA_1))(v_k(z) \otimes I_n) = (v_k(z) \otimes I_n)(A_0 + zA_1), \quad (v_k(z) \otimes I_n)^*(v_k(z) \otimes I_n) = \|v_k(z)\|^2 I_n ,$$

have been used in the derivation. This achieves the proof for the circle criterion.

To treat Popov criterion, one just needs to put $D_0 = D_1 = 0$ in the preceding computations and to add to (21) the term

$$\begin{aligned} & \left(\begin{array}{cc} v_{k-1}(z) \otimes I_n & 0_{(k-1)n \times p} \\ 0_{(k-1)p \times n} & v_{k-1}(z) \otimes I_p \end{array} \right)^* \left[\begin{array}{cc} 0_{(k-1)n} & 0_{(k-1)n \times (k-1)p} \\ (I_{k-1} \otimes \eta K(C_0 + zC_1)(A_0 + zA_1)) & (I_{k-1} \otimes \eta K(C_0 + zC_1)(B_0 + zB_1)) \end{array} \right) \\ & + \left(\begin{array}{cc} 0_{(k-1)n} & (I_{k-1} \otimes \eta K(C_0 + zC_1)(A_0 + zA_1))^* \\ 0_{(k-1)n \times (k-1)p} & (I_{k-1} \otimes \eta K(C_0 + zC_1)(B_0 + zB_1))^* \end{array} \right) \left(\begin{array}{cc} v_{k-1}(z) \otimes I_n & 0_{(k-1)n \times p} \\ 0_{(k-1)p \times n} & v_{k-1}(z) \otimes I_p \end{array} \right) , \end{aligned}$$

which amounts to add to (22) the term

$$\frac{k-1}{k} \|v_k(z)\|^2 \begin{pmatrix} 0_n & (\eta K(C_0 + zC_1)(A_0 + zA_1))^* \\ \eta K(C_0 + zC_1)(A_0 + zA_1) & \eta K(C_0 + zC_1)(B_0 + zB_1) + (\eta K(C_0 + zC_1)(B_0 + zB_1))^* \end{pmatrix} .$$

To obtain the previous formula, (19b) has to be employed, together with the fact that, for any $z \in \partial\mathbb{D}$, $\|v_{k-1}(z)\|^2 / \|v_k(z)\|^2 = (k-1)/k$.

Small adaptations of the same techniques permit to treat the other cases. For the \mathcal{H}_∞ performance analysis issue, it suffices to remark that

$$\begin{aligned} & \left(z\mathcal{Z}_k(z) \begin{pmatrix} I_{n+p} & & \\ f_k \otimes I_n & 0_{kn \times p} & \\ 0_{kp \times n} & f_k \otimes I_p & \end{pmatrix} \right)^* \left[\begin{pmatrix} (I_k \otimes C_0)^T \\ (I_k \otimes C_1)^T \\ (I_k \otimes D_0)^T \\ (I_k \otimes D_1)^T \end{pmatrix} \begin{pmatrix} (I_k \otimes C_0) & (I_k \otimes C_1) & (I_k \otimes D_0)^T & (I_k \otimes D_1) \end{pmatrix} \right. \\ & - \left. \left(\begin{array}{c|c} 0_{2kn} & 0_{2kn \times 2kp} \\ \hline 0_{2kp \times 2kn} & \begin{pmatrix} I_{kp} & 0_{kp} \\ 0_{kp} & 0_{kp} \end{pmatrix} \end{array} \right) \begin{pmatrix} I_{n+p} & & \\ z\mathcal{Z}_k(z) \begin{pmatrix} f_k \otimes I_n & 0_{kn \times p} \\ 0_{kp \times n} & f_k \otimes I_p \end{pmatrix} & & \end{pmatrix} \right) \\ & = \|v_k(z)\|^2 \left(\begin{pmatrix} 0_n & 0_{n \times p} \\ 0_{p \times n} & -I_p \end{pmatrix} + \begin{pmatrix} (C_0 + zC_1)^* \\ (D_0 + zD_1)^* \end{pmatrix} \begin{pmatrix} (C_0 + zC_1) & (D_0 + zD_1) \end{pmatrix} \right). \end{aligned}$$

□

6.3 Analytical solutions of LMIs

We have obtained in Section 6.2, conditions on existence of solution, in the set Ψ_n defined in Section 2, to the LMIs parametrized by z . We show here that, provided that there exist solutions to these problems for any $z \in \mathbb{D}$, then there indeed exist solutions with such regularity. The following result concerns complex analyticity of the solutions of LMIs.

