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*An integral representation theorem  
of  $g$ -expectations*

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# An integral representation theorem of $g$ -expectations

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**Abstract:** The notion of  $g$ -expectations was first introduced by S.Peng via a class of non-linear backward stochastic differential equations (BSDEs). He showed that  $g$ -expectations preserve many of the basic properties of the convenient mathematical expectations and conditional expectations except linearity. He also raised the question about the integral representation of  $g$ -expectations. In this paper, we try to investigate Peng's question and give a necessary and sufficient condition under which  $g$ -expectations can be represented by a Choquet integral.

**Key-words:** BSDE,  $g$ -expectation, convex capacity, Choquet integral, integral representation theorem.

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# Theorème de représentation intégrale pour les $g$ -espérances

**Résumé :** La notion de  $g$ -espérance a été introduite par S.Peng au moyen d'une classe d'équations différentielles stochastiques rétrogrades. La  $g$ -espérance conserve certaines propriétés de l'espérance et de l'espérance conditionnelle mais pas la linéarité. Dans ce papier on apporte des éléments de réponse à la question de Peng concernant la représentation intégrale de la  $g$ -espérance et on donne une condition suffisante sous laquelle la  $g$ -espérance peut être représentée par une intégrale de Choquet.

**Mots-clés :** EDSR,  $g$ -espérance, capacité, intégrale de Choquet, représentation intégrale, théorème de représentation.

## Introduction

The first published paper on nonlinear BSDEs by Pardoux and Peng [PP1] appeared in 1990. Since then, many researchers have been working on this subject, due to the connection of this subject with mathematical finance, stochastic control, partial differential equations, stochastic games, stochastic geometry and mathematical economics (see for example [KPQ, KQ, P1, P2, P3, H, MPY, HL, D, CE] and references therein). More recently, Peng [P1, P2] introduced the notions of  $g$ -expectations, conditional  $g$ -expectations and the corresponding  $g$ -martingales via BSDEs and set up a connection between BSDEs and mathematical expectations and martingales. In [P1,P2], he showed that  $g$ -expectations preserve many of the basic properties of the mathematical expectations and martingales except linearity (see [P1,P2,CP] for details).

It is well-known that for a given probability measure  $P$ , the expectation  $E\xi$  of a random variable  $\xi$ , can be computed as

$$E\xi = \int_{-\infty}^0 [P(\xi \geq t) - 1] dt + \int_0^{\infty} P(\xi \geq t) dt.$$

The Choquet integral was first introduced in the mathematical literature by Choquet [Ch 1955]. He extended the above Riemann integral to the case where  $P$  is a non-additive set function (capacity). Although the Choquet integral was originally motivated by potential theory in physics, it has found many applications in various fields. In particular, owing to Schmeidler's work [S1,S2], the Choquet integral has become an important tool in describing individuals' behavior under uncertainty in economics. A natural question is: Can  $g$ -expectation be represented by a Choquet integral? More precisely, let  $\mathcal{E}_g[\xi]$  be the  $g$ -expectation of a random variable  $\xi$  and  $P_g(A)$  be the  $g$ -probability defined by  $P_g(A) := \mathcal{E}_g[1_A]$  for a given event  $A$ . Under which conditions does the following integral representation hold ?

$$\mathcal{E}_g[\xi] = \int_{-\infty}^0 [P_g(\xi > t) - 1] dt + \int_0^{\infty} P_g(\xi > t) dt. \quad (*)$$

In this paper, we discuss this issue and provide a necessary and sufficient condition under which the integral representation (\*) holds. The paper is organized as follows: In Section 1, we recall briefly some basic lemmas of BSDEs and the notions of  $g$ -expectations and related basic properties. In Section 3, we give a necessary condition under which  $g$ -expectation can be represented as a Choquet integral. Finally in Section 4, we prove that such a necessary condition is also sufficient.

# 1 BSDEs and $g$ -expectation

Pardoux and Peng (1990) showed an existence and uniqueness theorem for nonlinear BSDEs. Furthermore, S. Peng (1997) [P1] introduced the notions of  $g$ -expectations via this class of BSDEs. In this section, we shall recall the basic lemmas of BSDEs, the notion of  $g$ -expectation and related basic properties. To be more specific, fix  $T \in [0, \infty)$ , let  $(W_t)_{0 \leq t \leq T}$  be a 1-dimensional standard Brownian motion defined on a completed probability space  $(\Omega, \mathcal{F}, P)$  and  $\{\mathcal{F}_t\}_{0 \leq t \leq T}$  be the natural filtration generated by  $(W_t)_{0 \leq t \leq T}$ , i.e.

$$\mathcal{F}_t = \sigma\{W_s; s \leq t\}.$$

We use the following notation:

$$\begin{aligned} L^2(0, T) &:= \left\{ (\mathcal{F}_t)\text{-adapted processes } V \text{ with } \|V\|_{L^2} := [E \int_0^T |V(s)|^2 ds]^{\frac{1}{2}} < \infty \right\}; \\ L^2(\Omega, \mathcal{F}_t, P) &:= \left\{ (\mathcal{F}_t)\text{-measurable random variables } \xi \text{ with } E|\xi|^2 < \infty, t \in [0, T] \right\}; \end{aligned}$$

Let  $g$  be a function from  $\mathbb{R}^2 \times [0, T]$  into  $\mathbb{R}$ . We shall say that  $g$  satisfies hypothesis (H1) and (H2) if:

- (H1) For any  $(y, z) \in \mathbb{R}^2$ ,  $g(y, z, t)$  is continuous in  $t$  and  $\int_0^T |g^2(y, z, s)| ds < \infty$ ;
- (H2)  $g$  satisfies the Lipschitz condition: there exists a constant  $\mu > 0$  such that  $|g(y_1, z_1, t) - g(y_2, z_2, t)| \leq \mu[|y_1 - y_2| + |z_1 - z_2|]$ ,  $\forall (y_i, z_i) \in \mathbb{R}^2, i = 1, 2$ .

Pardoux and Peng (1990) obtained the following lemma:

**Lemma 1** *Suppose that  $g$  satisfies (H1) and (H2). For any  $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ , there exists a unique pair of adapted process  $(y, z) \in L^2(0, T) \times L^2(0, T)$ , which depends on  $(\xi, g, T)$ , satisfying the following BSDE:*

$$y_t = \xi + \int_t^T g(y_s, z_s, s) ds - \int_t^T z_s dW_s, \quad 0 \leq t \leq T. \quad (1)$$

The following lemma can be found in [KPQ, P1, P2]:

**Lemma 2** *Suppose that  $g_1$  and  $g_2$  satisfy (H1) and (H2). For any  $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$ , let  $(y^i, z^i)$  ( $i = 1, 2$ ) be the solutions of BSDE (1) corresponding to  $\xi = \xi_1$  and  $g = g_1$ ,  $\xi = \xi_2$  and  $g = g_2$ , respectively. Then there exists a constant  $c > 0$  such that*

$$E \left[ \sup_{t \leq s \leq T} |y_s^1 - y_s^2|^2 + \int_t^T |z_s^1 - z_s^2|^2 ds | \mathcal{F}_t \right] \leq cE \left[ |\xi_1 - \xi_2|^2 + \left( \int_t^T |\bar{g}_s| ds \right)^2 | \mathcal{F}_t \right],$$

where  $\bar{g}_s := g_1(y_s^1, z_s^1, s) - g_2(y_s^1, z_s^1, s)$ .