Lemma 5. *Let $A(z) \in \mathbb{C}^{n \times n}$, $B(z) \in \mathbb{C}^{n \times p}$, $M(z) \in \mathbb{C}^{(n+p) \times (n+p)}$ be functions of z , analytic in z , z^* in every point of a neighborhood of $\overline{\mathbb{D}}$. Assume that, for any $z \in \overline{\mathbb{D}}$, there exists $P(z) \in \mathbb{C}^{n \times n}$, such that $P(z) = P(z)^* > 0$ and*

$$\begin{pmatrix} P(z)A(z) + A(z)^*P(z) & P(z)B(z) \\ B(z)^*P(z) & 0_p \end{pmatrix} + M(z) < 0. \quad (23)$$

Assume moreover that, for any $z \in \overline{\mathbb{D}}$, $M(z)^* = M(z)$ and

$$M_{11}(z) \stackrel{\text{def}}{=} \begin{pmatrix} I_n & 0_{n \times p} \end{pmatrix} M(z) \begin{pmatrix} I_n \\ 0_{p \times n} \end{pmatrix} \in \Phi_n^+, \quad M_{22}(z)^{-1} \stackrel{\text{def}}{=} \left(\begin{pmatrix} 0_{p \times n} & I_p \end{pmatrix} M(z) \begin{pmatrix} 0_{n \times p} \\ I_p \end{pmatrix} \right)^{-1} \in -\Phi_p^+. \quad (24)$$

Then, LMI (23) admits a solution $P \in \Psi_n$. ■

Proof. By continuity and compactness, there exists $\varepsilon > 0$ such that, for any $z \in \overline{\mathbb{D}}$, the left-hand side of (23) may be chosen (strictly) less than $-\varepsilon I_{n+p}$. Consider the stabilizing solution [26] $\tilde{P}(z) = \tilde{P}^*(z)$ of the Riccati equation

$$A^*(z)\tilde{P}(z) + \tilde{P}(z)A(z) + M_{11}(z) + \varepsilon I_n - (\tilde{P}(z)B(z) + M_{12}(z))(M_{22}(z) + \varepsilon I_p)^{-1}(\tilde{P}(z)B(z) + M_{12}(z)) = 0.$$

Using Schur complement, for any $z \in \overline{\mathbb{D}}$, the matrix $\tilde{P}(z)$ is positive definite, and solution of (23).

- 1ST STEP. Extending the result of Delchamps [9] as made by Kamen in Chapter 4 of [22, see Lemma p. 134], one proves that $\tilde{P}(z)$ is (complex) analytic, jointly in z and z^* . In other words, there exists a map $f(z_1, z_2)$, analytic in a neighborhood of $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$, such that $\tilde{P}(z) = f(z, z^*)$. One deduces that $\tilde{P} \in \Phi_n$. Also, the coefficients of the expansion of $\tilde{P}(z)$ are real matrices when the system is real.
- 2ND STEP. Now, prove that, for any $\tilde{M}(z) \in \Phi_n$, the product $\tilde{M}(z)^*\tilde{M}(z)$ is in Φ_n^+ . Indeed, matrix manipulations show that, for any $k \in \mathbb{N}$, there exists a matrix $\mathcal{M}_k \in \mathbb{R}^{(2k-1)n \times (2k-1)n}$ such that, for any $\tilde{M}_k \in \mathbb{R}^{kn \times kn}$,

$$\begin{aligned} & [(v_k(z) \otimes I_n)^* \tilde{M}_k(v_k(z) \otimes I_n)]^* (v_k(z) \otimes I_n)^* \tilde{M}_k(v_k(z) \otimes I_n) \\ & = (v_{2k-1}(z) \otimes I_{kn})^* (I_{2k-1} \otimes \tilde{M}_k)^T \mathcal{M}_k (I_{2k-1} \otimes \tilde{M}_k) (v_{2k-1}(z) \otimes I_{kn}). \end{aligned}$$

Taking $\tilde{M}_k = I_{kn}$ and identifying, one sees that $\mathcal{M}_k = \mathcal{D}_k \otimes I_{kn}$, where $\mathcal{D}_k \in \mathbb{R}^{(2k-1) \times (2k-1)}$ is the diagonal matrix whose i -th component is the coefficient of $|z|^{2(i-1)}$ in the expansion of $\|v_k(z)\|^4 = (1 + |z|^2 + \dots + |z|^{2(k-1)})^2$. In particular, $\mathcal{M}_k = \mathcal{M}_k^T > 0$, and $\tilde{M}(z)^* \tilde{M}(z) \in \Phi_n^+$.