**Remark 1** Lemma 2 is one of the important lemmas of this paper. It implies that, if  $\xi_2$  converges to  $\xi_1$  in  $L^2(\Omega, \mathcal{F}_T, P)$  and  $g_1(y^1, z^1, \cdot)$  converges to  $g_2(y^1, z^1, \cdot)$  in  $L^2(0, T)$ , then  $(y^2, z^2)$  converges to  $(y^1, z^1)$  in  $L^2(0, T) \times L^2(0, T)$ .

**Assumption A.** Let  $b(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, \sigma(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be continuous in  $(t, x)$  and uniformly Lipschitz continuous in  $x$ .

By the existence theorem of SDE, there exists a unique strong solution  $\{X_s^{t,x}\}$  satisfying the SDE:

$$\begin{cases} dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s, \\ X_t = x, \quad s \in [t, T]. \end{cases} \quad (2)$$

Let  $\Phi(x)$  be a continuous function defined on  $\mathbb{R}$  such that  $\Phi(X_T^{t,x}) \in L^2(\Omega, \mathcal{F}_T, P)$  and  $(y^{t,x}, z^{t,x})$  be the solution of the BSDE:

$$y_s = \Phi(X_T^{t,x}) + \int_s^T g(y_r, z_r, r)dr - \int_s^T z_r dW_r, \quad s \in [0, T]. \quad (3)$$

Pardoux and Peng (1992) proved the following lemma:

**Lemma 3** Suppose that the functions  $b, \sigma, \Phi, g$  are  $C^3$  with bounded derivatives, and let  $(y_s^{t,x}, z_s^{t,x})$  be the solution of BSDE (3), then

(i)  $u(t, x) := y_t^{t,x} \in C^{1,2}([0, T] \times \mathbb{R}, \mathbb{R})$  is the unique solution of the following partial differential equation (PDE):

$$\begin{cases} \partial_t u(t, x) + \mathcal{L}u(t, x) + g(t, u(t, x), \sigma^*(t, x) \partial_x u(t, x)) = 0, \\ u(T, x) = \Phi(x), \end{cases} \quad (4)$$

where  $\mathcal{L}u(t, x) := \frac{1}{2} \sigma^2(t, x) \partial_x^2 u(t, x) + b(t, x) \partial_x u(t, x)$ ,

(ii)  $z_s^{t,x} = \sigma(s, X_s^{t,x}) \partial_x u(s, X_s^{t,x})$ , a.e.  $s \in [t, T]$ , where  $\partial_x u$  is the partial derivative of  $u$ .

Using BSDE (1), Peng (1997) introduced the notion of  $g$ -expectation:

**Definition 1** Suppose that  $g$  satisfies (H1) and (H2). For any  $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ , let  $(y_t^{\xi, g, T}, z_t^{\xi, g, T})$  be the solution of BSDE (1).

(1) We call  $\mathcal{E}_g^T[\xi]$  and  $\mathcal{E}_g^T[\xi | \mathcal{F}_t]$  defined by

$$\mathcal{E}_g^T[\xi] := y_0^{\xi, g, T}; \quad \mathcal{E}_g^T[\xi | \mathcal{F}_t] := y_t^{\xi, g, T}, \quad 0 \leq t \leq T,$$

respectively the  $g$ -expectation and the conditional  $g$ -expectation of the random variable  $\xi$  generated by  $g$  on the time interval  $[0, T]$ .



(2) We call  $V_t(\xi, g)$  defined by

$$V_t(\xi, g) := z_t^{\xi, g, T}, \quad 0 \leq t \leq T,$$

the volatility of the random variable  $\xi$ .

(3) For any  $A \in \mathcal{F}_T$ , we call  $P_g(A)$  defined by

$$P_g(A) := \mathcal{E}_g^T[1_A]$$

the  $g$ -probability of the event  $A$ .

To simplify notation, we rewrite  $\mathcal{E}_g^T[\xi]$ ,  $\mathcal{E}_g^T[\xi|\mathcal{F}_t]$  and  $P_g(\cdot)$  as  $\mathcal{E}_a^T[\xi]$ ,  $\mathcal{E}_a^T[\xi|\mathcal{F}_t]$  and  $P_a(\cdot)$  respectively if  $g$  is of the form:  $g(y, z, t) := a(t)|z|$ .

**Remark 2 (1)** Obviously, if  $g = 0$ , then  $\mathcal{E}_g^T[\xi|\mathcal{F}_t] = E[\xi|\mathcal{F}_t]$ ,  $\mathcal{E}_g^T[\xi] = E\xi$  and  $P_g(A) = P(A)$ .

(2) The reason why  $V_t(\xi, g)$  is called volatility is because BSDE (1) can be rewritten as

$$\mathcal{E}_g^T[\xi|\mathcal{F}_t] = \mathcal{E}_g^T[\xi] - \int_0^t g(y_s, z_s, s)ds + \int_0^t V_s(\xi, g)dW_s,$$

which is the Martingale Representation Theorem if  $g \equiv 0$ . This notation will be very convenient to prove the theorems in next section.

(3) The  $P_g(\cdot)$ -probability is a nonlinear probability measure.

By Definition 1,  $(\mathcal{E}_g^T[\xi|\mathcal{F}_t], V_t(\xi, g))$  consists of a pair solution of BSDE (1). For convenience, in this paper, we shall use  $(y, z)$  and  $(\mathcal{E}_g^T[\xi|\mathcal{F}_t], V_t(\xi, g))$  alternately to express the solution of BSDE (1).

**Remark 3 (1)** The  $g$ -expectation  $\mathcal{E}_g^T[\cdot]$ , considered as a nonlinear operator on  $L^2(\Omega, \mathcal{F}_T, P)$ , usually depends not only on  $g$  but also on  $T$ . That is, if  $\xi$  is  $\mathcal{F}_r$ -measurable, it is certainly  $\mathcal{F}_t$ -measurable for all  $t > r$ , but the equality

$$\mathcal{E}_g^T[\xi] = \mathcal{E}_g^t[\xi], \quad 0 \leq r \leq t \leq T,$$

may fail.

For example, let  $g = 1$ , then  $\mathcal{E}_g^T[\xi] = E\xi + T$ , and  $\mathcal{E}_g^r[\xi] = E\xi + r$ , thus if  $T \neq r$ , then  $\mathcal{E}_g^T[\xi] \neq \mathcal{E}_g^r[\xi]$ . That is,  $\mathcal{E}_g^T[\cdot]$  depends on the time interval  $[0, T]$ .

- (2) The mathematical expectation  $E[\cdot]$ , viewed as a linear operator on  $L^1(\Omega, \mathcal{F}_T, P)$ , has the property that  $E[a|\mathcal{F}_t] = a$  for all constant  $a \in \mathbb{R}$ . The property usually is not true for  $\mathcal{E}_g[\cdot]$ . The following lemma provides conditions under which the  $g$ -expectation preserves such a property,

**Lemma 4** *The following conditions are equivalent:*

- (1)  $g(y, 0, t) = 0$ , for all  $(y, t) \in \mathbb{R} \times [0, T]$ ;
- (2)  $\mathcal{E}_g^T[a|\mathcal{F}_t] = a$  for all  $a \in \mathbb{R}$ ;
- (3)  $\eta := \mathcal{E}_g^T[\xi|\mathcal{F}_t]$  is the unique  $\mathcal{F}_t$ -measurable random variable such that

$$\mathcal{E}_g^T[1_A \xi] = \mathcal{E}_g^T[1_A \eta], \quad \forall A \in \mathcal{F}_t, \quad t \in [0, T].$$

**Proof.** (1) $\Rightarrow$  (3) follows from [P1]; (3) $\Rightarrow$  (2) is obvious.

Now let us prove (2) $\Rightarrow$  (1):

For any  $a \in \mathbb{R}$ , let  $y_t := \mathcal{E}_g^T[a|\mathcal{F}_t]$ , then by the definition of  $\mathcal{E}_g^T[\cdot]$ ,  $(y_t)$  is the solution of the BSDE:

$$y_t = a + \int_t^T g(y_s, z_s, s) ds - \int_t^T z_s dW_s, \quad 0 \leq t \leq T.$$

It follows by  $y_t = a$  that we have

$$\int_t^T g(a, z_s, s) ds - \int_t^T z_s dW_s = 0, \quad \forall t \in [0, T].$$

Consequently,  $z_t = 0$ , a.e.  $t \in [0, T]$  and  $g(a, 0, t) = 0$ , a.e.  $t \in [0, T]$ .

□

>From now on, we assume that  $g$  satisfies

$$(H3) \quad g(y, 0, t) = 0, \quad \forall (y, t) \in \mathbb{R} \times [0, T].$$

**Lemma 5** *For any  $r \in [0, T]$ , if  $\xi \in L^2(\Omega, \mathcal{F}_r, P)$  and  $g$  satisfies (H3), then*

$$\mathcal{E}_g^T[\xi] = \mathcal{E}_g^r[\xi], \quad \tau \in [r, T].$$

**Proof.** let  $(\mathcal{E}_g^r[\xi|\mathcal{F}_t], z_s^r)$  be the solution of the BSDE:

$$y_t^r = \xi + \int_t^r g(y_s^r, z_s^r, s) ds - \int_t^r z_s^r dW_s.$$

It is easy to check that if  $g$  satisfies (H3), then  $(y_t, z_t)$  defined by

$$y_t := \begin{cases} \xi, & t \geq \tau; \\ \mathcal{E}_g^\tau[\xi|\mathcal{F}_t], & t < \tau. \end{cases} \quad z_t := \begin{cases} 0, & t \geq \tau; \\ z_t^\tau, & t < \tau. \end{cases}$$

is the solution of BSDE:

$$y_t = \xi + \int_t^T g(y_s, z_s, s) ds - \int_t^T z_s dW_s.$$

By the uniqueness of the solution of the BSDE,

$$\mathcal{E}_g^T[\xi|\mathcal{F}_t] = \mathcal{E}_g^\tau[\xi|\mathcal{F}_t], \quad t \leq r \leq \tau \leq T.$$

In particular, if  $t = 0$ , we complete the proof.

Under assumption (H3), applying Lemma 5, without loss of generality, we shall write  $\mathcal{E}_g^T[\cdot]$  simple as  $\mathcal{E}_g[\cdot]$ .

S. Peng [P1] showed that the  $g$ -expectation and the conditional  $g$ -expectation of a random variable  $\xi$  preserve many of the basic properties of the mathematical expectation and conditional expectation (except linearity). For example,

**Property 1** (1)  $\mathcal{E}_g[\xi] = \mathcal{E}_g[\mathcal{E}_g[\xi|\mathcal{F}_t]]$ ,  $\mathcal{E}_g[a] = a, \forall a \in R$ .

(2)  $\mathcal{E}_g[\mathcal{E}_g[\xi|\mathcal{F}_t]|\mathcal{F}_r] = \mathcal{E}_g[\xi|\mathcal{F}_{r \wedge t}]$ ;

(3)  $\mathcal{E}_g[\xi + \eta|\mathcal{F}_t] = \xi + \mathcal{E}_g[\eta|\mathcal{F}_t]$ , if  $\xi$  is  $\mathcal{F}_t$ -measurable.

The following examples show a relation between  $g$ -expectations and mathematical expectations:

**Example 1** If  $\xi \in L^2(\Omega, \mathcal{F}_T, P)$  and  $g(y, z, t) = a(t)|z|$ , here  $a := \{a(t)\}$  is a square integrable process bounded by  $u$ , then

(i)  $g$ -expectation  $\mathcal{E}_a[\xi]$  :

$$\mathcal{E}_a[\xi] = \begin{cases} E\xi, & a = 0; \\ \inf_{Q \in \mathcal{P}} E_Q[\xi], & a < 0; \\ \sup_{Q \in \mathcal{P}} E_Q[\xi], & a > 0. \end{cases}$$

(ii) Conditional  $g$ -expectation:

$$\mathcal{E}_a[\xi|\mathcal{F}_t] = \begin{cases} E[\xi|\mathcal{F}_t], & a = 0; \\ \text{ess inf}_{Q \in \mathcal{P}} E_Q[\xi|\mathcal{F}_t], & a < 0 \\ \text{ess sup}_{Q \in \mathcal{P}} E_Q[\xi|\mathcal{F}_t], & a > 0. \end{cases}$$

(iii)  $g$ -probability :  $\forall A \in \mathcal{F}_T$ ,

$$P_a(A) = \begin{cases} P(A), & a = 0; \\ \inf_{Q \in \mathcal{P}} Q(A), & a < 0; \\ \sup_{Q \in \mathcal{P}} Q(A), & a > 0, \end{cases}$$

where  $\mathcal{P} := \{Q^v : \frac{dQ^v}{dP} := e^{-\frac{1}{2} \int_0^T |v_s|^2 ds + \int_0^T v_s dW_s}, |v_t| \leq |a_t|, a.e. t \in [0, T]\}$ .