• 3RD STEP. From the very definition, for any $z \in \mathbb{D}$,

$$\tilde{P}(z) = \int_0^{+\infty} e^{A^*(z)t} \left[M_{11}(z) + \varepsilon I_n - (\tilde{P}(z)B(z) + M_{12}(z))(M_{22}(z) + \varepsilon I_p)^{-1}(\tilde{P}(z)B(z) + M_{12}(z)) \right] e^{A(z)t} dt .$$

The previous integral converges uniformly wrt $z \in \overline{\mathbb{D}}$. This assertion is easily proved when remarking that, by compactness and continuity,

$$\sup_{z \in \overline{\mathbb{D}}} \{ \operatorname{Re} s : s \in \operatorname{spectrum}(A(z)) \} < 0 ,$$

so there exists a $T > 0$, independent of z , such that the matrix obtained when putting T instead of $+\infty$ in the upper bound of the integral, is still solution of (23) in Φ_n^+ . From this and the first two steps, $\tilde{P}(z)$ appears as a sum of three elements of Φ_n^+ (for small enough $\varepsilon > 0$, $-(M_{22}(z) + \varepsilon I_p)^{-1} = -M_{22}(z)^{-1} + \varepsilon M_{22}(z)^{-2} - \varepsilon^2 M_{22}(z)^{-3} + \dots$ is still in Φ_p^+). The latter set being stable by addition, $\tilde{P}(z)$ is itself an element of Φ_n^+ . The coefficients of its expansion are hermitian matrices, denoted \tilde{P}_k , $k \in \mathbb{N}$. Incidentally, $\tilde{P}_1 = \tilde{P}(0)$ (by convention (8)), so $\tilde{P}_1 > 0$.

• 4TH STEP. The matrix F_k being defined in Section 6.1, consider

$$P_k \stackrel{\text{def}}{=} \tilde{P}_k + (F_k \otimes I_n) \tilde{P}_k (F_k \otimes I_n)^T + \dots + (F_k \otimes I_n)^{k-1} \tilde{P}_k (F_k \otimes I_n)^{(k-1)T} . \quad (25)$$

The hermitian matrix P_k is obtained as a sum of k terms: the first term is \tilde{P}_k , the second term is the matrix whose $(k-1)n$ last lines and columns are the $(k-1)n$ first lines and columns of \tilde{P}_k (that is \tilde{P}_{k-1} by convention (8)), and zero otherwise. Similarly, the third is the matrix whose $(k-2)n$ last lines and columns form the matrix \tilde{P}_{k-2} , and so on ...

As, for any $i \in \mathbb{N}$, $0 \leq i \leq k-1$,

$$(F_k \otimes I_n)^{iT} (v_k(z) \otimes I_n) = z^i \left(\begin{pmatrix} v_{k-i}(z) \\ 0_{i \times 1} \end{pmatrix} \otimes I_n \right) ,$$

one sees in definitive that

$$(v_k(z) \otimes I_n)^* P_k (v_k(z) \otimes I_n) = \sum_{i=1}^k |z|^{k-i} (v_i(z) \otimes I_n)^* \tilde{P}_i (v_i(z) \otimes I_n) .$$

One then deduces that

$$\lim_{k \rightarrow +\infty} \frac{1}{\|v_k(\cdot)\|^2} (v_k(\cdot) \otimes I_n)^* P_k (v_k(\cdot) \otimes I_n) = \lim_{k \rightarrow +\infty} (v_k(\cdot) \otimes I_n)^* \tilde{P}_k (v_k(\cdot) \otimes I_n) = \tilde{P}(\cdot) ,$$

in the sense of the uniform convergence in $\overline{\mathbb{D}}$. In particular, beyond a certain rank, inequality (23) is fulfilled for any $z \in \overline{\mathbb{D}}$, with $P(z)$ defined by (20) and the matrices P_k by (25). From the fact that $\tilde{P}_k \geq 0$ and that the matrix at the intersection of the n first lines and columns of \tilde{P}_k is the *positive* matrix \tilde{P}_1 , one deduces that $P_k > 0$. This proves that, for small enough $\varepsilon > 0$, $\tilde{P} \in \Psi_n$, and achieves the proof of Lemma 5. \square

6.4 Proof of Theorem 1

• First, one shows that it is sufficient to check the frequency domain properties for $z \in \partial \mathbb{D}$ only, instead of the whole set $\overline{\mathbb{D}}$. Indeed, $\overline{\mathbb{C}^+} \rightarrow \overline{\mathbb{D}}$, $s_z \mapsto (1 - s_z)/(1 + s_z)$ is a one-to-one map and $s_z \mapsto \rho(e^{A_0 + (1-s_z)/(1+s_z)A_1})$ is subharmonic and continuous, hence [4] it attains its maximum on the complex axis, whence

$$\max_{z \in \overline{\mathbb{D}}} \rho(e^{A_0 + zA_1}) = \sup_{s_z \in \overline{\mathbb{C}^+}} \rho(e^{A_0 + (1-s_z)/(1+s_z)A_1}) = \sup_{s_z \in j\mathbb{R}} \rho(e^{A_0 + (1-s_z)/(1+s_z)A_1}) = \max_{z \in \partial \mathbb{D}} \rho(e^{A_0 + zA_1}) .$$

So $A_0 + zA_1$ is Hurwitz for any $z \in \overline{\mathbb{D}}$, provided that the property holds in $\partial\mathbb{D}$.

Similarly, each of the input-output properties may be expressed as the strict positive realness of a certain matrix, say $G(s, z)$, that is $\|(I - G(s, z))(I + G(s, z))^{-1}\|_\infty < 1$. For any $s \in \overline{\mathbb{C}^+}$, the map

$$s_z \mapsto \|(I - G(s, (1 - s_z)/(1 + s_z)))(I + G(s, (1 - s_z)/(1 + s_z)))^{-1}\|$$

is subharmonic and continuous on $\overline{\mathbb{C}^+}$, and hence maximal on the complex axis. In definitive,

$$\max_{z \in \overline{\mathbb{D}}} \|(I - G(s, z))(I + G(s, z))^{-1}\|_\infty = \max_{z \in \partial\overline{\mathbb{D}}} \|(I - G(s, z))(I + G(s, z))^{-1}\|_\infty .$$

- We now get ready to apply Lemma 5 to the inequalities (9a) to (13a). In the present case, among the hypotheses of Lemma 5, only verification of (24) necessitates special care. Using the notations as in the statement of Lemma 5, $M_{11}(z) = 0_p$ in (9a), (10a), (12a) and (13a), $M_{11}(z) = (C_0 + zC_1)^*(C_0 + zC_1)$ in (11a), and both values are in Φ_n^+ . Similarly, $M_{22}(z) = 0_p$ in (9a) and (13a), $M_{22}(z) = -(D_0 + zD_1) - (D_0 + zD_1)^*$ in (10a), $M_{22}(z) = -I_p + (D_0 + zD_1)^*(D_0 + zD_1)$ in (11a), $M_{22}(z) = -2I_p + (D_0 + zD_1) + (D_0 + zD_1)^*$ in (12a). For (9a) and (13a), one may replace $M_{22}(z)$ by $-\varepsilon I_p$ for small enough $\varepsilon > 0$. In this case, $-M_{22}(z)^{-1} \in \Phi_p^+$. For (11a), this is also clear, as

$$(I_p - (D_0 + zD_1)^*(D_0 + zD_1))^{-1} = I_p + (D_0 + zD_1)^*(D_0 + zD_1) + [(D_0 + zD_1)^*(D_0 + zD_1)]^2 + \dots$$

and every term of the sum is in Φ_p^+ , see the 3rd step in the proof of Lemma 5. One deduces that $-M_{22}(z)^{-1} \in \Phi_p^+$. Last, for the case of (10a) and (12a), loop shaping permits to transform the condition into $\|(I - G(s, z))(I + G(s, z))^{-1}\|_\infty < 1$, for which the assumption may be verified as for (11a).

- The equivalence between frequency domain conditions and solvability of LMIs is then established by the following steps, presented e.g. for \mathcal{H}_∞ performance.

$\forall z \in \overline{\mathbb{D}}$, S_z is internally stable with $\|H(s, z)\|_\infty < 1$

$\Rightarrow \forall z \in \overline{\mathbb{D}}$, (11a) admits a hermitian positive definite solution $P(z)$ (by Lemma 4)

\Rightarrow (11a) admits a hermitian positive definite solution $P(z) \in \Psi_n$ (by Lemma 5, whose assumptions are verified above)

$\Rightarrow \exists k \in \mathbb{N}, \exists P_k \in \mathbb{C}^{kn \times kn}, P_k = P_k^* > 0$, $P(z)$ defined by (20) is a hermitian positive definite solution of (11a) for any $z \in \overline{\mathbb{D}}$ (by truncation of the expansion of the solution in Ψ_n)

$\Rightarrow \exists k \in \mathbb{N}, \exists P_k \in \mathbb{C}^{kn \times kn}, P_k = P_k^* > 0$, $P(z)$ defined by (20) is a hermitian positive definite solution of (11a) for any $z \in \partial\mathbb{D}$

$\Rightarrow \exists k \in \mathbb{N}, \exists P_k \in \mathbb{C}^{kn \times kn}, P_k = P_k^* > 0, \exists Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}, Q_k^* = Q_k, R_{k,(11b)}(P_k, Q_k) < 0$ (by Lemma 4) .