*Proof.* We recall briefly the proof which can be found in [KQ, CP]. We just prove the claims (i),(ii) for  $a \leq 0$ , the rest can be proved in the same way.

If  $a \leq 0$ , for any  $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ , set  $(y, z) := (\mathcal{E}_a[\xi | \mathcal{F}_t], V_t(\xi, a))$ . By the definition of  $\mathcal{E}_g[\cdot]$ ,  $(y_t, z_t)$  is the unique solution of the BSDE:

$$y_t = \xi + \int_t^T a(s) |z_s| ds - \int_t^T z_s dW_s, \quad 0 \leq t \leq T. \quad (5)$$

Set  $a_s := a(s) \text{sgn } z_s$ , then BSDE (5) can be rewritten as follows:

$$y_t = \xi - \int_t^T z_s d\overline{W}_s, \quad (6)$$

where  $\overline{W}_t = W_t - \int_0^t a_s ds$ . By Girsanov's Theorem,  $\{\overline{W}_t\}$  is a  $Q$ -Brownian motion, where  $E[\frac{dQ}{dP} | \mathcal{F}_T] := e^{-\frac{1}{2} \int_0^T a_s^2 ds + \int_0^T a_s dW_s}$ . Taking conditional expectation  $E_Q[\cdot | \mathcal{F}_t]$  on both sides of the BSDE (6), we get

$$y_t = E_Q[\xi | \mathcal{F}_t] \geq \text{ess inf}_{Q \in \mathcal{P}} E_Q[\xi | \mathcal{F}_t],$$

that is ,

$$\mathcal{E}_a[\xi | \mathcal{F}_t] \geq \text{ess inf}_{Q \in \mathcal{P}} E_Q[\xi | \mathcal{F}_t].$$

On the other hand, let  $\{\theta_t\}$  be any process bounded by  $\{|a_t|\}$ , i.e.  $|\theta_t| \leq |a_t|$ . Applying Lemma 1, the following BSDE has a unique solution  $(y^\theta, z^\theta)$ ,

$$y_t^\theta = \xi + \int_t^T \theta_s z_s ds - \int_t^T z_s dW_s, \quad 0 \leq t \leq T. \quad (7)$$

Let  $E[\frac{dQ^\theta}{dP} | \mathcal{F}_T] := e^{-\frac{1}{2} \int_0^T \theta_s^2 ds + \int_0^T \theta_s dW_s}$ . Solving BSDE (7), we have

$$y_t^\theta = E_{Q^\theta}[\xi | \mathcal{F}_t].$$

Since  $a(t) \leq 0$ , thus  $\theta_t z \geq a(t)|z|$ , for all  $(z, t) \in \mathbb{R} \times [0, T]$ , comparing (5) with (7) and applying the comparison theorem ([P1], Theorem 35.3), we obtain

$$E_{Q^\theta}[\xi \mid \mathcal{F}_t] = y_t^\theta \geq y_t.$$

Consequently

$$\operatorname{ess\,inf}_{Q \in \mathcal{P}} E_Q[\xi \mid \mathcal{F}_t] \geq \mathcal{E}_a[\xi \mid \mathcal{F}_t].$$

Hence

$$\mathcal{E}_a[\xi \mid \mathcal{F}_t] = \operatorname{ess\,inf}_{Q \in \mathcal{P}} E_Q[\xi \mid \mathcal{F}_t].$$

This with Remark 2 (1) completes the proof of (ii) for  $a \leq 0$ . In particular, let  $t = 0$ , we obtain (i) for  $a \leq 0$ .  $\square$

**Definition 2** (1) The random variable  $\xi$  and  $\eta$  are called *comonotonic*, if

$$[\xi(\omega) - \xi(\omega')][\eta(\omega) - \eta(\omega')] \geq 0, \forall \omega, \omega' \in \Omega.$$

(2) A real function  $F$  defined on  $L^2(\Omega, \mathcal{F}_T, P)$  is called *comonotonic additive*, if

$$F(\xi + \eta) = F(\xi) + F(\eta), \text{ whenever } \xi \text{ and } \eta \text{ are comonotonic.}$$

(3) A real valued set function  $V : \mathcal{F}_T \rightarrow [0, 1]$  is called a *capacity* if

- (i)  $V(\emptyset) = 0, \quad V(\Omega) = 1$
- (ii)  $V(A) \leq V(B)$  for any  $A \subset B$ .

(4) Choquet integral: Let  $V$  be a capacity, for any  $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ , the functional  $F(\xi)$  defined by

$$F(\xi) := \int_{-\infty}^0 (V(\xi \geq t) - 1) dt + \int_0^\infty V(\xi \geq t) dt$$

is called the *Choquet integral* of  $\xi$  with respect to  $V$ .

## 2 Main Result.

Let  $H$  be the set of all  $\mathcal{F}_T$ -measurable random variables  $X_T \in L^2(\Omega, \mathcal{F}_T, P)$  which satisfy that there exist  $\sigma, b$  satisfying Assumption A, such that  $X_T$  is the value of the solution  $\{X_t\}$  of the following forward stochastic differential equation (SDE) at time  $T$ ,

$$\begin{cases} dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s, & 0 \leq s \leq T, \\ X_0 = x \in \mathbb{R}. \end{cases}$$

We denote by  $1_A$  the indicator function of the event  $A$  and denote  $V(A) := \mathcal{E}_g[1_A]$ .

**Theorem 1** Suppose that  $g$  satisfy (H1)(H2) and (H3).

$\mathcal{E}_g[\xi]$  can be represented as a Choquet integral for any  $\xi \in H$ , that is,

$$\mathcal{E}_g[\xi] = \int_{-\infty}^0 [V(\xi \geq t) - 1]dt + \int_0^{\infty} V(\xi \geq t)dt$$

if and only if there exist two continuous functions  $a, b$  such that  $g$  is of the form

$$g(y, z, t) := a(t)|z| + b(t)z.$$

We shall prove this theorem in the next sections.

### 3 Necessary condition for integral representation.

For a given capacity  $V$ , Choquet (1955) defined an integral which is called the Choquet integral. Dellacherie (1970) and Schmeidler (1989) for the general case showed that  $F(\cdot)$ , the operator of Choquet integral defined in Definiton 2 (4), is of comonotonic additivity, that is

$$F(\xi + \eta) = F(\xi) + F(\eta), \text{ whenever } \xi, \eta \text{ are comonotonic.}$$

We use here Dellacherie and Schmeidler's results to investigate a necessary condition for  $\mathcal{E}_g[\cdot]$  to be represented by a Choquet integral. We need the following lemmas. Lemma 6 extends the corresponding result in [BCHMP, CE].