On the other hand,

$\exists k \in \mathbb{N}, \exists P_k \in \mathbb{C}^{kn \times kn}, P_k = P_k^* > 0, \exists Q_k \in \mathbb{C}^{k(n+p) \times k(n+p)}, Q_k^* = Q_k, R_{k,(11b)}(P_k, Q_k) < 0$

$\Rightarrow \exists k \in \mathbb{N}, \exists P_k \in \mathbb{C}^{kn \times kn}, P_k = P_k^* > 0$, $P(z)$ defined by (20) is a hermitian positive definite solution of (11a) for any $z \in \partial\mathbb{D}$ (by Lemma 4)

$\Rightarrow \forall z \in \partial\mathbb{D}$, (11a) admits a hermitian positive definite solution $P(z)$ (by Lemma 4)

$\Rightarrow \forall z \in \partial\mathbb{D}$, S_z is internally stable with $\|H(s, z)\|_\infty < 1$ (by Lemma 4)

$\Rightarrow \forall z \in \overline{\mathbb{D}}$, S_z is internally stable with $\|H(s, z)\|_\infty < 1$ (see above) .

The other results are established similarly. Remark that, if the coefficients of S_z are real, then P_k, Q_k too are real (see Lemma 4 and 5).

- It remains, in order to achieve the proof of Theorem 1, to show that the solvability of the LMIs for index k implies the same property for $k + 1$. For this, one uses the parameter-dependent LMIs.

Consider for example the delay-independent circle criterion. Assume there exist a positive integer k and symmetric matrices P_k, Q_k with $P_k > 0$, such that $R_{k,(12b)}(P_k, Q_k) < 0$. Equivalently, by Lemma 4, inequality (12a) admits, for any $z \in \partial\mathbb{D}$, a solution given by (20), where P_k is the same matrix. Let us define

$$P_{k+1} \stackrel{\text{def}}{=} \frac{k+1}{2k} (\text{diag}\{P_k; 0_n\} + \text{diag}\{0_n; P_k\}) .$$

The matrix $P_{k+1} \in \mathbb{R}^{(k+1)n \times (k+1)n}$ is symmetric positive definite. Moreover, for any $z \in \partial\mathbb{D}$,

$$\begin{aligned} \frac{1}{\|v_{k+1}(z)\|^2} (v_{k+1}(z) \otimes I_n)^* P_{k+1} (v_{k+1}(z) \otimes I_n) &= \frac{1+|z|^2}{2} \frac{k+1}{k} \frac{1}{\|v_{k+1}(z)\|^2} (v_k(z) \otimes I_n)^* P_k (v_k(z) \otimes I_n) \\ &= \frac{1}{\|v_k(z)\|^2} (v_k(z) \otimes I_n)^* P_k (v_k(z) \otimes I_n) . \end{aligned}$$

Hence, using the result in Lemma 4 in the converse sense, one concludes that there exist symmetric matrices P_{k+1}, Q_{k+1} with $P_{k+1} > 0$, such that $R_{k+1,(12b)}(P_{k+1}, Q_{k+1}) < 0$.

The same principle applies to the other cases. For the uniform delay-independent Popov criterion, η has to be changed in $\frac{k^2-1}{k^2}\eta$. This achieves the proof of Theorem 1.

6.5 Proof of Theorem 2

One may found in [15, Theorem 2.4] the following equivalent form for delay-independent stability of S_h :

$$\forall (s, z) \in \mathbb{C}^2, s \in \overline{\mathbb{C}^+} \setminus \{0\}, z \in \overline{\mathbb{D}} \text{ or } s = 0, z \in \mathbb{D} \cup \{1\} \Rightarrow \det(sI_n - A_0 - zA_1) \neq 0 .$$

Hence if system S_h is delay-independently stable, but there exists $z \in \overline{\mathbb{D}}$ such that $A_0 + zA_1$ is *not* Hurwitz, then $\det(A_0 + zA_1) = 0$ for a certain $z \in \partial\mathbb{D} \setminus \{1\}$. So, for any $\varepsilon > 0$, $s = \varepsilon$ is a root of the equation $\det(sI_n - (A_0 + \varepsilon I_n) - zA_1) = 0$: infinitely close from the nominal system S_h , there exist unstable systems.

As the set of strongly delay-independently stable systems is obviously open, it is hence the largest open subset in the set of delay-independently stable systems, in other words, its *interior*.

The same argument shows that if an input-output property of S_h is fulfilled for any value of $h \geq 0$, but the corresponding property of S_z is not fulfilled for any $z \in \overline{\mathbb{D}}$, then the appropriate condition is violated for $s = 0$ and a certain $z \in \partial\mathbb{D} \setminus \{1\}$. In this case, small perturbations suppress the desired property. In other words, the set of matrices $(A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1)$ for which S_z fulfills a certain property for any $z \in \overline{\mathbb{D}}$ is the interior of the set of those matrices for which S_h fulfills the same property for any $h \geq 0$. It then suffices to apply Theorem 1, remarking (see the proof of this result) that realness of the coefficients implies realness of the matrices P_k and Q_k .