**Lemma 6** Suppose that  $\{X_t\}$  is of the form

$$X_t = x + \int_0^t a_s ds + \int_0^t b_s dW_s, \quad 0 \leq t \leq T, \quad (8)$$

where  $\{a_t\}$  and  $\{b_t\}$  are two continuous bounded processes. Then

(i)  $\mathcal{E}_g[X_\tau | \mathcal{F}_t] \rightarrow X_t, \quad V_t(X_\tau, g) \rightarrow b_t, \text{ as } \tau \rightarrow t.$

(ii)  $\lim_{\tau \rightarrow t^+} \frac{\mathcal{E}_g[\xi_\tau | \mathcal{F}_t] - E[X_\tau | \mathcal{F}_t]}{\tau - t} = g(X_t, b_t, t),$

where the limits are in the sense of  $L^2(\Omega, \mathcal{F}_t, P)$ .

**Proof.** Because  $a$  and  $b$  are bounded, for any  $\tau \in [0, T]$ ,  $X_\tau \in L^2(\Omega, \mathcal{F}_T, P)$ . Let  $(y_t, z_t) := (\mathcal{E}_g[X_\tau | \mathcal{F}_t], V_t(X_\tau, g))$ . Thanks to assumption (H3),  $(y_t, z_t)$  is the solution of the BSDE:

$$y_t = X_\tau + \int_t^T g(y_s, z_s, s)ds - \int_t^T z_s dW_s,$$

with

$$z_t \equiv 0, \text{ a.e. } t \in (\tau, T].$$

Hence the above BSDE can be written as

$$y_t = X_\tau + \int_t^\tau g(y_s, z_s, s)ds - \int_t^\tau z_s dW_s. \quad (9)$$

Applying Lemma 2, there exists a constant  $c > 0$  such that

$$E\left[\int_t^\tau |z_s|^2 ds | \mathcal{F}_t\right] \leq cE[|X_\tau|^2 | \mathcal{F}_t].$$

Taking the conditional expectation  $E[\cdot | \mathcal{F}_t]$  on both sides of BSDE(9), we obtain

$$\begin{aligned} |\mathcal{E}_g[X_\tau | \mathcal{F}_t] - E[X_\tau | \mathcal{F}_t]|^2 &= |y_t - E[X_\tau | \mathcal{F}_t]|^2 \\ &= |E[\int_t^\tau g(y_s, z_s, s)ds | \mathcal{F}_t]|^2 \\ &\leq u^2(\tau - t)E[\int_t^\tau |z_s|^2 ds | \mathcal{F}_t] \\ &\leq cu^2(\tau - t)E[\sup_{T \geq s \geq 0} |X_s|^2 | \mathcal{F}_t] \\ &\rightarrow 0 \text{ as } \tau \rightarrow t \text{ in } L^1(\bar{\Omega}, \mathcal{F}_t, P), \end{aligned}$$

which implies  $\mathcal{E}_g[X_\tau | \mathcal{F}_t] \rightarrow E[X_\tau | \mathcal{F}_t]$  as  $\tau \rightarrow t$  in  $L^2(\Omega, \mathcal{F}_T, P)$ .

On the other hand, from (8),

$$X_\tau = X_t + \int_t^\tau a_s ds + \int_t^\tau b_s dW_s,$$

thus

$$E[X_\tau | \mathcal{F}_t] - X_t = E\left[\int_t^\tau a_s ds | \mathcal{F}_t\right] \rightarrow 0 \text{ as } \tau \rightarrow t \text{ in } L^2(\Omega, \mathcal{F}_t, P).$$

Hence

$$\mathcal{E}_g[X_\tau | \mathcal{F}_t] \rightarrow X_t \text{ as } \tau \rightarrow t.$$

So

$$\langle \mathcal{E}_g[X_\tau | \mathcal{F}_t], W_t \rangle \rightarrow \langle X_t, W_t \rangle, \text{ as } \tau \rightarrow t.$$

But

$$\langle \mathcal{E}_g[X_\tau | \mathcal{F}_t], W_t \rangle = \int_0^t V_s(X_\tau, g)ds, \quad \langle X_t, W_t \rangle = \int_0^t b_s ds.$$

Hence

$$V_t(X_\tau, g) \rightarrow b_t, \text{ a.e., as } \tau \rightarrow t, \text{ in } L^2(\Omega, \mathcal{F}_t, P).$$

Here  $\langle X, Y \rangle$  is the covariance process associated to the semi-martingales  $X$  and  $Y$ . The proof of (i) is thus complete.

Applying (i) and the continuity of  $g$ , we get

$$\begin{aligned} \frac{\mathcal{E}_g[X_\tau|\mathcal{F}_t] - E[X_\tau|\mathcal{F}_t]}{\tau - t} &= \frac{1}{\tau - t} E \left[ \int_t^\tau g(\mathcal{E}_g[X_\tau|\mathcal{F}_s], V_s(X_\tau, g), s) ds | \mathcal{F}_t \right] \\ &\rightarrow g(X_t, b_t, t), \quad \text{as } \tau \rightarrow t \end{aligned}$$

in  $L^2(\Omega, \mathcal{F}_t, P)$ . The proof of (ii) is thus complete.

**Lemma 7** *If  $\mathcal{E}_g[\cdot]$  is comonotonic additive in  $H$ , that is*

$$\mathcal{E}_g[\xi + \eta] = \mathcal{E}_g[\xi] + \mathcal{E}_g[\eta], \text{ whenever } \xi \text{ and } \eta \in H \text{ are comonotonic,} \quad (10)$$

*then there exist two continuous functions  $a := \{a(t)\}$ ,  $b := \{b(t)\}$  such that  $g$  is of the form:*

$$g(y, z, t) = a(t)z^+ - b(t)z^-,$$

*where  $z^+ := \max\{z, 0\}$  and  $z^- := \min\{z, 0\}$ .*

**Proof.** For each  $i = 1, 2$ , let us choose  $(y_i, z_i, t) \in \mathbb{R}^2 \times [0, T]$  such that  $z_1 z_2 \geq 0$ .

For any  $\tau \in [t, T]$ , let  $\xi = y_1 + z_1(W_\tau - W_t)$  and  $\eta = y_2 + z_2(W_\tau - W_t)$ . Obviously  $\xi$  and  $\eta$  are comonotonic. Note that

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = \mathcal{E}_g[\xi], \quad \mathcal{E}_g[\eta|\mathcal{F}_t] = \mathcal{E}_g[\eta], \quad \mathcal{E}_g[\xi + \eta|\mathcal{F}_t] = \mathcal{E}_g[\xi + \eta]. \quad (11)$$

From (10), we have

$$\frac{\mathcal{E}_g[\xi + \eta|\mathcal{F}_t] - E[\xi + \eta|\mathcal{F}_t]}{\tau - t} = \frac{\mathcal{E}_g[\xi|\mathcal{F}_t] - E[\xi|\mathcal{F}_t]}{\tau - t} + \frac{\mathcal{E}_g[\eta|\mathcal{F}_t] - E[\eta|\mathcal{F}_t]}{\tau - t}. \quad (12)$$

Applying Lemma 6 (ii) and the fact that

$$V_t(\xi + \eta, g) \rightarrow y_1 + y_2, \quad V_t(\xi, g) \rightarrow y_1,$$

and

$$V_t(\eta, g) \rightarrow y_2, \quad \text{as } \tau \rightarrow t$$

we obtain

$$g(y_1 + y_2, z_1 + z_2, t) = g(y_1, z_1, t) + g(y_2, z_2, t), \quad \forall z_1 z_2 \geq 0, \quad (13)$$

this with (H3) imply that there exist two continuous functions  $a$  and  $b$  bounded by  $\{u\}$  such that

$$g(y, z, t) = a(t)z^+ - b(t)z^-. \quad (14)$$

Using the above method, we can easily get,

**Corollary 1** *The operator of  $g$ -expectation  $\mathcal{E}_g[\cdot]$  is linear iff  $g$  is linear.*



In [P1], Peng showed that if  $g$  is convex then  $\mathcal{E}_g[\cdot]$  is convex. Hence,

**Corollary 2**  $\mathcal{E}_g[\cdot]$  is convex, if and only if  $g$  is convex.

Using the above lemmas, we can give a necessary condition for the  $g$ -expectation to be represented by a Choquet integral.

**Theorem 2** If  $\mathcal{E}_g[\cdot]$  can be represented by a Choquet integral, then there exist two continuous functions  $\alpha$  and  $\beta$  such that  $g$  is of the form

$$g(y, z, t) = \alpha(t)|z| + \beta(t)z. \quad (15)$$

**Proof.** If  $\mathcal{E}_g[\cdot]$  can be represented by a Choquet integral, by Dellacherie's Theorem in [De, 1970] (for the general case, see Schmeidler [S]), then  $\mathcal{E}_g[\cdot]$  satisfies comonotonic additivity, that is  $\mathcal{E}_g[\xi + \eta] = \mathcal{E}_g[\xi] + \mathcal{E}_g[\eta]$  whenever  $\xi$  and  $\eta$  are comonotonic. Applying Lemma 7,  $g$  is of the form:  $g(y, z, t) = a(t)z^+ - b(t)z^-$ . Set  $\alpha(t) := \frac{a(t)-b(t)}{2}$  and  $\beta(t) := \frac{a(t)+b(t)}{2}$ , we complete the proof.

**Remark 4** If  $g$  is of the form (15), then  $y_t := \mathcal{E}_g[\xi | \mathcal{F}_t]$  is the solution of the BSDE:

$$y_t = \xi + \int_t^T [\alpha(s)|z_s| + \beta(s)z_s]ds - \int_t^T z_s dW_s,$$

which can be rewritten as

$$y_t = \xi + \int_t^T \alpha(s)|z_s|ds - \int_t^T z_s d\bar{W}_s,$$

where  $\bar{W}_t := W_t - \int_0^t \beta(s)ds$ , is a  $Q$ -Brownian motion under  $Q$  defined by

$$\frac{dQ}{dP} = \exp\left\{-\frac{1}{2} \int_0^T \beta^2(s)ds + \int_0^T \beta(s)dW_s\right\}.$$

Thus, without loss of generality, in the rest of this paper, we assume  $\beta \equiv 0$ . Otherwise we can consider our problem in the probability space  $(\Omega, \mathcal{F}, Q)$ .

## 4 Sufficient condition for integral representation

In this section, we will prove that the necessary condition in Theorem 2 is also sufficient. To do so, we need the following lemmas:

For any  $X_T \in H$ , let  $X_t$  be the value of  $\{X_t\}$ , the solution of the following SDE, at time  $T$  :

$$\begin{cases} dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s, 0 \leq s \leq T, \\ X_0 = x, x \in R, \end{cases} \quad (16)$$

where  $b, \sigma$  satisfy Assumption A.

**Lemma 8** Suppose that  $\Phi$  and  $\Psi$  are two monotonic functions such that  $\Phi(X_T)$  and  $\eta = \Psi(X_T) \in L^2(\Omega, \mathcal{F}, P)$ , and let  $(y^\Phi, z^\Phi)$  and  $(y^\Psi, z^\Psi)$  be the solutions of the BSDE:

$$y_t = \xi + \int_t^T g(y_s, z_s, s) ds - \int_t^T z_s dW_s, \quad (17)$$

with terminal values  $\xi = \Phi(X_T)$  and  $\xi = \Psi(X_T)$ , respectively. If  $b, \sigma, \Phi, \Psi$  and  $g$  are assumed to be  $C^3$ , then

$$z_t^\Phi z_t^\Psi \geq 0, \quad \text{a.e. } t \in [0, T].$$

**Proof.** Let  $\{X_s^{t,x}\}$  be the solution of the SDE:

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x}) dt + \sigma(s, X_s^{t,x}) dW_s, \\ X_t = x, \quad s \in [t, T]. \end{cases}$$

and let  $(y_s^{t,x,\Phi}, z_s^{t,x,\Phi})$  and  $(y_s^{t,x,\Psi}, z_s^{t,x,\Psi})$  be the solutions of the following BSDE corresponding to  $\xi = \Phi(X_T^{t,x}), g = g_1$  and  $\xi = \Psi(X_T^{t,x}), g = g_2$ , respectively:

$$y_t = \xi + \int_t^T g(y_s, z_s, s) ds - \int_t^T z_s dW_s,$$

then, obviously,  $X_T = X_T^{0,x}$  and  $z_s^\Phi = z_s^{0,x,\Phi}, z_s^\Psi = z_s^{0,x,\Psi}$ .