7 Conclusion

Delay-independent stability and input-output properties of linear delay systems and stability and input-output properties of linear systems with a complex parameter have been studied. Based on the proximity between the two classes of problems, the issues of internal stability, passivity, H_∞ performance analysis, circle criterion and Popov criterion are examined. Adequate extensions of the class of quadratic Lyapunov-Krasovskii functionals, a technique originated from delay systems, permitted to obtain, for each problem under study, a family, indexed by a positive integer k , of *sufficient criteria*. The criteria, expressed as the solvability of some simple LMI problem, are less and less conservative when k increases, and the results presented are also *necessary*, in the sense that: if the property under study is true, then the corresponding LMI problems are solvable from a certain k and beyond.

Numerical examples have shown that the derived algorithms are immediatly implementable in a MATLAB/SCILAB-like environment, and that the computation times are small, at least for small dimension systems.

Further research involves the following aspects.

1. Determination of a priori (upper) estimate on the least k , if any, for which the LMIs are solvable.

2. Extension of the results to the analysis of systems with independent delays (or with many complex parameters).
3. Extension of the results to the analysis of systems with real parameters.

A Appendix – Kalman-Yakubovich-Popov lemma

Recall that the continuous-time version is due to Yakubovich [40] and Kalman [19], and the discrete-time version has been derived from Szegö and Kalman [36]. We use the statement as expressed e.g. in [34]. A proof of the result in the complex case may be found in [28, Theorem 1.11.1 and Remark 1.11.1].

Let $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times p}$, $M = M^* \in \mathbb{C}^{(n+p) \times (n+p)}$.

Lemma A.1. *If $\det(I_n - zA) \neq 0$ for any $z \in \partial\mathbb{D}$, then the following two statements are equivalent.*

(i) *There exists $Q = Q^* \in \mathbb{C}^{n \times n}$ such that*

$$\begin{pmatrix} A^*QA - Q & A^*QB \\ B^*QA & B^*QB \end{pmatrix} + M < 0 .$$

(ii) *For any $z \in \partial\mathbb{D}$,*

$$\begin{pmatrix} z(I_n - zA)^{-1}B \\ I_p \end{pmatrix}^* M \begin{pmatrix} z(I_n - zA)^{-1}B \\ I_p \end{pmatrix} < 0 .$$

■

Lemma A.2. *If $\det(sI_n - A) \neq 0$ for $s \in j\mathbb{R}$, then the following two statements are equivalent.*

1. *There exists $P = P^* \in \mathbb{C}^{n \times n}$ such that*

$$\begin{pmatrix} A^*P + PA & PB \\ B^*P & 0_p \end{pmatrix} + M < 0 .$$

When A, B, M are real, P is real, symmetric.

2. *For any $s \in j\overline{\mathbb{R}}$,*

$$\begin{pmatrix} (sI_n - A)^{-1}B \\ I_p \end{pmatrix}^* M \begin{pmatrix} (sI_n - A)^{-1}B \\ I_p \end{pmatrix} < 0 .$$

■

Remark A.3. When A is Hurwitz and

$$\begin{pmatrix} I_n & 0_{n \times p} \end{pmatrix} M \begin{pmatrix} I_n \\ 0_{p \times n} \end{pmatrix} \geq 0 ,$$

then P in Lemma A.2 is definite positive.

Remark A.4. When in the statements the matrices A, B, M are real, then Q (resp, P) is real, symmetric.

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Value k of the index	Estimate $\lambda_k^{-1/2}$ of $\sup_{h>0} \ H(s, e^{-sh})\ _\infty$	CPU time Q_k symmetric	CPU time Q_k definite positive
$k = 1$	132	0.08 s	0.09 s
$k = 2$	64.3	0.25 s	0.31 s
$k = 3$	"	0.81 s	1.07 s
$k = 4$	"	2.22 s	3.74 s

Table 1: EXAMPLE 1 – Estimation of $\sup_{h>0} \|H(s, e^{-sh})\|_\infty$.

Value k of the index	Estimate K_k of the largest sector	CPU time
$k = 1$	0.01406	0.05 s
$k = 2$	0.03147	0.22 s
$k = 3$	"	0.67 s
$k = 4$	"	1.87 s

Table 2: EXAMPLE 2 – Circle criterion: estimation of the largest sector.

Value k of the index	Estimate K_k of the largest sector	Slope η_k of the Popov line	CPU time
$k = 1$	0.01406	0	0.07 s
$k = 2$	0.05233	-0.1873	0.23 s
$k = 3$	"	"	0.71 s
$k = 4$	"	"	2.18 s

Table 3: EXAMPLE 2 – Uniform Popov criterion: estimation of the largest sector.