But by Lemma 3(ii),

$$\begin{cases} z_s^{t,x,\Phi} = \sigma(s, X_s^{t,x}) \partial_x u(s, X_s^{t,x}), \text{ a.e. } s \in [0, T]; \\ z_s^{t,x,\Psi} = \sigma(s, X_s^{t,x}) \partial_x v(s, X_s^{t,x}), \text{ a.e. } s \in [0, T], \end{cases} \quad (18)$$

where  $u(t, x) := y_t^{t,x,\Phi}$  and  $v(t, x) := y_t^{t,x,\Psi}$ . Thus, if  $t = 0$ , it follows

$$\begin{aligned} z_s^\Phi z_s^\Psi &= z_s^{0,x,\Phi} z_s^{0,x,\Psi} \\ &= \sigma(s, X_s) \partial_x u(s, X_s) \sigma(s, X_s) \partial_x v(s, X_s) \\ &= \sigma^2(s, X_s) \partial_x u(s, X_s) \partial_x v(s, X_s), \text{ a.e. } s \in [0, T]. \end{aligned} \quad (19)$$

If  $\Phi, \Psi$  are monotonic, applying the Comparison Theorem for SDE and BSDE, it is easy to check that  $u(t, x)$  and  $v(t, x)$  are monotonic, from (19). The proof is complete.

**Remark 5** By Lemma 2, the assumption that  $b, \sigma, g, \Phi$  and  $\Psi$  are  $C^3$  in Lemma 8 is not necessary. Indeed, if there exist  $b^\epsilon, \sigma^\epsilon, g^\epsilon, \Phi_\epsilon$  and  $\Psi_\epsilon$  such that  $b^\epsilon, \sigma^\epsilon, g^\epsilon, \Phi_\epsilon, \Psi_\epsilon$  satisfy the assumptions of Lemma 8 and converge to  $b, \sigma, g, \Phi, \Psi$  as  $\epsilon \rightarrow 0$ , then by Lemma 2, the result of Lemma 8 is still true.

Let us now consider the case where  $\Phi$  and  $\Psi$  are indicator functions:

Suppose that  $\{X_t\}$  is the solution of SDE (16). For any  $b < c$ , set  $B = \{X_T \geq b\}$  and  $C = \{X_T \geq c\}$  and let  $(y^B, z^B)$  and  $(y^C, z^C)$  be the solutions of the following BSDE corresponding to  $1_B$  and  $1_C$  respectively:

$$y_t = \xi + \int_t^T g(y_s, z_s, s) ds - \int_t^T z_s dW_s.$$

Obviously,  $C \subset B$ , and we obtain,

**Lemma 9** *Using the above notation, suppose that  $b, \sigma, g$  satisfy the assumptions of Lemma 8, then*

$$z_t^C z_t^B \geq 0, \quad \text{a.e. } t \in [0, T]. \quad (20)$$

**Proof.** First, let us construct  $C^3$  functions which converge to indicator functions.

Indeed, for any  $n = 1, 2, \dots$ , we denote by  $\Phi_n(x, b)$  and  $\Phi_n(x, c)$  :

$$\Phi_n(x, b) := e^{-nd^{(b)}(x)}, \quad \Phi_n(x, c) := e^{-nd^{(c)}(x)},$$

where

$$d^{(b)}(x) = \begin{cases} (b-x)^3, & x < b, \\ 0, & x \geq b; \end{cases} ; d^{(c)}(x) = \begin{cases} (c-x)^3, & x < c, \\ 0, & x \geq c. \end{cases}$$

It is easy to check that for each  $n \geq 1$ ,  $\Phi_n(x, b), \Phi_n(x, c) \in C^3$  and

$$\Phi_n(x, a) \rightarrow 1_{(x \leq a)}, \quad \Phi_n(x, b) \rightarrow 1_{(x \geq b)}, \quad \Phi_n(x, c) \rightarrow 1_{(x \geq c)} \text{ as } n \rightarrow \infty.$$

Let us now prove (20):

Let  $(y^{n,b}, z^{n,b})$  and  $(y^{n,c}, z^{n,c})$  be the solutions of the following BSDE corresponding to  $\xi = \Phi_n(X_T, b)$  and  $\xi = \Phi_n(X_T, c)$  respectively,

$$y_t = \xi + \int_t^T g(y_s, z_s, z) ds - \int_t^T z_s dW_s.$$

Since  $\Phi_n(x, b)$  and  $\Phi_n(x, c)$  are monotonic, applying Lemma 8, we have

$$z_s^{n,b} z_s^{n,c} \geq 0, \quad \text{a.e. } s \in [0, T].$$

Note that  $\Phi_n(X_T, b) \rightarrow 1_B$  and  $\Phi_n(X_T, c) \rightarrow 1_C$  as  $n \rightarrow \infty$  in  $L^2(\Omega, \mathcal{F}, P)$ , by Lemma 2  $z^{n,b} \rightarrow z^B$  and  $z^{n,c} \rightarrow z^C$  as  $n \rightarrow \infty$  in  $L^2(0, T)$ . The proof is complete.

**Remark 6** *In Lemma 9, if we replace  $X_T$  by  $\Phi(X_T)$ , where  $\Phi$  is monotonic, then the corresponding result of Lemma 9 is still true.*

We now assume that  $g$  is of the form:

$$g(y, z, t) = a(t)|z|, \quad (21)$$

where  $a$  is a continuous function bounded by  $\mu$ .

**Lemma 10** Suppose that  $\xi \in H$  and  $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$  is a sequence in  $R$ . Let  $A_i := \{\omega : \xi(\omega) \geq \alpha_i\}$ ,  $i = 1, 2, \dots, n$ . Then for any positive sequence  $\{b_i\}_{i=1}^n$ ,

$$\mathcal{E}_a\left[\sum_{i=1}^n b_i 1_{A_i} | \mathcal{F}_t\right] = \sum_{i=1}^n b_i \mathcal{E}_a[1_{A_i} | \mathcal{F}_t], \quad t \in [0, T].$$

In particular,

$$\mathcal{E}_a\left[\sum_{i=1}^n b_i 1_{A_i}\right] = \sum_{i=1}^n b_i \mathcal{E}_a[1_{A_i}]. \quad (22)$$

**Proof.** For any  $\epsilon > 0$ , the function  $g_\epsilon$  defined by  $g_\epsilon(y, z, t) := a(t)\sqrt{|z|^2 + \epsilon}$ , is differentiable with respect to  $z$  and its derivative is uniformly bounded.