Value k of the index	Estimate ρ_k	CPU time
$k = 1$	0.2906	0.08 s
$k = 2$	0.6504	0.49 s
$k = 3$	"	2.18 s
$k = 4$	"	10.01 s

Table 4: EXAMPLE 3 – Estimation of the largest ρ preserving asymptotic stability of (17).

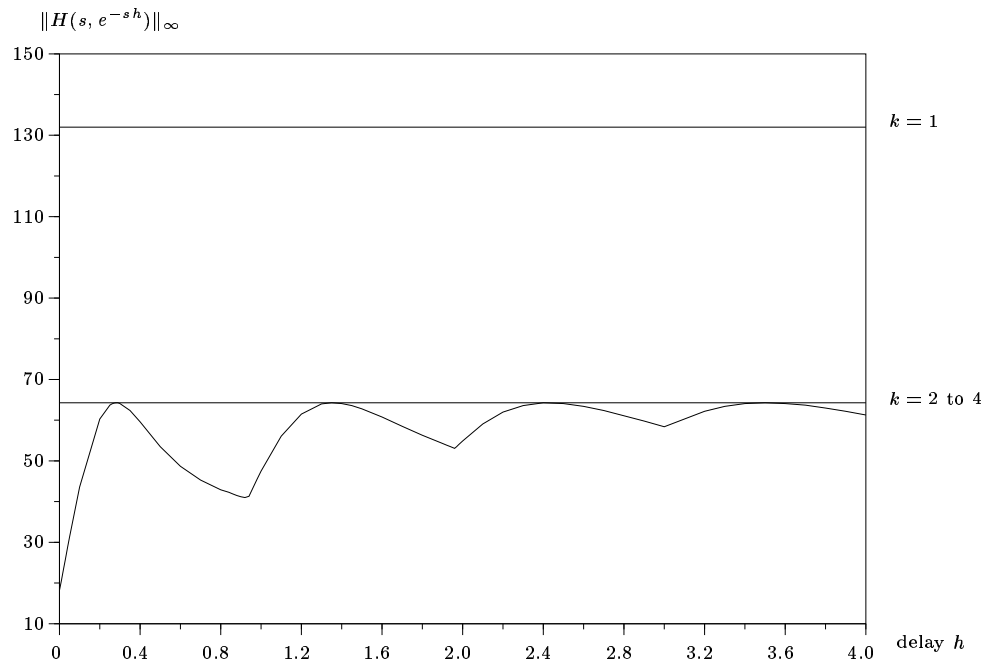


Figure 1: EXAMPLE 1 – \mathcal{H}_∞ norm $\|H(s, e^{-sh})\|_\infty$ as a function of h .

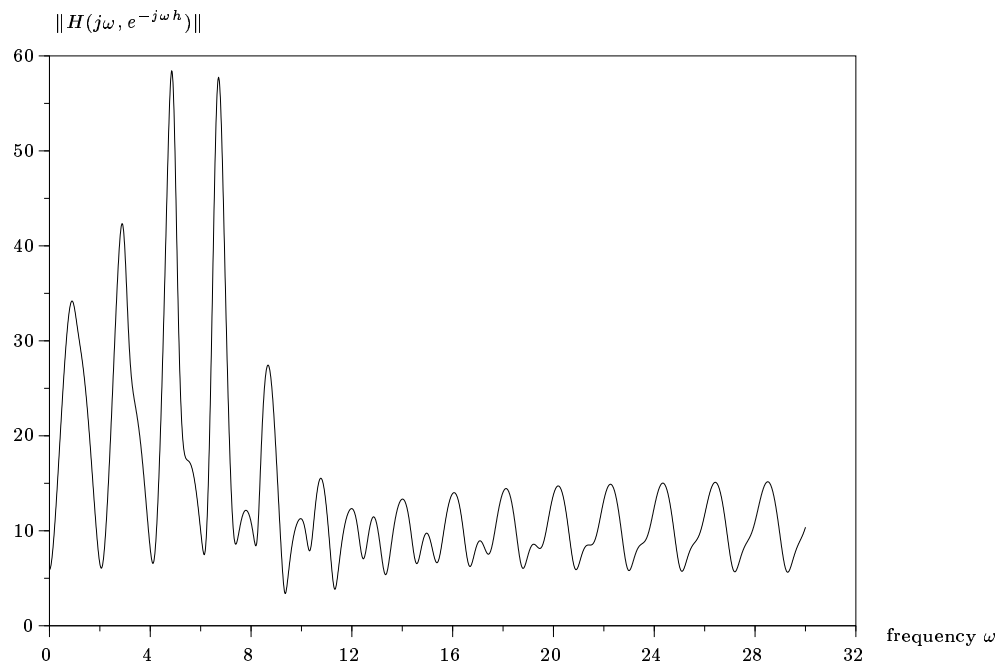


Figure 2: EXAMPLE 1 - $\|H(j\omega, e^{-j\omega h})\|$ as a function of ω , for $h = 3$.

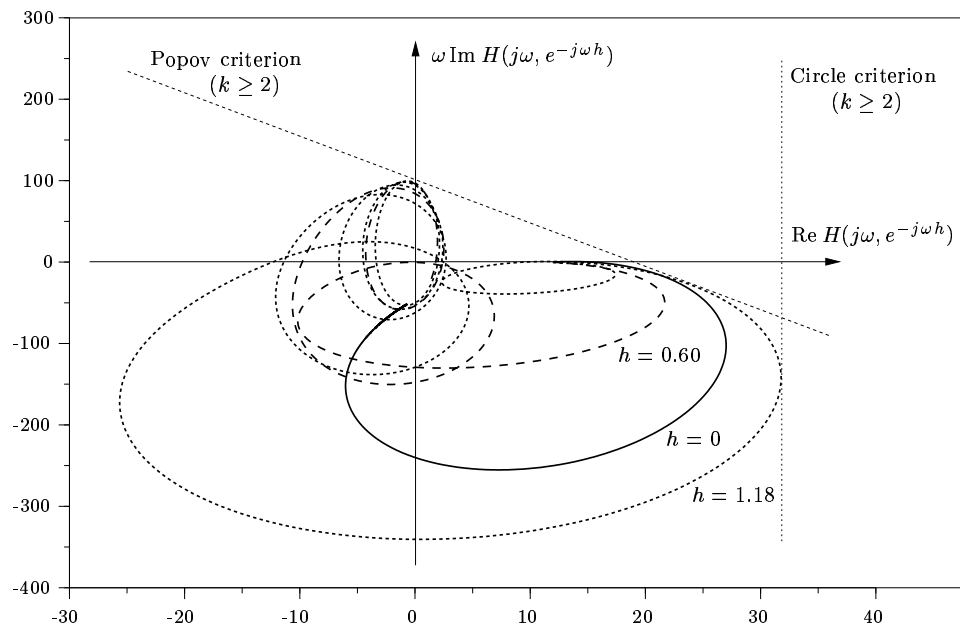


Figure 3: EXAMPLE 2 – Popov locii for $h = 0.00$, $h = 0.60$, $h = 1.18$.

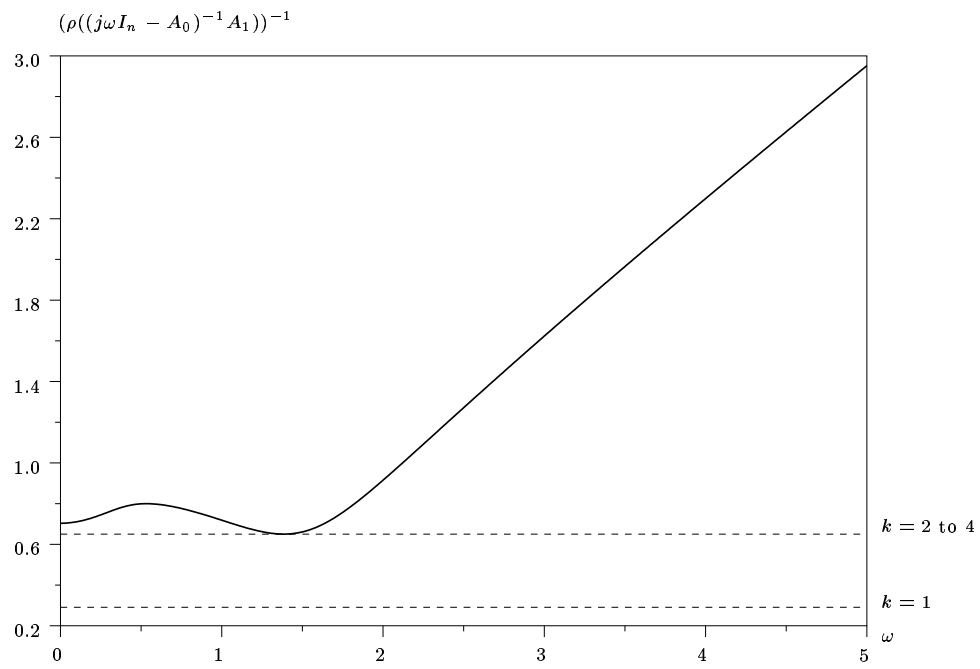


Figure 4: EXAMPLE 3 – Plot of $\omega \mapsto 1/\rho((j\omega I_n - A)^{-1} B)$ and the successive estimates of ρ .



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