Let  $(y_t^{i,\epsilon}, z_t^{i,\epsilon})$  be the solution of the BSDE

$$y_t = 1_{A_i} + \int_t^T a(s)\sqrt{|z_s|^2 + \epsilon}ds - \int_t^T z_s dW_s, \quad i = 1, 2, \dots, n.$$

Applying Lemma 2 again,  $(y_t^{i,\epsilon}, z_t^{i,\epsilon}) \rightarrow (y^i, z^i)$  as  $\epsilon \rightarrow 0$  in  $L^2(0, T) \times L^2(0, T)$ , where  $(y^i, z^i)$  are the solutions of the BSDE:

$$y_t^i = 1_{A_i} + \int_t^T a(s)|z_s^i|ds - \int_t^T z_s^i dW_s, \quad i = 1, 2, \dots, n. \quad (23)$$

Lemma 9 implies that for any  $i, j = 1, 2, \dots, n$ ,

$$z_t^{i,\epsilon} z_t^{j,\epsilon} \geq 0.$$

Hence,

$$z_t^i z_t^j \geq 0, \quad \text{a.e. } t \in [0, T], \quad i, j = 1, 2, \dots, n,$$

which implies

$$\left| \sum_{i=1}^n z_t^i \right| = \sum_{i=1}^n |z_t^i| \quad \text{a.e. } t \in [0, T]. \quad (24)$$

Multiplying by  $b_i$  both sides of BSDE (23), adding from  $i = 1$  to  $n$ , and applying (24), we obtain

$$\sum_{i=1}^n b_i y_t^i = \sum_{i=1}^n b_i 1_{A_i} + \int_t^T a(s) \left| \sum_{i=1}^n b_i z_s^i \right| ds - \int_t^T \sum_{i=1}^n b_i z_s^i dW_s,$$

which means that  $(\sum_{i=1}^n b_i y_t^i, \sum_{i=1}^n b_i z_t^i)$  is the solution of the BSDE:

$$y_t = \sum_{i=1}^n b_i 1_{A_i} + \int_t^T a(s) |z_s| ds - \int_t^T z_s dW_s.$$

By the uniqueness of the solution of the BSDE,

$$\mathcal{E}_a[\sum_{i=1}^n b_i 1_{A_i} | \mathcal{F}_t] = \sum_{i=1}^n b_i y_t^i = \sum_{i=1}^n b_i \mathcal{E}_a[1_{A_i} | \mathcal{F}_t], \quad t \in [0, T].$$

In particular, taking  $t = 0$ , we obtain (22)  $\square$

Using the lemmas above, we can show our main theorem in this section.

**Theorem 3** *Assume  $g$  is of the form (21), then, for any  $\xi \in H$ ,  $\mathcal{E}_a[\xi]$  can be represented as a Choquet integral, i.e.  $\mathcal{E}_a[\xi]$  can be represented in the form (\*).*

**Proof.** The proof is divided into two steps:

**Step 1:** Assume that  $\xi$  is bounded by  $N$ , i.e.  $|\xi| \leq N$ .

First, assume  $\xi \geq 0$ .

Set

$$\xi_-^{(n)} := \sum_{i=1}^{2^n} \frac{iN}{2^n} 1_{(\frac{iN}{2^n} \leq \xi < \frac{(i+1)N}{2^n})}; \quad \xi_+^{(n)} := \sum_{i=1}^{2^n} \frac{(i+1)N}{2^n} 1_{(\frac{iN}{2^n} \leq \xi < \frac{(i+1)N}{2^n})},$$

then  $\xi_-^{(n)} \leq \xi \leq \xi_+^{(n)}$ ,  $\xi_-^{(n)} \rightarrow \xi$ ,  $\xi_+^{(n)} \rightarrow \xi$  as  $n \rightarrow \infty$  in  $L^2(\Omega, \mathcal{F}_T, P)$  and

$$\lim_{n \rightarrow \infty} \mathcal{E}_a[\xi_-^{(n)}] = \lim_{n \rightarrow \infty} \mathcal{E}_a[\xi_+^{(n)}] = \mathcal{E}_a[\xi].$$

Note that it is easy to rewrite  $\xi_-^{(n)}$  and  $\xi_+^{(n)}$  as

$$\xi_-^{(n)} = \sum_{i=1}^{2^n} \frac{N}{2^n} 1_{(\xi \geq \frac{iN}{2^n})}, \quad \xi_+^{(n)} = \sum_{i=1}^{2^n} \frac{N}{2^n} 1_{(\xi \geq \frac{(i-1)N}{2^n})}.$$

Applying (22), remembering the definition of  $P_a(A) = \mathcal{E}_a[1_A]$ , we have

$$\mathcal{E}_a[\xi_-^{(n)}] = \sum_{i=1}^{2^n} \frac{N}{2^n} P_a(\xi \geq \frac{iN}{2^n}), \quad \mathcal{E}_a[\xi_+^{(n)}] = \sum_{i=1}^{2^n} \frac{N}{2^n} P_a(\xi \geq \frac{(i-1)N}{2^n}).$$

But

$$\sum_{i=1}^{2^n} \frac{N}{2^n} P_a(\xi \geq \frac{iN}{2^n}) \leq \int_0^N P_a(\xi \geq t) dt \leq \sum_{i=1}^{2^n} \frac{N}{2^n} P_a(\xi \geq \frac{(i-1)N}{2^n}).$$

Hence,

$$\mathcal{E}_a[\xi_-^{(n)}] \leq \int_0^N P_a(\xi \geq t) dt \leq \mathcal{E}_a[\xi_+^{(n)}].$$

Let  $n \rightarrow \infty$ ,

$$\mathcal{E}_a[\xi] = \int_0^N P_a(\xi \geq t) dt. \quad (25)$$

Thus  $\mathcal{E}_a[\xi]$  can be represented by the Choquet integral.

Second, if  $\xi$  is not positive, let  $\bar{\xi} = \xi + N$ , then  $0 \leq \bar{\xi} \leq 2N$ , Applying (25), we have

$$\mathcal{E}_a[\bar{\xi}] = \int_0^{2N} P_a(\bar{\xi} \geq t) dt = \int_{-N}^N P_a(\xi \geq t) dt.$$

By Property 1(3), we have

$$\mathcal{E}_a[\bar{\xi}] = \mathcal{E}_a[\xi + N] = \mathcal{E}_a[\xi] + N.$$

Consequently,

$$\mathcal{E}_a[\xi] = \mathcal{E}_a[\bar{\xi}] - N = \int_{-N}^0 [P_a(\xi \geq t) - 1] dt + \int_0^N P_a(\xi \geq t) dt. \quad (26)$$

**Step 2:** Let  $\Phi(x) = x \vee (-N) \wedge N$ , then  $\Phi$  is monotonic, and for any  $\xi \in H$ ,  $\xi^N := \Phi(\xi)$  is bounded by  $N$ . By Step 1,  $\mathcal{E}_a[\xi^N]$  is of the form (26). Let  $N \rightarrow \infty$  and note that  $\lim_{N \rightarrow \infty} \mathcal{E}_a[\xi^N] = \mathcal{E}_a[\xi]$ , we obtain

$$\mathcal{E}_a[\xi] = \int_{-\infty}^0 (P_a(\xi \geq t) - 1) dt + \int_0^{\infty} P_a(\xi \geq t) dt.$$

□

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